Feedback stabilization of discrete-time quantum systems subject to non-demolition measurements with imperfections and delays [*]

H. Amini[†]   R. Somaraju[‡]   I. Dotsenko[§]   C. Sayrin[¶]   M. Mirrahimi[‖]

P. Rouchon[**]

June 18, 2018

Abstract

We consider a controlled quantum system whose finite dimensional state is governed by a discrete-time nonlinear Markov process. In open-loop, the measurements are assumed to be quantum non-demolition (QND). The eigenstates of the measured observable are thus the open-loop stationary states: they are used to construct a closed-loop supermartingale playing the role of a strict control Lyapunov function. The parameters of this supermartingale are calculated by inverting a Metzler matrix that characterizes the impact of the control input on the Kraus operators defining the Markov process. The resulting state feedback scheme, taking into account a known constant delay, provides the almost sure convergence of the controlled system to the target state. This convergence is ensured even in the case where the filter equation results from imperfect measurements corrupted by random errors with conditional probabilities given as a left stochastic matrix. Closed-loop simulations corroborated by experimental data illustrate the interest of such nonlinear feedback scheme for the photon box, a cavity quantum electrodynamics system.

1 Introduction

Manipulating quantum systems allows one to accomplish tasks far beyond the reach of classical devices. Quantum information is paradigmatic in this sense: quantum computers will substantially outperform classical machines for several problems [20]. Though significant progress has been made recently, severe difficulties still remain, amongst which deco-

[†]Centre Automatique et Systèmes, Mathématiques et Systèmes, Mines ParisTech, 60 Boulevard Saint-Michel, 75272 Paris Cedex 6, France.

[‡]B-Phot, Department of Applied Physics and Photonics, Vrije Universiteit Brussel, Pleinlaan no. 2, Brussels - 1050 (Belgium).

[§]Laboratoire Kastler-Brossel, ENS, UPMC-Paris 6, CNRS, 24 rue Lhomond, 75005 Paris, France.

[¶]Laboratoire Kastler-Brossel, ENS, UPMC-Paris 6, CNRS, 24 rue Lhomond, 75005 Paris, France.

[‖]INRIA Paris-Rocquencourt, Domaine de Voluceau, BP 105, 78153 Le Chesnay Cedex, France.

[**]Centre Automatique et Systèmes, Mathématiques et Systèmes, Mines ParisTech, 60 Boulevard Saint-Michel, 75272 Paris Cedex 6, France.

herence is certainly the most important. Large systems consisting of many qubits must be prepared in fragile quantum states, which are rapidly destroyed by their unavoidable coupling to the environment. Measurement-based feedback and coherent feedback are possible routes towards the preparation, protection and stabilization of such states. For coherent feedback strategy, the controller is also a quantum system coupled to the original one (see [13, 16] and the references therein). This paper is devoted to measurement-based feedback where the controller and the control input are classical objects [30]. The results presented here are directly inspired by a recent experiment [25, 24] demonstrating that such a quantum feedback scheme achieves the on-demand preparation and stabilization of non-classical states of a microwave field.

Following [12] and relying on continuous-time Lyapunov techniques exploited in [19], an initial measurement-based feedback was proposed in [10]. This feedback scheme stabilizes photon-number states (Fock states) of a microwave field (see e.g. [15] for a physical description of such cavity quantum electrodynamics (CQED) systems). The controller consists of a quantum filter that estimates the state of the field from discrete-time measurements performed by probe atoms, and secondly a stabilizing state-feedback that relies on Lyapunov techniques. The discrete-time behavior is crucial for a possible real-time implementation of such controllers. Closed-loop simulations reported in [10] have been confirmed by the stability analysis performed in [2]. In the experimental implementation [25, 24], the state-feedback has been improved by considering a strict Lyapunov function: this ensures better convergence properties by avoiding the passage by high photon numbers during the transient regime. The goal of this paper is to present, for a class of discrete-time quantum systems, the mathematical methods underlying such improved Lyapunov design. Our main result is given in Theorem 4.2 where closed-loop convergence is proved in presence of delays and measurement imperfections.

The state-feedback scheme may be applied to generic discrete-time finite-dimensional quantum systems using *controlled* measurements in order to deterministically prepare and stabilize the system at some pre-specified target state. The dynamics of these systems may be expressed in terms of a (classical) nonlinear controlled Markov chain, whose state space consists of the set of density matrices on some Hilbert space. The jumps in this Markov chain are induced by quantum measurements and the associated jump probabilities are state-dependent. These systems are subject to a discrete-time sequence of positive operator valued measurements (POVMs [20, 15]) and we use these POVMs to stabilize the system at the target state. By controlled measurements, we mean that at each time-step the chosen POVM is not fixed but is a function of some classical control signal u, similar to [29]. However, we assume that when the control u is zero, the chosen POVM performs a quantum non-demolition (QND) measurement [15, 30] for some orthonormal basis that includes the target state. The feedback-law is based on a Lyapunov function that is a linear combination of a set of martingales corresponding to the open-loop QND measurements. This Lyapunov function determines a "distance" between the target state and the current state. The parameters of this Lyapunov function are given by inverting Metzler matrices characterizing the impact of the control input on the Kraus operators defining the Markov processes and POVMs. The (graph theoretic) properties of the Metzler matrices are used to construct families of open-loop supermartingales that become strict supermartingales in closed-loop. This fact provides directly the convergence to the target state without using the invariance principle.

A common problem that occurs in quantum feedback control is that of delays between the measurement process and the control process [21, 17]. In this paper we demonstrate, using a predictive quantum filter, that the proposed scheme works even in the presence of delays. Convergence analysis is done for perfect and imperfect measurements. For imperfect measurements, the dynamics of the system are governed by a nonlinear Markov chain given in [26].

In both the perfect and imperfect measurement situations we prove a robustness property of the feedback algorithm: the convergence of the closed-loop system is ensured even when the feedback law is based on the state of a quantum filter that is not initialized correctly. This robustness property, is similar

in spirit to the separation principle proven in [7, 8]. We use the fact that the state space is a convex set and the target state, being a pure state, is an extreme point of this convex state space and therefore cannot be expressed as a convex combination of any other states. One then uses the linearity of the conditional expectation to prove the robustness property. Our result is only valid for target quantum states that are pure states.

The paper is organized as follows. In Section 2, we describe the finite dimensional Markov model together with the main modeling assumptions in the case of perfect measurements and study the open-loop behavior (Theorem 2.1) which can be seen as a non-deterministic protocol for preparing a finite number of isolated and orthogonal quantum states. In Section 3, we present the main ideas underlying the construction of these control-Lyapunov functions W_ϵ: a Metzler matrix attached to the second derivative of the measurement operators and a technical lemma assuming this Metzler matrix is irreducible. Finally, Theorem 3.1 describes the stabilizing state feedback derived from W_ϵ. The same analysis is done for the case of imperfect measurements in Section 4. For the state estimations used in the feedback scheme we propose a brief discussion on the quantum filters and prove a rather general robustness property for perfect measurements in Section 3 with Theorem 3.2 and for imperfect ones in Section 4 with Theorem 4.2. Section 5 is devoted to the experimental implementation that has been done at Laboratoire Kastler-Brossel of Ecole Normale Supérieure de Paris. Closed-loop simulations and experimental data complementary to those reported in [25, 24] are presented.

2 System model and open-loop dynamics

2.1 The nonlinear Markov model

We consider a finite dimensional quantum system (the underlying Hilbert space $\mathcal{H} = \mathbb{C}^d$ is of dimension $d > 0$) being measured through a generalized measurement procedure at discrete-time intervals. The dynamics are described by a nonlinear controlled Markov chain. Here, we suppose perfect measurements and no decoherence. The system state is described by a density operator ρ belonging to $\mathcal{D}$, the set of non-negative Hermitian matrices of trace one:
$\mathcal{D} := \{\rho \in \mathbb{C}^{d \times d} \mid \rho = \rho^\dagger, \ \text{Tr}(\rho) = 1, \ \rho \geq 0\}$. To each measurement outcome $\mu \in \{1, \ldots, m\}$, m being the numbers of possible outcomes, is attached the Kraus operator $M_\mu^u \in \mathbb{C}^{d \times d}$ depending on μ and on a scalar control input $u \in \mathbb{R}$. For each u, $(M_\mu^u)_{\mu \in \{1, \ldots, m\}}$ satisfy the constraint $\sum_{\mu=1}^m M_\mu^{u\dagger} M_\mu^u = \mathbf{I}$, the identity matrix. The Kraus map $\mathbb{K}^u$ is defined by

$$\mathcal{D} \ni \rho \mapsto \mathbb{K}^u(\rho) = \sum_{\mu=1}^m M_\mu^u \rho M_\mu^{u\dagger} \in \mathcal{D}. \qquad (1)$$

The random evolution of the state $\rho_k \in \mathcal{D}$ at time-step k is modeled through the following dynamics:

$$\rho_{k+1} = \mathbb{M}_{\mu_k}^{u_{k-\tau}}(\rho_k) := \frac{M_{\mu_k}^{u_{k-\tau}} \rho_k \left(M_{\mu_k}^{u_{k-\tau}}\right)^\dagger}{\text{Tr}\left(M_{\mu_k}^{u_{k-\tau}} \rho_k \left(M_{\mu_k}^{u_{k-\tau}}\right)^\dagger\right)}, \qquad (2)$$

where $u_{k-\tau}$ is the control at step k, subject to a delay of $\tau > 0$ steps. This delay is usually due to delays in the measurement process that can also be seen as delays in the control process. μ_k is a random variable taking values μ in $\{1, \ldots, m\}$ with probability $p_{\mu, \rho_k}^{u_{k-\tau}} = \text{Tr}\left(M_\mu^{u_{k-\tau}} \rho_k \left(M_\mu^{u_{k-\tau}}\right)^\dagger\right)$. For each μ, $\mathbb{M}_\mu^u(\rho)$ is defined when $p_{\mu, \rho}^u = \text{Tr}\left(M_\mu^u \rho \left(M_\mu^u\right)^\dagger\right) \neq 0$.

We now state some assumptions that we will be using in the remainder of this paper.

Assumption 1. *For $u = 0$, all M_μ^0 are diagonal in the same orthonormal basis $\{|n\rangle \mid n \in \{1, \ldots, d\}\}$: $M_\mu^0 = \sum_{n=1}^d c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.*

Assumption 2. *For all $n_1 \neq n_2$ in $\{1, \ldots, d\}$, there exists $\mu \in \{1, \ldots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.*

Assumption 3. *All M_μ^u are C^2 functions of u.*

Assumption 1 means that when $u = 0$ the measurements are quantum non demolition (QND) measurements over the states $\{|n\rangle \mid n \in \{1, \ldots, d\}\}$:

when $u_k \equiv 0$, any $\rho = |n\rangle \langle n|$ (orthogonal projector on the basis vector $|n\rangle$) is a fixed point of (2). Since $\sum_{\mu=1}^{m} M_\mu^{0\dagger} M_\mu^0 = \mathbf{I}$, we have $\sum_{\mu=1}^{m} |c_{\mu,n}|^2 = 1$ for all $n \in \{1, \ldots, d\}$ according to Assumption 1. Assumption 2 means that there exists a μ such that the statistics when $u_k \equiv 0$ for obtaining the measurement result μ are different for the fixed points $|n_1\rangle \langle n_1|$ and $|n_2\rangle \langle n_2|$. This follows by noting that $\mathrm{Tr}\left(M_\mu^0 |n\rangle \langle n| M_\mu^{0\dagger} \right) = |c_{\mu,n}|^2$ for $n \in \{1, \ldots, d\}$. Assumption 3 is a technical assumption we will use in our proofs.

2.2 Convergence of the open-loop dynamics

When the control input vanishes ($u \equiv 0$), the dynamics are simply given by

$$\rho_{k+1} = \mathbb{M}_{\mu_k}^0(\rho_k), \tag{3}$$

where μ_k is a random variable with values in $\{1, \ldots, m\}$. The probability p_{μ,ρ_k}^0 to have $\mu_k = \mu$ depends on ρ_k: $p_{\mu,\rho_k}^0 = \mathrm{Tr}\left(M_\mu^0 \rho_k M_\mu^{0\dagger} \right)$.

Theorem 2.1. *Consider a Markov process ρ_k obeying the dynamics of (3) with an initial condition ρ_0 in $\mathcal{D}$. Then with probability one, ρ_k converges to one of the d states $|n\rangle \langle n|$ with $n \in \{1, \ldots, d\}$ and the probability of convergence towards the state $|n\rangle \langle n|$ is given by $\langle n| \rho_0 |n\rangle$.*

This Theorem is already proved in [2, 3] and also in [4] in a slightly different formulation. A direct proof is based on the following Lyapunov function:

$$\Gamma(\rho) := -\sum_{n=1}^{d} \frac{\left(\langle n| \rho |n\rangle \right)^2}{2}. \tag{4}$$

Since $\langle n| \rho |n\rangle$ is a martingale for any n and since

$$\Gamma(\rho_k) - \mathbb{E}\left(\Gamma(\rho_{k+1}) | \rho_k \right) = Q_1(\rho_k) :=$$

$$\sum_{n,\mu,\nu} \frac{p_{\mu,\rho_k}^0 p_{\nu,\rho_k}^0}{4} \left(\frac{|c_{\mu,n}|^2 \langle n|\rho_k|n\rangle}{p_{\mu,\rho_k}^0} - \frac{|c_{\nu,n}|^2 \langle n|\rho_k|n\rangle}{p_{\nu,\rho_k}^0} \right)^2 \tag{5}$$

3 Feedback stabilization with perfect measurements

In Subsection 3.1, we give an overview of the control method and then in Subsections 3.2 and 3.3, we prove the main results. Finally in Subsection 3.4, we prove a robustness principle that explains how we can ensure convergence even if the initial state ρ_0 is unknown.

3.1 Overview of the control method

Theorem 2.1 shows that the open-loop dynamics are stable in the sense that in each realization ρ_k converges (non-deterministically) to one of the pure states $|n\rangle \langle n|$ with probability $\langle n| \rho_0 |n\rangle$. The control goal is to make this convergence deterministic toward a chosen $\bar{n} \in \{1, \ldots, d\}$ playing the role of controller set point. We build on the ideas in [10, 2, 3, 27] to design a controller that is based on a strict Lyapunov function for the target state. In this paper we assume arbitrary controlled Kraus operators M_μ^u that cannot be decomposed into QND measurement operators M_μ^0 followed by a unique controlled unitary operator D_u with $D_0 = \mathbf{I}$ as assumed in [10, 2, 3, 27] where $M_\mu^u \equiv D_u M_\mu^0$. It can be argued that the control we are proposing is non-Hamiltonian control [22, 23], as the control parameter u is not necessarily a parameter in the interaction Hamiltonian and could indeed be any parameter of an auxiliary system such as the measurement device.

To convey the main ideas involved in the control design, we begin with the case where there are no delays ($\tau = 0$) and we assume the initial state is known. We wish to use the open-loop supermartingales to design a Lyapunov function for the closed-loop system. By an open-loop supermartingale we mean any function $V : \mathcal{D} \to \mathbb{R}$ such that $\mathbb{E}\left(V(\rho_{k+1}) | \rho_k = \rho, u_k = 0 \right) \leq V(\rho)$ for all $\rho \in \mathcal{D}$. The Lyapunov function underlying Theorem 2.1 demonstrates how we can construct such open-loop supermartingales. Now, at each time-step k, the feedback signal u_k is chosen by minimizing this

4

supermartingale V knowing the state ρ_k:

$$u_k = \hat{u}(\rho) := \underset{u \in [-\bar{u}, \bar{u}]}{\mathrm{argmin}} \left\{ \mathbb{E}\left(V(\rho_{k+1}) | \rho_k = \rho, u_k = u \right) \right\}.$$

Here $\bar{u}$ is some small positive number that needs to be determined. Because $0 \in [-\bar{u}, \bar{u}]$ and the control u_k is chosen to minimize V at each step, we directly have that V is a *closed-loop supermartingale*, i.e.,

$$\tilde{Q}(\rho) := \mathbb{E}\left(V(\rho_{k+1}) | \rho_k = \rho, u_k = \hat{u}(\rho) \right) - V(\rho) \le 0$$

for all $\rho \in \mathcal{D}$. If this supermartingale V is bounded from below then we can directly apply the convergence Theorem A.1 in the appendix to prove that ρ_k converges with probability one to the set $I_\infty := \{\rho : \tilde{Q}(\rho) = 0\}$.

What remains to be done is to choose an appropriate open-loop supermartingale V so that the set I_∞ is restricted to the target state $\{|\bar{n}\rangle \langle \bar{n}|\}$. The Lyapunov function Γ defined in (4) does not discriminate between the different basis vectors. We therefore use the Lyapunov function $V_\epsilon(\rho) = V_0(\rho) - \frac{\epsilon}{2} \sum_{n=1}^{d} \langle n| \rho |n\rangle^2$ where $V_0(\rho) = \sum_{n=1}^{d} \sigma_n \langle n| \rho |n\rangle$ and ϵ is a small positive constant. The weights σ_n are strictly positive numbers except for $\sigma_{\bar{n}} = 0$. This function V_ϵ is clearly a concave function of the open-loop martingales $\langle n| \rho |n\rangle$ and therefore is an open-loop supermartingale. Moreover, the weights σ_n can be used to quantify the distance of the state ρ from the target state $|\bar{n}\rangle \langle \bar{n}|$ (c.f. Figure 2 below which shows how σ_n are chosen for the experimental setting).

A state ρ is in the set I_∞ if and only if for all $u \in [-\bar{u}, \bar{u}]$, we have

$$\mathbb{E}\left(V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = u \right) - V_\epsilon(\rho) \ge 0. \quad (6)$$

Also from the fact that V_ϵ is an open-loop supermartingale, we have for all $\rho \in \mathcal{D}$

$$\mathbb{E}\left(V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = 0 \right) - V_\epsilon(\rho) \le 0. \quad (7)$$

We prove in Lemma 3.2 below that given any $\bar{n} \in \{1, \ldots, d\}$, we can always choose the weights $\sigma_1, \ldots, \sigma_d$ so that V_ϵ satisfies the following property: $\forall n \in \{1, \ldots, d\}$, $u \mapsto$

$\mathbb{E}\left(V_\epsilon(\rho_{k+1}) | \rho_k = |n\rangle \langle n|, u_k = u \right)$ has a strict local minimum at $u = 0$ for $n = \bar{n}$ and strict local maxima at $u = 0$ for $n \ne \bar{n}$. This combined with Equation (7) then ensures that for any $n \ne \bar{n}$, there is some $u \in [-\bar{u}, \bar{u}]$ such that $\mathbb{E}\left(V_\epsilon(\rho_{k+1}) | \rho_k = |n\rangle \langle n|, u_k = u \right) - V_\epsilon(|n\rangle \langle n|) < 0$. Therefore using Equation (6), we know that $|n\rangle \langle n|$ is in the limit set I_∞ if and only if $n = \bar{n}$.

This idea can easily be extended to the situations where the delay τ is non zero. Take $(\rho_k, u_{k-1}, \ldots, u_{k-\tau})$ as state at step k: denote by $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$ this state where β_r stands for the control input u delayed r steps. Then the state form of the delayed dynamics (2) is governed by the following Markov chain

$$\begin{cases} \rho_{k+1} = \mathbb{M}_{\mu_k}^{\beta_{\tau,k}}(\rho_k) \\ \beta_{1,k+1} = u_k, \quad \beta_{r,k+1} = \beta_{r-1,k} \text{ for } r = 2, \ldots \tau. \end{cases} \quad (8)$$

The goal is to design a feedback law $u_k = f(\chi_k)$ that globally stabilizes this Markov chain χ_k towards a chosen target state $\bar{\chi} = (|\bar{n}\rangle \langle \bar{n}|, 0, \ldots, 0)$ for some $\bar{n} \in \{1, \ldots, d\}$. In Theorem 3.1, we show how to design a feedback relying on the control Lyapunov function $W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots \ldots \mathbb{K}^{\beta_\tau}(\rho) \ldots)))$. The idea is to use a predictive filter to estimate the state of the system τ time-steps later.

Finally, we address the situation where the initial state of the system is not fully known but only estimated by ρ_0^{est}. We show under some assumptions on the initial condition that, the feedback law based on the state ρ_k^{est} of the miss-initialized filter still ensures the convergence of ρ_k^{est} as well as the well-initialized conditional state ρ_k towards $|\bar{n}\rangle \langle \bar{n}|$. This demonstrates how the control algorithm is robust to uncertainties in the initialization of the estimated state of the quantum system.

3.2 Choosing the weights σ_n

The construction of the control Lyapunov function relies on two lemmas.

Lemma 3.1. *Consider the $d \times d$ matrix R defined by*

$$R_{n_1,n_2} = \sum_\mu \left(2 \left| \langle n_1| \frac{dM_\mu^{u\dagger}}{du} \right|_{u=0} |n_2\rangle \right|^2 +$$

$$2\delta_{n_1,n_2} \Re\left(c_{\mu,n_1} \langle n_1| \frac{d^2 M_\mu^{u\dagger}}{du^2} \Big|_{u=0} |n_2\rangle \right) \right).$$

When $R \neq 0$, the non-negative $P = \mathbf{I} - R/\mathrm{Tr}\,(R)$ is a right stochastic matrix.

Proof. For $n_1 \neq n_2$, $R_{n_1,n_2} \geq 0$. Thus R is a Metzler matrix [1]. Let us prove that the sum of each row vanishes. This results from identity $\sum_\mu M_\mu^{u\dagger} M_\mu^u = \mathbf{I}$. Deriving twice versus u the relation

$$\sum_{\mu,n_2} \langle n_1| M_\mu^{u\dagger} |n_2\rangle \langle n_2| M_\mu^u |n_1\rangle =$$

$$\sum_\mu \langle n_1| M_\mu^{u\dagger} M_\mu^u |n_1\rangle = 1,$$

yields

$$\sum_{\mu,n_2} 2 \langle n_1| \frac{dM_\mu^{u\dagger}}{du} |n_2\rangle \langle n_2| \frac{dM_\mu^u}{du} |n_1\rangle +$$

$$\langle n_1| \frac{d^2 M_\mu^{u\dagger}}{du^2} |n_2\rangle \langle n_2| M_\mu^u |n_1\rangle +$$

$$\langle n_1| M_\mu^{u\dagger} |n_2\rangle \langle n_2| \frac{d^2 M_\mu^u}{du^2} |n_1\rangle = 0.$$

Since for $u = 0$, $\langle n_2| M_\mu^0 |n_1\rangle = \delta_{n_1,n_2} c_{\mu,n_1}$ the above sum corresponds to $\sum_{n_2} R_{n_1,n_2}$. Therefore, the diagonal elements of R are non-positive. If $R \neq 0$, then $\mathrm{Tr}\,(R) < 0$ and the matrix $P = \mathbf{I} - R/\mathrm{Tr}\,(R)$ is well defined with non-negative entries. Since the sum of each row of R vanished, the sum of each row of P is equal to 1. Thus P is a right stochastic matrix. $\square$

To the Metzler matrix R defined in Lemma 3.1, we associate its directed graph denoted by G. This graph admits d vertices labeled by $n \in \{1, \ldots, d\}$. To each strictly positive off-diagonal element of the matrix R, say, on the n_1'th row and the n_2'th column we associate an edge from vertex n_1 towards vertex n_2.

---

[1] A Metzler matrix is a matrix such that all the off-diagonal components are non-negative.

Lemma 3.2. *Assume the directed graph G of the matrix R defined in Lemma 3.1 is strongly connected, i.e., for any $n, n' \in \{1, \ldots, d\}$, $n \neq n'$, there exists a chain of r distinct elements $(n_j)_{j=1,\ldots,r}$ of $\{1, \ldots, d\}$ such that $n_1 = n$, $n_r = n'$ and for any $j = 1, \ldots r-1$, $R_{n_j,n_{j+1}} \neq 0$. Take $\bar{n} \in \{1, \ldots, d\}$. Then, there exist $d-1$ strictly positive real numbers e_n, $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, such that*

- *for any reals λ_n, $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, there exists a unique vector $\sigma = (\sigma_n)_{n \in \{1,\ldots,d\}}$ of $\mathbb{R}^d$ with $\sigma_{\bar{n}} = 0$ such that $R\sigma = \lambda$ where λ is the vector of $\mathbb{R}^d$ of components λ_n for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$ and $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$; if additionally $\lambda_n < 0$ for all $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, then $\sigma_n > 0$ for all $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$.*

- *for any vector $\sigma \in \mathbb{R}^d$, solution of $R\sigma = \lambda \in \mathbb{R}^d$, the function $V_0(\rho) = \sum_{n=1}^d \sigma_n \langle n| \rho |n\rangle$ satisfies*

$$\frac{d^2 V_0 \left(\mathbb{K}^u(|n\rangle \langle n|) \right)}{du^2} \Bigg|_{u=0} = \lambda_n \qquad \forall n \in \{1, \ldots, d\}.$$

Proof. Since the directed graph G coincides with the directed graph of the right stochastic matrix P defined in Lemma 3.1, P is irreducible. Since it is a right stochastic matrix, its spectral radius is equal to 1. By Perron-Frobenius theorem for non-negative irreducible matrices, this spectral radius, i.e., 1, is also an eigen-value of P and of P^T, with multiplicity one and associated to eigen-vectors having strictly positive entries: the right eigen-vector $(Pw = w)$ is obviously $w = (1, \ldots, 1)^T$; the left eigen-vector $e = (e_1, \ldots, e_n)^T$ $(P^T e = e)$ can be chosen such that $e_{\bar{n}} = 1$. Consequently, the rank of R is $d - 1$ with $\ker(R) = \mathrm{span}(w)$ and $\mathrm{Im}(R) = e^\perp$ where $e^\perp$ is the hyper-plane orthogonal to e. Since $e^T \lambda = 0$, $\lambda \in \mathrm{Im}(R)$, exists σ such that $R\sigma = \lambda$. Since $\ker(R) = \mathrm{span}(w)$, there is a unique σ solution of $R\sigma = \lambda$ such that $\sigma_{\bar{n}} = 0$. The fact that $\sigma_n > \sigma_{\bar{n}}$ when $\lambda_n < 0$ for $n \neq \bar{n}$, comes from elementary manipulations of $P\sigma = \sigma - \lambda/\mathrm{Tr}\,(R)$ showing that $\min_{n \neq \bar{n}} \sigma_n > \sigma_{\bar{n}}$.

$\forall n$, set $g_n^u = V_0(\mathbb{K}^u(|n\rangle \langle n|)) = \sum_l \sigma_l \langle l| \mathbb{K}^u(|n\rangle \langle n|) |l\rangle$. Set $P_l^u := \langle l| \mathbb{K}^u(|n\rangle \langle n|) |l\rangle$. Then $\frac{dP_l^u}{du}$ is given by $\sum_\mu \langle l| \frac{dM_\mu^u}{du} |n\rangle \langle n| M_\mu^{u\dagger} |l\rangle +$

$\langle l | M_\mu^u | n \rangle \langle n | \frac{dM_\mu^{u\,\dagger}}{du} | l \rangle$ and

$$\frac{d^2 P_l^u}{du^2} \Big|_{u=0} = \sum_\mu \Big(\langle l | \frac{d^2 M_\mu^u}{du^2} \Big|_{u=0} | n \rangle \langle n | M_\mu^{0\,\dagger} | l \rangle$$

$$+ \langle l | \frac{dM_\mu^u}{du} \Big|_{u=0} | n \rangle \langle n | \frac{dM_\mu^{u\,\dagger}}{du} \Big|_{u=0} | l \rangle$$

$$+ \langle l | M_\mu^0 | n \rangle \langle n | \frac{d^2 M_\mu^{u\,\dagger}}{du^2} \Big|_{u=0} | l \rangle \Big)$$

$$= \sum_\mu 2 \Big(\Big| \langle n | \frac{dM_\mu^{u\,\dagger}}{du} \Big|_{u=0} | l \rangle \Big|^2$$

$$+ 2\delta_{nl} \Re \big(c_{\mu,n} \langle n | \frac{d^2 M_\mu^{u\,\dagger}}{du^2} \Big|_{u=0} | l \rangle \big) \Big).$$

Therefore $\frac{d^2 P_l^u}{du^2} \Big|_{u=0} = R_{n,l}$ and $\frac{d^2 V_0 \big(\mathbb{K}^u (|n\rangle\langle n|) \big)}{du^2} \Big|_{u=0} = \sum_{l=1}^d R_{n,l} \sigma_l = \lambda_n.$ $\qquad \square$

3.3 The global stabilizing feedback

The main result of this section is expressed through the following theorem.

Theorem 3.1. *Consider the Markov chain* (8) *with Assumptions 1, 2 and 3. Take $\bar{n} \in \{1, \ldots, d\}$ and assume that the directed graph G associated to the Metzler matrix R of Lemma 3.1 is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}_+^d$ the solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$ (see Lemma 3.2) and $V_\epsilon(\rho) = \sum_{n=1}^d \sigma_n \langle n| \rho |n\rangle - \frac{\epsilon}{2} (\langle n| \rho |n\rangle)^2$. Take $\bar{u} > 0$ and consider the following feedback law*

$$u_k = \underset{\xi \in [-\bar{u},\bar{u}]}{argmin} \big(\mathbb{E} \left(W_\epsilon(\chi_{k+1}) | \chi_k, u_k = \xi \right) \big) := f(\chi_k)$$

where $W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots \ldots \mathbb{K}^{\beta_\tau}(\rho) \ldots)))$. Then there exists $u^* > 0$ such that, for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}} \right) \right]$, the closed-loop Markov chain of state χ_k with the feedback law of Theorem 3.1 converges almost surely towards $(|\bar{n}\rangle \langle \bar{n}|, 0, \ldots, 0)$ for any initial condition $\chi_0 \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau$.

Proof. For the sake of simplicity, first we demonstrate this Theorem for $\tau = 1$ and thus for $\chi = (\rho, \beta_1)$. We then explain how this proof may be extended to arbitrary $\tau > 1$. Here, $\mathbb{E} \left(W_\epsilon(\chi_{k+1}) | \chi_k, u_k \right)$ is given

by $\sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^{u_k}(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k)))$ that can also be presented as $\sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^0(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))) - Q_2(\chi_k)$ with $Q_2 \geq 0$:

$$Q_2(\chi_k) = \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^0(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))) -$$
$$\min_{\xi \in [-\bar{u},\bar{u}]} \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^\xi(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))). \quad (9)$$

Since $V_\epsilon(\mathbb{K}^0(\rho)) = V_\epsilon(\rho)$ for any $\rho \in \mathcal{D}$, we have

$$\mathbb{E} \left(W_\epsilon(\chi_{k+1}) | \chi_k, u_k \right) = W_\epsilon(\chi_k) - Q_1(\chi_k) - Q_2(\chi_k) \tag{10}$$

where Q_1 is defined in (5). Thus, $W_\epsilon(\chi_k)$ is a supermartingale. According to Theorem A.1, the ω-limit set I_∞ of χ_k is included in

$$\{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid Q_1(\rho, \beta_1) = 0, \ Q_2(\rho, \beta_1) = 0\}.$$

We show in 3 steps that I_∞ is restricted to the target state.

Step 1. For all $\delta > 0$, there exists $\bar{u} > 0$ such that

$$I_\infty \subset \bigcup_{n=1}^d \Big\{ (\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta \Big\}.$$

Proof of step 1. Step 1 follows from $Q_1(\chi) = 0$ for all $(\rho, \beta_1) \in I_\infty$ and technical Lemma A.1 of the appendix.

Step 2. There exist $\delta' > 0$ and $\bar{u} > 0$ such that $I_\infty \subset \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle \bar{n}|\rho|\bar{n}\rangle \geq 1 - \delta'\}$.

Proof of step 2. If $Q_2(\rho, \beta_1) = 0$, the minimum of the function $[-\bar{u}, \bar{u}] \ni \xi \mapsto F_{\rho,\beta_1}(\xi) = \sum_\mu p_{\mu,\rho}^{\beta_1} V_\epsilon(\mathbb{K}^\xi(\mathbb{M}_\mu^{\beta_1}(\rho)))$ is attained at $\xi = 0$. By construction of V_ϵ, we know that, for $\beta_1 = 0$, $\rho = |n\rangle \langle n|$, $\frac{d^2 F_{|n\rangle\langle n|,0}(\xi)}{d\xi^2} \Big|_{\xi=0} < 0$ if $n \neq \bar{n}$, and $\epsilon \leq \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}} \right)$. The dependence of $d^2 F_{\rho,\beta_1}/d\xi^2$ versus ρ and β_1 is continuous from Assumption 3. Therefore, $\exists \delta', \bar{u} > 0$ such that for all (ρ, β_1) in $\bigcup_{n \neq \bar{n}} \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta'\}$ we have $\frac{d^2 F_{\rho,\beta_1}(\xi)}{d\xi^2} \Big|_{\xi=0} < 0$. Therefore, such (ρ, β_1) and $\xi = 0$, cannot minimize $F_{\rho,\beta_1}(\xi)$ and we have $I_\infty \cap \bigcup_{n \neq \bar{n}} \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta'\} = \emptyset$. Therefore we get,

$$I_\infty \subset \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle \bar{n}|\rho|\bar{n}\rangle \geq 1 - \delta'\},$$

by setting $\delta = \delta'$ in Step 1, for $\bar{u}$ small enough.

Step 3. $\bar{u}$ can be chosen small enough so that $I_\infty = \{(|\bar{n}\rangle\langle\bar{n}|, 0)\}$.

Proof of step 3. By construction of V_ϵ, we have for $\rho = |\bar{n}\rangle\langle\bar{n}|$ and $\beta_1 = 0$, $\left.\frac{d^2 F_{|\bar{n}\rangle\langle\bar{n}|,0}(\xi)}{d\xi^2}\right|_{\xi=0} > 0$. By continuity of $d^2 F_{\rho,\beta_1}/d\xi^2$ with respect to ρ and β_1, we can choose $\bar{\delta}$ and $\bar{u}$ small enough such that for all $\beta_1 \in [-\bar{u}, \bar{u}]$ and ρ satisfying $\langle\bar{n}|\rho|\bar{n}\rangle \geq 1 - \bar{\delta}$ we have $\left.\frac{d^2 F_{\rho,\beta_1}(\xi)}{d\xi^2}\right|_{\xi=0} > 0$. This implies that on the domain $\{(\rho, \beta_1, \xi) \in \mathcal{D} \times [-\bar{u}, \bar{u}]^2 \mid \langle n|\rho|n\rangle \geq 1 - \bar{\delta}\}$, F is a uniformly strongly convex function of $\xi \in [-\bar{u}, \bar{u}]$. Thus the argument of its minimum over $\xi \in [-\bar{u}, \bar{u}]$ is a continuous function of ρ and β_1. Now we choose $\delta'' = \min\{\bar{\delta}/2, \delta'/2\}$, where δ' is as in step 2. Take a convergent subsequence of $\{\chi_k\}_{k=1}^\infty$ (that we still denote by χ_k for simplicity sakes). Its limit $(\rho_\infty, \beta_{1,\infty})$ belongs to I_∞ and also to $\{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \bar{\delta}\}$. Therefore, $u_k = \underset{\xi\in[-\bar{u},\bar{u}]}{\operatorname{argmin}} F_{\rho_k,\beta_{1,k}}(\xi)$ converges to $\underset{\xi\in[-\bar{u},\bar{u}]}{\operatorname{argmin}} F_{\rho_\infty,\beta_{1,\infty}}(\xi) = 0$ since $Q_2(\rho_\infty, \beta_{1,\infty}) = 0$. Because $\beta_{1,k} = u_{k-1}$, $\beta_{1,k}$ tends almost surely towards 0. We know that $Q_1(\rho_k, \beta_{1,k})$ and $Q_2(\rho_k, \beta_{1,k})$ tend also almost surely to 0 (Theorem A.1) and by uniform (in ρ) continuity with respect to $\beta_{1,k}$, $Q_1(\rho_k, 0)$ and $Q_2(\rho_k, 0)$ tend almost surely to 0. Since $Q_2(\rho, 0) = Q_1(\rho, 0) = 0$ implies $\rho = |\bar{n}\rangle\langle\bar{n}|$, ρ_k converges almost surely towards $|\bar{n}\rangle\langle\bar{n}|$. This completes the proof for $\tau = 1$.

Extension to $\tau > 1$. For $\tau > 1$ and $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$, the proof is very similar. We still have (10) with Q_1 and Q_2 given by formulae analogous to (5) and (9): $\beta_{1,k}$ is replaced by $\beta_{\tau,k}$; $\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k)$ is replaced by $\mathbb{K}^{\beta_{1,k}} \ldots \mathbb{K}^{\beta_{\tau-1,k}}(\mathbb{M}_\mu^{\beta_{\tau,k}}(\rho_k))$; $p_{\mu,\rho_k}^{\beta_{1,k}}$ is replaced by $\operatorname{Tr}\left(\mathbb{K}^{\beta_{1,k}} \ldots \mathbb{K}^{\beta_{\tau-1,k}}(M_\mu^{\beta_{\tau,k}} \rho_k M_\mu^{\beta_{\tau,k}\dagger})\right)$ since Kraus maps are trace preserving. The analogue of technical Lemma A.1, whose proof relies on similar continuity and compactness arguments, has now the following statement: $\exists C > 0$ *such that* $\forall(\rho, \beta_1, \ldots, \beta_\tau) \in \mathcal{D} \times [-\tilde{u}, \tilde{u}]^\tau$ *satisfying* $Q_1(\rho, \beta_1, \ldots, \beta_\tau) = 0$, $\exists n \in \{1, \ldots, d\}$ *such that* $\rho_{n,n} \geq 1 - C(|\beta_1| + |\beta_2| + \ldots +$ $|\beta_\tau|)$.

The last part of the proof showing that, for $\bar{u}$ small enough, the control u_k is a continuous function of χ_k when χ_k is in the neighborhood of the ω-limit set $\{\chi \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau \mid Q_1(\chi) = Q_2(\chi) = 0\}$ remains almost the same. $\square$

3.4 Quantum filter and robustness property of filter

When the measurement process is fully efficient (i.e., the detectors are ideal with detection efficiency equal to one and they observe all the quantum jumps) and the jump model (2) admits no error, the Markov process (8) represents a natural choice for estimating the hidden state ρ. Indeed, the estimate ρ^{est} of ρ satisfies the following recursive dynamics

$$\rho_{k+1}^{\text{est}} = \mathbb{M}_{\mu_k}^{u_{k-\tau}}(\rho_k^{\text{est}}), \tag{11}$$

where the measurement outcome μ_k is driven by (2). In practice, the control u_k defined in Theorem 3.1 could only depend on this estimation ρ_k^{est} replacing ρ_k in χ_k. If $\rho_0^{\text{est}} = \rho_0$, then $\rho_k^{\text{est}} \equiv \rho_k$ and Theorem 3.1 ensures convergence towards the target state. Otherwise, the following result guaranties the convergence of such observer/controller scheme when $\ker(\rho_0^{\text{est}}) \subset \ker(\rho_0)$.

Theorem 3.2. *Consider the recursive Equation* (2) *and assume that the assumptions of Theorem 3.1 are satisfied. For each measurement outcome μ_k given by* (2), *consider the estimation ρ_k^{est} given by* (11) *with an initial condition ρ_0^{est}. Set $u_k = f(\rho_k^{\text{est}}, u_{k-1}, \ldots, u_{k-\tau})$ where f is given in Theorem 3.1. Then there exists $u^* > 0$ such that, for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, ρ_k and ρ_k^{est} converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$ as soon as $\ker(\rho_0^{\text{est}}) \subset \ker(\rho_0)$.*

Proof. Set $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$. The main idea is that $\mathbb{E}\left(\operatorname{Tr}(\rho_k\bar{\rho}) | \rho_0, \rho_0^{\text{est}}\right)$ (where we take the expectation over all jump realizations) depends linearly on ρ_0 even though we apply the feedback control. Since $u_{k-\tau}$ is a function of $(\rho_0^{\text{est}}, \mu_0, \ldots, \mu_{k-\tau-1})$, $\mathbb{E}\left(\operatorname{Tr}(\rho_k\bar{\rho}) | \rho_0, \rho_0^{\text{est}}\right)$ is given

8

by $\sum_{\mu_0,\ldots,\mu_{k-1}} \mathrm{Tr}\left(\widetilde{M}_{\mu_{k-1}}^{u_{k-\tau}}\left(\ldots\left(\widetilde{M}_{\mu_0}^{u_{-\tau}}(\rho_0)\right)\right)\bar{\rho}\right)$, where $\widetilde{M}_\mu^u(\rho) = M_\mu^u \rho M_\mu^{u\dagger}$. The linearity of $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right)$ with respect to ρ_0 is thus verified. Now we apply the assumption $\ker(\rho_0^{\mathrm{est}}) \subset \ker(\rho_0)$ and therefore one can find a constant $\gamma \in]0,1[$ and a well-defined density matrix ρ_0^c in $\mathcal{D}$, such that $\rho_0^{\mathrm{est}} = \gamma\rho_0 + (1-\gamma)\rho_0^c$. Applying the dominated convergence theorem: $\lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0^{\mathrm{est}}, \rho_0^{\mathrm{est}}\right) = 1$. By the linearity of $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right)$ with respect to ρ_0, $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0^{\mathrm{est}}, \rho_0^{\mathrm{est}}\right)$ is given by $\gamma\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right) + (1-\gamma)\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0^c, \rho_0^{\mathrm{est}}\right)$. As $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right) \le 1$ and $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0^c, \rho_0^{\mathrm{est}}\right) \le 1$, we necessarily have that both of them converge to 1 since $0 < \gamma < 1$: $\lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right) = 1$. This implies the almost sure convergence of ρ_k towards the pure state $\bar{\rho}$. We have also $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k^{\mathrm{est}} \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right)$ depending linearly on ρ_0. Thus $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k^{\mathrm{est}} \bar{\rho}\right) | \rho_0^{\mathrm{est}}, \rho_0^{\mathrm{est}}\right)$ is given by $\gamma\mathbb{E}\left(\mathrm{Tr}\left(\rho_k^{\mathrm{est}} \bar{\rho}\right) | \rho_0, \rho_0^{\mathrm{est}}\right) + (1-\gamma)\mathbb{E}\left(\mathrm{Tr}\left(\rho_k^{\mathrm{est}} \bar{\rho}\right) | \rho_0^c, \rho_0^{\mathrm{est}}\right)$. Using the fact that when $\rho_0 = \rho_0^{\mathrm{est}}$, $\rho_k = \rho_k^{\mathrm{est}}$ for all k, we conclude similarly that ρ_k^{est} converges almost surely towards $\bar{\rho}$ even if ρ_0 and ρ_0^{est} do not coincide. $\square$

4   Imperfect Measurements

We now consider the feedback control problem in the presence of classical measurement imperfections with the possibility of detection errors. This model is a direct generalization of the ones used in [11, 10, 25] (see also e.g., [6, 8] for an introduction to quantum filtering). The imperfections in the measurement process are described by a classical probabilistic model relying on a left stochastic matrix $(\eta_{\mu',\mu})$, $\mu' \in \{1,\ldots,m'\}$ and $\mu \in \{1,\ldots,m\}$: $\eta_{\mu',\mu} \ge 0$ and for any μ, $\sum_{\mu'=1}^{m'} \eta_{\mu',\mu} = 1$. The integer m' corresponds to the number of imperfect outcomes and $\eta_{\mu',\mu}$ is the probability of having the imperfect outcome μ' knowing the perfect one μ. Set

$$\widehat{\rho}_k = \mathbb{E}\left(\rho_k | \rho_0, \mu_0', \ldots, \mu_{k-1}', u_{-\tau}, \ldots, u_{k-\tau-1}\right). \quad (12)$$

Since ρ_k follows (2), $\widehat{\rho}_k$ is also governed by a recursive equation [26]:

$$\widehat{\rho}_{k+1} = \mathbb{L}_{\mu_k'}^{u_{k-\tau}}(\widehat{\rho}_k), \quad (13)$$

where for each μ', $\mathbb{L}_{\mu'}^u$ is the super-operator defined by $\widehat{\rho} \mapsto \mathbb{L}_{\mu'}^u(\widehat{\rho}) = \frac{\mathbf{L}_{\mu'}^u(\widehat{\rho})}{\mathrm{Tr}\left(\mathbf{L}_{\mu'}^u(\widehat{\rho})\right)}$ with $\mathbf{L}_{\mu'}^u(\widehat{\rho}) = \sum_{\mu=1}^m \eta_{\mu',\mu} M_\mu^u \widehat{\rho} M_\mu^{u\dagger}$, and where μ_k' is a random variable taking values μ' in $\{1,\ldots,m'\}$ with probability $p_{\mu',\widehat{\rho}_k}^{u_{k-\tau}} = \mathrm{Tr}\left(\mathbf{L}_{\mu'}^{u_{k-\tau}}(\widehat{\rho}_k)\right)$. Since $\eta_{\mu',\mu}$ is a left stochastic matrix, $\mathbb{E}(\widehat{\rho}_{k+1}|\widehat{\rho}_k = \rho, u_{k-\tau} = u) = \mathbb{K}^u(\widehat{\rho})$, which is precisely the Kraus map (1) associated with the Markov process ρ_k. By Assumption 1, each pure state $|n\rangle\langle n|$, $n \in \{1,\ldots,d\}$ remains a fixed point of the Markov process (13) when $u \equiv 0$.

We now consider the dynamics of the filter state $\widehat{\rho}$ in the presence of delays in the feedback control. Similar to the case with perfect measurements, let $\widehat{\chi}_k = (\widehat{\rho}_k, \beta_{1,k}, \ldots, \beta_{\tau,k})$ be the filter state at step k, where $\beta_{l,k} = u_{k-l}$ is the feedback control at time-step k delayed l steps. Then the delay dynamics are determined by the following Markov chain

$$\begin{cases} \widehat{\rho}_{k+1} = \mathbb{L}_{\mu_k'}^{\beta_{\tau,k}}(\widehat{\rho}_k) \\ \beta_{1,k+1} = u_k, \quad \beta_{r,k+1} = \beta_{r-1,k} \text{ for } r = 2,\ldots\tau. \end{cases} \quad (14)$$

Instead of Assumption 2 we now assume

Assumption 4. *For all $n_1 \ne n_2$ in $\{1,\ldots,d\}$, there exists $\mu' \in \{1,\ldots,m'\}$ such that $\sum_{\mu=1}^m \eta_{\mu',\mu}|c_{\mu,n_1}|^2 \ne \sum_{\mu=1}^m \eta_{\mu',\mu}|c_{\mu,n_2}|^2$.*

Assumption 4 means that there exists a μ' such that the statistics when $u \equiv 0$ for obtaining the measurement result μ' are different for the fixed points $|n_1\rangle\langle n_1|$ and $|n_2\rangle\langle n_2|$. This follows by noting that $\mathrm{Tr}\left(\mathbf{L}_{\mu'}^0(|n\rangle\langle n|)\right) = \sum_{\mu=1}^m \eta_{\mu',\mu}|c_{\mu,n}|^2$ for $n \in \{1,\ldots,d\}$.

We now state the analogue of Theorem 3.1 in the case of imperfect measurements.

Theorem 4.1. *Consider the Markov chain (14) with Assumptions 1, 3 and 4. Take $\bar{n} \in \{1,\ldots,d\}$. Assume that the directed graph of Metzler matrix R of Lemma 3.1 is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}_+^d$*

solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$ (see Lemma 3.2) and set $V_\epsilon(\widehat{\rho}) = \sum_{n=1}^d \sigma_n \langle n| \widehat{\rho} |n\rangle - \frac{\epsilon}{2} (\langle n| \widehat{\rho} |n\rangle)^2$. Take $\bar{u} > 0$ and consider the following feedback law $u_k = \underset{\xi \in [-\bar{u}, \bar{u}]}{argmin} \left(\mathbb{E} \left(W_\epsilon(\widehat{\chi}_{k+1}) | \widehat{\chi}_k, u_k = \xi \right) \right) = f(\widehat{\chi}_k)$, where $W_\epsilon(\widehat{\chi}) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots \ldots \mathbb{K}^{\beta_\tau}(\widehat{\rho}) \ldots)))$. Then there exists $u^* > 0$ such that, for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}} \right)\right]$, the closed-loop Markov chain of state $\widehat{\chi}_k$ with the feedback law of Theorem 4.1 converges almost surely towards $(|\bar{n}\rangle \langle \bar{n}|, 0, \ldots, 0)$ for any initial condition $\widehat{\chi}_0 \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau$.

The proof of this theorem is almost identical to that of Theorem 3.1 with $\mathbb{M}_\mu^u$ replaced by $\mathbb{L}_{\mu'}^u$ and $p_{\mu,\rho}^u$ replaced by $p_{\mu',\widehat{\rho}}^u$ and we do not give the details of this proof.

For estimating the hidden state $\widehat{\rho}$ needed for the feedback design of Theorem 4.1, let us consider the estimate $\widehat{\rho}^{\text{est}}$ given by

$$\widehat{\rho}_{k+1}^{\text{est}} = \mathbb{L}_{\mu'_k}^{u_{k-\tau}}(\widehat{\rho}_k^{\text{est}}) \tag{15}$$

where μ'_k corresponds to the imperfect outcome detected at step k. Such μ'_k is correlated to the perfect and hidden outcome μ_k of (2) through the classical stochastic process attached to $(\eta_{\mu',\mu})$: for each μ_k, μ'_k is a random variable to be equal to $\mu' \in \{1, \ldots, m'\}$ with probability η_{μ',μ_k}. In practice, the control u_k defined in Theorem 4.1 could only depend on this estimation $\widehat{\rho}_k^{\text{est}}$ replacing $\widehat{\rho}_k$ in $\widehat{\chi}_k = (\widehat{\rho}_k, u_{k-1}, \ldots, u_{k-\tau})$. The following result guaranties the convergence of the feedback scheme when $\ker(\widehat{\rho}_0^{\text{est}}) \subset \ker(\rho_0)$.

Theorem 4.2. *Consider the recursive Equation (2) and take assumptions of Theorem 4.1. Consider the estimation $\widehat{\rho}_k^{\text{est}}$ given by (15) with an initial condition $\widehat{\rho}_0^{\text{est}}$. Set $u_k = f(\widehat{\rho}_k^{\text{est}}, u_{k-1}, \ldots, u_{k-\tau})$ where f is given by Theorem 4.1. Then there exists $u^* > 0$ such that, for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}} \right)\right]$, ρ_k and $\widehat{\rho}_k^{\text{est}}$ converge almost surely towards the target state $|\bar{n}\rangle \langle \bar{n}|$ as soon as $\ker(\widehat{\rho}_0^{\text{est}}) \subset \ker(\rho_0)$.*

Proof. Let us first prove that $\widehat{\rho}_k$ defined by (12) converges almost surely towards $|\bar{n}\rangle \langle \bar{n}|$. Since $\widehat{\rho}_0 = \rho_0$, we have $\ker(\widehat{\rho}_0^{\text{est}}) \subset \ker(\widehat{\rho}_0)$. Thus,

there exist $\rho_c \in \mathcal{D}$ and $\gamma \in]0, 1[$, such that, $\widehat{\rho}_0^{\text{est}} = \gamma \widehat{\rho}_0 + (1 - \gamma) \widehat{\rho}_0^c$. Similarly to the proof of Theorem 3.2, $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0, \widehat{\rho}_0^{\text{est}} \right)$ depends linearly on $\widehat{\rho}_0$: $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0^{\text{est}}, \widehat{\rho}_0^{\text{est}} \right)$ is given by $\gamma \mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0, \widehat{\rho}_0^{\text{est}} \right) + (1 - \gamma) \mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0^c, \widehat{\rho}_0^{\text{est}} \right)$. Moreover by Theorem 4.1, $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0^{\text{est}}, \widehat{\rho}_0^{\text{est}} \right)$ converges almost surely towards 1, thus $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0, \widehat{\rho}_0^{\text{est}} \right)$ and $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k \bar{\rho} \right) | \widehat{\rho}_0^c, \widehat{\rho}_0^{\text{est}} \right)$ converge also towards 1. Consequently $\widehat{\rho}_k$ converges almost surely towards $|\bar{n}\rangle \langle \bar{n}|$. Since $\widehat{\rho}_k$ is the conditional expectation of ρ_k knowing the past imperfect outcomes and control inputs and since its limit $|\bar{n}\rangle \langle \bar{n}|$ is a pure state, ρ_k converges necessarily towards the same pure state almost surely. Convergence of $\widehat{\rho}_k^{\text{est}}$ relies on similar arguments exploiting the linearity of $\mathbb{E} \left(\text{Tr} \left(\widehat{\rho}_k^{\text{est}} \bar{\rho} \right) | \widehat{\rho}_0, \widehat{\rho}_0^{\text{est}} \right)$ versus $\widehat{\rho}_0$. □

5 The photon box

In this section, we give the explicit expression of the feedback controller which has been experimentally tested in Laboratoire Kastler-Brossel (LKB) at Ecole Normal Supérieure (ENS) de Paris. We briefly summarize how the control design elucidated in this paper is applied to the LKB experiment. We refer the interested reader to [24, 25] for more details. This feedback controller has been obtained by the Lyapunov design discussed in previous sections.

5.1 Experimental system

Figure 1 is a sketch of the experimental setup considered in [10]. Two-level atoms, with state space spanned by the vectors $|e\rangle$ and $|g\rangle$ (excited and ground state respectively), act as qubits. The atoms, emitted from box B, interact individually with the electromagnetic microwave field stored in the cavity C and get entangled with it. The cavity is placed in between two additional cavities, R_1 and R_2, where the atomic state can be manipulated at will. Together, they form an atomic interferometer similar to those used in atomic clocks. The atomic state, $|e\rangle$ or $|g\rangle$, is eventually measured in the detection device D. Using the outcomes of those measurements, a quantum filter, run by a real-time computer, estimates
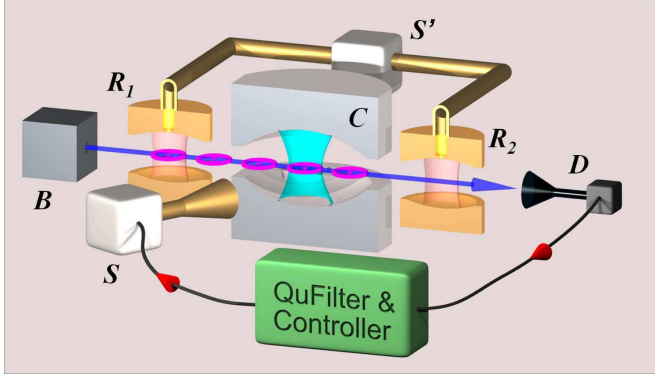
Figure 1: Scheme of the experimental setup.

the density matrix ρ of the microwave field trapped in the cavity C. It also uses all the available knowledge on the experimental setup, especially the limited detection efficiency ε (percentage of atoms that are actually detected), the detection errors η_e and η_g ($\eta_s, s \in \{e, g\}$ is the probability to detect an atom in the wrong state). A controller eventually calculates (state feedback) the amplitude $u \in \mathbb{R}$ of the classical microwave field that is injected into C by an external source S so as to bring the cavity field closer to the target state. The succession of the atomic detection, the state estimation and the microwave injection define one iteration of our feedback loop. It corresponds to a sampling time of around 80 μs. For a more detailed description see [15, 24, 25].

5.2 The controlled Markov process and quantum filter

The state to be stabilized is the cavity state. The underlying Hilbert space is $\mathcal{H} = \mathbb{C}^{n_{\mathrm{ph}}^{\max}+1}$ which is assumed to be finite-dimensional (truncations to $n_{\mathrm{ph}}^{\max}$ photons). It admits $(|0\rangle, |1\rangle, \ldots, \left|n_{\mathrm{ph}}^{\max}\right\rangle)$ as an orthonormal basis. Each basis vector $|n_{\mathrm{ph}}\rangle \in \mathbb{C}^{n_{\mathrm{ph}}^{\max}+1}$ corresponds to a pure state, called photon-number state (Fock state), with precisely n_{ph} photons in the cavity, $n_{\mathrm{ph}} \in \{0, \ldots, n_{\mathrm{ph}}^{\max}\}$. With notations of Subsection 2.1, $d = n_{\mathrm{ph}}^{\max} + 1$ and the index $n = n_{\mathrm{ph}} + 1$. The three main processes that drive the evolution of our

system are decoherence, injection, and measurement. Their action onto the system's state can be described by three super-operators $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{P}$, respectively. We refer to [10] for more details. All operators are expressed in the Fock-basis $(|n_{\mathrm{ph}}\rangle)_{n_{\mathrm{ph}}=0,\ldots,n_{\mathrm{ph}}^{\max}}$ truncated to $n_{\mathrm{ph}}^{\max}$ photons.

After taking into account our full knowledge about the experiment, we finally get the following state estimate at step $k + 1$:

$$\widehat{\rho}_{k+1}^{\mathrm{est}} = \mathbb{P}_{\mu'_k}(\mathbb{D}_{u_{k-\tau}}(\mathbb{T}_\theta(\widehat{\rho}_k^{\mathrm{est}}))) \equiv \mathbb{L}_{\mu'_k}^{u_{k-\tau},\theta}(\widehat{\rho}_k^{\mathrm{est}}), \quad (16)$$

with the following short descriptions for the super-operators $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{P}$.

- The decoherence manifests itself through spontaneous loss or capture of a photon to or from the environment which is described as follows:

$$\rho \mapsto \mathbb{T}_\theta(\rho) = L_0\rho L_0^\dagger + L_-\rho L_-^\dagger + L_+\rho L_+^\dagger. \quad (17)$$

where $L_0 = \mathbf{I} - \theta(1/2 + n_{\mathrm{th}})\mathbf{N} - (\theta n_{\mathrm{th}}/2)\mathbf{I}$, $L_- = \sqrt{\theta(1 + n_{\mathrm{th}})}\mathbf{a}$, and $L_+ = \sqrt{\theta n_{\mathrm{th}}}\mathbf{a}^\dagger$, with $\theta \ll 1$, $\mathbf{a}$ and $\mathbf{a}^\dagger$ photon annihilation and creation operators ($\mathbf{a}|n_{\mathrm{ph}}\rangle = \sqrt{n_{\mathrm{ph}}}|n_{\mathrm{ph}}-1\rangle$ and $\mathbf{a}^\dagger|n_{\mathrm{ph}}\rangle = \sqrt{n_{\mathrm{ph}}+1}|n_{\mathrm{ph}}+1\rangle$) and with $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ the photon number operator ($\mathbf{N}|n_{\mathrm{ph}}\rangle = n_{\mathrm{ph}}|n_{\mathrm{ph}}\rangle$). Besides, n_{th} is the mean number of photons in the cavity mode at thermal equilibrium with its environment.

- The evolution of the state ρ after the control injection is modeled through

$$\rho \mapsto \mathbb{D}_u(\rho) = D_u\rho D_{-u}, \quad (18)$$

with $D_u = \exp(u\mathbf{a}^\dagger - u\mathbf{a})$. In reality, the control at step k, u_k, is subject to a delay of $\tau > 0$ steps which corresponds to the number of flying atoms between the cavity C and the detector D.

- In the real experiment, the atom source is probabilistic and is characterized by a truncated Poisson probability distribution $P_a(n_a) \geq 0$ to have $n_a \in \{0, 1, 2\}$ atom(s) in a sample (we neglect events with more than 2 atoms). This expands the set of the possible detection outcomes to $m = 7$ values $\mu \in \{\emptyset, g, e, gg, eg, ge, ee\}$,

related to the following measurement operators, $L_\emptyset = \sqrt{P_a(0)}\,\mathbf{I}$, $L_g = \sqrt{P_a(1)}\,\cos(\phi_\mathbf{N})$, $L_e = \sqrt{P_a(1)}\,\sin(\phi_\mathbf{N})$,
$L_{gg} = \sqrt{P_a(2)}\,\cos^2(\phi_\mathbf{N})$, $L_{ee} = \sqrt{P_a(2)}\,\sin^2(\phi_\mathbf{N})$ and $L_{ge} = L_{eg} = \sqrt{P_a(2)}\,\cos(\phi_\mathbf{N})\sin(\phi_\mathbf{N})$ where $\phi_\mathbf{N} = \frac{\phi_r + \phi_0(\mathbf{N}+1/2)}{2}$, ϕ_r and ϕ_0 are physical parameters.

The real measurement process is not perfect: the detection efficiency is limited to $\varepsilon < 1$ and the state detection errors are non-zero ($0 < \eta_{e/g} < 1$). These imperfections are taken into account by considering the left stochastic matrix $\eta_{\mu',\mu}$ which is given in [26]. Consequently, the optimal state estimate after measurement outcome μ' gets the following form:

$$\mathbb{P}_{\mu'}(\rho) = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} L_\mu \rho L_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} L_\mu \rho L_\mu^\dagger\right)}. \qquad (19)$$

The measurement operators $(L_\mu)_{1 \le \mu \le m}$ are diagonal in the Fock basis $\{|n_{\mathrm{ph}}\rangle\}$, illustrating their quantum non-demolition nature with respect to the photon number operator and thus fulfilling Assumption 1. Besides, Assumption 4 can also be fulfilled by a proper choice of the experimental parameters ϕ_r and ϕ_0.

5.3 Feedback controller

For $\theta = 0$ (no cavity decoherence) the Markov model of density matrix $\widehat{\rho}$ associated to the filter (16) is exactly of the form (14) with $(|n_{\mathrm{ph}}\rangle\langle n_{\mathrm{ph}}|)_{n_{\mathrm{ph}}=0,\ldots,n_{\mathrm{ph}}^{\max}}$ being fixed-points in open-loop. Similarly, the underlying Markov process of the true cavity state ρ, which is unobservable in practice because of detection errors and delays, admits the same fixed points in open-loop. With parameters given in Subsection 5.4 (except $\theta = 0$), these Markov processes satisfy Assumptions 1-2-3 for ρ and Assumptions 1-3-4 for $\widehat{\rho}$. Moreover the Metzler matrix R of Lemma 3.2 is irreducible. Consequently the assumptions of Theorem 4.1 are satisfied. The feedback law proposed in Theorem 4.2 and relying on the filter state $\widehat{\rho}^{\mathrm{est}}$ will stabilize globally the unobservable state ρ towards

the target photon-number state $|\bar{n}_{\mathrm{ph}}\rangle\langle\bar{n}_{\mathrm{ph}}|$. Numerous closed-loop simulations show that taking $\epsilon = 0$ in the feedback law does not destroy stability and does not affect the convergence rates. This explains why in the simulations and experiments, we set $\epsilon = 0$ despite the fact that Theorem 4.2 guaranties convergence only for arbitrary small but strictly positive ϵ.

For θ positive and small, the $|n_{\mathrm{ph}}\rangle\langle n_{\mathrm{ph}}|$s are no more fixed-points for ρ and $\widehat{\rho}$ in open-loop. Let us detail how to adapt the feedback scheme of Theorem 4.2. At each step of an ideal experiment the control u_k minimizes the Lyapunov function $V_0(\tilde{\rho}_k) = \sum_{n_{\mathrm{ph}}} \sigma_{n_{\mathrm{ph}}} \langle n_{\mathrm{ph}}| \tilde{\rho}_k |n_{\mathrm{ph}}\rangle$ (ϵ is set to zero) calculated for state $\tilde{\rho}_k = \mathbb{D}_{u_k}(\rho_k)$. In our real experiment however, we also take into account decoherence and τ flying not-yet-detected samples and therefore choose u_k to minimize $V_0(\tilde{\rho}_k)$ for

$$\tilde{\rho}_k = \mathbb{D}_{u_k}(\mathbb{T}_\theta(\mathbb{K}^{u_{k-1},\theta}(\mathbb{K}^{u_{k-2},\theta}(\ldots\mathbb{K}^{u_{k-\tau},\theta}(\widehat{\rho}_k^{\mathrm{est}})\ldots)))))$$

with $\widehat{\rho}^{\mathrm{est}}$ given by (16). Here we have introduced the Kraus map of the real experiment $\mathbb{K}^{u,\theta}(\rho) = \sum_{\mu=1}^{m} L_\mu(\mathbb{D}_u(\mathbb{T}_\theta(\rho)))L_\mu^\dagger$. The control u minimizing V_0 is approximated by $\underset{\xi \in [-\bar{u},\bar{u}]}{\mathrm{argmax}}\left(a_1\xi + a_2\xi^2/2\right)$ with $a_1 = \mathrm{Tr}([\mathbf{a}^\dagger - \mathbf{a}, \sigma_\mathbf{N}]\rho)$ and $a_2 = \mathrm{Tr}([[\mathbf{a}^\dagger - \mathbf{a}, \sigma_\mathbf{N}], \mathbf{a}^\dagger - \mathbf{a}]\rho)$, where $\sigma_\mathbf{N}$ is the diagonal operator $\sum_{n_{\mathrm{ph}}} \sigma_{n_{\mathrm{ph}}} |n_{\mathrm{ph}}\rangle\langle n_{\mathrm{ph}}|$. The coefficients $\sigma_{n_{\mathrm{ph}}}$ are computed using Lemma 3.2 where, for $n_{\mathrm{ph}} \ne \bar{n}_{\mathrm{ph}}$, $\lambda_{n_{\mathrm{ph}}}$ are chosen negative and with a decreasing modulus versus n_{ph} in order to compensate cavity decay. For $\bar{n}_{\mathrm{ph}} = 3$, we have compared in simulations different setting and selected the profile displayed in Figure 2.

5.4 Simulations and experimental results

Closed-loop simulation of Figure 3 shows a typical Monte-Carlo trajectory of the feedback loop aiming to stabilize the 3-photon state $|\bar{n}_{\mathrm{ph}} = 3\rangle$. The experimental parameters used in the simulations are the following: $n_{\mathrm{ph}}^{\max} = 8$, $\phi_0 = 0.245\,\pi$, $\phi_r = \pi/2 - \phi_0(\bar{n}_{\mathrm{ph}} + 1/2)$, $\langle n_{\mathrm{a}} \rangle = 0.6$, $\varepsilon = 0.35$, $\eta_e = 0.13$, $\eta_g = 0.11$, $\theta = 0.014$, $n_{\mathrm{th}} = 0.05$, and $\tau = 4$. For the feedback, $\sigma_{n_{\mathrm{ph}}}$ are given in Figure 2, $\epsilon = 0$ and $\bar{u} = 1/10$.
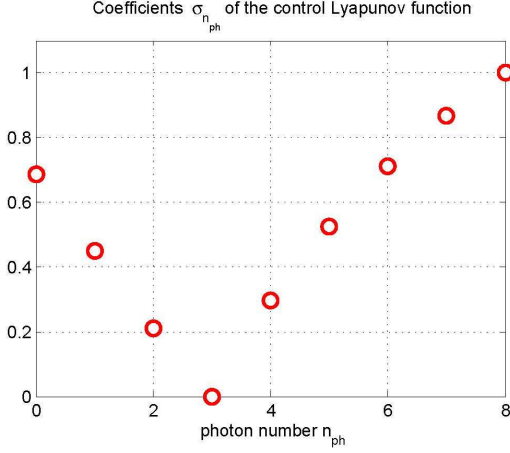
Figure 2: Coefficients of the Lyapunov function $V_0(\rho)$ used in the feedback for the simulation of Figure 3 and for the experiment of Figure 4.

The initial states ρ_0 and $\widehat{\rho}_0^{\text{est}}$ take the following form: $\mathbb{D}_{\sqrt{\bar{n}_{\text{ph}}}}(|0\rangle\langle 0|)$.

The results of the experimental implementation of the feedback scheme are presented in Figure 4. Figure 4(e) shows that the average fidelity of the target state is about 47%. Besides, the asymmetry between the distributions for $n_{\text{ph}} < \bar{n}_{\text{ph}}$ and $n_{\text{ph}} > \bar{n}_{\text{ph}}$ indicates the presence of quantum jumps occurring preferentially downwards ($\bar{n}_{\text{ph}} \to \bar{n}_{\text{ph}} - 1$). Contrarily to the simulations of Figure 3, the cavity photon number probabilities relying on ρ are not accessible in the experimental data of Figure 4 since we do not have access to the detection errors and to the cavity decoherence jumps [14, 9, 28]. Nevertheless, green curves in simulations of Figures 3(d) and 3(e) indicate that when $\langle \bar{n}_{\text{ph}}| \widehat{\rho}^{\text{est}} |\bar{n}_{\text{ph}}\rangle$ exceeds 8/10, ρ coincides, with high probability, with $|\bar{n}_{\text{ph}}\rangle\langle \bar{n}_{\text{ph}}|$.

6 Conclusion

We have proposed a Lyapunov design for state-feedback stabilization of a discrete-time finite-dimensional quantum system with QND measurements. Extensions of this design are possible in different directions such as
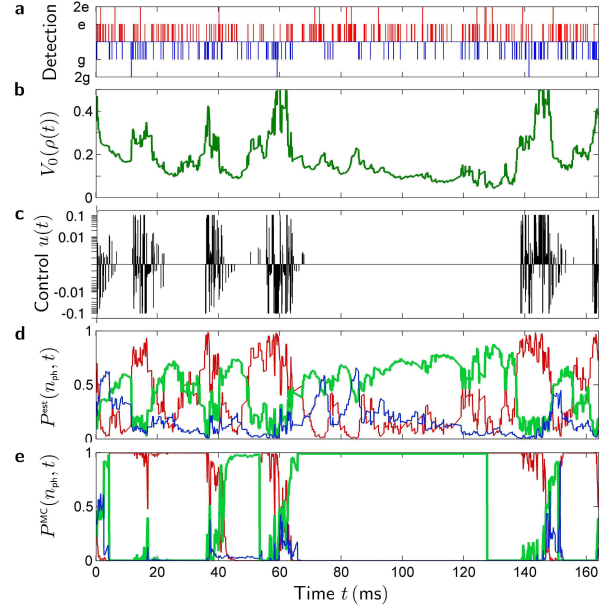

Figure 3: Simulation of one Monte-Carlo trajectory in closed-loop with the target state $|\bar{n}_{\text{ph}} = 3\rangle$. (a) Detection results. (b) Control Lyapunov function. (c) Control input. (d) Estimated photon number probabilities: $\sum_{n_{\text{ph}} < \bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \widehat{\rho}^{\text{est}} |n_{\text{ph}}\rangle$ in red, $\langle \bar{n}_{\text{ph}}| \widehat{\rho}^{\text{est}} |\bar{n}_{\text{ph}}\rangle$ in thick green, and $\sum_{n_{\text{ph}} > \bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \widehat{\rho}^{\text{est}} |n_{\text{ph}}\rangle$ in blue. (e) Cavity photon number probabilities: $\sum_{n_{\text{ph}} < \bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \rho |n_{\text{ph}}\rangle$ in red, $\langle \bar{n}_{\text{ph}}| \rho |\bar{n}_{\text{ph}}\rangle$ in thick green, and $\sum_{n_{\text{ph}} > \bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \rho |n_{\text{ph}}\rangle$ in blue.
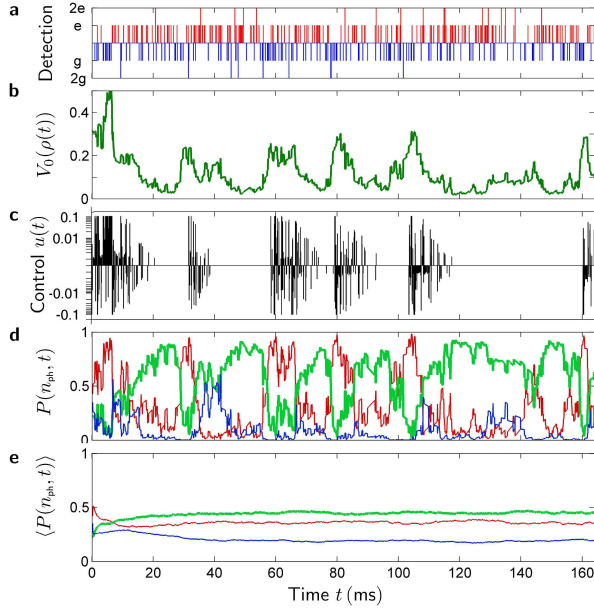
Figure 4: Single experimental trajectory of the feedback loop with the target state $|\bar{n}_{\mathrm{ph}} = 3\rangle$. (a) Detection results. (b) Evolution of the control function. (c) Control injection. (d) Estimated photon number probabilities: $\sum_{n_{\mathrm{ph}} < \bar{n}_{\mathrm{ph}}} \langle n_{\mathrm{ph}} | \widehat{\rho}^{\mathrm{est}} | n_{\mathrm{ph}} \rangle$ in red, $\langle \bar{n}_{\mathrm{ph}} | \widehat{\rho}^{\mathrm{est}} | \bar{n}_{\mathrm{ph}} \rangle$ in thick green, and $\sum_{n_{\mathrm{ph}} > \bar{n}_{\mathrm{ph}}} \langle n_{\mathrm{ph}} | \widehat{\rho}^{\mathrm{est}} | n_{\mathrm{ph}} \rangle$ in blue. (e) Estimated photon number probabilities averaged over 4000 experimental trajectories of the feedback loop.

- replacing the continuous and one-dimensional input u by a multi-dimensional one $(u_1, \ldots, u_p)$;

- assuming that u belongs to a finite set of discrete values;

- taking an infinite dimensional state space as in [27] where the truncation to finite photon numbers is removed;

- considering continuous-time systems similar to the ones investigated in [19];

- ensuring convergence towards a sub-space [5] instead of a pure-state and thus achieving a goal similar to error correction code as already proposed in [1].

References

[1] C. Ahn, A. C. Doherty, and A. J. Landahl. Continuous quantum error correction via quantum feedback control. *Phys. Rev. A*, 65:042301, Mar 2002.

[2] H. Amini, M. Mirrahimi, and P. Rouchon. Stabilization of a delayed quantum system: the Photon Box case-study. *IEEE Transactions on Automatic Control*, 57(8):1918–1930, 2012.

[3] H. Amini, P. Rouchon, and M. Mirrahimi. Design of strict control-Lyapunov functions for quantum systems with QND measurements. In *50th IEEE Conference on Decision and Control and European Control Conference*, pages 8193–8198, 2011.

[4] Michel Bauer and Denis Bernard. Convergence of repeated quantum nondemolition measurements and wave-function collapse. *Phys. Rev. A*, 84:044103, Oct 2011.

[5] S. Bolognani and F. Ticozzi. Engineering stable discrete-time quantum dynamics via a canonical QR decomposition . *IEEE Transactions on Automatic Control*, 55(12):2721–2734, 2010.

[6] L. Bouten, R. van Handel, and M. R. James. An introduction to quantum filtering. *SIAM Journal on Control and Optimization*, 46(6):2199–2241, 2007.

[7] L. Bouten and R. van Handel. *Quantum Stochastics and Information: Statistics, Filtering and Control*, chapter On the separation principle of quantum control. World Scientific, 2008. (see also: arXiv:math-ph/0511021)

[8] L. Bouten, R. van Handel, and M. R. James. A discrete invitation to quantum filtering and feedback control. *SIAM Review 51*, 51:239–316, 2009.

[9] M. Brune, J. Bernu, C. Guerlin, S. Deleglise, C. Sayrin, S. Gleyzes, S. Kuhr, I. Dotsenko, J-M. Raimond, S. Haroche. Process tomography of field damping and measurement of Fock state lifetimes by quantum nondemolition photon counting in a cavity. *Phys. Rev. Lett.*, 101, 240402, 2008.

[10] I. Dotsenko, M. Mirrahimi, M. Brune, S. Haroche, J.-M. Raimond, and P. Rouchon. Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. *Phys. Rev. A*, 80: 013805-013813, 2009.

[11] C. W. Gardiner and P. Zoller. Quantum Noise. *Springer, Berlin*, 2000.

[12] J.M. Geremia. Deterministic and nondestructively verifiable preparation of photon number states. *Phys. Rev. Lett.*, 97(073601), 2006.

[13] J. Gough and M.R. James. The series product and its application to quantum feedforward and feedback networks. *IEEE Transactions on Automatic Control*, 54(11):2530 –2544, 2009.

[14] C. Guerlin, J. Bernu, S. Deléglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J-M. Raimond and S. Haroche. Progressive field-state collapse and quantum non-demolition photon counting. *Nature*, 448: 889–894, 2007.

[15] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford University Press, 2006.

[16] M.R. James and J.E. Gough. Quantum dissipative systems and feedback control design by interconnection. *IEEE Transactions on Automatic Control*, 55(8):1806 –1821, 2010.

[17] K. Kashima and N. Yamamoto. Control of Quantum Systems Despite Feedback Delay. *IEEE Transactions on Automatic Control*, 54(4):876 –881, 2009.

[18] H.J. Kushner. *Introduction to Stochastic Control.* Holt, Rinehart and Wilson, INC., 1971.

[19] M. Mirrahimi and R. Van Handel. Stabilizing feedback controls for quantum systems. *SIAM Journal on Control and Optimization*, 46(2):445–467, 2007.

[20] M.A. Nielsen and I.L. Chuang. Quantum Computation and Quantum Information. *Cambridge University Press*, 2000.

[21] K. Nishio, K. Kashima, and J. Imura. Effects of time delay in feedback control of linear quantum systems. *Phys. Rev. A*, 79:062105, Jun 2009.

[22] R. Romano, D. D'Alessandro. Incoherent control and entanglement for two-dimensional coupled systems. *Phys. Rev. A*, 73:022323, 2006.

[23] R. Romano, D. D'Alessandro. Environment-mediated control of a quantum system. *Phys. Rev. Lett.*, 97:080402, 2006.

[24] C. Sayrin. Préparation et stabilisation d'un champ non classique en cavité par rétroaction quantique. *PhD thesis, Université Paris VI*, 2011.

[25] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, Th. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.M. Raimond, and S. Haroche. Real-time quantum feedback prepares and stabilizes photon number states. *Nature*, 477:73–77, 2011. doi:10.1038/nature10376.

[26] A. Somaraju, I. Dotsenko, C. Sayrin, and P. Rouchon. Design and stability of discrete-time quantum filters with measurement imperfections. *American Control Conference*, 5084–5089, 2011.

[27] A. Somaraju, M. Mirrahimi, and P. Rouchon. Approximate stabilization of an infinite dimensional quantum stochastic system. *Review of Mathematical Physcis*,25:1350001,2013.

[28] H. Wang, M. Hofheinz, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O'Connell, D. Sank, J. Wenner, A. N. Cleland, and John M. Martinis. Measurement of the decay of Fock states in a superconducting quantum circuit. *Phys. Rev. Lett.*, 101: 240401, 2008.

[29] H.M. Wiseman and A.C. Doherty. Optimal unravellings for feedback control in linear quantum systems. *Phys. Rev. Lett.*, 94(7):070405, 2005.

[30] H.M. Wiseman and G.J. Milburn. Quantum Measurement and Control. *Cambridge University Press*, 2009.

A Appendix

The following theorem is just an application of Theorem 1 in [18, Ch. 8].

Theorem A.1. Let X_k be a Markov chain on the compact state space S. Suppose, there exists a continuous function $V(X)$ satisfying

$$\mathbb{E}\left(V(X_{k+1})|X_k\right) - V(X_k) = -Q(X_k), \qquad (20)$$

where $Q(X)$ is a non-negative continuous function of X, then the ω-limit set Ω (in the sense of almost sure convergence) of X_k is contained by the following set $I_\infty = \{X|\quad Q(X) = 0\}$.

Lemma A.1. *Consider the function Q_1 defined by (5) and $\widetilde{u} > 0$. Then there exists $C > 0$ such that for all $(\rho, \beta_1) \in \mathcal{D} \times [-\widetilde{u}, \widetilde{u}]$ satisfying $Q_1(\rho, \beta_1) = 0$, there exists $n \in \{1, \ldots, d\}$ such that $\rho_{n,n} = \langle n| \rho |n\rangle \geq 1 - C|\beta_1|$.*

The proof is the following. For all n, μ, ν, condition $Q_1 = 0$ implies that $p_{\nu,\rho}^{\beta_1} \langle n| M_\mu^{\beta_1} \rho M_\mu^{\beta_1 \dagger} |n\rangle = p_{\mu,\rho}^{\beta_1} \langle n| M_\nu^{\beta_1} \rho M_\nu^{\beta_1 \dagger} |n\rangle$. Taking the sum over all ν, we get $\langle n| M_\mu^{\beta_1} \rho M_\mu^{\beta_1 \dagger} |n\rangle = p_{\mu,\rho}^{\beta_1} \langle n| \mathbb{K}^{\beta_1}(\rho) |n\rangle$ for all n and μ. Since $M_\mu^{\beta_1}$ and $\mathbb{K}^{\beta_1}$ are C^2-function of β_1, these relations read $|c_{\mu,n}|^2 \rho_{n,n} = \left(\sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'}\right) \rho_{n,n} + \beta_1\, b_{\mu,n}(\rho, \beta_1)$ where ρ_{n_1,n_2} stands for $\langle n_1| \rho |n_2\rangle$ and the scalar functions $b_{\mu,n}$ depend continuously on ρ and β_1.
Let us finish the proof by contradiction. Assume that for all $C > 0$, there exists $(\rho^C, \beta_1^C) \in \mathcal{D} \times [-\widetilde{u}, \widetilde{u}]$ satisfying $Q_1(\rho^C, \beta_1^C) = 0$, such that $\forall n \in \{1, \ldots, d\}$, $\rho_{n,n}^C \leq 1 - C|\beta_1^C|$. Take C tending towards $+\infty$. Since ρ^C and β_1^C remain in a compact set, we can assume, up to some extraction process, that ρ^C and β_1^C converge towards ρ^* and β_1^* in $\mathcal{D}$ and $[-\widetilde{u}, \widetilde{u}]$. Since $|\beta_1^C| \leq (1 - \rho_{n,n}^C)/C \leq 1/C$, we have $\beta_1^* = 0$. Since

$$|c_{\mu,n}|^2 \rho_{n,n}^C = \rho_{n,n}^C \sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'}^C$$
$$+ \beta_1^C\, b_{\mu,n}(\rho^C, \beta_1^C) \quad (21)$$

we have by continuity for C tending to $+\infty$ and for all n and μ: $|c_{\mu,n}|^2 \rho_{n,n}^* = \left(\sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'}^*\right) \rho_{n,n}^*$. Thus there exists $n^* \in \{1, \ldots, d\}$ such that $\rho^* = |n^*\rangle \langle n^*|$ (see the proof of Theorem 2.1). Since $\rho_{n^*,n^*}^* = 1$, for C large enough, $\rho_{n^*,n^*}^C > 1/2$ and thus

$$\sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'}^C = |c_{\mu,n^*}|^2 - \beta_1^C\, \frac{b_{\mu,n^*}(\rho^C, \beta_1^C)}{\rho_{n^*,n^*}^C}. \quad (22)$$

Taking $n \neq n^*$, by Assumption 2, there exists μ such that $|c_{\mu,n}|^2 \neq |c_{\mu,n^*}|^2$. Replacing (22) in (21) yields: $\left(|c_{\mu,n}|^2 - |c_{\mu,n^*}|^2 + \beta_1^C\, \frac{b_{\mu,n^*}(\rho^C, \beta_1^C)}{\rho_{n^*,n^*}^C}\right) \rho_{n,n}^C = \beta_1^C\, b_{\mu,n}(\rho^C, \beta_1^C)$. Thus, there exists $C_0 > 0$, such that for $n \neq n^*$ and C large enough $\rho_{n,n}^C \leq C_0|\beta_1^C|$. But $\rho_{n^*,n^*}^C = 1 - \sum_{n\neq n^*} \rho_{n,n}^C \geq 1 - C_0(d-1)|\beta_1^C|$. This is in contradiction with $\rho_{n^*,n^*}^C \leq 1 - C|\beta_1^C|$ as soon as $C > C_0(d-1)$.