Automatica 49 (2013) 906–914

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

# Cooperative distributed MPC for tracking ☆

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ARTICLE INFO

*Article history:*
Received 5 December 2011
Received in revised form
28 May 2012
Accepted 30 October 2012
Available online 14 February 2013

*Keywords:*
Model predictive control
Distributed control
Cooperative games
Setpoint tracking
Stability

ABSTRACT

This paper proposes a cooperative distributed linear model predictive control (MPC) strategy for tracking changing setpoints, applicable to any finite number of subsystems. The proposed controller is able to drive the whole system to any admissible setpoint in an admissible way, ensuring feasibility under any change of setpoint. It also provides a larger domain of attraction than standard distributed MPC for regulation, due to the particular terminal constraint. Moreover, the controller ensures convergence to the centralized optimum, even in the case of coupled constraints. This is possible thanks to the *warm start* used to initialize the optimization Algorithm, and to the design of the cost function, which integrates a Steady-State Target Optimizer (SSTO). The controller is applied to a real four-tank plant.



## 1. Introduction

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industries (Camacho & Bordons, 2004), capable of providing controllers ensuring stability, robustness, constraint satisfaction and tractable computation for linear and for nonlinear systems (Mayne, Rawlings, Rao, & Scokaert, 2000; Rawlings & Mayne, 2009).

Most of the results on MPC consider the regulation problem: that is, steering the system to the origin or to a fixed setpoint. When this setpoint changes, the stability of the controller may be lost, and the controller may fail to track the reference (Ferramosca, 2011). This happens because the stabilizing design of the controller and the feasibility of the optimization problem depend on the steady state. Therefore, this may require an on-line redesign of the controller for each setpoint or a choice for a larger prediction horizon $N$, which can be computationally expensive. In Ferramosca, Limon, Alvarado, Alamo, and Camacho (2009) and Limon, Alvarado, Alamo, and Camacho (2008), an MPC for tracking for constrained linear systems is proposed, which ensures convergence of the closed-loop system under any change of the setpoint, maintaining feasibility.

In the process industries, plants are usually considered as large-scale systems, consisting of linked units of operations. Therefore, they can be divided into a number of subsystems, connected by networks of different nature, such as material, energy or information streams (Stewart, Venkat, Rawlings, Wright, & Pannocchia, 2010). The overall control of these plants by means of a centralized controller is not easy to realize, due to the difficult coordination and maintenance of a centralized control scheme.

A common way to control a large-scale plant is given by decentralized controllers (Magni & Scattolini, 2006). In this formulation, each subsystem is controlled independently, without interchange of information between different subsystems. The information that flows in the network is usually considered as a disturbance by each subsystem. The drawback of this control formulation is the big loss of information when the interactions between subsystems are strong (Cui & Jacobsen, 2002). A possible solution is coordination, which uses a coordinating controller in order to improve the performance, taking into account the closed-loop response of the network (Liu, Muñoz de la Peña, & Christofides, 2009).

Distributed control is a control strategy based on different agents – instead of a centralized controller – controlling each subsystem, which may or may not share information. There are different distributed control strategies proposed in the literature; they differs in the way the open-loop information is used, allowing one to define basically two kinds of distributed control formulation:

☆ This work has been funded by the European FP7 STREP project HD-MPC (FP7-ICT-223854) and MEC-Spain (under Grant DPI2010-21589-c05-01). The material in this paper was partially presented at the 18th IFAC World Congress, August 28–September 2, 2011, Milan, Italy. This paper was recommended for publication in revised form by Associate Editor Tamas Keviczky under the direction of Editor Frank Allgöwer.

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doi:10.1016/j.automatica.2013.01.019

noncooperative controllers and cooperative controllers. In noncooperative controllers, each agent makes decisions on a single subsystem, considering other subsystems information only locally (Camponogara, Jia, Krogh, & Talukdar, 2002; Dunbar, 2007). This strategy is usually referred to as a noncooperative dynamic game, and the performance of the plant converges to a Nash equilibrium (Başar & Olsder, 1999). Cooperative distributed controllers, on the other hand, consider the effect of all the control actions on all subsystems in the network. Each controller optimizes an overall plant object function, such as the centralized object. Cooperative control makes the system converge to the Pareto optimum, which is the centralized performance. Cooperative control is a form of suboptimal control for the overall plant, and therefore stability is proved resorting to suboptimal control theory (Pannocchia, Rawlings, & Wright, 2011; Scokaert, Mayne, & Rawlings, 1999; Stewart et al., 2010).

Another interesting approach to distributed control is dual decomposition (Rantzer, 2009; Wakasa, Arakawa, Tanaka, & Akashi, 2008), which uses Lagrange multipliers in order to relax the coupling between different agents. These multipliers can be seen as prices in a market mechanism, by means of which an agreement between the solutions of the different subproblems is achieved. In Negenborn, Schutter, and Hellendoorn (2008), a comparison of parallel versus serial schemes is presented.

This paper deals with the formulation of a stabilizing cooperative distributed MPC when the setpoint of the controlled plant changes. In particular, the controller presented in Ferramosca et al. (2009) and Limon et al. (2008) is extended to the case of distributed systems, considering a cooperative game.

The paper is organized as follows. In Section 2, the constrained tracking problem is stated. In Section 3, the proposed cooperative distributed MPC for tracking is presented. Section 4 presents the steady-state optimization property of the proposed controller. In Section 5, the application of the proposed controller to a real four-tank plant is presented. Finally, the conclusions of this work are given in Section 6. The proofs of the Lemmas and Theorem can be found in the Appendix.

*Notation:* For a given symmetric matrix $P > 0$, $\|x\|_P$ denotes the weighted Euclidean norm of $x$, i.e. $\|x\|_P = \sqrt{x'Px}$. Consider $a \in \mathbb{R}^{n_a}$ and $b \in \mathbb{R}^{n_b}$; for a set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, the projection of $\Gamma$ onto $a$ is defined as $Proj_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b},\ (a, b) \in \Gamma\}$. A vector $\mathbf{u}$ in bold denotes a finite sequence of vectors, that is, a vector defined as $(u(0), u(1), \ldots, u(N))$, where $N$ is deduced from the context. A matrix $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$ denotes a matrix of zeros, and $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. For a given $\lambda$, $\lambda\mathcal{U} \triangleq (\lambda I_n)\mathcal{U}$. Given two integers, $l \le r$, the set $\mathbb{I}_{l:r}$ is defined as $\mathbb{I}_{l:r} = \{l, l+1, \ldots, r-1, r\}$.

## 2. Problem statement

Consider a system described by a linear invariant discrete time model

$$x^+ = Ax + Bu \qquad (1)$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and $x^+$ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state $x$ is denoted as $x(j) = \varphi(j; x, \mathbf{u})$, where $x = \varphi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time $k$ are denoted as $x(k)$ and $u(k)$, respectively. The system is subject to hard constraints on the state and control:

$$x(k) \in X, \qquad u(k) \in U \qquad (2)$$

for all $k \ge 0$. $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact convex polyhedra containing the origin in their interior. It is assumed that the following hypothesis holds.

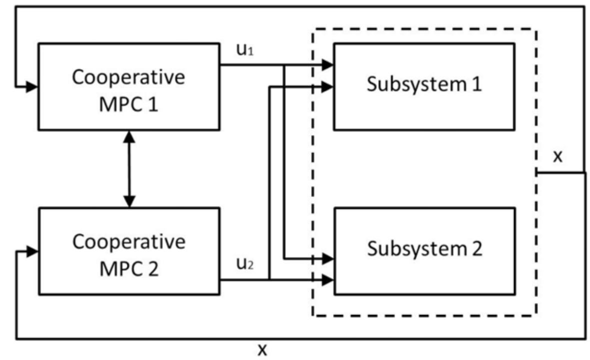



**Fig. 1.** Interaction between subsystems in a cooperative distributed control scheme (two-player game).

**Assumption 1.** The pair $(A, B)$ is stabilizable, and the state is measured at each sampling time.

The objective of the paper is to design a controller such that the output of the system is driven to the target provided by an upper-layer real-time optimizer (RTO), that is, $y(k) \to y_t$ as $k \to \infty$, in an admissible way.

### 2.1. Characterization of the equilibrium points of the plant

The steady state, input and output of the plant $(x_s, u_s, y_s)$ are such that (1) is fulfilled, i.e. $x_s = Ax_s + Bu_s$, and $y_s = Cx_s + Du_s$.

The steady conditions of the system can be determined by a suitable parameterization.

Under Assumption 1 and Lemma 1.14 in Rawlings and Mayne (2009, p. 83), the steady state and input $(x_s, u_s)$ of system (1) can be parameterized by their associated steady output $y_s$; that is, every solution of the following equation,

$$\begin{bmatrix} A - I_n & B & \mathbf{0}_{p,1} \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ y_s \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n,1} \\ \mathbf{0}_{p,1} \end{bmatrix} \qquad (3)$$

is given by $(x_s, u_s) = M_y y_s$, where $M_y$ is a suitable matrix.

If Lemma 1.14 in Rawlings and Mayne (2009, p. 83) does not hold, then another parameterization has to be used. In Limon et al. (2008), the authors state that the steady state and input $(x_s, u_s)$ are univocally defined by a vector $\theta \in \mathbb{R}^m$, in such a way that $(x_s, u_s) = M_\theta \theta$, where $M_\theta$ is a suitable matrix.

We define the sets of admissible equilibrium states, inputs and outputs as

$$\mathcal{Z}_s = \{(x, u) \in X \times U \mid x = Ax + Bu\} \qquad (4)$$
$$\mathcal{X}_s = \{x \in X \mid \exists u \in U \text{ such that } (x, u) \in \mathcal{Z}_s\} \qquad (5)$$
$$\mathcal{Y}_s = \{y = Cx + Du \mid (x, u) \in \lambda \mathcal{Z}_s\}, \qquad (6)$$

where $\lambda \in (0, 1)$ is a parameter chosen arbitrarily close to 1. Notice that $\mathcal{X}_s$ is the projection of $\mathcal{Z}_s$ onto $X$.

### 2.2. Distributed model of the plant

The plant given by (1) can be considered as a collection of coupled subsystems, connected by networks of different nature. (See Fig. 1.)

Given model (1), without loss of generality, it is considered that $u = (u_1, \ldots, u_M)$, where $M \le m$. Then, by virtue of Stewart et al. (2010, Section 3.1.1) and Rawlings and Mayne (2009, Chapter 6, pp. 421–422), model (1) is partitioned into $M$ subsystems, coupled by the control inputs, and modeled as follows:

$$x_i^+ = A_i x_i + \sum_{j=1}^{M} B_{ij} u_j \tag{7}$$

$$y_i = C_i x_i + \sum_{j=1}^{M} D_{ij} u_j$$

where $x_i \in \mathbb{R}^{n_i}$, $u_j \in \mathbb{R}^{m_j}$, $y_i \in \mathbb{R}^{p_i}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_{ij} \in \mathbb{R}^{n_i \times m_j}$, $C_i \in \mathbb{R}^{p_i \times n_i}$ and $D_{ij} \in \mathbb{R}^{p_i \times m_j}$.

The solution of this system, given the sequences of control inputs $(\mathbf{u}_1, \ldots, \mathbf{u}_M)$ and initial state $x = (x_1, \ldots, x_M)$, is denoted as $x(j) = \phi(j; x, \mathbf{u}_1, \ldots, \mathbf{u}_M)$, where $x = \phi(0; x, \mathbf{u}_1, \ldots, \mathbf{u}_M)$.

As proved in Stewart et al. (2010), any plant can be partitioned as proposed for a certain definition of $x_i$. If the couple $(C_i, A_i)$ is observable, the inner state of the partition can be calculated or estimated from the measured output of the subsystem $y_i$.

**Remark 1.** The use of linear models in the design of controllers for tracking is very common. The existing mature theory on estimation, identification and controller design for linear systems can be exploited, and the controllers obtained have demonstrated themselves to be successful thanks to the capability of linear models to capture the dynamics of the plant and to the feedback structure, which reduces the effect of model mismatches. Model mismatches and offset cancellation can be dealt with by offset-free techniques (Pannocchia & Kerrigan, 2005).

## 3. Cooperative MPC for tracking

The distributed control scheme proposed in this section extends the MPC for tracking presented in Ferramosca et al. (2009) and Limon et al. (2008) to a cooperative distributed framework (Rawlings & Mayne, 2009; Stewart et al., 2010, Chapter 6, p. 433) where players share a common (and hence coupled) objective, which can be considered as the overall plant objective.

As in Limon et al. (2008), an artificial equilibrium point of the plant $(\hat{x}_s, \hat{u}_s, \hat{y}_s)$, characterized by $\hat{y}_s$, is added as decision variable, and the following modified cost function is considered:

$$V_N(x, y_t; \mathbf{u}, \hat{y}_s) = \sum_{j=0}^{N-1} \|x(j) - \hat{x}_s\|_Q^2 + \|u(j) - \hat{u}_s\|_R^2 + \|x(N) - \hat{x}_s\|_P^2 + V_O(\hat{y}_s, y_t),$$

where $x = (x_1, \ldots, x_M)$, $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_M)$ and $(\hat{x}_s, \hat{u}_s, \hat{y}_s)$ is the artificial equilibrium point of the plant given by $\hat{y}_s$. The function $V_O(\hat{y}_s, y_t)$ is the so-called *offset cost function*, and it is defined as follows.

**Definition 2.** Let $V_O(\hat{y}_s, y_t)$ be a convex and positive definite function in $(\hat{y}_s - y_t)$ such that it has a unique minimizer given by

$$y_s = \arg\min_{\hat{y}_s \in \mathcal{Y}_s} V_O(\hat{y}_s, y_t).$$

The following assumptions are considered to prove the stability of the controller.

**Assumption 2.** (1) Let $R \in \mathbb{R}^{m \times m}$ be a positive definite matrix and $Q \in \mathbb{R}^{n \times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2}, A)$ is observable.

(2) Let $K \in \mathbb{R}^{m \times n}$ be a stabilizing control gain for the centralized system, such that $(A + BK)$ has all the eigenvalues in the unit circle.

(3) Let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix for the centralized system such that

$$(A + BK)'P(A + BK) - P = -(Q + K'RK). \tag{8}$$

(4) Let $\Omega_\lambda \subseteq \mathbb{R}^{n+p}$ be an admissible polyhedral invariant set for tracking for system (1) subject to (2), for a given gain $K$ (Limon et al., 2008).

That is, given the extended state $w = (x, \hat{y}_s)$, for all $w \in \Omega_\lambda$, then $w^+ = A_w w \in \Omega_\lambda$, where $A_w$ is the closed-loop matrix given by

$$A_w = \begin{bmatrix} A + BK & BL \\ 0 & I_p \end{bmatrix}$$

and $L = [-K, \; I_p] M_y$. Furthermore, $\Omega_\lambda$ must be contained in the polyhedral set $W_\lambda$ given by

$$W_\lambda = \{(x, \hat{y}_s) \in X \times \mathcal{Y}_s : Kx + L\hat{y}_s \in U\}.$$

**Remark 3.** As in Limon et al. (2008), the invariant set for tracking $\Omega_\lambda$ is calculated following the algorithm presented in Gilbert and Tan (1991). The computational cost of the calculation of this set increases with the dimension of the system. However, this computation is made off-line, so it has no effect on the MPC control problem.

Moreover, in case of high-dimension systems, the computation of $\Omega_\lambda$ can be avoided by using a terminal equality constraint: it is sufficient to impose the last predicted state to be an arbitrary equilibrium point, that is, $(x(N), \hat{y}_s) \in (\mathcal{X}_s \times \mathcal{Y}_s)$, which is invariant.

In cooperative distributed MPC (Stewart et al., 2010), each agent $i$ calculates its corresponding input $u_i$ by solving an iterative decentralized optimization problem, given an initial feasible solution $\mathbf{u}_i^{[0]}$. The solution of agent $i$ at iteration $p$ will be denoted as $\mathbf{u}_i^{[p]}$. Based on this, the solution of each agent at the next iteration $p + 1$ is calculated from the solution of the following optimization problem for the $i$th agent $P_i(x, y_t; \mathbf{u}^{[p]}, \hat{y}_{s,i}^{[p]})$, which depends on the state $x$, the target $y_t$ and the solution of the $p$th iteration $(\mathbf{u}^{[p]}, \hat{y}_s^{[p]})$. This optimization problem is given by

$$(\mathbf{u}_i^*, \hat{y}_{s,i}^*) = \arg\min_{\mathbf{u}_i, \hat{y}_s} V_N(x, y_t; \mathbf{u}, \hat{y}_s) \tag{9a}$$

$$s.t. \tag{9b}$$

$$x_q(j+1) = A_q x_q(j) + \sum_{\ell=1}^{M} B_{q\ell} u_\ell(j), \quad q \in \mathbb{I}_{1:M} \tag{9c}$$

$$x_q(0) = x_q, \quad q \in \mathbb{I}_{1:M} \tag{9d}$$

$$(\mathbf{u}_1^{[p]}, \ldots, \mathbf{u}_M^{[p]}) = \mathbf{u}^{[p]}, \tag{9e}$$

$$u_\ell(j) = u_\ell^{[p]}(j) \quad \ell \in \mathbb{I}_{1:M} \setminus i, \tag{9f}$$

$$(x_1(j), \ldots, x_M(j)) \in X, \quad j \in \mathbb{I}_{0:N-1} \tag{9g}$$

$$(u_1(j), \ldots, u_M(j)) \in U, \quad j \in \mathbb{I}_{0:N-1} \tag{9h}$$

$$(x(N), \hat{y}_s) \in \Omega_\lambda. \tag{9i}$$

Based on the solution of this optimization problem for each agent, namely $\mathbf{u}_i^*$, the solution of the $p + 1$th iteration is given by

$$\mathbf{u}_i^{[p+1]} = w_i \mathbf{u}_i^*(\mathbf{u}_\ell^{[p]}, \hat{y}_s^{[p]}) + (1 - w_i)\mathbf{u}_i^{[p]}, \quad i \in \mathbb{I}_{1:M}, \; \ell \in \mathbb{I}_{1:M} \setminus i \tag{10a}$$

$$\hat{y}_s^{[p+1]} = \sum_{i=1}^{M} w_i \hat{y}_{s,i}^*(\mathbf{u}_\ell^{[p]}, \hat{y}_s^{[p]}) \quad \ell \in \mathbb{I}_{1:M} \setminus i \tag{10b}$$

$$\sum_{i=1}^{M} w_i = 1, \quad w_i > 0, \; i \in \mathbb{I}_{1:M}.$$

At time $k$, the iterative method finishes at the iteration $\bar{p}$, once the computation time is expired or a given accuracy of the solution is achieved. Then the best available solution $u(k) = (u_1^{[\bar{p}]}(0;k), \dots, u_M^{[\bar{p}]}(0;k))$ is applied to the plant.

This distributed optimization scheme is of Gauss–Jacobi type (Bertsekas & Tsitsiklis, 1997, pp. 219–223). Therefore, the overall predictive controller can be considered as a suboptimal MPC, since the distributed solution is a suboptimal solution of the centralized MPC problem.

To proceed with the analysis of the proposed controller, we will denote

$$\mathbf{v} = (\mathbf{u}_1, \dots, \mathbf{u}_M, \hat{y}_s).$$

$\mathbf{v}$ is said to be feasible at $(x, y_t)$ if each optimization problem $P_i(x, y_t; \mathbf{u}, \hat{y}_s)$ is feasible for all $i \in \mathbb{I}_{1:M}$. The set of states for which there exists a feasible $\mathbf{v}$, denoted as $\mathcal{X}_N$, is given by

$$\begin{aligned}\mathcal{X}_N = \{&x \in X \mid \exists \mathbf{v} = (\mathbf{u}_1, \dots, \mathbf{u}_M, \hat{y}_s), (u_1(j), \dots, u_M(j)) \in U,\\ &j \in \mathbb{I}_{0:N-1}, \hat{y}_s \in \mathcal{Y}_s, \text{ s.t. } (x, \mathbf{v}) \in \mathcal{Z}_N\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{Z}_N = \{&(x, \mathbf{v}) \mid \mathbf{v} = (\mathbf{u}_1, \dots, \mathbf{u}_M, \hat{y}_s), (u_1(j), \dots, u_M(j)) \in U,\\ &\hat{y}_s(j) \in \mathcal{Y}_s, \phi(j; x, \mathbf{u}) \in X \text{ for all } j \in \mathbb{I}_{0:N-1},\\ &\phi(N; x, \mathbf{u}) \in \Omega_\lambda\}.\end{aligned}$$

Notice that this set is equal to the feasible set of the centralized MPC for tracking (Limon et al., 2008), i.e. the set of states that can be admissibly steered to $proj_x(\Omega_\lambda)$ in $N$ steps. Besides, we will denote $V_N(x, y_t; \mathbf{v}) = V_N(x, y_t; \mathbf{u}, \hat{y}_s)$.

In order to precisely define the proposed cooperative control scheme, the initial solution $\mathbf{v}^{[0]}$ of the iterative procedure (10) must be defined. Since the proposed distributed MPC can be considered as a suboptimal formulation of the centralized MPC, this initialization plays the role of the *warm start* of the suboptimal MPC and determines recursive feasibility and convergence of the control algorithm. The following algorithm calculates a *warm start* that ensures convergence to the optimal centralized target and controllability of the solution.

**Algorithm 1.** Given the solution $\mathbf{v}(k)$, the objective is to calculate the warm start at sampling time $k+1$, denoted as

$$\mathbf{v}(k+1)^{[0]} = (\mathbf{u}_1(k+1)^{[0]}, \dots, \mathbf{u}_M(k+1)^{[0]}, \hat{y}_s(k+1)^{[0]}).$$

- Define the first candidate initial solution:

  $$\tilde{\mathbf{u}}_i(k+1) = \{u_i(1;k), \dots, u_i(N-1;k), u_{c,i}(N)\},$$

  where

  $$u_c(N) = (u_{c,1}(N), \dots, u_{c,M}(N)) = Kx(N) + L\hat{y}_s(k)$$

  is the centralized solution given by the centralized terminal control law, and $x(N) = \phi(N; x(k), \mathbf{u}_1(k), \dots, \mathbf{u}_M(k))$.
- Define the second candidate initial solution:

  $$\hat{\mathbf{u}}_i(k+1) = \{\hat{u}_{c,i}(0), \dots, \hat{u}_{c,i}(N-1)\},$$

  where $(\hat{u}_{c,1}(j), \dots, \hat{u}_{c,M}(j)) = \hat{u}_c(j)$ and

  $$\begin{aligned}&\hat{x}(0) = x(k+1)\\ &\hat{x}(j+1) = (A+BK)\hat{x}(j) + BL\hat{y}_s(k), \quad j \in \mathbb{I}_{1:N-2}\\ &\hat{u}_c(j) = K\hat{x}(j) + L\hat{y}_s(k).\end{aligned}$$

- **if** $(x(k+1), \hat{y}_s(k)) \in \Omega_\lambda$
  *and*
  $V_N(x(k+1), y_t; \hat{\mathbf{u}}, \hat{y}_s(k)) \le V_N(x(k+1), y_t; \tilde{\mathbf{u}}, \hat{y}_s(k))$
  **then**
  $\mathbf{v}(k+1)^{[0]} = (\hat{\mathbf{u}}_1(k+1), \dots, \hat{\mathbf{u}}_M(k+1), \hat{y}_s(k))$
  **else**
  $\mathbf{v}(k+1)^{[0]} = (\tilde{\mathbf{u}}_1(k+1), \dots, \tilde{\mathbf{u}}_M(k+1), \hat{y}_s(k))$
  **end if**

As usual in the suboptimal MPC optimization algorithm, the proposed *warm start* for the first optimization iteration $p = 0$ is given by the previous optimal sequence, shifted by one position, with the last control move given by the centralized terminal control law applied to the predicted terminal state of the overall plant and the same artificial steady output; that is, $(\tilde{\mathbf{u}}_1(k+1), \dots, \tilde{\mathbf{u}}_M(k+1), \hat{y}_s(k))$. But, according to the algorithm, when the state of the system reaches the invariant set for tracking, that is, $(x(k+1), \hat{y}_s(k)) \in \Omega_\lambda$, it is desirable that the distributed MPC achieves a *better* cost than the one provided by using the centralized terminal controller. If this is not possible, that is, $V_N(x(k+1), y_t; \hat{\mathbf{u}}, \hat{y}_s(k)) \le V_N(x(x+1), y_t; \tilde{\mathbf{u}}, \hat{y}_s(k))$, then the centralized terminal control law is chosen as *warm start*.

**Remark 4.** In this work, we are considering suboptimality in the sense that the proposed distributed solution is a suboptimal solution of a centralized optimization problem. On the other hand, we assume that the optimal solution of the optimization problem of each agent at each iteration $P_i(x, y_t; \mathbf{u}^{[p]}, \hat{y}_s^{[p]})$ is achieved. In the case of suboptimality of this solution, the method proposed in Zeilinger (2011) can be used.

### 3.1. Stability analysis

At each sampling time $k$, the initial *warm start* $\mathbf{v}^{[0]}(k)$ is calculated using Algorithm 1, which depends on $x(k-1)$ and $\mathbf{v}(k-1)$. Then, $\mathbf{v}^{[p]}(k)$ is obtained from the iterative procedure given by (9) and (10). At a certain number of iterations $\bar{p}$, the final solution, denoted as

$$\mathbf{v}(k) = \mathbf{v}^{[\bar{p}]}(k) = (\mathbf{u}_1^{[\bar{p}]}(k), \dots, \mathbf{u}_M^{[\bar{p}]}(k), \hat{y}_s^{[\bar{p}]}(k)),$$

is achieved. This solution is a function of (i) the current state $x(k)$, (ii) the initial feasible solution $\mathbf{v}^{[0]}(k)$ and (iii) the number of iterations $\bar{p}$. Since $x(k)$ and $\mathbf{v}^{[0]}(k)$ are functions of $x(k-1)$ and $\mathbf{v}(k-1)$, the overall control law can be written as

$$\mathbf{v}(k) = g(x(k-1), \mathbf{v}(k-1), \bar{p}).$$

This difference equation, together with the equations of system dynamics (1), forms the overall closed-loop system, which can be posed as follows:

$$\begin{aligned}&x(k+1) = Ax(k) + BH\mathbf{v}(k)\\ &\mathbf{v}(k+1) = g(x(k), \mathbf{v}(k), \bar{p}),\end{aligned}$$

where $H$ is a suitable matrix such that $u(k) = H\mathbf{v}(k)$. As $\bar{p}$ can take any value depending on the computation time of the optimization problem and the communication network delay, this can be considered as $\bar{p} \in \mathbb{N}$, and then the closed-loop dynamics can be modeled as a set-valued map

$$\mathbf{z}(k+1) \in F(\mathbf{z}(k)), \tag{11}$$

where $\mathbf{z}(k) = (x(k), \mathbf{v}(k))$.

The stabilizing properties of this controller are stated in the following theorem.

**Theorem 1** (*Asymptotic Stability*)**.** *Consider that Assumptions 1 and 2 hold. Then, for all $(x(0), \mathbf{v}(0)) \in \mathcal{Z}_N$, and for all $y_t$, the closed-loop system is asymptotically stable and converges to an equilibrium point $(x_s, u_s) = M_y y_s$ such that*

$$y_s = \arg\min_{\hat{y}_s \in \mathcal{Y}_s} V_O(\hat{y}_s, y_t).$$

*Moreover, if $y_t \in \mathcal{Y}_s$, then $y_s = y_t$.*

**Proof.** Considering Lemmas 5, 6 and 8, and combining Lemma 9 for convergence with Lemma 10 for stability, Theorem 1 is proved. □

### 3.2. Properties

The proposed controller provides the following properties to the closed-loop system.

- *Enlargement of the domain of attraction:*
  The domain of attraction of the proposed distributed MPC is the set of states that can be admissible steered to $\Omega_\lambda$ in $N$ steps. Since this set is not defined for the target, but for *any* equilibrium point, the domain of attraction of the proposed controller is (potentially) larger than the one of the (distributed) MPC for regulation (Limon et al., 2008).
- *Changing operation points:*
  Considering that the optimization problem is feasible for any $y_t$, the proposed controller is able to track changing operation points maintaining the recursive feasibility and constraint satisfaction.
- *Local optimality:*
  The addition of an artificial reference as a decision variable means that the local optimality property of the controller may be lost even in the case of optimality of the centralized solution. But, if the offset cost function is such that

  $$\|V_O(\hat{y}_s, y_t) - V_O(y_t, y_t)\| \geq \alpha \|\hat{y}_s - y_t\|,$$

  then there exists a constant $\alpha_{\min}$ such that for all $\alpha > \alpha_{\min}$ the local optimality property also holds for this controller (Ferramosca et al., 2009).

## 4. Integration of the steady-state target optimizer

Process industries are characterized by a hierarchical control structure (Engell, 2007): at the top, an economic scheduler and planner determines the whole plant production (level, quality, etc.). The outputs of this layer are sent to a real-time optimizer (RTO) – based on a complex nonlinear stationary model of the plant – which is devoted to computing the stationary targets, $y_t$, according to economic criteria. The targets computed by the RTO are sent to the MPC control level which calculates the control actions necessary for the plant to reach the targets, taking into account a simplified dynamic model of the plant and constraints. One well-known drawback of this hierarchical control structure is that the communication between the economic/stationary and the dynamic layers may be inconsistent, producing in this way problems that go from unreachability of the targets to poor economic performance.

A way to avoid this problem is the so-called two-layer structure (Rao & Rawlings, 1999): an upper optimization level is added in between the RTO and the MPC. This level, referred to as the steady-state target optimizer (SSTO), calculates the steady state $(x_s, u_s, y_s)$ to which the system has to be stabilized, solving a linear or quadratic programming problem and taking into account the target $y_t$ calculated by the RTO. The plant model used in this intermediate level is the same as the MPC one, thus reducing inconsistencies (Engell, 2007).

The steady-state optimizer is usually of the form

$$(x_s, u_s) = \arg\min_{x,u} \ell_{ss}(y - y_t) \tag{12}$$

$$s.t$$

$$x \in X, \quad u \in U$$

$$x = Ax + Bu, \quad y = Cx + Du,$$

where $\ell_{ss}$ is a local convex approximation of the RTO economic function, typically a norm (or a square norm) of the distance $(y - y_t)$.

In distributed MPC, the target problem, that is, the SSTO problem, is typically solved in a distributed way: there is a steady-state optimizer for each agent (Rawlings & Mayne, 2009, Section 6.3.4). If the constraints of each subsystem are decoupled, then the distributed target problem ensures that the distributed controller steers the system to the calculated setpoint. But if the constraints of the problem are coupled, then the optimality of the target problem might be lost, and the controller might fail to steer the plant to the desired setpoint, driving the plant to a suboptimal one. In this case, it is recommended to use the centralized approach to solve the target problem (Rawlings & Mayne, 2009, Section 6.3.4). The proposed controller integrates the SSTO into the control law in a natural way. In effect, from Theorem 1, it can be immediately seen that, taking

$$V_O(\hat{y}_s, y_t) = \ell_{ss}(\hat{y}_s - y_t),$$

then the distributed controller steers the plant to the equilibrium point $(x_s, u_s) = M_y y_s$ such that

$$y_s = \arg\min_{\hat{y}_s \in \mathcal{Y}_s} \ell_{ss}(\hat{y}_s, y_t),$$

which is equivalent to the SSTO problem (12).

Then the controller integrates the two-layer structure given by the steady-state optimizer and the MPC controller in just one layer. Notice also that, since every agent solves an optimization problem with a centralized offset cost function, the proposed controller ensures convergence to the optimal equilibrium point of the centralized SSTO, even in the case of coupled constraints or a small number of iterations $\bar{p}$.

## 5. Application to the four-tank system

In this section, the experimental results of the application of the proposed controller to a real four-tank plant are presented.

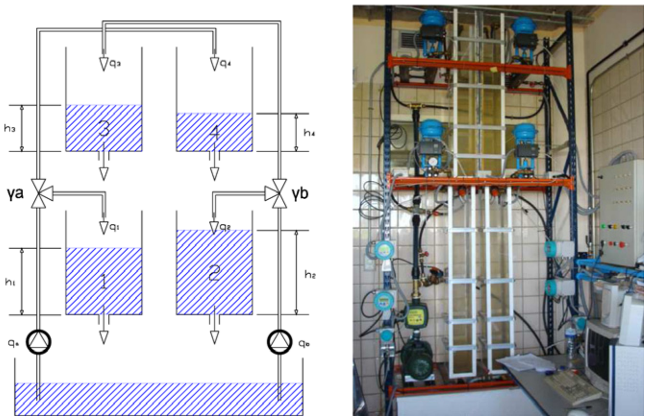
The four-tank plant (Johansson, 2000) is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics, and is subject to state and input constraints. A scheme of this plant is presented in Fig. 2(a). The inputs are the voltages of the two pumps and the outputs are the water levels in the lower two tanks.

A real experimental plant developed at the University of Seville is presented in Fig. 2(b). This real plant has been implemented using industrial instrumentation and a PLC (Programmable Logic Controller) for the low-level control. Supervision and control of the plant are carried out in a PC by means of OPC (Ole for Process Control), which allows one to connect the plant with a wide range of control programs such as LabView, Matlab or an industrial SCADA.

A state–space continuous-time nonlinear model of the quadruple-tank process system is given in Johansson (2000). The linearized model, at the operating point given by $h^0 = (0.67, 0.66, 0.55, 0.58)$, is given by

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{A_1} & 0 \\ 0 & \frac{\gamma_b}{A_2} \\ 0 & \frac{(1-\gamma_b)}{A_3} \\ \frac{(1-\gamma_a)}{A_4} & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x,$$




(a) Scheme of the four-tank process. (b) The real plant.

**Fig. 2.** The four-tank process.

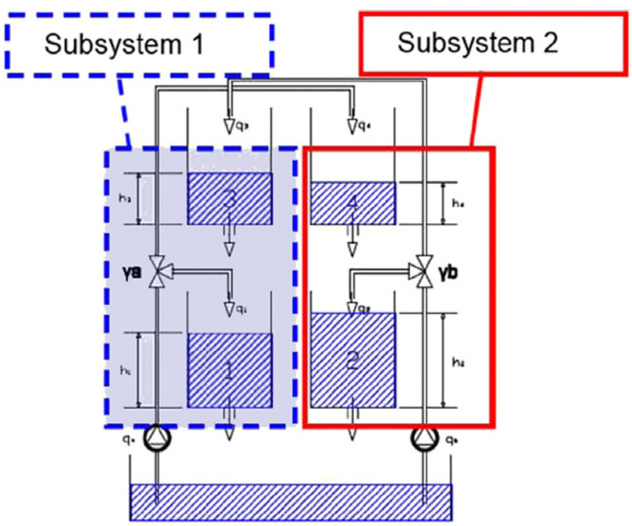



**Fig. 3.** The four-tank process: two distributed subsystems.

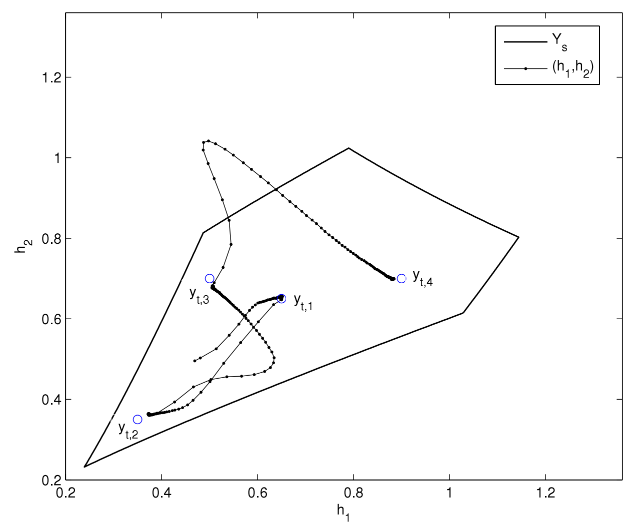



**Fig. 4.** State–space evolution of the outputs $h_1$ and $h_2$.

where $x_i = h_i - h_i^o$, $u_j = q_j - q_j^o$, $j = a, b$ and $i = 1, \dots, 4$. $\tau_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1, \dots, 4$, are the time constants of each tank. This model has been discretized using the zero-order hold method with a sampling time of 15 s.

The plant parameters, estimated on the real plant, are given in Alvarado et al. (2011, Table 1).

### 5.1. Experimental results

The proposed controller has been applied to the four-tank plant, following the guidelines of the HD-MPC project Benchmark (Alvarado et al., 2011). To this aim, the original plant has been divided into two subsystems (Fig. 3), coupled through the control action, that is, the flows from the pumps.

The experiment has been run considering four changes of reference: $y_{t,1} = (0.65, 0.65)$, $y_{t,2} = (0.35, 0.35)$, $y_{t,3} = (0.50, 0.70)$ and $y_{t,4} = (0.90, 0.70)$. The initial state is $x_0 = (0.47, 0.49, 0.44, 0.46)$. Notice also that the constraints on the model are coupled due to the dynamic. The setup parameters for the distributed predictive controller are $Q = I_4$, $R = 0.01I_2$, $w_{1,2} = 0.5$. The prediction horizon has been taken as $N = 5$.

The number of iterations of the suboptimal optimization algorithm has been chosen as $\bar{p} = 1$. The gain $K$ is chosen as the one of the LQR (Linear Quadratic Regulator), and the matrix $P$ is the solution of the Riccati equation. The invariant set for tracking has been calculated for the gain matrix $K$. The chosen offset cost function is $V_O(y_s, y_t) = \|y_s - y_t\|_T^2$, where $T = 100I$. The optimization has been run in Matlab. The calculated control inputs have been injected into the plant by means of OPC.

The result of the experiment are presented in Figs. 4–6.

In particular, in Fig. 4, the set of admissible equilibrium outputs $\mathcal{Y}_s$ and the state–space evolution of the output are depicted. The dots represent the desired setpoints. Notice how the controller always steers the system to the desired target.

In Fig. 5, the time evolution of the output is presented. The desired setpoint $y_t$, the artificial references $\hat{y}_s$ and the real output $y$ are depicted respectively in blue dashed–dotted, red dashed and black solid lines. Notice how the controller steers the system to the desired setpoint, always fulfilling the constraints. Notice also the role played by the artificial reference in maintaining feasibility when the setpoint changes. See in particular in Fig. 5, the fourth change of reference of the output $h_1$.

Notice that the offset between references and output is due to the mismatches between the nonlinear plant and the linearized model used for predictions. This offset can be corrected using an offset-free technique (Pannocchia & Kerrigan, 2005; Rawlings & Mayne, 2009). However, this was not the objective of this experiment.

In Fig. 6, the time evolution of $h_3$ and $h_4$ and the control input $q_a$ and $q_b$ are presented.

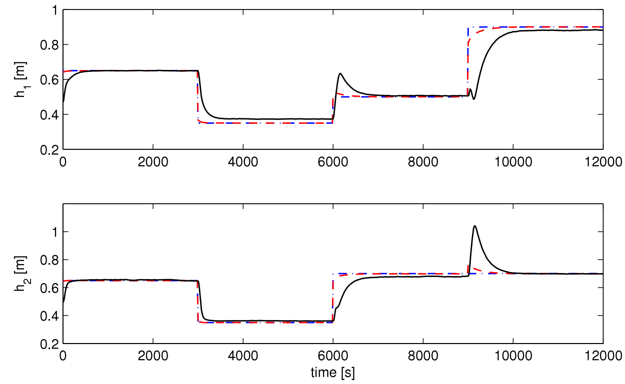



**Fig. 5.** Time evolution of the outputs $h_1$ and $h_2$.

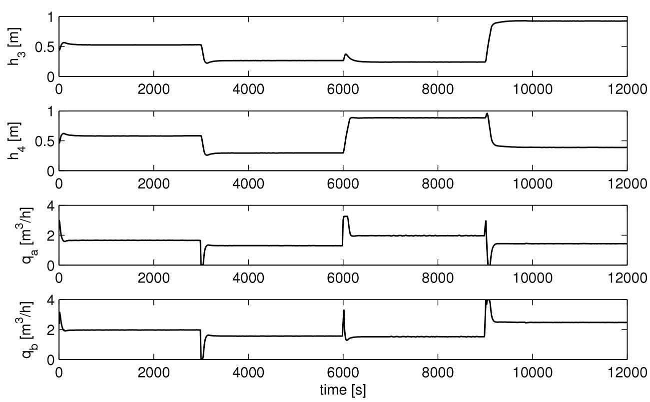



**Fig. 6.** Time evolution of $h_3$ and $h_4$ and of the control inputs $q_a$ and $q_b$.

## 6. Conclusion

In this paper, a cooperative distributed linear model predictive control strategy for tracking changing nonzero setpoints has been proposed, applicable to any finite number of subsystems. The proposed controller is able to steer the system to any admissible setpoint in an admissible way. Feasibility under any changing of the target steady state and convergence to the centralized optimum are ensured, thanks to the controller design and the *warm start* used to initialize the iterative optimization algorithm. The proposed controller also provides a larger domain of attraction than standard cooperative MPC for regulation, due to the centralized invariant set for tracking used as the terminal constraint of the MPC problem. The controller has been applied to a real four-tank plant.

## Acknowledgment

The authors would like to thank Professor James B. Rawlings for helpful discussions and collaboration in the writing of this paper.

## Appendix. Technical lemmas

**Lemma 5** (*Recursive Feasibility*)**.** *For any* $\mathbf{z}(0) \in \mathcal{Z}_N$, $\mathbf{z}(0) = (x(0), \mathbf{v}(0))$*, the evolution of system* (11) *is such that* $\mathbf{z}(k) \in \mathcal{Z}_N$ *for all* $k \geq 0$.

**Proof.** Recursive feasibility of the time instant $k$ is proved by showing that, if $\mathbf{z}(k) \in \mathcal{Z}_N$, then $(x(k+1), \mathbf{v}^{[0]}(k+1)) \in \mathcal{Z}_N$, where $\mathbf{v}^{[0]}(k+1)$ is calculated by Algorithm 1.

If $(x(k+1), \hat{y}_s^*(k)) \in \Omega_\lambda$, and $V_N(x(k+1), y_t, \hat{\mathbf{u}}) \leq V_N(x(k+1), y_t, \tilde{\mathbf{u}})$, then $\mathbf{v}^{[0]}(k+1)$ is feasible since the centralized terminal control law provides a feasible solution. Otherwise, the standard shifted solution is used, which is feasible thanks to the feasibility of the terminal controller.

Recursive feasibility of the iteration $p$ is proved by showing that, if $(x, \mathbf{v}^{[0]}) \in \mathcal{Z}_N$, then $(x, \mathbf{v}^{[p]}) \in \mathcal{Z}_N$ for all $p \in \mathbb{N}$.

Since $X$, $U$ and $\Omega_\lambda$ are convex sets, if

$$
\begin{aligned}
&(\mathbf{u}_1^*(x, y_t, \mathbf{v}^{[p]}), \ldots, \mathbf{u}_M^{[p]}, \hat{y}_{s,1}^{[p]}(x, y_t, \mathbf{v}^{[p]}))\\
&\vdots\\
&(\mathbf{u}_1^{[p]}, \ldots, \mathbf{u}_M^*(x, y_t, \mathbf{v}^{[p]}), \hat{y}_{s,M}^{[p]}(x, y_t, \mathbf{v}^{[p]}))
\end{aligned}
$$

are feasible, then $\mathbf{v}^{[p+1]} = (\mathbf{u}_1^{[p+1]}, \ldots, \mathbf{u}_M^{[p+1]}, \hat{y}_s^{[p+1]})$ which results from a convex combination of these solutions is also feasible. Hence, by induction, this is proved for any $p \in \mathbb{N}$. $\square$

**Lemma 6** (*Convergence of the Algorithm*)**.** *For any* $k \geq 0$ *and* $\forall p \in \mathbb{N}$*, the obtained cost function is such that*

$$V_N(x(k), y_t; \mathbf{v}^{[p+1]}(k)) \leq V_N(x(k), y_t; \mathbf{v}^{[p]}(k)).$$

**Proof.** In this proof, the time dependence has been removed for the sake of simplicity. Given the solution $\mathbf{v}^{[p]}$, the following solutions are computed:

$$
\begin{aligned}
&\mathbf{v}_a = (\mathbf{u}_1^*(x, y_t, \mathbf{u}^{[p]}), \ldots, \mathbf{u}_M^{[p]}, \hat{y}_s^{[p]})\\
&\vdots\\
&\mathbf{v}_m = (\mathbf{u}_1^{[p]}, \ldots, \mathbf{u}_M^*(x, y_t, \mathbf{u}^{[p]}), \hat{y}_s^{[p]}).
\end{aligned}
$$

From the definition of $P_i(x, y_t; \mathbf{u}^{[p]}, \hat{y}^{[p]})$, these solutions are feasible for this optimization problem, and they provide a lower cost than $\mathbf{v}^{[p]}$.

Then, from convexity of the optimal cost function and the fact that $\mathbf{v}^{[p+1]}$ is the optimal solution of $P_i(x, y_t; \mathbf{u}^{[p]}, \hat{y}^{[p]})$, we have that

$$
\begin{aligned}
V_N(x, y_t; \mathbf{v}^{[p+1]}) &\leq w_1 V_N(x, y_t; \mathbf{v}_a) + \cdots + w_M V_N(x, y_t; \mathbf{v}_m)\\
&\leq w_1 V_N(x, y_t; \mathbf{v}^{[p]}) + \cdots + w_M V_N(x, y_t; \mathbf{v}^{[p]})\\
&= V_N(x, y_t; \mathbf{v}^{[p]}). \quad \square
\end{aligned}
$$

**Corollary 7.** *For all* $k \geq 0$ *and* $\bar{p} \in \mathbb{N}$*, the cost function is such that*

$$V_N(x(k), y_t; \mathbf{v}^{[\bar{p}]}(k)) \leq V_N(x(k), y_t; \mathbf{v}^{[0]}(k)).$$

**Lemma 8** (*Local Bounding*)**.** *Let* $k$ *be an instant such that* $(x(k), \hat{y}_s^{[0]}(k)) \in \Omega_\lambda$*. Then,* $\forall p \in \mathbb{N}$,

$$V_N(x(k), y_t; \mathbf{v}^{[p]}(k)) \leq \|x(k) - \hat{x}_s^{[0]}(k)\|_P^2 + V_O(\hat{y}_s^{[0]}(k), y_t).$$

**Proof.** Since $(x(k), \hat{y}_s^{[0]}(k)) \in \Omega_\lambda$, Algorithm 1 ensures that

$$
\begin{aligned}
&V_N(x(k), y_t; \mathbf{v}^{[0]}(k))\\
&= \sum_{j=0}^{N-1} \overbrace{\|x(j) - \hat{x}_s^{[0]}(k)\|_Q^2 + \|u(j) - \hat{u}_s^{[0]}(k)\|_R^2}^{\|x(j)-\hat{x}_s^{[0]}(k)\|^2_{(Q+K'RK)}}\\
&\quad + \|x(N) - \hat{x}_s^{[0]}(k)\|_P^2 + V_O(\hat{y}_s^{[0]}(k), y_t)\\
&\leq \|x(k) - \hat{x}_s^{[0]}(k)\|_P^2 + V_O(\hat{y}_s^{[0]}(k), y_t).
\end{aligned}
$$

This fact and Lemma 6 prove the lemma. $\square$

Let $(x_s, u_s) = My_s$ be the equilibrium point corresponding to $y_s$ defined in Theorem 1, and define $\mathbf{u}_{s,i} = (u_{s,i}, \ldots, u_{s,i})$, $i \in \mathbb{I}_{1:M}$, where $u_s = (u_{s,1}, \ldots, u_{s,M})$. Denote $\mathbf{v}_s = (\mathbf{u}_{s,1}, \ldots, \mathbf{u}_{s,M}, y_s)$ and $\mathbf{z}_s = (x_s, \mathbf{v}_s)$. Then in the following lemma it is proved that $\mathbf{z}_s$ is the equilibrium point of the closed-loop system where this converges to.

**Lemma 9** (*Convergence*). *Let Assumption 2 hold. For any initial feasible solution* $\mathbf{z}(0) = (x(0), \mathbf{v}(0)) \in \mathcal{Z}_N$, *system (11) converges to the equilibrium point* $\mathbf{z}_s$.

**Proof.** Using standard MPC procedures (Pannocchia et al., 2011), Lemma 5 and Lemma 6, it is easy to demonstrate that, if $\mathbf{z}(k) \in \mathcal{Z}_N$, then $(x(k+1), \mathbf{v}^{[0]}(k+1)) \in \mathcal{Z}_N$, and that $\mathbf{z}(k+1) = (x(k+1), \mathbf{v}(k+1))$ is such that

$$V_N(x(k+1), y_t; \mathbf{v}(k+1)) \le V_N(x(k), y_t; \mathbf{v}(k)) - \|x(k) - \hat{x}_s(k)\|_Q^2 - \|u(k) - \hat{u}_s(k)\|_R^2.$$

By virtue of the positive definite nature of the cost function, it is derived that

$$\lim_{k\to\infty} \|x(k) - \hat{x}_s^*(x(k), y_t)\| = 0, \quad \lim_{k\to\infty} \|u(k) - \hat{u}_s^*(x(k), y_t)\| = 0.$$

Then, by continuity of the model function, the system converges to an equilibrium point

$$(x_s, u_s) = (\hat{x}_s^*(x_s, y_t), \hat{u}_s^*(x_s, y_t)).$$

Notice that, in this case, the warm start is given by the equilibrium point; that is, $\mathbf{v}^{[0]}(\infty) = (\mathbf{u}_{s,1}, \dots, \mathbf{u}_{s,M}, y_s)$. Furthermore, the solution of the optimization problem for the $i$th agent is $(\mathbf{u}_{s,i}, y_s)$, and then $\mathbf{v}^{[p]}(\infty) = \mathbf{v}^{[0]}(\infty)$ for all $p \in \mathbb{N}$. Therefore system (11) converges to $\mathbf{z}_s$.

Besides, it can be shown that

$$V_N(x_s, y_t; \mathbf{v}(\infty)) = V_O(y_s, y_t);$$

that is, the cost function is equal to the optimal solution of the centralized optimization control problem.

Now, it will be proved that $(x_s, u_s)$ is the steady state given by Theorem 1.

This result is obtained by contradiction. First of all, suppose that $(x_s, u_s, y_s)$ is not the optimal steady state. Hence, there exists an equilibrium point $(\tilde{x}_s, \tilde{u}_s, \tilde{y}_s)$ such that $V_O(\tilde{y}_s, y_t) < V_O(y_s, y_t)$.

Since $V_O(\hat{y}_s, y_t)$ is convex in $(\hat{y}_s - y_t)$, it can be proved that there exists a $\hat{\beta} \in [0, 1)$ such that, for every $\beta \in [\hat{\beta}, 1)$, the equilibrium point parameterized by

$$\tilde{y}_s^+ = \beta y_s + (1-\beta)\tilde{y}_s$$

is such that the control law $u = Kx + L\tilde{y}_s^+$, where $L = [-K, \ I_p]M_y$, steers the system from $x_s$ to $\tilde{x}_s^+$ fulfilling the constraints.

Defining as $\tilde{\mathbf{u}}$ the sequence of control actions derived from the control law $u = Kx + L\tilde{y}_s^+$, it is inferred that $(\tilde{\mathbf{u}}, \tilde{y}_s^+)$ is a feasible solution for $P_N(x_s, y_t)$ (Limon et al., 2008).

Since $V_O(y_s, y_t)$ is the centralized optimal cost at $x_s$, from Assumption 2, we have that

$$\begin{aligned} V_O(y_s, y_t) &\le V_N(x_s, y_t; \tilde{\mathbf{u}}, \tilde{y}_s^+) \\ &= \sum_{j=0}^{N-1} \overbrace{\|x(j) - \tilde{x}_s^+\|_Q^2 + \|K(x(j) - \tilde{x}_s^+)\|_R^2}^{\|x(j)-\tilde{x}_s^+\|^2_{(Q+K'RK)}} \\ &\quad + \|x(N) - \tilde{x}_s^+\|_P^2 + V_O(\tilde{y}_s^+, y_t) \\ &= \|x_s - \tilde{x}_s^+\|_P^2 + V_O(\tilde{y}_s^+, y_t) \\ &= (1-\beta)^2\|x_s - \tilde{x}_s\|_P^2 + V_O(\tilde{y}_s^+, y_t). \end{aligned}$$

Define now $W(x_s, y_t, \beta) = (1-\beta)^2\|x_s - \tilde{x}_s\|_P^2 + V_O(\tilde{y}_s^+, y_t)$, and notice that $W(x_s, y_t, 1) = V_N^*(x_s, y_t) = V_O(y_s, y_t)$.

The partial of $W(x_s, y_t, \beta)$ about $\beta$ is

$$\frac{\partial W(x_s, y_t, \beta)}{\partial \beta} = -2(1-\beta)\|x_s - \tilde{x}_s\|_P^2 + g'(y_s - \tilde{y}_s),$$

where $g' \in \partial V_O(\tilde{y}_s^+, y_t)$, defining $\partial V_O(\tilde{y}_s^+, y_t)$ as the subdifferential of $V_O(\tilde{y}_s^+, y_t)$. Evaluating this partial for $\beta = 1$, we obtain that

$$\left.\frac{\partial W(x_s, y_t, \beta)}{\partial \beta}\right|_{\beta=1} = \bar{g}'(y_s - \tilde{y}_s),$$

where $\bar{g}' \in \partial V_O(y_s, y_t)$, defining $\partial V_O(y_s, y_t)$ as the subdifferential of $V_O(y_s, y_t)$. Taking into account that $V_O$ is a convex function, and hence subdifferentiable, we can state that

$$\bar{g}'(y_s - \tilde{y}_s) \ge V_O(y_s, y_t) - V_O(\tilde{y}_s, y_t).$$

Considering that $V_O(y_s, y_t) - V_O(\tilde{y}_s, y_t) > 0$, it can be derived that

$$\left.\frac{\partial W(x_s, y_t, \beta)}{\partial \beta}\right|_{\beta=1} \ge V_O(y_s, y_t) - V_O(\tilde{y}_s, y_t) > 0.$$

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that the value of $W(x_s, y_t, \beta)$ is smaller than the value of $W(x_s, y_t, 1) = V_N^*(x_s, y_t) = V_O(y_s, y_t)$. But previously we stated that $W(x_s, y_t, \beta) \ge V_O(y_s, y_t) = V_N^*(x_s, y_t)$.

This contradicts the optimality of the solution, and hence it is proved that $y_s = \arg\min_{\hat{y}_s \in \mathcal{Y}_s} V_O(\hat{y}_s, y_t)$. $\square$

**Lemma 10** (*Lyapunov Stability*). *Let Assumption 2 hold. Then system (11) is Lyapunov stable at the equilibrium point* $\mathbf{z}_s$.

**Proof.** Define $\mathbf{z} = (x, \mathbf{v})$, $\Phi(\mathbf{z}) = V_N(x, y_t; \mathbf{v}) - V_O(y_s, y_t)$, and define also $\mathcal{Z}_c = \{\mathbf{z} | \Phi(\mathbf{z}) \le c\}$, $c > 0$, containing $\mathbf{z}_s$ in its interior and such that $\mathcal{Z}_c \subseteq \mathcal{Z}_N$. Such a set exists because $(x_s, u_s)$ is in the interior of $Z_s$.

Notice that $\Phi(\mathbf{z})$ is defined on $\mathcal{Z}_c$, it is positive definite with respect to $\mathbf{z}_s$ because $\Phi(\mathbf{z}_s) = 0$, it is positive away from $\mathbf{z} = \mathbf{z}_s$ due to the nonnegativity of the stage cost, terminal cost functions and offset cost function, and it is continuous by Assumption 2. Due to these facts, there exists a couple of $\mathcal{K}$-functions $\alpha$ and $\beta$ such that

$$\alpha(\|\mathbf{z} - \mathbf{z}_s\|) \le \Phi(\mathbf{z}) \le \beta(\|\mathbf{z} - \mathbf{z}_s\|) \quad \mathbf{z} \in \mathcal{Z}_c.$$

Following the same arguments as Rawlings and Mayne (2009, p. 608), choose $\epsilon > 0$ and define $\delta = \beta^{-1}(\alpha(\epsilon)) > 0$. Since $V_N(x(k+1), y_t; \mathbf{v}(k+1)) - V_N(x(k), y_t; \mathbf{v}(k)) \le 0$ for all $\mathbf{z}(k) \in \mathcal{Z}_c$, we have that, for all $\mathbf{z}(0) \in \mathcal{Z}_c$ such that $\|\mathbf{z}(0) - \mathbf{z}_s\| \le \delta$, then

$$\alpha(\|\mathbf{z}(k) - \mathbf{z}_s\|) \le \Phi(\mathbf{z}(k)) \le \Phi(\mathbf{z}(0)) \le \beta(\|\mathbf{z}(0) - \mathbf{z}_s\|).$$

Then $\alpha(\|\mathbf{z}(k) - \mathbf{z}_s\|) \le \beta(\delta) = \alpha(\epsilon)$, and hence $\|\mathbf{z}(k) - \mathbf{z}_s\| \le \epsilon$, for all $k > 0$. This fact establishes the stability of $\mathbf{z}_s$ for a constrained system. $\square$

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