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The Generalised Discrete Algebraic Riccati Equation in Linear-Quadratic optimal control [★]

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Abstract

This paper investigates the properties of the solutions of the generalised discrete algebraic Riccati equation arising from the classic infinite-horizon linear quadratic (LQ) control problem. In particular, a geometric analysis is used to study the relationship existing between the solutions of the generalised Riccati equation and the output-nulling subspaces of the underlying system and the corresponding reachability subspaces. This analysis reveals the presence of a subspace that plays an important role in the solution of the related optimal control problem, which is reflected in the generalised eigenstructure of the corresponding extended symplectic pencil. In establishing the main results of this paper, several ancillary problems on the discrete Lyapunov equation and spectral factorisation are also addressed and solved.

Key words: Generalised discrete algebraic Riccati equation, LQ optimal control, output-nulling subspaces, reachability subspaces.

1 Introduction

Due to their ubiquitousness in optimal control and filtering problems, as well as in linear factorisation and stochastic realisation problems, Riccati equations are universally regarded as a cornerstone of modern control theory. Several monographs have been entirely devoted to providing a general and systematic framework for the study of Riccati equations, see e.g. [8,7,1].

The classic solution of the discrete-time infinite-horizon LQ problem is traditionally expressed in terms of the solution X of the Riccati equation

$$X = A^T X A - (A^T X B + S)(R + B^T X B)^{-1} (B^T X A + S^T) + Q, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$ and $R \in$

$\mathbb{R}^{m \times m}$ are such that the *Popov matrix* Π satisfies

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \Pi^T \geq 0. \quad (2)$$

The set of matrices $\Sigma = (A, B; Q, R, S)$ is often referred to as *Popov triple*, see e.g. [7]. Equation (1) is the so-called Discrete Riccati Algebraic Equation DARE(Σ).

Nevertheless, an LQ problem may have solutions even if DARE(Σ) has no solutions, and the optimal control can be written in this case as a state feedback given in terms of a matrix X such that $R + B^T X B$ is singular and satisfies the more general Riccati equation

$$X = A^T X A - (A^T X B + S)(R + B^T X B)^\dagger (B^T X A + S^T) + Q, \quad (3)$$

$$\ker(R + B^T X B) \subseteq \ker(A^T X B + S), \quad (4)$$

where the matrix inverse in DARE(Σ) has been replaced by the Moore-Penrose pseudo-inverse, see [10]. Eq. (3) is known as the *generalised discrete-time algebraic Riccati equation* GDARE(Σ). The GDARE(Σ) with the additional constraint (4) is sometimes referred to as *constrained generalised discrete-time algebraic Riccati equation*, herein denoted by CGDARE(Σ). It is obvious that (3) is a generalisation of the classic DARE(Σ), in the sense that any solution of DARE(Σ) is also a solution of GDARE(Σ) – and therefore

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also of CGDARE(Σ) because $\ker(R+B^T X B) = 0_m$ – but the *vice-versa* is not true in general. Despite its generality, this type of Riccati equation has only been marginally studied in the monographs [11,7,1] and in the paper [3]. The only contributions entirely devoted to the study of the solutions of this equation are [6] and [12]. The former investigates conditions under which the GDARE(Σ) admits a stabilising solution in terms of the deflating subspaces of the extended symplectic pencil. The latter studies the connection between the solutions of this equation and the rank-minimising solutions of the so-called Riccati linear matrix inequality. In pursuing this task, the authors of [12] derived a series of results that shed some light into the fundamental role played by the term $R_X \stackrel{\text{def}}{=} R+B^T X B$. An example is the important observation according to which the inertia of this matrix R_X is independent of the solution X of CGDARE(Σ), [12, Theorem 2.4]. Hence, **i**) if X is a solution of DARE(Σ), then all solutions of CGDARE(Σ) will also satisfy DARE(Σ) and, **ii**) if X is a solution of CGDARE(Σ) such that R_X is singular, then DARE(Σ) does not admit solutions. The results presented in [12] are established in the very general setting in which the Popov matrix Π is not necessarily positive semidefinite as in (2).

It is often taken for granted that GDARE(Σ) generalises the standard DARE(Σ) in the solution of the infinite LQ optimal control problem in the same way in which [10] established that the generalised Riccati difference equation generalises the standard Riccati difference equation in the solution of the finite-horizon LQ problem. However, to the best of the authors' knowledge, this fact has never been presented in a direct, self-contained and rigorous way. Thus, the first aim of this paper is to show the connection of the CGDARE(Σ) and the solution of the standard infinite-horizon LQ optimal control problem. The second aim of this paper is to provide a geometric picture describing the structure of the solutions of the CGDARE(Σ) in terms of the output nulling subspaces and the corresponding reachability subspaces of Σ . Indeed, when $\Pi \geq 0$, the null-space of R_X is independent of the solution X of CGDARE(Σ), and is linked to the presence of a subspace which plays an important role in the characterisation of the solutions of CGDARE(Σ) and in the solution of the related optimal control problem. This subspace does not depend on the particular solution X , nor does the closed-loop matrix restricted to this subspace. This new geometric analysis reveals that the spectrum of the closed-loop system is divided into a part that depends on the solution X of CGDARE(Σ), and one – coinciding with the eigenvalues of the closed-loop restricted to this subspace – which is independent of it. At first sight, this fact seems to constitute a limitation in the design of the optimal feedback, because regardless of the solution of the generalised Riccati equation chosen for the implementation of the optimal feedback, the closed-loop matrix will always present a certain fixed eigenstructure as part of its spectrum. However, when R_X is singular, the set of optimal controls presents a further degree of freedom – which is also identified in [11, Remark 4.2.3] – that allows to place all the closed-loop poles at the

desired locations without changing the cost.

2 Linear Quadratic optimal control and CGDARE

In this section we analyse the connections between LQ optimal control and CGDARE. Most of the results presented here are considered “common wisdom”. However, we have not been able to find a place where they have been explicitly derived, so we believe that this section may be useful. Consider the discrete linear time-invariant system governed by

$$x_{t+1} = Ax_t + Bu_t, \quad (5)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, and let the initial state $x_0 \in \mathbb{R}^n$ be given. The problem is to find a sequence of inputs u_t , with $t = 0, 1, \dots, \infty$, minimising the cost function

$$J(x_0, u) \stackrel{\text{def}}{=} \sum_{t=0}^{\infty} \begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}. \quad (6)$$

Before we introduce the solution of the optimal control problem, we recall some well-known classic linear algebra results which will be useful in the sequel, see e.g. [3,5].

Lemma 2.1 *Consider the symmetric positive semidefinite matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$. Then,*

- (i)** $\ker P_{12} \supseteq \ker P_{22}$, **(iii)** $P_{12}(I - P_{22}^\dagger P_{22}) = 0$,
- (ii)** $P_{12} P_{22}^\dagger P_{22} = P_{12}$, **(iv)** $P_{11} - P_{12} P_{22}^\dagger P_{12}^T \geq 0$.

Given $X = X^T \in \mathbb{R}^{n \times n}$ we define

$$Q_X \stackrel{\text{def}}{=} Q + A^T X A - X, \quad S_X \stackrel{\text{def}}{=} A^T X B + S, \quad R_X \stackrel{\text{def}}{=} R + B^T X B, \quad (7)$$

$$G_X \stackrel{\text{def}}{=} I_m - (R + B^T X B)^\dagger (R + B^T X B) = I_m - R_X^\dagger R_X, \quad (8)$$

$$K_X \stackrel{\text{def}}{=} (R + B^T X B)^\dagger (B^T X A + S^T) = R_X^\dagger S_X^T, \quad (9)$$

$$A_X \stackrel{\text{def}}{=} A - B(R + B^T X B)^\dagger (B^T X A + S^T) = A - B K_X, \quad (10)$$

$$\Pi_X \stackrel{\text{def}}{=} \begin{bmatrix} Q_X & S_X \\ S_X^T & R_X \end{bmatrix}. \quad (11)$$

The term $R_X^\dagger R_X$ is the orthogonal projector that projects onto $\text{im} R_X^\dagger = \text{im} R_X$ so that G_X is the orthogonal projector that projects onto $\ker R_X$. Hence, $\ker R_X = \text{im} G_X$. When X is a solution of CGDARE(Σ), then K_X is the corresponding gain matrix, A_X the associated closed-loop matrix, and Π_X is the so-called *dissipation matrix*. All symmetric and positive semidefinite solutions of GDARE(Σ) satisfy (4), and are therefore solutions of CGDARE(Σ). In fact, if $X \geq 0$, we find $\begin{bmatrix} Q_X + X & S_X \\ S_X^T & R_X \end{bmatrix} = \begin{bmatrix} A^T \\ B^T \end{bmatrix} X \begin{bmatrix} A & B \end{bmatrix} + \Pi \geq 0$. Therefore, applying Lemma 2.1 we find (4), that can be rewritten as $\ker R_X \subseteq \ker S_X$ and also as $S_X G_X = 0$.

Theorem 2.1 Suppose that for every x_0 there exists an input $u_t \in \mathbb{R}^m$, with $t \in \mathbb{N}$, such that $J(x_0, u)$ is finite. Then we have:

- (1) CGDARE(Σ) admits symmetric solutions: a solution $\bar{X} = \bar{X}^T \geq 0$ may be obtained as the limit of the sequence of matrices generated by iterating the generalised Riccati difference equation (see (14)) with zero initial condition.
- (2) The value of the optimal cost is $x_0^T \bar{X} x_0$.
- (3) \bar{X} is the minimum positive semidefinite solution of CGDARE(Σ).
- (4) The set of all optimal controls minimising (6) is

$$u_t = -K_{\bar{X}} x_t + G_{\bar{X}} v_t, \quad v_t \text{ arbitrary.} \quad (12)$$

Proof: (1). Consider the finite horizon LQ problem consisting in the minimisation of the performance index

$$J_T \stackrel{\text{def}}{=} \sum_{t=0}^{T-1} \begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \Pi \begin{bmatrix} x_t \\ u_t \end{bmatrix} \quad (13)$$

subject to (5) with assigned initial state $x_0 \in \mathbb{R}^n$. The optimal control is obtained by iterating, backward in time starting from the terminal condition $P_T(T) = 0$, the generalised Riccati difference equation $P_T(t) = \mathbf{R}[P_T(t+1)]$, [10], where $\mathbf{R}[\cdot]$ is the Riccati operator defined as

$$\mathbf{R}[P] \stackrel{\text{def}}{=} A^T P A - (A^T P B + S)(R + B^T P B)^{\dagger} (B^T P A + S^T) + Q \quad (14)$$

and the optimal value of the cost is $J_T^*(x_0) = x_0^T P_T(0) x_0$. Consider the ‘‘reverse time’’ sequence of matrices defined as $X_t \stackrel{\text{def}}{=} P_t(0)$. Since $P_t(t) = P_{\tau-t}(0)$ for all $t \leq \tau$, the sequence $\{X_t\}_{t \in \mathbb{N}}$ is obtained by iterating the generalised Riccati difference equation forward with initial condition $X_0 = 0$. The sequence $\{J_t^*(x_0) \stackrel{\text{def}}{=} x_0^T X_t x_0\}_{t \in \mathbb{N}}$ is obviously monotonically non-decreasing (it is the sequence of optimal costs over intervals of increasing lengths t). Hence, $\{X_t\}_{t \in \mathbb{N}}$, and $\{R + B^T X_t B\}_{t \in \mathbb{N}}$ are monotonically non-decreasing sequences of positive semidefinite matrices. We now show that these sequences are bounded. Assume, by contradiction, that $\lim_{t \rightarrow +\infty} \|X_t\| = +\infty$. The sequence $\{X_t^1 = \frac{X_t}{\|X_t\|}\}_{t \in \mathbb{N}}$ is bounded. Thus, there exists a converging sub-sequence $\{X_{t_i}^1\}$. Let \bar{X}^1 be its limit. Clearly $\|\bar{X}^1\| = 1$: let $x_0^1 \in \mathbb{R}^n$ be such that $\|x_0^1\| = 1$ and $(x_0^1)^T \bar{X}^1 x_0^1 = 1$. Since we assumed that for any x_0 there exists a trajectory that renders J in (6) finite, there exist a constant m_0 and an input trajectory u^1 such that $J_{t_i}^*(x_0^1) \leq J(x_0^1, u^1) \leq m_0$, where the first inequality follows from the optimality of the cost $J_{t_i}^*(x_0^1)$ and the fact that, for a given u^1 , the index (6) is a sum of infinite non-negative terms which is greater than or equal to the sum of the first t_i terms of the sum. On the other hand we have $J_{t_i}^*(x_0^1) = \|X_{t_i}\| (x_0^1)^T X_{t_i}^1 x_0^1 \rightarrow +\infty$, which is a contradiction. Since $\{X_t\}_{t \in \mathbb{N}}$ is non-decreasing and bounded, it admits a limit \bar{X} for $t \rightarrow \infty$. Then, $\lim_{t \rightarrow \infty} X_t = \lim_{t \rightarrow \infty} X_{t+1} = \lim_{t \rightarrow \infty} \mathbf{R}[X_t] = \bar{X}$. Thus, if $\lim_{t \rightarrow \infty} \mathbf{R}[X_t] = \mathbf{R}[\bar{X}]$, then $\mathbf{R}[\bar{X}] = \bar{X}$, i.e. \bar{X} is a positive semidefinite solution of CGDARE(Σ). To prove that this is indeed the case, it is

sufficient to show that $\lim_{t \rightarrow \infty} R_{X_t}^{\dagger} = R_{\bar{X}}^{\dagger}$. In fact, the pseudo-inverse is the only possible source of discontinuity in the Riccati iteration. To prove the latter equality, consider the sequence $\{R + B^T X_t B\}_{t \in \mathbb{N}}$. Since it is a monotonically non-decreasing sequence of positive semidefinite matrices, the chain of inclusions

$$\ker(R + B^T X_0 B) \supseteq \ker(R + B^T X_1 B) \supseteq \ker(R + B^T X_2 B) \supseteq \dots$$

holds. Clearly, there exist a \bar{t} such that for any $t \geq \bar{t}$ this chain becomes stationary, i.e., for any $t \geq \bar{t}$ there holds $\ker(R + B^T X_t B) = \ker(R + B^T X_{t+1} B)$. This implies that a change of coordinates independent of t exists such that in the new basis $R_{X_t} = R + B^T X_t B = \text{diag}\{R_t^0, O\}$, where $\{R_t^0\}_{t \geq \bar{t}}$ is a non-decreasing sequence of positive definite matrices. Clearly, $\lim_{t \rightarrow \infty} R_{X_t} = R_{\bar{X}}$, so that, in this basis, $R_{\bar{X}}$ has the form $R_{\bar{X}} = R + B^T \bar{X} B = \text{diag}\{R^0, O\}$, where $R^0 \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} R_t^0$. Moreover, since the sequence $\{R_t^0\}$ is non-decreasing, R^0 is also nonsingular, so that $(R_t^0)^{-1} \rightarrow (R^0)^{-1}$. Thus, in the chosen basis we have indeed

$$R_{X_t}^{\dagger} = (R + B^T X_t B)^{\dagger} = \begin{bmatrix} (R_t^0)^{-1} & O \\ O & O \end{bmatrix} \rightarrow \begin{bmatrix} (R^0)^{-1} & O \\ O & O \end{bmatrix} = R_{\bar{X}}^{\dagger}.$$

(2). Let

$$J^{\circ}(x_0) \stackrel{\text{def}}{=} \inf_u J(x_0, u). \quad (15)$$

Clearly, $J^{\circ}(x_0) \geq J_t^*(x_0) = x_0^T X_t x_0$. Then, by taking the limit, we get $J^{\circ}(x_0) \geq x_0^T \bar{X} x_0$. We now show that the time-invariant feedback control $u_t^* \stackrel{\text{def}}{=} -K_{\bar{X}} x_t$ yields the cost $x_0^T \bar{X} x_0$, which is therefore the optimal value of the cost. Consider the cost index $J_{T, \bar{X}} \stackrel{\text{def}}{=} J_T + x_0^T \bar{X} x_T$, where J_T is defined in (13). It follows from [10, Section II], that an optimal control for this index is given by the time-invariant feedback $u_t^* = -K_{\bar{X}} x_t$ and the optimal cost does not depend on the length T of the time interval and is given by $J_{T, \bar{X}}^* = x_0^T \bar{X} x_0$. For this conclusion we only need the fact that \bar{X} is a positive semidefinite solution of CGDARE(Σ). Now we have

$$\begin{aligned} x_0^T \bar{X} x_0 &\leq J^{\circ}(x_0) \leq J(x_0, u^*) = \sum_{t=0}^{\infty} \begin{bmatrix} x_t^T & (u_t^*)^T \end{bmatrix} \Pi \begin{bmatrix} x_t \\ u_t^* \end{bmatrix} \\ &= \lim_{T \rightarrow \infty} \sum_{t=0}^T \begin{bmatrix} x_t^T & (u_t^*)^T \end{bmatrix} \Pi \begin{bmatrix} x_t \\ u_t^* \end{bmatrix} \\ &= \lim_{T \rightarrow \infty} J_{T, \bar{X}}^* - x_T^T \bar{X} x_T \leq \lim_{T \rightarrow \infty} x_0^T \bar{X} x_0 = x_0^T \bar{X} x_0. \end{aligned} \quad (16)$$

Comparing the first and last term of the latter expression we see that all the inequalities are indeed equalities, so that the infimum in (15) is a minimum and its value is indeed $x_0^T \bar{X} x_0$.

(3). Suppose by contradiction that there exist another positive semidefinite solution \bar{X} of CGDARE(Σ) and a vector $x_0 \in \mathbb{R}^n$ such that $x_0^T \bar{X} x_0 < x_0^T \bar{X} x_0$. Take the time-invariant feedback $\tilde{u}_t = -K_{\bar{X}} x_t$. The same argument that led to (16) now gives $J(x_0, \tilde{u}) \leq x_0^T \bar{X} x_0 < x_0^T \bar{X} x_0$, which is a contradiction because we have shown that $x_0^T \bar{X} x_0$ is the optimal value of the cost function J .

(4). Let \mathcal{U}_0 be the set of optimal control inputs at time $t = 0$. Let $u_0 \in \mathbb{R}^m$ and $x_1 = Ax_0 + Bu_0$ be the corresponding state at $t = 1$. Clearly the optimal cost can be written as

$$J^* = x_0^T \bar{X} x_0 = \begin{bmatrix} x_0^T & u_0^T \end{bmatrix} \begin{bmatrix} \bar{X} & O \\ O & O \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.$$

Moreover, $u_0 \in \mathcal{U}_0$ if and only if the optimal cost can be written in the following alternative form:

$$\begin{aligned} J^* &= x_1^T \bar{X} x_1 + \begin{bmatrix} x_0^T & u_0^T \end{bmatrix} \Pi \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \\ &= \begin{bmatrix} x_0^T & u_0^T \end{bmatrix} \left(\begin{bmatrix} A^T \\ B^T \end{bmatrix} \bar{X} \begin{bmatrix} A & B \end{bmatrix} + \Pi \right) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}. \end{aligned}$$

By subtracting the first expression from the second, we get that $u_0 \in \mathcal{U}_0$ if and only if $\begin{bmatrix} x_0^T & u_0^T \end{bmatrix} \begin{bmatrix} Q_{\bar{X}} & S_{\bar{X}} \\ S_{\bar{X}}^T & R_{\bar{X}} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$. Since

\bar{X} , and hence $\Pi_{\bar{X}} = \begin{bmatrix} Q_{\bar{X}} & S_{\bar{X}} \\ S_{\bar{X}}^T & R_{\bar{X}} \end{bmatrix}$, are positive semidefinite, this is equivalent to $\begin{bmatrix} Q_{\bar{X}} & S_{\bar{X}} \\ S_{\bar{X}}^T & R_{\bar{X}} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$. Finally, this is equivalent to $u_0 = -R_{\bar{X}}^\dagger S_{\bar{X}}^T x_0 + G_{\bar{X}} v_0$, where $v_0 \in \mathbb{R}^m$ is arbitrary, because the columns of $G_{\bar{X}}$ form a basis for $\ker R_{\bar{X}}$. By iterating this argument for all $t = 1, 2, \dots$, we get (12). ■

3 Preliminary technical results

In this section, we present several technical results of independent interest that will be used in the sequel.

3.1 The Hermitian Stein equation

In this section, we give some important results on the solutions X of the so-called Hermitian Stein equation (known also as the discrete-time Lyapunov equation):

$$X = A^T X A + Q, \quad (17)$$

where $A, Q \in \mathbb{R}^{n \times n}$ and $Q = Q^T \geq 0$.

Lemma 3.1 *Let X be a solution of the Hermitian Stein equation (17). Then, $\ker X$ is A -invariant and is contained in the null-space of Q .*

Proof: Let $\lambda \in \mathbb{C}$ be on the unit circle and such that $(A + \lambda I_n)$ is invertible. We can re-write (17) as $X = A^T X (A + \lambda I_n) - \lambda A^T X + Q$, so that $(\lambda A^T + I_n)X = \lambda A^T X (\lambda^* A + I_n) + Q$, since λ is on the unit circle (which implies that $\lambda^* = \lambda^{-1}$). This is equivalent to

$$\begin{aligned} X(\lambda^* A + I_n)^{-1} \\ = \lambda (\lambda A^T + I_n)^{-1} A^T X + (\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1}. \end{aligned} \quad (18)$$

Let $\xi \in \ker X$. On pre-multiplying (18) by ξ^* and post-multiplying it by ξ , we obtain $\xi^* (\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1} \xi = 0$, and since $(\lambda A^T + I_n)^{-1} Q (\lambda^* A + I_n)^{-1}$ is Hermitian and positive semidefinite, we get

$$Q(\lambda^* A + I_n)^{-1} \xi = 0. \quad (19)$$

By post-multiplying (18) by ξ , we get $X(\lambda^* A + I_n)^{-1} \xi = 0$, which means that $\ker X$ is $(\lambda^* A + I_n)^{-1}$ -invariant. Hence, it is also $(\lambda^* A + I_n)$ -invariant and therefore A -invariant. In view of (19), $\ker X = (\lambda^* A + I_n)^{-1} \ker X$ is also contained in $\ker Q$. ■

We recall that equation (17) has a unique solution if and only if A is *unmixed*, i.e. for all pairs $\lambda_1, \lambda_2 \in \sigma(A)$ we have $\lambda_1 \lambda_2 \neq 1$. In this case, we have (see [14]):

Lemma 3.2 (Lemma 5.1 in [14]) . *Let A be unmixed and X be the unique solution of (17) where $Q = Q^T \geq 0$. Then, $\ker X$ is the unobservable subspace of the pair (A, Q) .*

Lemma 3.3 *Let $A \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and assume $X \in \mathbb{R}^{n \times n}$ is such that*

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} X F = \begin{bmatrix} X \\ O \end{bmatrix}. \quad (20)$$

Then, $B^T (A^T)^k X = 0$ for all $k \geq 0$, i.e., $\text{im } X$ is contained in the unobservable subspace of the pair (A^T, B^T) .

Proof: We first prove that $B^T X = 0$. Let us choose a basis in which $F = \text{diag}\{N, F_I\}$, where N is nilpotent and F_I is invertible. Let us decompose X accordingly, i.e., $X = [X_1 \ X_2]$. Then, $A^T X_1 N = X_1$ implies $X_1 = 0$. In fact, multiplying such equation by A^T and N to the left and to the right, respectively, we obtain $X_1 = (A^T)^k X_1 N^k$ for all $k \geq 0$. Choosing k to be greater than the nilpotency index of N , we get $X_1 = (A^T)^k X_1 N^k = 0$. From (20) we also obtain $B^T X_2 F_I = 0$, which gives $B^T X_2 = 0$ since F_I is invertible. Thus, $B^T X = 0$. The same argument can be iterated to prove that $B^T (A^T)^k X = 0$ for all $k \geq 0$. Indeed, by pre-multiplying the first of (20) by A^T we get $A^T (A^T X) F = A^T X$. By pre-multiplying the same equation by B^T , we get $B^T (A^T X) F = 0$ since $B^T X = 0$. Hence, we can write these two equations as $\begin{bmatrix} A^T \\ B^T \end{bmatrix} (A^T X) F = \begin{bmatrix} A^T X \\ O \end{bmatrix}$ and re-apply the same argument used above to show that $B^T A^T X = 0$, and so on. ■

3.2 Spectral Factorisation

Since $\Pi = \Pi^T \geq 0$, we can consider the factorisation

$$\Pi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}, \quad (21)$$

where $Q = C^T C$, $S = C^T D$ and $R = D^T D$. Let us define the rational matrix $W(z) \stackrel{\text{def}}{=} C(zI_n - A)^{-1} B + D$. The spectrum

(or Popov function) $\Phi(z) \stackrel{\text{def}}{=} W^\sim(z)W(z)$ – where $W^\sim(z) \stackrel{\text{def}}{=} W^T(z^{-1})$ – associated with Σ can be written as

$$\Phi(z) = \begin{bmatrix} B^T(z^{-1}I_n - A^T)^{-1} & I_n \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (zI_n - A)^{-1}B \\ I_n \end{bmatrix}.$$

The matrix inequality for an unknown matrix $X = X^T$ of the form $\Pi_X \geq 0$ is called the *discrete linear matrix inequality*, and is denoted by DLMi(Σ). Let us define

$$L(X) \stackrel{\text{def}}{=} \Pi_X - \Pi = \begin{bmatrix} A^T X A - X & A^T X B \\ B^T X A & B^T X B \end{bmatrix}.$$

Lemma 3.4 ([12, p.322], see e.g. [2] for a detailed proof). *For any $X = X^T \in \mathbb{R}^{n \times n}$, there holds*

$$\Phi(z) = \begin{bmatrix} B^T(z^{-1}I_n - A^T)^{-1} & I_n \end{bmatrix} \Pi_X \begin{bmatrix} (zI_n - A)^{-1}B \\ I_n \end{bmatrix}. \quad (22)$$

Theorem 3.1 *Let r denote the normal rank of the spectrum $\Phi(z)$.¹ If X is a solution of CGDARE(Σ), the rank of R_X is equal to r . If X is a solution of DRLMI(Σ), the rank of R_X is at most equal to r .*

Proof: Consider $X = X^T$ such that $\Pi_X \geq 0$. By Lemma 2.1, in particular (i) R_X is positive semidefinite, (ii) $\ker S_X \supseteq \ker R_X$, and (iii) $Q_X - S_X R_X^\dagger S_X^T$ is positive semidefinite. Note that (iii) means that X satisfies the Riccati inequality

$$\begin{aligned} \mathcal{D}(X) &\stackrel{\text{def}}{=} A^T X A \\ &- (A^T X B + S)(R + B^T X B)^\dagger (B^T X A + S^T) + Q - X \geq 0. \end{aligned}$$

Therefore, we can write $\mathcal{D}(X) = H_X^T H_X$ for some matrix H_X , which leads to the expression

$$\Pi_X = \begin{bmatrix} S_X R_X^\dagger \\ I \end{bmatrix} R_X \begin{bmatrix} R_X^\dagger S_X^T & I \end{bmatrix} + \begin{bmatrix} H_X^T \\ O \end{bmatrix} \begin{bmatrix} H_X & O \end{bmatrix}. \quad (23)$$

Plugging (23) into (22), $\Phi(z) = W^\sim(z)W(z)$ becomes

$$\Phi(z) = W_1^\sim(z)W_1(z) + W_2^\sim(z)W_2(z) = \begin{bmatrix} W_1^\sim(z) & W_2^\sim(z) \end{bmatrix} \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix}$$

where $W_1(z)$ is given by

$$W_1(z) = R_X^{\frac{1}{2}} \begin{bmatrix} R_X^\dagger S_X^T & I \end{bmatrix} \begin{bmatrix} (zI_n - A)^{-1}B \\ I_n \end{bmatrix},$$

¹ The normal rank of a rational matrix $M(z)$ is defined as $\text{normrank } M(z) \stackrel{\text{def}}{=} \max_{z \in \mathbb{C}} \text{rank } M(z)$.

i.e., $W_1(z) = R_X^{\frac{1}{2}}(R_X^\dagger S_X^T(zI_n - A)^{-1}B + I_m)$, and $W_2(z)$ is

$$W_2(z) = \begin{bmatrix} H_X & O \end{bmatrix} \begin{bmatrix} (zI_n - A)^{-1}B \\ I_n \end{bmatrix} = H_X(zI_n - A)^{-1}B.$$

Notice that $W_1(z) = R_X^{\frac{1}{2}}T_X(z)$, where $T_X(z) \stackrel{\text{def}}{=} R_X^\dagger S_X^T(zI_n - A)^{-1}B + I_m$ is square and invertible for all but finitely many $z \in \mathbb{C}$. Its inverse can be written as $T_X^{-1}(z) = I_m - R_X^\dagger S_X^T(zI_n - A_X)^{-1}B$. Thus, the normal rank of

$$T_X^{-\sim}(z)\Phi(z)T_X^{-1}(z) = \begin{bmatrix} R_X^{\frac{1}{2}} & T_X^{-\sim}(z)W_2^\sim(z) \end{bmatrix} \begin{bmatrix} R_X^{\frac{1}{2}} \\ W_2(z)T_X^{-1}(z) \end{bmatrix}$$

is equal to the normal rank r of $\Phi(z)$. Then, the rank of $R_X^{\frac{1}{2}}$, which equals that of R_X , is not greater than r . Now consider the case where $X = X^T$ is a solution of CGDARE(Σ). In this case, the term H_X in (23) is zero, and therefore so is the rational function $W_2(z)$. As such, $W_1(z)$ is a square spectral factor of $\Phi(z)$, i.e., $W^\sim(z)W(z) = W_1^\sim(z)W_1(z)$. Moreover, $T_X^{-\sim}(z)\Phi(z)T_X^{-1}(z) = R_X$, which implies that when $X = X^T$ is a solution of CGDARE(Σ), the rank of R_X is exactly r . ■

Remark 3.1 Theorem 3.1 is strictly related to Theorem 2.4 in [12]. The latter has been derived in the very general setting in which the Popov matrix may not be positive semidefinite. In that case, $\text{rank } R_X = r$ for any solution X of CGDARE. Since we are assuming (2), a stronger result holds. Namely, $\text{rank } R_X \leq r$ for any solution X of the linear matrix inequality.

4 Geometric properties of the solutions of GDARE

Now we show that, given a solution X of GDARE(Σ)

- (a) $\ker X$ is an output-nulling subspace for the quadruple (A, B, C, D) , i.e., $\begin{bmatrix} A \\ C \end{bmatrix} \ker X \subseteq (\ker X \oplus 0_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}$;
- (b) the gain K_X is such that $-K_X$ is a *friend* of $\ker X$, i.e., $\begin{bmatrix} A - BK_X \\ C - DK_X \end{bmatrix} \ker X \subseteq \ker X \oplus 0_p$.

In the case where $X = X^T$ is the solution of GDARE(Σ) corresponding to the optimal cost, these properties are intuitive. Now we prove that the following stronger result holds.

Proposition 4.1 *Let X be the minimal positive semidefinite solution of GDARE(Σ). Then $\ker X$ is the largest output-nulling subspace of the quadruple (A, B, C, D) . Moreover, $-K_X$ is the corresponding friend.*

Proof: Let $x_0 \in \ker X$. Since the corresponding optimal cost is $J = x_0^T X x_0 = 0$, the initial state x_0 must belong to the largest output-nulling subspace of the quadruple (A, B, C, D) . Vice-versa, if we take x_0 on the largest output-nulling subspace \mathcal{V}^* of (A, B, C, D) , we can find u_k ($k \geq 0$) such that

the state lies on \mathcal{V}^* by maintaining the output at zero, so the corresponding value of the cost is zero. Hence, $x_0^\top X x_0 = 0$ implies $x_0 \in \ker X$. The fact that $-K_X$ is a friend of $\ker X$ follows from the fact that if $x_0 \in \ker X$ and we assume by contradiction that $(A - BK_X)x_0 \notin \ker X$, the corresponding trajectory is not optimal because it is associated with a strictly positive cost. Moreover, since the optimal cost is zero, we must have $(C - DK_X)\ker X = 0_p$. ■

Our aim is to prove that (a) and (b) hold for any symmetric solution X of GDARE(Σ).

Theorem 4.1 *Let X be a solution of GDARE(Σ). Then, $\ker X$ is an output-nulling subspace of the quadruple (A, B, C, D) and $-K_X$ is a friend of $\ker X$.*

Proof: Since X is a solution of GDARE(Σ), the identity

$$X = A_X^\top X A_X + Q_{0X} \quad (24)$$

holds, where $Q_{0X} \stackrel{\text{def}}{=} \begin{bmatrix} I_n & -S_X R_X^\dagger \\ S_X^\top & R \end{bmatrix} \begin{bmatrix} I_n & S \\ -R_X^\dagger S_X^\top \end{bmatrix} \geq 0$. From Lemma 3.1, $\ker X$ is A_X -invariant and is contained in $\ker Q_{0X}$. By factorising Π as in (21), we get $Q_{0X} = C_X^\top C_X$ where

$$C_X \stackrel{\text{def}}{=} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} I_n \\ -R_X^\dagger S_X^\top \end{bmatrix} = C - DR_X^\dagger S_X^\top. \quad (25)$$

Hence, $\ker X$ is also contained in $\ker C_X$ so that $\ker X$ is output-nulling and $-K_X$ is a friend of $\ker X$. ■

Now we provide a characterisation of the reachable subspace on $\ker X$. We show that this subspace plays a crucial role in the solution of the associated optimal control problem. We recall that the reachable subspace \mathcal{R}_Y^* on an output-nulling subspace \mathcal{V} is the subspace of points of \mathcal{V} that can be reached from the origin along trajectories contained on \mathcal{V} by at the same time maintaining the output at zero.

We will show that the reachable subspace $\mathcal{R}_{\ker X}^*$ on $\ker X$, coincides with the reachable subspace of the pair $(A_X, B G_X)$. To prove this fact, we first need to give some additional results on the solutions of CGDARE(Σ). In particular, we now focus our attention on the term R_X . Clearly, when X is positive semidefinite, the null-space of R_X is the intersection of the null-space of R with that of $X B$. This result, which is intuitive and easy to prove for positive semidefinite solutions of CGDARE(Σ), holds for any solution. However, in this case the proof – which is divided between Lemma 4.1 and Lemma 4.2 presented below – is more involved, and requires the machinery constructed in the first part of the paper.

Lemma 4.1 *Let $X = X^\top$ be a solution of CGDARE(Σ), C_X be defined by (25) and*

$$\mathcal{R}_0 \stackrel{\text{def}}{=} \text{im} \begin{bmatrix} B G_X & A_X B G_X & A_X^2 B G_X & \dots & A_X^{n-1} B G_X \end{bmatrix}. \quad (26)$$

Then, $\ker R_X \subseteq \ker R$, and $\mathcal{R}_0 \subseteq \ker C_X$. (27)

Proof: Since the columns of G_X span $\ker R_X$, we need to show that $R G_X = 0$. Recall from the proof of Theorem 3.1 that when $X = X^\top$ is a solution of CGDARE(Σ), $\Phi(z)$ can be written as $\Phi(z) = W^\sim(z) W(z) = T_X^\sim(z) R_X T_X(z)$ where $T_X(z) = R_X^\dagger S_X^\top (z I_n - A)^{-1} B + I_m$ is square and invertible for all but finitely many $z \in \mathbb{C}$. Hence, we have $R_X = [W(z) T_X^{-1}(z)]^\sim [W(z) T_X^{-1}(z)]$ so that $R_X G_X = 0$ implies $W(z) T_X^{-1}(z) G_X \equiv 0$. Recall from the proof of Theorem 3.1 that $T_X^{-1}(z) = I_m - R_X^\dagger S_X^\top (z I_n - A_X)^{-1} B$, so that

$$W(z) T_X^{-1}(z) = (C(z I_n - A)^{-1} B + D)(I_m - R_X^\dagger S_X^\top (z I_n - A_X)^{-1} B).$$

Consider the following term of the product:

$$H(z) = C(z I_n - A)^{-1} B R_X^\dagger S_X^\top (z I_n - A_X)^{-1} B.$$

By noticing that $B R_X^\dagger S_X^\top = A - A_X = (z I_n - A_X) - (z I_n - A)$, we obtain $H(z) = C(z I_n - A)^{-1} B - C(z I_n - A_X)^{-1} B$. Hence,

$$\begin{aligned} W(z) T_X^{-1}(z) &= W(z) - D R_X^\dagger S_X^\top (z I_n - A_X)^{-1} B - H(z) \\ &= D + C_X (z I_n - A_X)^{-1} B. \end{aligned}$$

Since $W(z) T_X^{-1}(z) G_X$ is identically zero, it must be zero also when $z \rightarrow \infty$. In particular, $D G_X = 0$, so that $R G_X = 0$, which yields the first of (27). From $W(z) T_X^{-1}(z) G_X \equiv 0$ we also get $C_X (z I - A_X)^{-1} B G_X \equiv 0$ so that the reachable subspace of the pair $(A_X, B G_X)$, i.e. (26), is contained in $\ker C_X$ so that also the second of (27) holds. ■

In Lemma 4.1 we have shown that $\ker R_X \subseteq \ker R$. Since $R_X = R + B^\top X B$, it also straightforwardly follows that $\ker R_X \subseteq \ker(B^\top X B)$ for any solution X of CGDARE(Σ). However, a stronger result holds, which says that $\ker R_X \subseteq \ker(X B)$. This is an obvious consequence of Lemma 4.1 for any solution $X \geq 0$, while it is a quite surprising and deep geometric result in the general case.

Lemma 4.2 *Let $X = X^\top$ be a solution of CGDARE(Σ). Then,*

$$\ker R_X \subseteq \ker(X B). \quad (28)$$

Proof: From Lemma 4.1, if $v \in \ker R_X$, then $v \in \ker R \cap \ker(B^\top X B)$. We can select a change of coordinates in the input space \mathbb{R}^m induced by the $m \times m$ orthogonal matrix $T_X = \begin{bmatrix} T_{1X} & T_{2X} \end{bmatrix}$ where $\text{im} T_{1X} = \text{im} R_X$ and $\text{im} T_{2X} = \text{im} G_X = \ker R_X$. In this basis R_X is block-diagonal, with the first block being non-singular and the second being zero. Since $\ker R \supseteq \ker R_X$ as proved in Lemma 4.1, matrix R in this basis has the form $R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$. In the same basis, matrix B can be partitioned accordingly as $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, so that $\text{im} B_2 = \text{im}(B G_X)$. We must show that $X B_2 = 0$. Since $\ker R_X \subseteq$

$\ker(B^T X B)$, in this basis we find

$$B^T X B_2 = \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} X B_2 = 0. \quad (29)$$

Moreover, since $\ker R \subseteq \ker S$, in the selected basis S takes the form $S = \begin{bmatrix} S_1 & 0 \end{bmatrix}$. Thus, $S_X = A^T X B + S = \begin{bmatrix} A^T X B_1 + S_1 & A^T X B_2 \end{bmatrix}$. From $\ker R_X \subseteq \ker S_X$ it now follows that $A^T X B_2 = 0$ which, together with (29), yields

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} X B_2 = 0. \quad (30)$$

If A is non-singular or, more in general, if the zero eigenvalue of A , when present, is controllable from B , then clearly $X B_2 = 0$. However, this result is true in general, without any assumption. To prove this, let us consider \mathcal{R}_0 defined in (26) which, in the chosen input space basis, is the reachable subspace of the pair (A_X, B_2) .² Let us consider a basis of the state-space where the pair (A_X, B_2) are in Kalman controllability form. In such a basis, the subspace \mathcal{R}_0 is spanned by the columns of the matrix $\begin{bmatrix} I \\ 0 \end{bmatrix}$ and we have

$$A_X = \begin{bmatrix} A_{X,11} & A_{X,12} \\ 0 & A_{X,22} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad (31)$$

where $(A_{X,11}, B_{21})$ is reachable. In this basis, $C_X = [O \mid C_{X,1}]$ in view of the second of (27). Since $A_X = A - BK_X$, we can re-write (30) as $\begin{bmatrix} A_X^T + K_X^T B^T \\ B^T \end{bmatrix} X B_2 = 0$ or, equivalently, as $\begin{bmatrix} A_X^T \\ B^T \end{bmatrix} X B_2 = 0$. We can write this equation as

$$\begin{bmatrix} A_{X,11}^T & 0 \\ A_{X,12}^T & A_{X,22}^T \\ B_{11}^T & B_{12}^T \\ B_{21}^T & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} B_{21} \\ 0 \end{bmatrix} = 0. \quad (32)$$

We want to show that $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} B_{21} \\ 0 \end{bmatrix} = 0$, i.e., that $X_{11} B_{21} = 0$ and $X_{12}^T B_{21} = 0$. From (32) we find

$$A_{X,11}^T X_{11} B_{21} = 0, \quad (33)$$

$$B_{21}^T X_{11} B_{21} = 0. \quad (34)$$

Since the pair $(A_{X,11}, B_{21})$ is reachable by construction, $X_{11} B_{21} = 0$. It remains to show that $X_{12}^T B_{21} = 0$. In this basis,

² In the symbol denoting this subspace we dropped the subscript X because, as it will be proved in the sequel, this subspace is independent of the particular solution of the CGDARE(Σ).

equation (24) – which is exactly GDARE(Σ) – now reads as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} = \begin{bmatrix} A_{X,11}^T & 0 \\ A_{X,12}^T & A_{X,22}^T \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{X,11} & A_{X,12} \\ 0 & A_{X,22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_{X,1}^T C_{X,1} \end{bmatrix}$$

from which we find in particular $X_{11} = A_{X,11}^T X_{11} A_{X,11}$. This equation can be written together with (33) and (34) as

$$\begin{bmatrix} A_{X,11}^T X_{11} A_{X,11} - X_{11} & A_{X,11}^T X_{11} B_{21} \\ B_{21}^T X_{11} A_{X,11} & B_{21}^T X_{11} B_{21} \end{bmatrix} = 0.$$

Since the pair $(A_{X,11}, B_{21})$ is reachable, we can apply Lemma 2.9 in [12], which guarantees that X_{11} is zero. Now we can re-write (24) as

$$\begin{bmatrix} 0 & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} = \begin{bmatrix} A_{X,11}^T & 0 \\ A_{X,12}^T & A_{X,22}^T \end{bmatrix} \begin{bmatrix} 0 & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{X,11} & A_{X,12} \\ 0 & A_{X,22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_{X,1}^T C_{X,1} \end{bmatrix}$$

In particular, we get

$$X_{12} = A_{X,11}^T X_{12} A_{X,22}. \quad (35)$$

Plugging $X_{11} = 0$ into (32), we obtain

$$B_{21}^T X_{12} A_{X,22} = 0. \quad (36)$$

Equations (35) and (36) can be re-written as

$$\begin{bmatrix} A_{X,11}^T \\ B_{21}^T \end{bmatrix} X_{12} A_{X,22} = \begin{bmatrix} X_{12} \\ 0 \end{bmatrix}. \quad (37)$$

Applying Lemma 3.3 for $k = 0$, we get $B_{21}^T X_{12} = 0$. ■

Remark 4.1 In the last line of the proof of Lemma 4.2 we can apply Lemma 3.3 for $k \geq 0$, and obtain $X_{12}^T A_{X,11}^k B_{21} = 0$ for all $k \geq 0$. Since the pair $(A_{X,11}, B_{21})$ is reachable, this yields $X_{12} = 0$. Therefore the following stronger result holds.

Proposition 4.2 *Let $X = X^T$ be a solution of CGDARE(Σ) and \mathcal{R}_0 be defined by (26). Then, $X \mathcal{R}_0 = 0_n$.*

Remark 4.2 As an obvious corollary of Lemmas 4.1 and 4.2, we have $\ker R_X = \ker(XB) \cap \ker R = \ker \begin{bmatrix} XB \\ R \end{bmatrix}$.

Remark 4.3 The result established in Lemma 4.2 does not continue to hold if we only assume that $X = X^T$ is a solution of GDARE(Σ). Consider e.g. $A = \begin{bmatrix} -1 & 0 \\ -5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$, $C = [0 \ 1]$ and $D = [4 \ 0]$. It can be easily verified that $X = \text{diag}\{-1, 1\}$ is a solution of the GDARE(Σ) but not of the CGDARE(Σ). Here $\ker R_X = \text{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \ker \begin{bmatrix} XB \\ R \end{bmatrix} = 0_m$.

Remark 4.4 The result established in Lemma 4.2 does not hold when Π is not positive semidefinite. Consider e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. A solution of CGDARE(Σ) is given by $X = \text{diag}\{1, -1\}$. Indeed, X satisfies (4) since $S_X = 0$ and $\ker R_X = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Changing coordinates in the input space as shown in the proof of Lemma 4.2 leads to $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, this time $X B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Theorem 4.2 Let $X = X^T$ be a solution of CGDARE(Σ). Let \mathcal{R}_0 denote the reachable subspace of the pair $(A_X, B G_X)$ as defined in (26), and $\mathcal{R}_{\ker X}^*$ be the largest reachability subspace on the output-nulling subspace $\ker X$. Then,

$$\mathcal{R}_{\ker X}^* = \mathcal{R}_0. \quad (38)$$

Proof: Let us first show that

$$\text{im}(B G_X) = \ker X \cap B \ker D. \quad (39)$$

We recall that $\text{im} G_X = \ker R_X$. From Remark 4.2 we know that $\ker R_X = \ker(XB) \cap \ker R$. Then $\text{im}(B G_X) = B \ker R_X = B(\ker(XB) \cap \ker R) = \ker X \cap B \ker R = \ker X \cap B \ker D$. Subspace \mathcal{R}_0 is by definition the smallest A_X -invariant subspace containing $\text{im}(B G_X) = \ker X \cap B \ker D$. On the other hand, $\mathcal{R}_{\ker X}^*$ is characterised as follows [9, p. 424]: Let F be an arbitrary friend of $\ker X$, i.e., F is any feedback matrix such that $(A + BF) \ker X \subseteq \ker X$ and $(C + DF) \ker X = 0_p$. Then $\mathcal{R}_{\ker X}^*$ is the smallest $(A + BF)$ -invariant subspace containing $\ker X \cap B \ker D$. Note that $\mathcal{R}_{\ker X}^*$ does not depend on the choice of the friend F , [13, Theorem 7.18]. We have seen in Theorem 4.1 that $F = -K_X$ is a particular friend of $\ker X$. For this choice of F , we have $A + BF = A - BK_X = A_X$, so that $\mathcal{R}_{\ker X}^*$ is the smallest A_X -invariant subspace containing $\ker X \cap B \ker D$, which is the definition of \mathcal{R}_0 . ■

In [12] it is proved that the inertia of R_X is independent of the solution $X = X^T$ of CGDARE(Σ). Here, we show that much more is true when $\Pi \geq 0$, i.e., $\ker R_X$ is independent of the solution $X = X^T$ of CGDARE(Σ).

Theorem 4.3 Let X_1, X_2 be two solutions of CGDARE(Σ). Then, $\ker R_{X_1} = \ker R_{X_2}$.

Proof: Consider two solutions $X_1 = X_1^T$ and $X_2 = X_2^T$ of CGDARE(Σ). In particular, X_1 and X_2 also satisfy the generalised Riccati inequality, so that $\Pi_{X_1} \geq 0$ and $\Pi_{X_2} \geq 0$. In other words, with the same notation used in the proof of Theorem 3.1, $\Pi_{X_i} = L(X_i) + \Pi \geq 0$ for $i \in \{1, 2\}$. The set of solutions of the generalised Riccati inequality is a convex set, i.e., by taking $\alpha \in (0, 1)$, then $\alpha \Pi_{X_1} + (1 - \alpha) \Pi_{X_2}$ is positive semidefinite because it is a convex combination of positive semidefinite terms. For our purposes, it is sufficient

to fix an arbitrary value of α , say $\alpha = \frac{1}{2}$. Then,

$$0 \leq \frac{1}{2} (\Pi_{X_1} + \Pi_{X_2}) = \Pi_{\frac{1}{2}(X_1 + X_2)} = \Pi + L \left(\frac{1}{2}(X_1 + X_2) \right),$$

where the last equality holds in view of the linearity of $L(\cdot)$. This means that $Y \stackrel{\text{def}}{=} \frac{1}{2}(X_1 + X_2)$ satisfies the Riccati inequality $\Pi_Y \geq 0$. By virtue of Theorem 3.1, the rank of $R_Y \stackrel{\text{def}}{=} R + B^T Y B$ is not greater than r . On the other hand,

$$R_Y = R + \frac{1}{2} B^T (X_1 + X_2) B = \frac{1}{2} (R_{X_1} + R_{X_2}). \quad (40)$$

Hence, since X_1 and X_2 are both solutions of CGDARE(Σ), the ranks of R_{X_1} and R_{X_2} are exactly equal to r . Thus, the rank of R_Y is greater or equal to r . This means that the rank of R_Y must be exactly equal to r , i.e., from (40) we have that R_{X_1} and R_{X_2} must have the same null-space. ■

Now we want to prove that the subspace $\mathcal{R}_{\ker X}^*$ is independent of the particular solution $X = X^T$ of CGDARE(Σ). Moreover, A_X restricted to this subspace does not depend on the particular solution $X = X^T$ of CGDARE(Σ).

Theorem 4.4 Let X and Y be two solutions of CGDARE(Σ). Let A_X and A_Y be the corresponding closed-loop matrices. Then, $\mathcal{R}_{\ker X}^* = \mathcal{R}_{\ker Y}^*$, and $A_X|_{\mathcal{R}_{\ker X}^*} = A_Y|_{\mathcal{R}_{\ker Y}^*}$.

Proof: Let $\Delta \stackrel{\text{def}}{=} Y - X$. Since $\ker R_X$ coincides with $\ker R_Y$ by virtue of Theorem 4.3, we have $R_X^\dagger = R_Y^\dagger R_Y R_X^\dagger$ so that

$$A_X - A_Y = B (R_Y^\dagger S_Y^T - R_X^\dagger S_X^T) = B R_Y^\dagger (S_Y^T - R_Y R_X^\dagger S_X^T). \quad (41)$$

Plugging $S_Y^T = B^T Y A + S^T = B^T \Delta A + S_X^T$ and $R_Y = R_X + B^T \Delta B$ into (41) yields $A_X - A_Y = B R_Y^\dagger (B^T \Delta A - B^T \Delta B R_X^\dagger S_X^T) = B R_Y^\dagger B^T \Delta A_X$. This means that $A_Y = A_X - B R_Y^\dagger B^T \Delta A_X$. We already know that in a suitable basis of the state space such that the first coordinates span $\mathcal{R}_{\ker X}^*$ and a suitable orthogonal basis of the input space such that the second group of coordinates span $\ker R_X$, matrices A_X and B can be written as in (31), see Lemma 4.2. The reachable subspace of the pair $(A_X, B G_X)$ is written in this basis as $\mathcal{R}_0 = \begin{bmatrix} I \\ 0 \end{bmatrix}$. We want to show that, in the same basis, we also have

$$A_Y = \begin{bmatrix} A_{X,11} & A_{Y,12} \\ 0 & A_{Y,22} \end{bmatrix}, \quad \text{and} \quad B_2 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix}. \quad (42)$$

In fact, if this is the case, the reachable subspace of the pair $(A_Y, B G_Y)$ exactly coincides with the reachable subspace of the pair $(A_X, B G_X)$, i.e., with \mathcal{R}_0 , because the pair $(A_{Y,22}, 0)$ is completely non controllable. In the chosen basis, the difference $\Delta = Y - X$ can be written as $\Delta = \text{diag}\{0, \Delta_2\}$ in view

of [12, Theorem 2.10].³ Thus

$$A_Y = A_X - BR_Y^\dagger B^T \begin{bmatrix} O & O \\ O & \Delta_2 \end{bmatrix} \begin{bmatrix} A_{X,11} & A_{X,12} \\ O & A_{X,22} \end{bmatrix} = \begin{bmatrix} A_{X,11} & \star \\ O & \star \end{bmatrix}.$$

Therefore, the reachable subspace of the pair (A_Y, BG_Y) is exactly $\mathcal{R}_0 = \begin{bmatrix} I \\ O \end{bmatrix}$, and $A_X|_{\mathcal{R}_{\ker X}^*} = A_Y|_{\mathcal{R}_{\ker Y}^*} = A_{X,11}$. ■

5 Stabilisation

In the previous sections, we have observed that the eigenvalues of A_X restricted to \mathcal{R}_0 are independent of the solution $X = X^T$ of CGDARE(Σ). This means that these eigenvalues are present in the closed-loop regardless of the solution $X = X^T$ of CGDARE(Σ) that we consider. On the other hand, \mathcal{R}_0 coincides with the subspace $\mathcal{R}_{\ker X}^*$, which is by definition the smallest $(A - BK_X)$ -invariant subspace containing $\text{im}(BG_X)$. Then, we can always find a matrix L that assigns all the eigenvalues of $(A_X + BG_X L)$ restricted to $\mathcal{R}_{\ker X}^*$, by adding a further term $BG_X Lx_k$ to the feedback control law, because this does not change the value of the cost with respect to the one obtained by $u_k = -K_X x_k$. In doing so, we can stabilise the closed-loop if $\ker X$ is externally stabilised by $-K_X$. We show this fact in the following example.

Example 5.1 Consider a Popov triple in which $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The matrix $X = \text{diag}\{0, 1\}$ is the only solution of GDARE(Σ) but not a solution of DARE(Σ), since $R + B^T X B$ is singular. Hence, DARE(Σ) does not admit solutions. The closed-loop matrix is $A_X = \text{diag}\{1, 0\}$, and the resulting closed-loop system is not asymptotically stable. However, the solution X of GDARE(Σ) is optimal for the LQ problem, because it leads to the cost $J^* = x_2^2(0)$ which cannot be decreased. Now, consider the gain $K = B^{-1}A$, which leads to the closed-loop matrix $A_{CL} = A - BK = 0$, and the value of the performance index associated with this closed-loop is again $J = x_2^2(0) = J^*$. Therefore, this is another optimal solution of the LQ problem, which differently from X is also stabilising. However, this optimal solution is not associated with any solution of GDARE(Σ), since X is the only solution of GDARE(Σ). This example shows that there can be an optimal control which

³ The result in [12, Theorem 2.10] is shown in a basis that is the same considered here. Indeed, the basis of the state space considered in [12, Theorem 2.10] has the first coordinates spanning the largest controllability subspace of the quadruple (A, B, S_X^T, R_X) . However, this subspace coincides with the largest controllability subspace of a quadruple obtained from the previous one by applying the control input $u_t = -K_X x_t + H_X v_t$, where $\text{im} H_X = \ker R_X$. The quadruple thus obtained is exactly $(A_X, BG_X, S_X^T - R_X R_X^\dagger S_X^T, 0) = (A_X, BG_X, 0, 0)$, and the corresponding largest controllability subspace is indeed $\mathcal{R}_{\ker X}^*$.

is stabilising, but no stabilising solutions of GDARE(Σ) exist. This fact can be explained on the basis of the fact that the set of all solutions of the LQ problem is given by $\mathcal{U}_k = \{-K_X x_k + G_X v_k \mid v_k \in \mathbb{R}^m\}$, where X is the optimizing solution of GDARE(Σ) and $G_X = (I_m - R_X^\dagger R_X) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Therefore, the problem becomes that of using the degree of freedom given by v_k to find a closed-loop solution that is optimal and also stabilising, i.e., to determine a matrix L in

$$x_{t+1} = (A - BK_X)x_t + BG_X Lx_t = A_X x_t + BG_X Lx_t$$

such that the closed-loop $A_{CL} = A_X + BG_X L$ is stabilised. It is easy to see that, in general, the set of all optimal closed loop matrices $A_{CL} = A_X + BG_X L$ are parameterised by $A_{CL} = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ where α and β can be arbitrarily chosen by selecting a suitable L . In fact, since $BG_X = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, by choosing $L = \begin{bmatrix} \alpha-1 & \beta \\ 0 & 0 \end{bmatrix}$ we obtain the desired form for the closed-loop matrix. Hence, in particular, we can obtain a zero or nilpotent closed-loop matrix. In both cases, the cost is the same and is equal to $J^* = x_2^2(0)$.

In other words, there is only one solution to GDARE(Σ) and is not stabilising, and all the optimal solutions of the optimal control problem are given by the closed-loop matrix $A_X + BG_X L$, where L is a degree of freedom. By using this degree of freedom, we have found solutions of the optimal control problem that are stabilising but which do not correspond to stabilising solutions of GDARE(Σ), because GDARE(Σ) does not have stabilising solutions. □

Example 5.2 Consider an infinite-horizon LQ problem with $A = \begin{bmatrix} 0 & 0 \\ -7 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 5 & 9 \end{bmatrix}$, $C = [-3 \quad -5]$, and $D = [0 \quad -3]$. Let $Q = C^T C$, $S = C^T D$ and $R = D^T D$. This system has one invariant zero at the origin. Moreover, it is not left-invertible. Thus, DARE(Σ) cannot be solved. However, the 2×2 zero matrix is a solution of CGDARE(Σ). The corresponding closed-loop matrix is $A_X = \begin{bmatrix} 0 & 0 \\ -16 & -19 \end{bmatrix}$. The eigenvalue at the origin is fixed because it is an invariant zero of the system. The remaining eigenvalue -19 , which is unstable, is an eigenvalue of A_X restricted to $\mathcal{R}_{\ker X}^*$, and it can therefore be placed arbitrarily in the complex plane without affecting the cost. Matrix $G_X = I_m - R_X^\dagger R_X = \text{diag}\{1, 0\}$ is such that $\text{im}(BG_X) = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as expected. By using the standard procedure to assign the eigenvalue $1/2$ we obtain

$$L = \begin{bmatrix} 84/273 & 39/10 \\ -1 & -5/3 \end{bmatrix}.$$

It is easy to check that the eigenvalues of $A_X + BG_X L$ are exactly $\{0, 0.5\}$. □

Concluding remarks

In this paper we presented a self-contained analysis of some structural properties of the CGDARE that arises in infinite-horizon discrete LQ optimal control. The considerations that emerged from this analysis revealed that a subspace \mathcal{R}_0 can be identified that is independent of the particular solution of CGDARE considered. Even more importantly, it has been shown that the closed-loop matrix restricted to this subspace does not depend on the particular solution of CGDARE, and has been shown to be fixed for any state-feedback control constructed from a solution of the CGDARE. On the other hand, if such subspace is not zero, in the optimal control a further term can be added to the state-feedback generated from the solution of the Riccati equation that does not modify the value of the cost. This term can in turn be expressed in state-feedback form, and acts as a degree of freedom that can be employed to stabilise the closed-loop even in cases in which no stabilising solutions exists of the Riccati equation.

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