

Discrete time mean-field stochastic linear-quadratic optimal control problems*

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Abstract

This paper first presents necessary and sufficient conditions for the solvability of discrete time, mean-field, stochastic linear-quadratic optimal control problems. Then, by introducing several sequences of bounded linear operators, the problem becomes an operator stochastic LQ problem, in which the optimal control is a linear state feedback. Furthermore, from the form of the optimal control, the problem changes to a matrix dynamic optimization problem. Solving this optimization problem, we obtain the optimal feedback gain and thus the optimal control. Finally, by completing the square, the optimality of the above control is validated.

1 Introduction

In this paper we consider a class of stochastic linear-quadratic (LQ) optimal control problems of mean-field type. The system equation is the following linear stochastic difference equation with $k \in \{0, 1, 2, \dots, N-1\} \equiv \mathbb{N}$,

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k) w_k, \\ x_0 = \zeta, \end{cases} \quad (1.1)$$

where $A_k, \bar{A}_k, C_k, \bar{C}_k \in \mathbb{R}^{n \times n}$, and $B_k, \bar{B}_k, D, \bar{D}_k \in \mathbb{R}^{n \times m}$ are given deterministic matrices, and \mathbb{E} is the expectation operator. Denote $\{0, 1, 2, \dots, N\}$ by $\bar{\mathbb{N}}$. In (1.1), $\{x_k, k \in \bar{\mathbb{N}}\}$ and $\{u_k, k \in \mathbb{N}\}$ are the state process and control process, respectively. $\{w_k, k \in \mathbb{N}\}$, defined on a probability space (Ω, \mathcal{F}, P) , represents the stochastic disturbances, which is assumed to be a martingale difference sequence

$$\mathbb{E}[w_{k+1} | \mathcal{F}_k] = 0, \quad \mathbb{E}[(w_{k+1})^2 | \mathcal{F}_k] = 1, \quad (1.2)$$

where \mathcal{F}_k is the σ -algebra generated by $\{\zeta, w_l, l = 0, 1, \dots, k\}$. The initial value ζ and $\{w_k, k \in \mathbb{N}\}$ are assumed to be independent of each other. The cost functional associated with (1.1) is

$$\begin{aligned} J(\zeta, u) = & \mathbb{E} \left[\sum_{k=0}^{N-1} (x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k) \right] \\ & + \mathbb{E} (x_N^T G_N x_N) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N, \end{aligned} \quad (1.3)$$

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where $Q_k, \bar{Q}_k, R_k, \bar{R}_k, k \in \mathbb{N}, G_N, \bar{G}_N$ are deterministic symmetric matrices with appropriate dimensions.

We introduce the following admissible control set

$$\mathcal{U}_{ad} \equiv \{u = (u_0, u_1, \dots, u_{N-1}) : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^m \mid u_k \in \mathcal{F}_k, \mathbb{E}|u_k|^2 < \infty\}.$$

The optimal control problem considered in this paper is stated as follows:

Problem (MF-LQ). For any given square-integrable initial value ζ , find $u^o \in \mathcal{U}_{ad}$ such that

$$J(\zeta, u^o) = \inf_{u \in \mathcal{U}_{ad}} J(\zeta, u). \quad (1.4)$$

We then call u^o an optimal control for Problem (MF-LQ).

Unlike the classical stochastic LQ problem, the expectation $\mathbb{E}x_k$ of x_k appears in the system equation (1.1) and the cost functional (1.3). Therefore, the above problem is called a mean-field LQ problem. This problem is a combination of mean field theory and a LQ problem. Mean-field theory was developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamic systems. According to mean-field theory, the interactions among agents are modelled by a mean-field term. Letting the number of individuals go to the infinity, the mean-field term will approach the expectation. To see this, assume the dynamics of particle i ($i = 1, \dots, L$) are

$$x_{k+1}^{i,L} = \left(A_k x_k^{i,L} + \bar{A}_k \frac{1}{L} \sum_{j=1}^L x_k^{j,L} + B_k u_k \right) + \left(C_k x_k^{i,L} + \bar{C}_k \frac{1}{L} \sum_{j=1}^L x_k^{j,L} + D_k u_k \right) w_k.$$

Under appropriate conditions and letting $L \rightarrow \infty$, we have

$$x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k) + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k) w_k. \quad (1.5)$$

An exact derivation of (1.5) follows the classical McKean-Vlasov argument; for the McKean-Vlasov argument, readers may see, for example, [7][28][30] and references therein. Clearly, (1.1) is a natural extension of (1.5). For the motivation for including $\mathbb{E}x_k$ and $\mathbb{E}u_k$ in the cost functional (1.3), [32] points out that it is natural to introduce variations $\text{var}(x_k)$ and $\text{var}(u_k)$ to the cost functional so as to the state process and the control process could be not too sensitive to the random events. An example of this case is the known finance mean-variance problem. For system (1.1) without mean-field terms, we refer to papers such as [3][22][23][26].

The continuous-time counterpart of (1.1) is a mean-field stochastic differential equation (SDE), whose investigation goes back to the McKean-Vlasov SDE proposed in 1950s ([27][28]). Since then, many researches study McKean-Vlasov type SDEs and applications ([12][14][15][16]). For recent development of mean-field SDEs, readers may refer to [10][11][13] and references therein. The control problems for mean-field SDEs are investigated by many authors; see, for example, [1][2][4][7][9][29][32][17]. On the other hand, another class of problems is mean-field games in terms of its ability to model the collective behavior of individuals due to their mutual interactions. Comparing the above mentioned papers, this class of problems may be viewed as decentralized control problems, that is, the controls are selected to achieve each individual's own goal. For other aspects of mean field games, readers may refer to [7][18][19][20][21][24][25][31] and references therein.

The problem considered in this paper is related to that of [32], which deals with a continuous time mean-field LQ problem. In [32], by a variational method, the optimality system is derived, which is a mean-field forward-backward SDE. Furthermore, by a decoupling technique, two Riccati differential equations are obtained. This gives the feedback representation of the optimal control. Our discussion of the mean-field LQ optimal control problem differs from [32] in the following ways. First, we consider the discrete time case. There are several situations when it is necessary, or natural, to describe a system by a discrete time model. A typical case is when the signal values are only available for measurement or manipulation at certain times, for example, because a continuous time system is sampled at certain times. Another case arises from discretization of the dynamics of continuous time problems. The second difference between this paper and [32] is that methodology differs. In this paper, we first convert Problem (MF-LQ) to a quadratic optimization

problem in Hilbert space. This gives necessary and sufficient conditions for the solvability of Problem (MF-LQ). The optimal control obtained in this case has an abstract form, which may be viewed as open loop. Secondly, by introducing several sequences of bounded linear operators, Problem (MF-LQ) becomes an operator LQ optimal control problem. This operator LQ problem shows that the optimal control is a linear state feedback, which is clearly closed-loop. Thirdly, by the linearity of the optimal control, Problem (MF-LQ) becomes a matrix dynamic optimization problem. Using the matrix minimum principle ([5]), we derive the optimal feedback gains and thus the optimal control is obtained. Finally, by completing the square, we validate the correctness of the obtained optimal control.

The paper is organized as follows. The next section presents the preliminaries and two abstract considerations—quadratic optimization in Hilbert space and the operator LQ problem. In Section 3, optimal control via Riccati equations is presented. Section 4 gives an example to calculate the solutions to Riccati equations and the optimal feedback gains. Section 5 gives some conclude remarks.

2 Preliminaries and abstract considerations

Some standard notion is introduced.

Definition 2.1 (i). *Problem (MF-LQ) is said to be finite for ζ if*

$$\inf_{u \in \mathcal{U}_{ad}} J(\zeta, u) > -\infty.$$

Problem (MF-LQ) is said to be finite if it is finite for any ζ .

(ii). *Problem (MF-LQ) is said to be (uniquely) solvable for ζ if there exists a (unique) $u^o \in \mathcal{U}_{ad}$ such that (1.4) holds for ζ . Problem (MF-LQ) is said to be (uniquely) solvable if it is solvable for any ζ .*

In this subsection, we shall consider Problem (MF-LQ) using two methods. The first converts Problem (MF-LQ) to a quadratic optimization problem in Hilbert space, which gives necessary and sufficient conditions for the solvability of Problem (MF-LQ). The second considers Problem (MF-LQ) in the language of an operator LQ problem. This reveals that the optimal control is a linear state feedback.

Introduce the following spaces:

$$\mathcal{X}_k = L^2_{\mathcal{F}_k}(\mathbb{R}^n) = \{ \xi : \Omega \mapsto \mathbb{R}^n \mid \xi \text{ is } \mathcal{F}_k\text{-measurable, } \mathbb{E}|\xi|^2 < \infty \}, \quad k \in \bar{\mathbb{N}},$$

$$\mathcal{X}[0, k] = \left\{ (x_0, \dots, x_k) \mid x_k \in \mathcal{X}_k, \text{ and is } \mathcal{F}_k\text{-measurable, } \sum_{l=0}^k \mathbb{E}|x_l|^2 < \infty \right\}, \quad k \in \bar{\mathbb{N}},$$

$$\mathcal{U}_k = L^2_{\mathcal{F}_k}(\mathbb{R}^m) = \{ \eta : \Omega \mapsto \mathbb{R}^m \mid \eta \text{ is } \mathcal{F}_k\text{-measurable, } \mathbb{E}|\eta|^2 < \infty \}, \quad k \in \mathbb{N}.$$

Clearly, for any $k \in \bar{\mathbb{N}}, l \in \mathbb{N}$, $\mathcal{X}[0, k]$ and $\mathcal{H} = \mathcal{X}_k, \mathcal{U}_l$ are Hilbert spaces under the usual inner products

$$\langle x, z \rangle = \mathbb{E}(x^T z), \quad \text{for any } x, z \in \mathcal{H}, \quad (2.1)$$

and

$$\langle x, z \rangle = \mathbb{E} \left(\sum_{p=0}^k x_p^T z_p \right), \quad \text{for any } x = (x_0, \dots, x_k), z = (z_0, \dots, z_k) \in \mathcal{X}[0, k]. \quad (2.2)$$

For any variable z in \mathcal{X}_k or \mathcal{U}_k , the expectation of z , i.e., $\mathbb{E}z$, is clearly well defined. If we consider \mathbb{E} as an operator, it is clear that the domain and range of \mathbb{E} may differ from place to place. For example, the domain may be $\mathcal{X}_k, \mathcal{U}_k$, and the range is \mathbb{R}^n and \mathbb{R}^m , respectively. Therefore, the domain and range of the adjoint

operator \mathbb{E}^* of \mathbb{E} may differ. For example, if \mathbb{E} maps \mathcal{X}_k to \mathbb{R}^n , then \mathbb{E}^* is defined from \mathbb{R}^n to \mathcal{X}_k and is defined as

$$\langle \mathbb{E}^* r_n, x \rangle = \langle r_n, \mathbb{E}x \rangle, \quad \text{for any } r_n \in \mathbb{R}^n, x \in \mathcal{X}_k.$$

If \mathbb{E} maps \mathcal{U}_k to \mathbb{R}^m , then \mathbb{E}^* is defined from \mathbb{R}^m to \mathcal{U}_k , and is defined as

$$\langle \mathbb{E}^* r_m, z \rangle = \langle r_m, \mathbb{E}z \rangle, \quad \text{for any } r_m \in \mathbb{R}^m, z \in \mathcal{U}_k.$$

Consequently, \mathbb{E} and \mathbb{E}^* could be denoted by $\mathbb{E}_{\mathcal{H}}$ and $\mathbb{E}_{\mathcal{H}}^*$ in order to emphasize \mathcal{H} , which is the domain of \mathbb{E} and thus the range of \mathbb{E}^* . In this paper, \mathcal{H} may be $\mathcal{X}_k, \mathcal{U}_k, k \in \mathbb{N}$, and $\mathbb{E}_{\mathcal{H}}^*$ always appears accompanying $\mathbb{E}_{\mathcal{H}}$ in the form of $\mathbb{E}_{\mathcal{H}}^* K \mathbb{E}_{\mathcal{H}}$ with K being a generic square matrix. Clearly, for any $z \in \mathcal{H}$, $\mathbb{E}_{\mathcal{H}}^* K \mathbb{E}_{\mathcal{H}} z$ is in \mathcal{H} . Another form is $\mathbb{E}_{\mathcal{U}_k}^* M \mathbb{E}_{\mathcal{X}_k}$, $k \in \mathbb{N}$, with $M \in \mathbb{R}^{m \times n}$. To simplify the notation, throughout this paper, $\mathbb{E}_{\mathcal{H}}^* K \mathbb{E}_{\mathcal{H}} z$ with $z \in \mathcal{H}$, and $\mathbb{E}_{\mathcal{U}_k}^* M \mathbb{E}_{\mathcal{X}_k} x$ with $x \in \mathcal{X}_k$, will be denoted by $\mathbb{E}^* K \mathbb{E} z$ and $\mathbb{E}^* M \mathbb{E} x$, respectively. The meanings will be understood during the context.

Taking expectations in (1.1), we have

$$\begin{cases} \mathbb{E}x_{k+1} = (A_k + \bar{A}_k)\mathbb{E}x_k + (B_k + \bar{B}_k)\mathbb{E}u_k, & k \in \mathbb{N}, \\ \mathbb{E}x_0 = \mathbb{E}\zeta. \end{cases} \quad (2.3)$$

Let

$$\begin{cases} \bar{\Phi}(k, l) = (A_k + \bar{A}_k)(A_{k-1} + \bar{A}_{k-1}) \cdots (A_l + \bar{A}_l), & k \geq l, \\ \bar{\Phi}(k, l) = I, & k < l. \end{cases}$$

Then we have

$$\mathbb{E}x_{k+1} = \bar{\Phi}(k, 0)\mathbb{E}\zeta + \sum_{l=1}^k \bar{\Phi}(k, l)(B_{l-1} + \bar{B}_{l-1})\mathbb{E}u_{l-1}, \quad k \in \mathbb{N} \setminus \{0\}.$$

On the other hand, let

$$\begin{cases} \Phi(k, l) = (A_k + w_k C_k)(A_{k-1} + w_{k-1} C_{k-1}) \cdots (A_{l-1} + w_{l-1} C_{l-1}), & k \geq l, \\ \Phi(k, l) = I, & k < l. \end{cases}$$

By (1.1), we have for $k \in \mathbb{N}$

$$\begin{aligned} x_{k+1} &= \Phi(k, 0)\zeta + \sum_{l=1}^k \Phi(k, l) \left[(\bar{A}_{l-1} + w_{l-1} \bar{C}_{l-1})\mathbb{E}x_{l-1} + (B_{l-1} + w_{l-1} D_{l-1})u_{l-1} + (\bar{B}_{l-1} + w_{l-1} \bar{D}_{l-1})\mathbb{E}u_{l-1} \right] \\ &= \Phi(k, 0)\zeta + \sum_{l=1}^k \Phi(k, l) (\bar{A}_{l-1} + w_{l-1} \bar{C}_{l-1}) \bar{\Phi}(l-2, 0) \mathbb{E}\zeta \\ &\quad + \sum_{l=1}^k \left[\Phi(k, l) (\bar{A}_{l-1} + w_{l-1} \bar{C}_{l-1}) \sum_{i=1}^{l-2} \bar{\Phi}(l-2, i) (B_{i-1} + \bar{B}_{i-1}) \mathbb{E}u_i \right] \\ &\quad + \sum_{l=1}^k \Phi(k, l) (B_{l-1} + w_{l-1} D_{l-1}) u_{l-1} + \sum_{l=1}^k \Phi(k, l) (\bar{B}_{l-1} + w_{l-1} \bar{D}_{l-1}) \mathbb{E}u_{l-1}. \end{aligned}$$

Now define the following operators for any $\zeta \in \mathcal{X}_0, u \in \mathcal{U}_{ad}$:

$$\left\{ \begin{array}{l} (\Gamma\zeta)(\cdot) = \Phi(\cdot - 1, 0)\zeta, \\ \hat{\Gamma}\zeta = (\Gamma\xi)(N), \\ (\bar{\Gamma}\zeta)(\cdot) = \sum_{l=1}^{-1} \Phi(\cdot - 1, l)(\bar{A}_{l-1} + w_{l-1}\bar{C}_{l-1})\bar{\Phi}(l - 2, 0)\mathbb{E}\zeta, \\ \hat{\bar{\Gamma}}\zeta = (\bar{\Gamma}\zeta)(N), \\ (Lu)(\cdot) = \sum_{l=1}^{-1} \Phi(\cdot - 1, l)(B_{l-1} + w_{l-1}D_{l-1})u_{l-1}, \\ \hat{L}u = (Lu)(N), \\ (\bar{L}u)(\cdot) = \sum_{l=1}^{-1} \Phi(\cdot - 1, l)(\bar{A}_{l-1} + w_{l-1}\bar{C}_{l-1}) \sum_{i=1}^{l-2} \bar{\Phi}(l - 2, i)(B_{i-1} + \bar{B}_{i-1})\mathbb{E}u_{i-1} \\ \quad + \sum_{l=1}^{-1} \Phi(\cdot - 1, l)(\bar{B}_{l-1} + w_{l-1}\bar{D}_{l-1})\mathbb{E}u_{l-1}, \\ \hat{\bar{L}}u = (\bar{L}u)(N). \end{array} \right.$$

Then

$$x_k = (\Gamma\zeta)(k) + (\bar{\Gamma}\zeta)(k) + (Lu)(k) + (\bar{L}u)(k).$$

Clearly, the operators

$$\left\{ \begin{array}{l} \Gamma, \bar{\Gamma} : \mathcal{X}_0 \mapsto \mathcal{X}[0, N], \quad \hat{\Gamma} : \mathcal{X}_0 \mapsto \mathcal{X}_N, \\ L, \bar{L} : \mathcal{U}_{ad} \mapsto \mathcal{X}[0, N], \quad \hat{L} : \mathcal{U}_{ad} \mapsto \mathcal{X}_N, \end{array} \right. \quad (2.4)$$

are all bounded and linear. Notice that the spaces in (2.4) are all Hilbert space. Therefore, the corresponding adjoint operators uniquely exist. Further, in what follows, we use the convention

$$\left\{ \begin{array}{l} (Qx)(\cdot) = Q_k x., \quad \forall x \in \mathcal{X}[0, N - 1], \\ (\bar{Q}\varphi)(\cdot) = \bar{Q}_k \varphi., \quad \forall \varphi = (\varphi_0, \dots, \varphi_{N-1}) \text{ with } \varphi_k \in \mathbb{R}^n \text{ such that } \sum_{k=0}^{N-1} |\varphi_k|^2 < \infty, \\ (Ru)(\cdot) = R_k u., \quad \forall u \in \mathcal{U}_{ad}, \\ (\bar{R}\psi)(\cdot) = \bar{R}_k \psi., \quad \forall \psi = (\psi_0, \dots, \psi_{N-1}) \text{ with } \psi_k \in \mathbb{R}^m \text{ such that } \sum_{k=0}^{N-1} |\psi_k|^2 < \infty. \end{array} \right.$$

Consequently, the cost functional $J(\zeta, u)$ has the following form

$$\begin{aligned} J(\zeta, u) &= \langle Q(\Gamma\zeta + \bar{\Gamma}\zeta + Lu + \bar{L}u), \Gamma\zeta + \bar{\Gamma}\zeta + Lu + \bar{L}u \rangle + \langle Ru, u \rangle \\ &\quad + \langle \bar{Q}\mathbb{E}(\Gamma\zeta + \bar{\Gamma}\zeta + Lu + \bar{L}u), \mathbb{E}(\Gamma\zeta + \bar{\Gamma}\zeta + Lu + \bar{L}u) \rangle + \langle \bar{R}\mathbb{E}u, \mathbb{E}u \rangle \\ &\quad + \langle G_T(\hat{\Gamma}\zeta + \hat{\bar{\Gamma}}\zeta + \hat{L}u + \hat{\bar{L}}u), \hat{\Gamma}\zeta \\ &\quad + \hat{\bar{\Gamma}}\zeta + \hat{L}u + \hat{\bar{L}}u \rangle + \langle \bar{G}_T\mathbb{E}(\hat{\Gamma}\zeta + \hat{\bar{\Gamma}}\zeta + \hat{L}u + \hat{\bar{L}}u), \mathbb{E}(\hat{\Gamma}\zeta + \hat{\bar{\Gamma}}\zeta + \hat{L}u + \hat{\bar{L}}u) \rangle \\ &= \langle \Theta_1 u, u \rangle + 2\langle \Theta_2 \xi, u \rangle + \langle \Theta_3 \xi, \xi \rangle. \end{aligned}$$

Here

$$\begin{aligned} \Theta_1 &= R + \mathbb{E}^* \bar{R}\mathbb{E} + (L + \bar{L})^* Q(L + \bar{L}) + (L + \bar{L})^* \mathbb{E}^* Q\mathbb{E}(L + \bar{L}) + (\hat{L} + \hat{\bar{L}})^* G(\hat{L} + \hat{\bar{L}}) \\ &\quad + (\hat{L} + \hat{\bar{L}})^* \mathbb{E}^* G\mathbb{E}(\hat{L} + \hat{\bar{L}}), \\ \Theta_2 &= (L + \bar{L})^* Q(\Gamma + \bar{\Gamma}) + (L + \bar{L})^* \mathbb{E}^* \bar{Q}\mathbb{E}(\Gamma + \bar{\Gamma}) + (\hat{L} + \hat{\bar{L}})^* G_T(\hat{\Gamma} + \hat{\bar{\Gamma}}) + (\hat{L} + \hat{\bar{L}})^* \mathbb{E}^* \bar{G}_T\mathbb{E}(\hat{\Gamma} + \hat{\bar{\Gamma}}), \\ \Theta_3 &= (\Gamma + \bar{\Gamma})^* Q(\Gamma + \bar{\Gamma}) + (\Gamma + \bar{\Gamma})^* \mathbb{E}^* \bar{Q}\mathbb{E}(\Gamma + \bar{\Gamma}) + (\hat{\Gamma} + \hat{\bar{\Gamma}})^* G_T(\hat{\Gamma} + \hat{\bar{\Gamma}}) + (\hat{\Gamma} + \hat{\bar{\Gamma}})^* \mathbb{E}^* \bar{G}_T\mathbb{E}(\hat{\Gamma} + \hat{\bar{\Gamma}}), \end{aligned}$$

and the inner products are understood from the context. Hence, for any $\zeta, u \mapsto J(\zeta, u)$ is a quadratic functional on the Hilbert space \mathcal{U}_{ad} , and the original problem (MF-LQ) is transformed to a minimization problem of a functional over \mathcal{U}_{ad} . We then have the following results.

Proposition 2.1 (i). If $J(\zeta, u)$ has a minimum, then

$$\Theta_1 \geq 0.$$

(ii). Problem (MF-LQ) is (uniquely) solvable if and only if $\Theta_1 \geq 0$ and there exists a (unique) u such that

$$\Theta_1 u + \Theta_2 \zeta = 0.$$

(iii). If $\Theta_1 > 0$, then for any ζ , $J(\zeta, u)$ admits a pathwise unique minimizer u° given by

$$u_k^\circ = -(\Theta_1^{-1} \Theta_2 \zeta)(k), \quad k \in \mathbb{N}. \quad (2.5)$$

In addition, if

$$Q_k, Q_k + \bar{Q}_k \geq 0, \quad R_k, R_k + \bar{R}_k > 0, \quad k \in \mathbb{N}, \quad G_N, G_N + \bar{G}_N \geq 0, \quad (2.6)$$

then $\Theta_1 > 0$.

Proof. The proofs of (i), (ii) and the first part of (iii) are well known, and are omitted here; readers may refer to [32][33][34] for solutions of similar problems of quadratic functional optimization on Hilbert space. Clearly, from (2.6), we have that $(L + \bar{L})^* Q (L + \bar{L}) + (L + \bar{L})^* \mathbb{E}^* Q \mathbb{E} (L + \bar{L}) + (\hat{L} + \hat{\bar{L}})^* G (\hat{L} + \hat{\bar{L}}) \geq 0$. Also for any nonzero $u \in \mathcal{U}_{ad}$,

$$\begin{aligned} \langle Ru, u \rangle + \langle \mathbb{E}^* \bar{R} \mathbb{E} u, u \rangle &= \sum_{k=0}^{N-1} \mathbb{E} [u_k^T R_k u_k + (\mathbb{E} u_k)^T \bar{R}_k \mathbb{E} u_k] \\ &= \sum_{k=0}^{N-1} \left[\mathbb{E} \left[(u_k - \mathbb{E} u_k)^T R_k (u_k - \mathbb{E} u_k) \right] + (\mathbb{E} u_k)^T (R_k + \bar{R}_k) (\mathbb{E} u_k) \right] > 0. \end{aligned}$$

This implies that $R + \mathbb{E}^* \bar{R} \mathbb{E}$ is positive definite. Therefore, Θ_1 is positive definite. This completes the proof. \square

Remark 2.1 Proposition 2.1 presents necessary and sufficient conditions for the solvability of Problem (MF-LQ); it also shows that the optimal control (2.5) is a linear functional of initial value ζ . Note that for any $k \in \mathbb{N}$, u_k does not depend on the current state x_k explicitly, and it depends on ζ and k . Therefore, the optimal control (2.5) may be viewed as an open-loop control.

We shall show that under condition (2.6), the unique optimal control (2.5) is, indeed, a linear feedback of current state, i.e., a closed-loop control. First, introduce several sequences of operators

$$\begin{cases} \mathcal{A}_k x = A_k x + \bar{A}_k \mathbb{E} x, & x \in \mathcal{X}_k, \quad k \in \mathbb{N}, \\ \mathcal{B}_k u = B_k u + \bar{B}_k \mathbb{E} u, & u \in \mathcal{U}_k, \quad k \in \mathbb{N}, \\ \mathcal{C}_k x = C_k x + \bar{C}_k \mathbb{E} x, & x \in \mathcal{X}_k, \quad k \in \mathbb{N}, \\ \mathcal{D}_k u = D_k u + \bar{D}_k \mathbb{E} u, & u \in \mathcal{U}_k, \quad k \in \mathbb{N}. \end{cases} \quad (2.7)$$

Clearly, $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k, k \in \mathbb{N}$, are all bounded linear operators, defined from \mathcal{X}_k and \mathcal{U}_k to \mathcal{X}_k and \mathcal{U}_k , respectively. Consequently, (1.1) may be rewritten as

$$x_{k+1} = (\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k. \quad (2.8)$$

Further, the performance functional may be represented as

$$\begin{aligned}
J(\zeta, u) &= \sum_{k=0}^{N-1} (\langle Q_k x_k, x_k \rangle + \langle \bar{Q}_k \mathbb{E} x_k, \mathbb{E} x_k \rangle + \langle R_k u_k, u_k \rangle + \langle \bar{R}_k \mathbb{E} u_k, \mathbb{E} u_k \rangle) \\
&\quad + \langle G_N x_N, x_N \rangle + \langle \bar{G}_N \mathbb{E} x_N, \mathbb{E} x_N \rangle \\
&= \sum_{k=0}^{N-1} (\langle (Q_k + \mathbb{E}^* \bar{Q}_k \mathbb{E}) x_k, x_k \rangle + \langle (R_k + \mathbb{E}^* \bar{R}_k \mathbb{E}) u_k, u_k \rangle) + \langle (G_N + \mathbb{E}^* \bar{G}_N \mathbb{E}) x_N, x_N \rangle.
\end{aligned}$$

Here $\langle Q_k x_k, x_k \rangle$ denotes $\mathbb{E} (x_k^T Q_k x_k)$, $(Q_k + \mathbb{E}^* \bar{Q}_k \mathbb{E}) x_k = Q_k x_k + \mathbb{E}^* \bar{Q}_k \mathbb{E} x_k$, with similar meanings for related notation. Define

$$\begin{cases} \mathcal{Q}_k x = (Q_k + \mathbb{E}^* \bar{Q}_k \mathbb{E}) x, & x \in \mathcal{X}_k, \quad k \in \mathbb{N}, \\ \mathcal{R}_k u = (R_k + \mathbb{E}^* \bar{R}_k \mathbb{E}) u, & u \in \mathcal{U}_k, \quad k \in \mathbb{N}, \\ \mathcal{G}_N x = (G_N + \mathbb{E}^* \bar{G}_N \mathbb{E}) x, & x \in \mathcal{X}_N. \end{cases}$$

Then

$$J(\zeta, u) = \sum_{k=0}^{N-1} (\langle \mathcal{Q}_k x_k, x_k \rangle + \langle \mathcal{R}_k u_k, u_k \rangle) + \langle \mathcal{G}_N x_N, x_N \rangle. \quad (2.9)$$

Therefore, Problem (MF-LQ) may be rewritten as the following

$$\begin{cases} \text{minimize (2.9)} \\ \text{subject to } u \in \mathcal{U}_{ad}, \text{ with } (x, u) \text{ satisfying (2.8)}. \end{cases} \quad (2.10)$$

Clearly, (2.10) is an operator stochastic LQ problem in discrete time. Operator LQ problems in the deterministic and continuous-time cases have been studied thoroughly. Generally speaking, an operator LQ problem is a problem of infinite dimensional control theory. For general infinite dimensional control theory, readers may refer to [6] and related literature. In this paper, by transforming Problem (MF-LQ) to (2.10), it is possible to obtain the closed-loop form of the optimal control.

Suppose we have a sequence of self-adjoint bounded linear operators $\{\mathcal{P}_k : \mathcal{X}_k \mapsto \mathcal{X}_k; k \in \bar{\mathbb{N}}\}$, which are determined below. Noting that $\mathcal{X}_l \subseteq \mathcal{X}_k$, for $l \leq k$, we assume that

$$\mathcal{P}_k \mathcal{X}_l \equiv \{\mathcal{P}_k x \mid x \in \mathcal{X}_l\} \subseteq \mathcal{X}_l, \quad l \leq k. \quad (2.11)$$

This will be proved below. By adding $\langle \mathcal{P}_N x_N, x_N \rangle - \langle \mathcal{P}_0 x_0, x_0 \rangle$ to both sides of (2.9), we have

$$\begin{aligned}
&J(\zeta, u) + \langle \mathcal{P}_N x_N, x_N \rangle - \langle \mathcal{P}_0 x_0, x_0 \rangle \\
&= \sum_{k=0}^{N-1} (\langle \mathcal{Q}_k x_k, x_k \rangle + \langle \mathcal{R}_k u_k, u_k \rangle + \langle \mathcal{P}_{k+1} x_{k+1}, x_{k+1} \rangle - \langle \mathcal{P}_k x_k, x_k \rangle) + \langle \mathcal{G}_N x_N, x_N \rangle.
\end{aligned} \quad (2.12)$$

To proceed, we need some calculations.

$$\begin{aligned}
&\langle \mathcal{P}_{k+1} x_{k+1}, x_{k+1} \rangle \\
&= \langle \mathcal{P}_{k+1} [(\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k], [(\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k] \rangle \\
&= \mathbb{E} [(\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k))] + \mathbb{E} [(\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k)] \\
&\quad + \mathbb{E} [(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k) w_k)] + \mathbb{E} [(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k^2)].
\end{aligned}$$

Clearly, by (2.11), we have

$$\begin{aligned}
& \mathbb{E} [(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k) w_k)] \\
&= \mathbb{E} [\mathbb{E} (w_k | \mathcal{F}_{k-1}) (\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k))] = 0, \\
& \mathbb{E} [(\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k)) w_k] \\
&= \mathbb{E} [\mathbb{E} (w_k | \mathcal{F}_{k-1}) (\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k))] = 0, \\
& \mathbb{E} [(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k) w_k^2)] \\
&= \mathbb{E} [\mathbb{E} (w_k^2 | \mathcal{F}_{k-1}) (\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T (\mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k))] \\
&= \mathbb{E} [(\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T \mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k)].
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\langle \mathcal{P}_{k+1} x_{k+1}, x_{k+1} \rangle &= \mathbb{E} [(\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T \mathcal{P}_{k+1} (\mathcal{A}_k x_k + \mathcal{B}_k u_k) + (\mathcal{C}_k x_k + \mathcal{D}_k u_k)^T \mathcal{P}_{k+1} (\mathcal{C}_k x_k + \mathcal{D}_k u_k)] \\
&= \langle \mathcal{A}_k^* \mathcal{P}_{k+1} \mathcal{A}_k x_k, x_k \rangle + \langle \mathcal{C}_k^* \mathcal{P}_{k+1} \mathcal{C}_k x_k, x_k \rangle + \langle \mathcal{A}_k^* \mathcal{P}_{k+1} \mathcal{B}_k u_k, x_k \rangle + \langle \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k x_k, u_k \rangle \\
&\quad + \langle \mathcal{C}_k^* \mathcal{P}_{k+1} \mathcal{D}_k u_k, x_k \rangle + \langle \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k x_k, u_k \rangle + \langle \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{B}_k u_k, u_k \rangle + \langle \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{D}_k u_k, u_k \rangle.
\end{aligned} \tag{2.13}$$

Substituting (2.13) in (2.12), we have

$$\begin{aligned}
J(x_0, u) &= \sum_{k=0}^{N-1} [\langle (\mathcal{Q}_k + \mathcal{A}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{C}_k^* \mathcal{P}_{k+1} \mathcal{C}_k - \mathcal{P}_k) x_k, x_k \rangle + 2 \langle (\mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k) x_k, u_k \rangle \\
&\quad + \langle (\mathcal{R}_k + \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{B}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{D}_k) u_k, u_k \rangle] + \langle (\mathcal{G}_N - \mathcal{P}_N) x_N, x_N \rangle + \langle \mathcal{P}_0 x_0, x_0 \rangle \\
&= \sum_{k=0}^{N-1} [\langle \Theta_{0k} x_k, x_k \rangle + 2 \langle \Theta_{1k} x_k, u_k \rangle + \langle \Theta_{2k} u_k, u_k \rangle] + \langle (\mathcal{G}_N - \mathcal{P}_N) x_N, x_N \rangle + \langle \mathcal{P}_0 x_0, x_0 \rangle,
\end{aligned} \tag{2.14}$$

where for any $k \in \mathbb{N}$

$$\begin{cases} \Theta_{0k} = \mathcal{Q}_k + \mathcal{A}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{C}_k^* \mathcal{P}_{k+1} \mathcal{C}_k - \mathcal{P}_k, \\ \Theta_{1k} = \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k, \\ \Theta_{2k} = \mathcal{R}_k + \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{B}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{D}_k. \end{cases} \tag{2.15}$$

Consequently, we have the following result:

Proposition 2.2 *Under the condition that*

$$Q_k, Q_k + \bar{Q}_k \geq 0, \quad R_k, R_k + \bar{R}_k > 0, \quad k \in \mathbb{N}, \quad G_N, G_N + \bar{G}_N \geq 0, \tag{2.16}$$

the unique optimal control for Problem (MF-LQ) is

$$u_k^o = -(\mathcal{R}_k + \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{B}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{D}_k)^{-1} (\mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k) x_k, \tag{2.17}$$

where

$$\begin{cases} \mathcal{P}_k = \mathcal{Q}_k + \mathcal{A}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{C}_k^* \mathcal{P}_{k+1} \mathcal{C}_k \\ \quad - (\mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k)^* (\mathcal{R}_k + \mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{B}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{D}_k)^{-1} (\mathcal{B}_k^* \mathcal{P}_{k+1} \mathcal{A}_k + \mathcal{D}_k^* \mathcal{P}_{k+1} \mathcal{C}_k), \quad k \in \mathbb{N}, \\ \mathcal{P}_N = \mathcal{G}_N. \end{cases} \tag{2.18}$$

Proof. By (2.14), if (2.18) is well posed, we have that

$$\begin{aligned}
J(\zeta, u) &= \sum_{k=0}^{N-1} [\langle \Theta_{2k} (u_k + \Theta_{2k}^{-1} \Theta_{1k} x_k), (u_k + \Theta_{2k}^{-1} \Theta_{1k} x_k) \rangle + \langle \Theta_{0k} - \Theta_{1k}^* \Theta_{2k}^{-1} \Theta_{1k} x_k, x_k \rangle] \\
&\quad + \langle (\mathcal{G}_N - \mathcal{P}_N) x_N, x_N \rangle + \langle \mathcal{P}_0 \zeta, \zeta \rangle \\
&= \sum_{k=0}^{N-1} [\langle \Theta_{2k} (u_k + \Theta_{2k}^{-1} \Theta_{1k} x_k), (u_k + \Theta_{2k}^{-1} \Theta_{1k} x_k) \rangle] + \langle \mathcal{P}_0 \zeta, \zeta \rangle.
\end{aligned}$$

Therefore, the optimal control is given by $u_k^o = -\Theta_{2k}^{-1}\Theta_{1k}x_k$, $k \in \mathbb{N}$, which is (2.17). We shall now prove that (2.18) is well posed. First, for any $k \in \mathbb{N}$ and $u_k \in \mathcal{U}_k$ and $u_k \neq 0$, it is clear that

$$\begin{aligned}
\langle \mathcal{R}_k u_k, u_k \rangle &= \mathbb{E} (u_k^T R_k u_k) + (\mathbb{E} u_k)^T \bar{R}_k (\mathbb{E} u_k) \\
&= \mathbb{E} \left[(u_k - \mathbb{E} u_k)^T R_k (u_k - \mathbb{E} u_k) \right] + (\mathbb{E} u_k)^T (R_k + \bar{R}_k) (\mathbb{E} u_k) \\
&\geq \lambda_1^{(k)} \mathbb{E} |u_k - \mathbb{E} u_k|_m^2 + \lambda_2^{(k)} |\mathbb{E} u_k|_m^2 \\
&= \lambda_1^{(k)} (\mathbb{E} |u_k|_m^2 - |\mathbb{E} u_k|_m^2) + \lambda_2^{(k)} |\mathbb{E} u_k|_m^2 \\
&\geq \lambda^{(k)} (\mathbb{E} |u_k|_m^2 - |\mathbb{E} u_k|_m^2 + |\mathbb{E} u_k|_m^2) \\
&= \lambda^{(k)} \|u_k\|_m^2.
\end{aligned} \tag{2.19}$$

Here $|\cdot|_m$ denotes the norm in \mathbb{R}^m ; $\lambda_1^{(k)}, \lambda_2^{(k)}$ are the smallest eigenvalues of matrix R_k and $R_k + \bar{R}_k$, respectively, and $\lambda^{(k)} = \min\{\lambda_1^{(k)}, \lambda_2^{(k)}\}$; $\|\cdot\|_m$ is the norm induced by inner product in \mathcal{U}_k . By (2.19), we know that $\mathcal{R}_k \geq \lambda^{(k)} I$, where I is the identical operator defined on \mathcal{U}_k , for each $k \in \mathbb{N}$. Under assumption (2.16), we know that $\lambda^{(k)} > 0$. Thus we have that \mathcal{R}_k is positive definite. Furthermore, by known results, we have that

$$\mathcal{R}_k^* = (R_k + \mathbb{E}^* \bar{R}_k \mathbb{E})^* = R_k^* + (\mathbb{E}^* \bar{R}_k \mathbb{E})^* = R_k^* + \mathbb{E}^* \bar{R}_k^* (\mathbb{E}^*)^* = R_k + \mathbb{E}^* \bar{R}_k \mathbb{E} = \mathcal{R}_k,$$

which means that \mathcal{R}_k is self-adjoint. Similarly, we have that \mathcal{G}_N is self-adjoint and positive semi-definite. Therefore, it follows

$$\begin{aligned}
&\langle (\mathcal{R}_{N-1} + \mathcal{B}_{N-1}^* \mathcal{P}_N \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^* \mathcal{P}_N \mathcal{D}_{N-1}) u_{N-1}, u_{N-1} \rangle \\
&= \langle \mathcal{R}_{N-1} u_{N-1}, u_{N-1} \rangle + \langle \mathcal{G}_N (\mathcal{B}_{N-1} u_{N-1}), \mathcal{B}_{N-1} u_{N-1} \rangle + \langle \mathcal{G}_N (\mathcal{D}_{N-1} u_{N-1}), \mathcal{D}_{N-1} u_{N-1} \rangle \\
&\geq \langle \mathcal{R}_{N-1} u_{N-1}, u_{N-1} \rangle \geq \lambda^{(N-1)} \|u_k\|_m^2.
\end{aligned} \tag{2.20}$$

Then, $\Theta_{2(N-1)} = \mathcal{R}_{N-1} + \mathcal{B}_{N-1}^* \mathcal{P}_N \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^* \mathcal{P}_N \mathcal{D}_{N-1}$ is self-adjoint and positive definite, and invertible. Clearly, $\Theta_{2(N-1)}$ is bounded and linear. By the inverse operator theorem, we have that $\Theta_{2(N-1)}^{-1}$ is bounded and linear. Consequently, (2.18) is well-posed for $k = N - 1$. By conditions in the paragraph before (2.3), we know that for any $l \leq N - 1$, $z \in \mathcal{X}_l$, $K \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{m \times n}$, we have that $\mathbb{E}^* K \mathbb{E} z \in \mathcal{X}_l$ and $\mathbb{E}^* M \mathbb{E} z \in \mathcal{U}_k$. Therefore,

$$(\mathcal{B}_{N-1}^* \mathcal{P}_N \mathcal{A}_{N-1} + \mathcal{D}_{N-1}^* \mathcal{P}_N \mathcal{C}_{N-1}) z \in \mathcal{U}_l, \quad z \in \mathcal{X}_l, \quad l \leq N - 1, \tag{2.21}$$

and

$$(\mathcal{R}_{N-1} + \mathcal{B}_{N-1}^* \mathcal{P}_N \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^* \mathcal{P}_N \mathcal{D}_{N-1}) z \in \mathcal{U}_l, \quad z \in \mathcal{U}_l, \quad l \leq N - 1. \tag{2.22}$$

From (2.22), it follows that

$$(\mathcal{R}_{N-1} + \mathcal{B}_{N-1}^* \mathcal{P}_N \mathcal{B}_{N-1} + \mathcal{D}_{N-1}^* \mathcal{P}_N \mathcal{D}_{N-1})^{-1} z \in \mathcal{U}_l, \quad z \in \mathcal{U}_l, \quad l \leq N - 1. \tag{2.23}$$

By (2.21) and (2.23), we have that u_{N-1}^o is \mathcal{F}_{N-1} -adapted and thus in \mathcal{U}_{N-1} . Further, we have

$$\mathcal{P}_{N-1} z \in \mathcal{X}_l, \quad \text{for any } z \in \mathcal{X}_l, \quad l \leq N - 1, \tag{2.24}$$

which is (2.11) for $k = N - 1$. Similar to (2.14), we have for any $x_{N-1} \in \mathcal{F}_{N-1}$

$$\begin{aligned}
&\mathbb{E} \left[x_{N-1}^T Q_{N-1} x_{N-1} + (\mathbb{E} x_{N-1})^T \bar{Q}_{N-1} \mathbb{E} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1} + (\mathbb{E} u_{N-1})^T \bar{R}_{N-1} \mathbb{E} u_{N-1} \right] \\
&+ \mathbb{E} \left[u_{N-1}^T R_{N-1} u_{N-1} + (\mathbb{E} u_{N-1})^T \bar{R}_{N-1} \mathbb{E} u_{N-1} \right] \\
&= \langle \Theta_{2(N-1)} (u_{N-1} + \Theta_{2(N-1)}^{-1} \Theta_{1(N-1)} x_{N-1}), (u_{N-1} + \Theta_{2(N-1)}^{-1} \Theta_{1(N-1)} x_{N-1}) \rangle + \langle \mathcal{P}_{N-1} x_{N-1}, x_{N-1} \rangle \\
&\geq \langle \mathcal{P}_{N-1} x_{N-1}, x_{N-1} \rangle,
\end{aligned}$$

and the equality is achieved by u_{N-1}^o . This implies that

$$\begin{aligned} & \langle \mathcal{P}_{N-1} x_{N-1}, x_{N-1} \rangle \\ = & \mathbb{E} \left[x_{N-1}^T Q_{N-1} x_{N-1} + (\mathbb{E} x_{N-1})^T \bar{Q}_{N-1} \mathbb{E} x_{N-1} + u_{N-1}^{oT} R_{N-1} u_{N-1}^o + (\mathbb{E} u_{N-1}^o)^T \bar{R}_{N-1} \mathbb{E} u_{N-1}^o \right] \geq 0. \end{aligned}$$

Thus, \mathcal{P}_{N-1} is positive semi-definite. Clearly, by (2.18), \mathcal{P}_{N-1} is linear and self-adjoint. Now, we are able to prove that \mathcal{P}_{N-1} is bounded. As noted by above, $\Theta_{2(N-1)}^{-1}$ is bounded, so we can easily assert that \mathcal{P}_{N-1} is bounded. Therefore, we may prove by induction that $\{\mathcal{P}_k, k \in \mathbb{N}\}$, are all self-adjoint, positive semi-definite and bounded linear operators, and that the optimal control is given by (2.17) with $u_k^o \in \mathcal{U}_k$, $k \in \mathbb{N}$. This completes the proof. \square

3 Solution by Riccati equations

The results presented in previous section are mathematically pleasing, but they are not in a form which can be implemented, as (2.5) and the operator Riccati difference equation (2.18) are referenced. However, Proposition 2.2 provides us with the information that the optimal control of Problem (MF-LQ) is a linear state feedback of operator form. Therefore, it is reasonable to assume that the optimal control u^o takes the form

$$u_k^o = L_k^o x_k + \bar{L}_k^o \mathbb{E} x_k, \quad k \in \mathbb{N}, \quad (3.1)$$

with $L_k^o, \bar{L}_k^o \in \mathbb{R}^{m \times n}$. To compute the optimal feedback gains L_k^o, \bar{L}_k^o , we start from a generic linear feedback control

$$u_k = L_k x_k + \bar{L}_k \mathbb{E} x_k, \quad L_k, \bar{L}_k \in \mathbb{R}^{m \times n}, \quad k \in \mathbb{N}. \quad (3.2)$$

Under (3.2), the closed loop system (1.1) becomes

$$\begin{cases} x_{k+1} = [(A_k + B_k L_k) x_k + [B_k \bar{L}_k + \bar{A}_k + \bar{B}_k (L_k + \bar{L}_k)] \mathbb{E} x_k] \\ \quad + [(C_k + D_k L_k) x_k + [D_k \bar{L}_k + \bar{C}_k + \bar{D}_k (L_k + \bar{L}_k)] \mathbb{E} x_k] w_k, \\ x_0 = \zeta, \end{cases} \quad (3.3)$$

and the cost functional (1.3) is

$$\begin{aligned} J(\zeta, u) &= \sum_{k=0}^{N-1} \mathbb{E} [x_k^T Q_k x_k + (\mathbb{E} x_k)^T \bar{Q}_k \mathbb{E} x_k + (L_k x_k + \bar{L}_k \mathbb{E} x_k)^T R_k (L_k x_k + \bar{L}_k \mathbb{E} x_k) \\ & \quad + ((L_k + \bar{L}_k) \mathbb{E} x_k)^T \bar{R}_k (L_k + \bar{L}_k) \mathbb{E} x_k] + \mathbb{E} (x_N^T G_N x_N) + (\mathbb{E} x_N)^T \bar{G}_N \mathbb{E} x_N \\ &= \sum_{k=0}^{N-1} \mathbb{E} [x_k^T (Q_k + L_k^T R_k L_k) x_k + (\mathbb{E} x_k)^T \bar{\Phi}_k \mathbb{E} x_k] + \mathbb{E} (x_N^T G_N x_N) + (\mathbb{E} x_N)^T \bar{G}_N \mathbb{E} x_N \quad (3.4) \\ &= \sum_{k=0}^{N-1} \{ Tr [(Q_k + L_k^T R_k L_k) \mathbb{E} (x_k x_k^T)] + Tr [\bar{\Phi}_k (\mathbb{E} x_k (\mathbb{E} x_k)^T)] \} \\ & \quad + Tr [G_N \mathbb{E} (x_N x_N^T)] + Tr [\bar{G}_N (\mathbb{E} x_N (\mathbb{E} x_N)^T)], \end{aligned}$$

where

$$\bar{\Phi}_k = \bar{Q}_k + (L_k + \bar{L}_k)^T \bar{R}_k (L_k + \bar{L}_k) + L_k^T R_k \bar{L}_k + \bar{L}_k^T R_k L_k + \bar{L}_k^T R_k \bar{L}_k.$$

From the form (3.2) of the control, we may view $\{(L_k, \bar{L}_k), k \in \mathbb{N}\}$ as the new control input. Also (3.4) reminds us that $\mathbb{E} x_k (\mathbb{E} x_k)^T$, $\mathbb{E} (x_k x_k^T)$ may be considered as the new system states. Write $X_k =$

$\mathbb{E}(x_k x_k^T)$, $\bar{X}_k = \mathbb{E}x_k(\mathbb{E}x_k)^T$. Then by (3.3), we have

$$\left\{ \begin{array}{l} X_{k+1} = (A_k + B_k L_k)X_k(A_k + B_k L_k)^T + (A_k + B_k L_k)\bar{X}_k[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k(L_k + \bar{L}_k)]^T \\ \quad + [\bar{A}_k + B_k \bar{L}_k + \bar{B}_k(L_k + \bar{L}_k)]\bar{X}_k(A_k + B_k L_k)^T \\ \quad + [\bar{A}_k + B_k \bar{L}_k + \bar{B}_k(L_k + \bar{L}_k)]\bar{X}_k[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k(L_k + \bar{L}_k)]^T \\ \quad + (C_k + D_k L_k)X_k(C_k + D_k L_k)^T + (C_k + D_k L_k)\bar{X}_k[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k(L_k + \bar{L}_k)]^T \\ \quad + [\bar{C}_k + D_k \bar{L}_k + \bar{D}_k(L_k + \bar{L}_k)]\bar{X}_k(C_k + D_k L_k)^T \\ \quad + [\bar{C}_k + D_k \bar{L}_k + \bar{D}_k(L_k + \bar{L}_k)]\bar{X}_k[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k(L_k + \bar{L}_k)]^T \\ \equiv \mathcal{X}_k(L_k, \bar{L}_k), \\ X_0 = \mathbb{E}(\zeta \zeta^T), \end{array} \right. \quad (3.5)$$

and

$$\left\{ \begin{array}{l} \bar{X}_{k+1} = [(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k + \bar{L}_k)]\bar{X}_k[(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k + \bar{L}_k)]^T \\ \equiv \bar{\mathcal{X}}_k(L_k, \bar{L}_k), \\ \bar{X}_0 = \mathbb{E}\zeta(\mathbb{E}\zeta)^T. \end{array} \right. \quad (3.6)$$

In the language of \bar{X} and \bar{X} , $J(\zeta, u)$ with u defined in (3.2) may be represented as

$$\begin{aligned} J(\zeta, u) &= \sum_{k=0}^{N-1} \{Tr[(Q_k + L_k^T R_k L_k) X_k] + Tr(\bar{\Phi}_k \bar{X}_k)\} + Tr(G_N X_N) + Tr(\bar{G}_N \bar{X}_k) \\ &\equiv \mathcal{J}(X_0, \bar{X}_0, \mathcal{L}), \end{aligned} \quad (3.7)$$

where $\mathcal{L} \equiv \{L_k, \bar{L}_k, k \in \mathbb{N}\}$. Therefore, Problem (MF-LQ) is equivalent to the following problem:

$$\left\{ \begin{array}{l} \min_{L_k, \bar{L}_k \in R^{m \times n}, k \in \mathbb{N}} \mathcal{J}(X_0, \bar{X}_0, \mathcal{L}) \\ \text{subject to (3.5)(3.6)}. \end{array} \right. \quad (3.8)$$

Clearly, this is a matrix dynamic optimization problem. A natural way to deal with this class of problems is by the matrix minimum principle ([5]). Following the framework above, we have the following results.

Theorem 3.1 *For Problem (MF-LQ), under the condition*

$$Q_k, Q_k + \bar{Q}_k \geq 0, \quad R_k, R_k + \bar{R}_k > 0, \quad k \in \mathbb{N}, \quad G_N, G_N + \bar{G}_N \geq 0, \quad (3.9)$$

the unique optimal control is

$$\begin{aligned} u_k^o &= -(W_k^{(1)})^{-1} H_k^{(1)} x_k + \left[-(W_k^{(2)})^{-1} H_k^{(2)} + (W_k^{(1)})^{-1} H_k^{(1)} \right] \mathbb{E}x_k \\ &\equiv L_k^o x_k + \bar{L}_k^o \mathbb{E}x_k, \quad k \in \mathbb{N}. \end{aligned} \quad (3.10)$$

Here,

$$\left\{ \begin{array}{l} W_k^{(1)} = R_k + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k, \\ W_k^{(2)} = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1}) (B_k + \bar{B}_k) + (D_k + \bar{D}_k)^T P_{k+1} (D_k + \bar{D}_k), \\ H_k^{(1)} = B_k^T P_{k+1} A_k + D_k^T P_{k+1} C_k, \\ H_k^{(2)} = (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1}) (A_k + \bar{A}_k) + (D_k + \bar{D}_k)^T P_{k+1} (C_k + \bar{C}_k), \end{array} \right. \quad (3.11)$$

with

$$\left\{ \begin{array}{l} P_k = Q_k + L_k^{oT} R_k L_k^o + (A_k + B_k L_k^o)^T P_{k+1} (A_k + B_k L_k^o) + (C_k + D_k L_k^o)^T P_{k+1} (C_k + D_k L_k^o), \\ P_N = G_N, \end{array} \right. \quad (3.12)$$

$$\left\{ \begin{array}{l}
\bar{P}_k = \bar{Q}_k + L_k^{oT} R_k \bar{L}_k^o + \bar{L}_k^{oT} R_k L_k^o + \bar{L}_k^{oT} R_k \bar{L}_k^o \\
+ (L_k^o + \bar{L}_k^o)^T \bar{R}_k (L_k^o + \bar{L}_k^o) \\
+ (A_k + B_k L_k^o)^T P_{k+1} [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)] \\
+ [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} (A_k + B_k L_k^o) \\
+ [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)] \\
+ (C_k + D_k L_k^o)^T P_{k+1} [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)] \\
+ [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} (C_k + D_k L_k^o) \\
+ [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)] \\
+ [A_k + \bar{A}_k + (B_k + \bar{B}_k)(L_k^o + \bar{L}_k^o)]^T \bar{P}_{k+1} [A_k + \bar{A}_k + (B_k + \bar{B}_k)(L_k^o + \bar{L}_k^o)], \\
\bar{P}_N = \bar{G}_N,
\end{array} \right. \quad (3.13)$$

having the property

$$P_k, P_k + \bar{P}_k \geq 0, \quad k \in \bar{\mathbb{N}}. \quad (3.14)$$

Proof. Introduce the Lagrangian function associated with Problem (3.8),

$$\begin{aligned}
\mathfrak{L} &= \sum_{k=0}^{n-1} \mathfrak{L}_k + Tr(G_N X_N) + Tr(\bar{G}_N \bar{X}_k) \\
&= \sum_{k=0}^{n-1} \mathfrak{L}_k + Tr \left[(G_N \quad \bar{G}_N) \begin{pmatrix} X_N \\ \bar{X}_N \end{pmatrix} \right],
\end{aligned} \quad (3.15)$$

where

$$\begin{aligned}
\mathfrak{L}_k &= Tr \left[(Q_k + L_k^T R_k L_k) X_k \right] + Tr(\bar{\Phi}_k \bar{X}_k) \\
&\quad + Tr \left[P_{k+1} (\mathcal{X}_k(L_k, \bar{L}_k) - X_{k+1}) \right] + Tr \left[\bar{P}_{k+1} (\bar{\mathcal{X}}_k(L_k, \bar{L}_k) - \bar{X}_{k+1}) \right] \\
&= Tr \left[(Q_k + L_k^T R_k L_k) X_k \right] + Tr(\bar{\Phi}_k \bar{X}_k) + Tr \left[(P_{k+1} \quad \bar{P}_{k+1}) \begin{pmatrix} \mathcal{X}_k(L_k, \bar{L}_k) - X_{k+1} \\ \bar{\mathcal{X}}_k(L_k, \bar{L}_k) - \bar{X}_{k+1} \end{pmatrix} \right],
\end{aligned} \quad (3.16)$$

and $(P_{k+1} \quad \bar{P}_{k+1}), k \in \bar{\mathbb{N}}$ are the Lagrangian multipliers. Denote $\mathbb{P}_{k+1} = (P_{k+1} \quad \bar{P}_{k+1}), \mathbb{X}_k = \begin{pmatrix} X_k \\ \bar{X}_k \end{pmatrix}$.

Clearly, by the matrix minimum principle ([5]), the optimal feedback gains $(L_k^o, \bar{L}_k^o), k \in \mathbb{N}$ and Lagrangian multipliers $\mathbb{P}_{k+1}, k \in \mathbb{N}$ satisfy the following first-order necessary conditions

$$\begin{cases} \frac{\partial \mathfrak{L}_k}{\partial L_k} = 0, & \frac{\partial \mathfrak{L}_k}{\partial \bar{L}_k} = 0, & \mathbb{P}_k = \frac{\partial \mathfrak{L}_k}{\partial \mathbb{X}_k}, \quad k \in \mathbb{N}, \\ \mathbb{P}_N = (G_N \quad \bar{G}_N), \end{cases}$$

i.e.,

$$\begin{cases} \frac{\partial \mathfrak{L}_k}{\partial L_k} = 0, & \frac{\partial \mathfrak{L}_k}{\partial \bar{L}_k} = 0, & P_k = \frac{\partial \mathfrak{L}_k}{\partial X_k}, & \bar{P}_k = \frac{\partial \mathfrak{L}_k}{\partial \bar{X}_k}, \quad k \in \mathbb{N}, \\ P_N = G_N, & \bar{P}_N = \bar{G}_N. \end{cases} \quad (3.17)$$

Now, we should calculate several gradient matrices. Noting that for any matrix Y , $\frac{\partial}{\partial Y} \text{Tr}(AYB) = A^T B^T$ if AYB is meaningful, we have

$$\begin{aligned}
\frac{\partial \mathcal{L}_k}{\partial L_k} &= \{2R_k L_k X_k + 2[R_k \bar{L}_k + \bar{R}_k(L_k + \bar{L}_k)] \bar{X}_k\} \\
&+ 2\{(B_k + \bar{B}_k)^T \bar{P}_{k+1}(A_k + \bar{A}_k) + (B_k + \bar{B}_k)^T \bar{P}_{k+1}(B_k + \bar{B}_k)(L_k + \bar{L}_k)\} \bar{X}_k \\
&+ 2\{B_k^T P_{k+1}(A_k + B_k L_k) X_k + B_k^T P_{k+1}(\bar{A}_k + B_k \bar{L}_k) \bar{X}_k + \bar{B}_k^T P_{k+1}(A_k + B_k L_k) \bar{X}_k \\
&+ B_k^T P_{k+1} \bar{B}_k(L_k + \bar{L}_k) \bar{X}_k + \bar{B}_k^T P_{k+1}(\bar{A}_k + B_k \bar{L}_k) \bar{X}_k + \bar{B}_k^T P_{k+1} \bar{B}_k(L_k + \bar{L}_k) \bar{X}_k\} \\
&+ 2\{D_k^T P_{k+1}(C_k + D_k L_k) X_k + D_k^T P_{k+1}(\bar{C}_k + D_k \bar{L}_k) \bar{X}_k + \bar{D}_k^T P_{k+1}(C_k + D_k L_k) \bar{X}_k \\
&+ D_k^T P_{k+1} \bar{D}_k(L_k + \bar{L}_k) \bar{X}_k + \bar{D}_k^T P_{k+1}(\bar{C}_k + D_k \bar{L}_k) \bar{X}_k + \bar{D}_k^T P_{k+1} \bar{D}_k(L_k + \bar{L}_k) \bar{X}_k\} \\
&= 2[R_k L_k + B_k^T P_{k+1}(A_k + B_k L_k) + D_k^T P_{k+1}(C_k + D_k L_k)] X_k \\
&+ 2\{[R_k + \bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(B_k + \bar{B}_k) + (D_k + \bar{D}_k)^T P_{k+1}(D_k + \bar{D}_k)] \bar{L}_k \\
&+ [\bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(B_k + \bar{B}_k) - B_k^T P_{k+1} B_k + \bar{D}_k^T P_{k+1} D_k \\
&+ D_k^T P_{k+1} \bar{D}_k + \bar{D}_k^T P_{k+1} \bar{D}_k] L_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(A_k + \bar{A}_k) \\
&+ (D_k + \bar{D}_k)^T P_{k+1}(C_k + \bar{C}_k) - B_k^T P_{k+1} A_k - D_k^T P_{k+1} C_k\} \bar{X}_k \\
&= 2(W_k^{(1)} L_k + H_k^{(1)}) X_k + 2(W_k^{(2)} \bar{L}_k + W_k^{(3)} L_k + H_k^{(3)}) \bar{X}_k \\
&= 2(W_k^{(1)} L_k + H_k^{(1)})(X_k - \bar{X}_k) + 2(W_k^{(2)}(\bar{L}_k + L_k) + H_k^{(2)}) \bar{X}_k,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_k}{\partial \bar{L}_k} &= 2\{[R_k + \bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(B_k + \bar{B}_k) + (D_k + \bar{D}_k)^T P_{k+1}(D_k + \bar{D}_k)] (\bar{L}_k + L_k) \\
&+ (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(A_k + \bar{A}_k) + (D_k + \bar{D}_k)^T P_{k+1}(C_k + \bar{C}_k)\} \bar{X}_k \\
&= 2(W_k^{(2)}(L_k + \bar{L}_k) + H_k^{(2)}) \bar{X}_k.
\end{aligned} \tag{3.19}$$

Here $W_k^{(i)}, H_k^{(j)}, i, j = 1, 2$, are defined in (3.11), and

$$\begin{cases} W_k^{(3)} = \bar{R}_k + (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(B_k + \bar{B}_k) - B_k^T P_{k+1} B_k + \bar{D}_k^T P_{k+1} D_k \\ \quad + D_k^T P_{k+1} \bar{D}_k + \bar{D}_k^T P_{k+1} \bar{D}_k, \\ H_k^{(3)} = (B_k + \bar{B}_k)^T (P_{k+1} + \bar{P}_{k+1})(A_k + \bar{A}_k) + (D_k + \bar{D}_k)^T P_{k+1}(C_k + \bar{C}_k) \\ \quad - B_k^T P_{k+1} A_k - D_k^T P_{k+1} C_k. \end{cases}$$

The following properties are used

$$W_k^{(2)} = W_k^{(1)} + W_k^{(3)}, \quad H_k^{(2)} = H_k^{(1)} + H_k^{(3)}.$$

Combining (3.17)-(3.19), the optimal feedback gains L_k^o and \bar{L}_k^o must satisfy

$$\begin{cases} (W_k^{(1)} L_k + H_k^{(1)})(X_k - \bar{X}_k) + (W_k^{(2)}(\bar{L}_k + L_k) + H_k^{(2)}) \bar{X}_k = 0, \\ (W_k^{(2)}(L_k + \bar{L}_k) + H_k^{(2)}) \bar{X}_k = 0. \end{cases} \tag{3.20}$$

Note that (3.20) holds for any initial values $X_0 - \bar{X}_0 = \mathbb{E}[(\zeta - \mathbb{E}\zeta)(\zeta - \mathbb{E}\zeta)^T]$ and $\bar{X}_0 = \mathbb{E}\zeta(\mathbb{E}\zeta)^T$. Therefore, (3.20) reduces to

$$\begin{cases} W_k^{(1)} L_k + H_k^{(1)} = 0, \\ W_k^{(2)}(L_k + \bar{L}_k) + H_k^{(2)} = 0, \end{cases} \tag{3.21}$$

which is obtained by letting coefficients be zero in (3.20). Clearly, we obtain the optimal feedback gains

$$\begin{cases} L_k^o = -(W_k^{(1)})^{-1}H_k^{(1)}, \\ \bar{L}_k^o = -(W_k^{(2)})^{-1}H_k^{(2)} + (W_k^{(1)})^{-1}H_k^{(1)}. \end{cases}$$

We now derive the equations that P_k and \bar{P}_k satisfy. By (3.17), we have

$$\begin{aligned} P_k &= \left. \frac{\partial \mathcal{L}_k}{\partial X_k} \right|_{L_k=L_k^o} = Q_k + (L_k^o)^T R_k L_k^o + (A_k + B_k L_k^o)^T P_{k+1} (A_k + B_k L_k^o) \\ &\quad + (C_k + D_k L_k^o)^T P_{k+1} (C_k + D_k L_k^o), \\ \bar{P}_k &= \left. \frac{\partial \mathcal{L}_k}{\partial \bar{X}_k} \right|_{L_k=L_k^o, \bar{L}_k=\bar{L}_k^o} = \bar{\Phi}_k^T + (A_k + B_k L_k^o)^T P_{k+1} [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)] \\ &\quad + [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} (A_k + B_k L_k^o) \\ &\quad + [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} [\bar{A}_k + B_k \bar{L}_k^o + \bar{B}_k (L_k^o + \bar{L}_k^o)] \\ &\quad + (C_k + D_k L_k^o)^T P_{k+1} [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)] \\ &\quad + [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} (C_k + D_k L_k^o) \\ &\quad + [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)]^T P_{k+1} [\bar{C}_k + D_k \bar{L}_k^o + \bar{D}_k (L_k^o + \bar{L}_k^o)] \\ &\quad + [(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k^o + \bar{L}_k^o)]^T \bar{P}_{k+1} [(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k^o + \bar{L}_k^o)], \end{aligned}$$

which are (3.12) and (3.13). The final thing is to assure that for any $k \in \mathbb{N}$, $P_k, P_k + \bar{P}_k \geq 0$. We prove this by induction. Clearly, $P_N, P_N + \bar{P}_N \geq 0$ by definition. For $k = N - 1$, P_{N-1} is positive semi-definite, while

$$\begin{aligned} P_{N-1} + \bar{P}_{N-1} &= Q_{N-1} + \bar{Q}_{N-1} + (L_{N-1}^o + \bar{L}_{N-1}^o)^T R_{N-1} (L_{N-1}^o + \bar{L}_{N-1}^o) \\ &\quad + [A + \bar{A} + (B + \bar{B})(L_{N-1}^o + \bar{L}_{N-1}^o)]^T (P_N + \bar{P}_N) [A + \bar{A} + (B + \bar{B})(L_{N-1}^o + \bar{L}_{N-1}^o)] \\ &\quad + [C + \bar{C} + (D + \bar{D})(L_{N-1}^o + \bar{L}_{N-1}^o)]^T P_N [C + \bar{C} + (D + \bar{D})(L_{N-1}^o + \bar{L}_{N-1}^o)] \geq 0. \end{aligned}$$

In addition, by (3.10), we have

$$\begin{aligned} &J_{N-1}^N(x_{N-1}, u^o) \\ &= \mathbb{E} (x_{N-1}^T Q_{N-1} x_{N-1}) + (\mathbb{E} x_{N-1})^T \bar{Q}_{N-1} \mathbb{E} x_{N-1} + \mathbb{E} (u_{N-1}^T R_{N-1} u_{N-1}) \\ &\quad + (\mathbb{E} u_{N-1})^T \bar{R}_{N-1} \mathbb{E} u_{N-1} + \mathbb{E} (x_{N-1}^T G_N x_N) + (\mathbb{E} x_N)^T \bar{G}_N \mathbb{E} x_N \\ &= \mathbb{E} [x_{N-1}^T (Q_{N-1} + L_{N-1}^{oT} R_{N-1} L_{N-1}^o) x] \\ &\quad + (\mathbb{E} x)^T [\bar{Q}_{N-1} + L_{N-1}^{oT} R_{N-1} \bar{L}_{N-1}^o + \bar{L}_{N-1}^{oT} R_{N-1} L_{N-1}^o + \bar{L}_{N-1}^{oT} R_{N-1} \bar{L}_{N-1}^o \\ &\quad + (L_{N-1}^o + \bar{L}_{N-1}^o)^T \bar{R}_{N-1} (L_{N-1}^o + \bar{L}_{N-1}^o)] \mathbb{E} x_{N-1} \\ &\quad + \mathbb{E} \left[x_{N-1}^T \left((A + B L_{N-1}^o)^T P_N (A + B L_{N-1}^o) + (C + D L_{N-1}^o)^T P_N (C + D L_{N-1}^o) \right) x_{N-1} \right] \\ &\quad + (\mathbb{E} x_{N-1})^T \left[(A + B L_{N-1}^o)^T P_N [\bar{A} + B \bar{L}_{N-1}^o + \bar{B} (L_{N-1}^o + \bar{L}_{N-1}^o)] \right. \\ &\quad + [\bar{A} + B \bar{L}_{N-1}^o + \bar{B} (L_{N-1}^o + \bar{L}_{N-1}^o)]^T P_N (A + B L_{N-1}^o) \\ &\quad + [\bar{A} + B \bar{L}_{N-1}^o + \bar{B} (L_{N-1}^o + \bar{L}_{N-1}^o)]^T P_N [\bar{A} + B \bar{L}_{N-1}^o + \bar{B} (L_{N-1}^o + \bar{L}_{N-1}^o)] \\ &\quad + (C + D L_{N-1}^o)^T P_N [\bar{C} + D \bar{L}_{N-1}^o + \bar{D} (L_{N-1}^o + \bar{L}_{N-1}^o)] \\ &\quad + [\bar{C} + D \bar{L}_{N-1}^o + \bar{D} (L_{N-1}^o + \bar{L}_{N-1}^o)]^T P_N (C + D L_{N-1}^o) \\ &\quad + [\bar{C} + D \bar{L}_{N-1}^o + \bar{D} (L_{N-1}^o + \bar{L}_{N-1}^o)]^T P_N [\bar{C} + D \bar{L}_{N-1}^o + \bar{D} (L_{N-1}^o + \bar{L}_{N-1}^o)] \\ &\quad \left. + [A + \bar{A} + (B + \bar{B})(L_{N-1}^o + \bar{L}_{N-1}^o)]^T \bar{P}_N [A + \bar{A} + (B + \bar{B})(L_{N-1}^o + \bar{L}_{N-1}^o)] \right] \mathbb{E} x_{N-1} \\ &= \mathbb{E} (x_{N-1}^T P_{N-1} x_{N-1}) + (\mathbb{E} x_{N-1})^T \bar{P}_{N-1} (\mathbb{E} x_{N-1}). \end{aligned}$$

Noting that

$$J_k^N(x_k, u^o|_{\{k, k+1, \dots, N-1\}}) = \mathbb{E} \left[(x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + (u_k^o)^T R_k u_k^o + (\mathbb{E}u_k^o)^T \bar{R}_k \mathbb{E}u_k^o) \right] \\ + J_{k+1}^N(x_{k+1}, u^o|_{\{k+1, k+2, \dots, N-1\}}),$$

by induction, we have that $P_k, P_k + \bar{P}_k \geq 0$, for any $k \in \mathbb{N}$. This completes the proof. \square

We may view Problem (MF-LQ) another way. For any $z \in \mathcal{H} = \mathcal{X}_k, \mathcal{U}_k$, we easily see that $\mathbb{E}z$ is orthogonal to $z - \mathbb{E}z$, as

$$\langle \mathbb{E}z, z - \mathbb{E}z \rangle = \mathbb{E} \left[(\mathbb{E}z)^T (z - \mathbb{E}z) \right] = 0.$$

Thus, (3.2) is equivalent to

$$u_k = (L_k + \bar{L}_k) \mathbb{E}x_k + L_k(x_k - \mathbb{E}x_k) \equiv M_k \mathbb{E}x_k + L_k(x_k - \mathbb{E}x_k), \quad (3.22)$$

which is a function of $\mathbb{E}x_k$ and $x_k - \mathbb{E}x_k$. Clearly, in (3.22), we may design M_k and L_k independently. This reminds us that we may study Problem (MF-LQ) using coordinates $\{\mathbb{E}x_k, x_k - \mathbb{E}x_k\}$ for any $k \in \bar{\mathbb{N}}$. The system equations that $\mathbb{E}x_k, x_k - \mathbb{E}x_k$ satisfy are

$$\begin{cases} \mathbb{E}x_{k+1} = (A_k + \bar{A}_k) \mathbb{E}x_k + (B_k + \bar{B}_k) \mathbb{E}u_k, \\ \mathbb{E}x_0 = \mathbb{E}\zeta, \end{cases} \quad (3.23)$$

$$\begin{cases} x_{k+1} - \mathbb{E}x_{k+1} = [A_k(x_k - \mathbb{E}x_k) + B_k(u_k - \mathbb{E}u_k)] + [C_k(x_k - \mathbb{E}x_k) + (C_k + \bar{C}_k) \mathbb{E}x_k \\ \quad + D_k(u_k - \mathbb{E}u_k) + (D_k + \bar{D}_k) \mathbb{E}u_k] w_k, \\ x_0 - \mathbb{E}x_0 = \zeta - \mathbb{E}\zeta. \end{cases} \quad (3.24)$$

The cost functional $J(\zeta, u)$ may be represented as

$$J(\zeta, u) = \sum_{k=0}^{N-1} \mathbb{E} \left[(x_k - \mathbb{E}x_k)^T Q_k (x_k - \mathbb{E}x_k) + (\mathbb{E}x_k)^T (Q_k + \bar{Q}_k) \mathbb{E}x_k + (u_k - \mathbb{E}u_k)^T R_k (u_k - \mathbb{E}u_k) \right] \\ + (\mathbb{E}u_k)^T (R_k + \bar{R}_k) \mathbb{E}u_k \Big] + \mathbb{E} \left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N) + (\mathbb{E}x_N)^T (G_N + \bar{G}_N) \mathbb{E}x_N \right]. \quad (3.25)$$

To follow the classical method of completing the square, we introduce two sequences of symmetric matrices $\{S_k, k \in \bar{\mathbb{N}}\}$ and $\{T_k, k \in \bar{\mathbb{N}}\}$, which are determined below. Then,

$$\begin{aligned}
& J(\zeta, u) + (x_N - \mathbb{E}x_N)^T S_N (x_N - \mathbb{E}x_N) - (x_0 - \mathbb{E}x_0)^T S_0 (x_0 - \mathbb{E}x_0) + (\mathbb{E}x_N)^T T_N \mathbb{E}x_N - (\mathbb{E}x_0)^T T_0 \mathbb{E}x_0 \\
= & \sum_{k=0}^{N-1} \mathbb{E} \left[(x_k - \mathbb{E}x_k)^T Q_k (x_k - \mathbb{E}x_k) + (\mathbb{E}x_k)^T (Q_k + \bar{Q}_k) \mathbb{E}x_k + (u_k - \mathbb{E}u_k)^T R_k (u_k - \mathbb{E}u_k) \right. \\
& + (\mathbb{E}u_k)^T (R_k + \bar{R}_k) \mathbb{E}u_k + (x_{k+1} - \mathbb{E}x_{k+1})^T S_{k+1} (x_{k+1} - \mathbb{E}x_{k+1}) - (x_k - \mathbb{E}x_k)^T S_k (x_k - \mathbb{E}x_k) \\
& \left. + (\mathbb{E}x_{k+1})^T T_{k+1} \mathbb{E}x_{k+1} - (\mathbb{E}x_k)^T T_k \mathbb{E}x_k \right] + \mathbb{E} \left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N) \right. \\
& \left. + (\mathbb{E}x_N)^T (G_N + \bar{G}_N) \mathbb{E}x_N \right] \\
= & \sum_{k=0}^{N-1} \mathbb{E} \left[(x_k - \mathbb{E}x_k)^T (Q_k + A_k^T S_{k+1} A_k + C_k^T S_{k+1} C_k) (x_k - \mathbb{E}x_k) \right. \\
& + 2(x_k - \mathbb{E}x_k)^T (A_k^T S_{k+1} B_k + C_k^T S_{k+1} D_k - S_k) (u_k - \mathbb{E}u_k) \\
& + (u_k - \mathbb{E}u_k)^T (R_k + B_k^T S_{k+1} B_k + D_k^T S_{k+1} D_k) (u_k - \mathbb{E}u_k) \\
& + (\mathbb{E}x_k)^T (Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T T_{k+1} (A_k + \bar{A}_k) - T_k) \mathbb{E}x_k \\
& + 2(\mathbb{E}x_k)^T ((C_k + \bar{C}_k)^T S_{k+1} (D_k + \bar{D}_k) + (A_k + \bar{A}_k)^T T_{k+1} (B_k + \bar{B}_k)) \mathbb{E}u_k \\
& \left. + (\mathbb{E}u_k)^T (R_k + \bar{R}_k + (D_k + \bar{D}_k)^T S_{k+1} (D_k + \bar{D}_k) + (B_k + \bar{B}_k)^T T_{k+1} (B_k + \bar{B}_k)) \mathbb{E}u_k \right] \\
& + \mathbb{E} \left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N) + (\mathbb{E}x_N)^T (G_N + \bar{G}_N) \mathbb{E}x_N \right] \\
= & \sum_{k=0}^{N-1} \mathbb{E} \left[(x_k - \mathbb{E}x_k)^T \left(Q_k + A_k^T S_{k+1} A_k + C_k^T S_{k+1} C_k - S_k - (\bar{H}_k^{(1)})^T (\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)} \right) (x_k - \mathbb{E}x_k) \right. \\
& + \left(u_k - \mathbb{E}u_k + (\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)} (x_k - \mathbb{E}x_k) \right)^T \bar{W}_k^{(1)} \left(u_k - \mathbb{E}u_k + (\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)} (x_k - \mathbb{E}x_k) \right) \\
& + (\mathbb{E}x_k)^T \left(Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T T_{k+1} (A_k + \bar{A}_k) - T_k \right) \mathbb{E}x_k \\
& - (\mathbb{E}x_k)^T \left((\bar{H}_k^{(2)})^T (\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)} \right) \mathbb{E}x_k \\
& \left. + \left(\mathbb{E}u_k + (\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)} \mathbb{E}x_k \right)^T \bar{W}_k^{(2)} \left(\mathbb{E}u_k + (\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)} \mathbb{E}x_k \right) \right] \\
& + \mathbb{E} \left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N) + (\mathbb{E}x_N)^T (G_N + \bar{G}_N) \mathbb{E}x_N \right].
\end{aligned}$$

Here

$$\begin{cases} \bar{W}_k^{(1)} = R_k + B_k^T S_{k+1} B_k + D_k^T S_{k+1} D_k, \\ \bar{H}_k^{(1)} = B_k^T S_{k+1} A_k + D_k^T S_{k+1} C_k, \\ \bar{W}_k^{(2)} = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1} (B_k + \bar{B}_k) + (D_k + \bar{D}_k)^T S_{k+1} (D_k + \bar{D}_k), \\ \bar{H}_k^{(2)} = (B_k + \bar{B}_k)^T T_{k+1} (A_k + \bar{A}_k) + (D_k + \bar{D}_k)^T S_{k+1} (C_k + \bar{C}_k). \end{cases} \quad (3.26)$$

Letting

$$\begin{cases} S_k = Q_k + A_k^T S_{k+1} A_k + C_k^T S_{k+1} C_k - (\bar{H}_k^{(1)})^T (\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)}, \\ T_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T T_{k+1} (A_k + \bar{A}_k) - (\bar{H}_k^{(2)})^T (\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)}, \\ S_N = G_N, \quad T_N = G_N + \bar{G}_N. \end{cases} \quad (3.27)$$

and

$$\begin{cases} \bar{u}_k - \mathbb{E}\bar{u}_k = -(\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)} (x_k - \mathbb{E}x_k), \quad k \in \mathbb{N}, \\ \mathbb{E}\bar{u}_k = -(\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)} \mathbb{E}x_k, \quad k \in \mathbb{N}, \end{cases}$$

that is,

$$\bar{u}_k = -(\bar{W}_k^{(2)})^{-1} \bar{H}_k^{(2)} \mathbb{E}x_k - (\bar{W}_k^{(1)})^{-1} \bar{H}_k^{(1)} (x_k - \mathbb{E}x_k), \quad k \in \mathbb{N}, \quad (3.28)$$

we have

$$J(\zeta, \bar{u}) = \mathbb{E} \left[(\zeta - \mathbb{E}\zeta)^T S_0 (\zeta - \mathbb{E}\zeta) + (\mathbb{E}\zeta)^T T_0 \mathbb{E}\zeta \right] \leq J(\zeta, u)$$

for any $u = (u_0, \dots, u_{N-1})$ with $u_k \in \mathcal{U}_k$. This means that (3.28) is an optimal control. In the following, we show that (3.28) is equal to (3.10). We need firstly to show that for any $k \in \bar{\mathbb{N}}$, $P_k = S_k, T_k = P_k + \bar{P}_k$. In fact, by substituting (3.10) into (3.12), we have

$$\begin{aligned} P_k &= Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k + (L_k^o)^T (R_k + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k) L_k^o \\ &\quad + (L_k^o)^T H_k^{(1)} + \left(H_k^{(1)} \right)^T L_k^o \\ &= Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - (H_k^{(1)})^T (W_k^{(1)})^{-1} H_k^{(1)}. \end{aligned} \quad (3.29)$$

Noting that $P_N = G_N = S_N$, by (3.11)(3.26)(3.27)(3.29), we have that

$$S_k = P_k, \quad \bar{W}_k^{(1)} = W_k^{(1)}, \quad \bar{H}_k^{(1)} = H_k^{(1)}, \quad k \in \bar{\mathbb{N}}.$$

Similarly, we have by (3.13)(3.12)

$$\begin{aligned} \bar{P}_k &= \bar{Q}_k - L_k^{oT} R_k L_k^o - (A_k + B_k L_k^o)^T S_{k+1} (A_k + B_k L_k^o) - (C_k + D_k L_k^o)^T S_{k+1} (C_k + D_k L_k^o) \\ &\quad + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T (P_{k+1} + \bar{P}_{k+1}) (A_k + \bar{A}_k) - (H_k^{(2)})^T (W_k^{(2)})^{-1} H_k^{(2)} \\ &= Q_k + \bar{Q}_k - P_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T (P_{k+1} + \bar{P}_{k+1}) (A_k + \bar{A}_k) \\ &\quad - (H_k^{(2)})^T (W_k^{(2)})^{-1} H_k^{(2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} P_k + \bar{P}_k &= Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T (P_{k+1} + \bar{P}_{k+1}) (A_k + \bar{A}_k) \\ &\quad - (H_k^{(2)})^T (W_k^{(2)})^{-1} H_k^{(2)}. \end{aligned} \quad (3.30)$$

Comparing this with (3.27), as $T_N = G_N + \bar{G}_N = P_N + \bar{P}_N$, we have that

$$T_k = P_k + \bar{P}_k, \quad \bar{W}_k^{(2)} = W_k^{(2)}, \quad \bar{H}_k^{(2)} = H_k^{(2)}, \quad k \in \bar{\mathbb{N}}.$$

Therefore, (3.28) equals to (3.10).

To summarize, Theorem 3.1 may be rewritten in a more compact form as the following results:

Theorem 3.2 For Problem (MF-LQ), under the condition

$$Q_k, Q_k + \bar{Q}_k \geq 0, \quad R_k, R_k + \bar{R}_k > 0, \quad k \in \mathbb{N}, \quad G_N, G_N + \bar{G}_N \geq 0, \quad (3.31)$$

the optimal control is

$$u_k^o = -(W_k^{(2)})^{-1} H_k^{(2)} \mathbb{E}x_k - (W_k^{(1)})^{-1} H_k^{(1)} (x_k - \mathbb{E}x_k) \equiv M_k^o \mathbb{E}x_k + L_k^o (x_k - \mathbb{E}x_k). \quad (3.32)$$

Here,

$$\left\{ \begin{array}{l} W_k^{(1)} = R_k + B_k^T S_{k+1} B_k + D_k^T S_{k+1} D_k, \\ W_k^{(2)} = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1} (B_k + \bar{B}_k) + (D_k + \bar{D}_k)^T S_{k+1} (D_k + \bar{D}_k), \\ H_k^{(1)} = B_k^T S_{k+1} A_k + D_k^T S_{k+1} C_k, \\ H_k^{(2)} = (B_k + \bar{B}_k)^T T_{k+1} (A_k + \bar{A}_k) + (D_k + \bar{D}_k)^T S_{k+1} (C_k + \bar{C}_k), \\ S_k = Q_k + A_k^T S_{k+1} A_k + C_k^T S_{k+1} C_k - (H_k^{(1)})^T (W_k^{(1)})^{-1} H_k^{(1)}, \\ T_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T S_{k+1} (C_k + \bar{C}_k) + (A_k + \bar{A}_k)^T T_{k+1} (A_k + \bar{A}_k) - (H_k^{(2)})^T (W_k^{(2)})^{-1} H_k^{(2)}, \\ S_N = G_N, \quad T_N = G_N + \bar{G}_N. \end{array} \right. \quad (3.33)$$

with the following property

$$P_k, T_k \geq 0, \quad k \in \bar{\mathbb{N}}. \quad (3.34)$$

4 Numerical results

We consider a 4-period numerical example

$$\begin{aligned} \min_{u_0, u_1, u_2, u_3} \quad & \mathbb{E} \left[\sum_{k=0}^3 (x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k) \right] \\ & + \mathbb{E} (x_4^T G_4 x_4) + (\mathbb{E}x_4)^T \bar{G}_4 \mathbb{E}x_4, \\ \text{subject to} \quad & \begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k) w_k, \\ x_0 \in \mathbb{R}^3, \end{cases} \quad k = 0, 1, 2, 3, \end{aligned}$$

with coefficients for $k = 0, 1, 2, 3$ as follows

$$\begin{aligned} A_k &= \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0 & 0.2 & 0.6 \\ 0.6 & 0.4 & 0.2 \end{bmatrix}, & \bar{A}_k &= \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0 & 0.2 & 0.7 \\ 0.6 & 0.5 & 0.2 \end{bmatrix}, \\ B_k &= \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}, & \bar{B}_k &= \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}, \\ C_k &= \begin{bmatrix} 0.2 & 0.4 & 0.6 \\ 0.4 & 0.2 & 0.6 \\ 0.2 & 0.4 & 0.2 \end{bmatrix}, & \bar{C}_k &= \begin{bmatrix} 0.3 & 0.4 & 0.6 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}, \\ D_k &= \begin{bmatrix} 0.2 & 0.6 \\ 0.6 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}, & \bar{D}_k &= \begin{bmatrix} 0.3 & 0.5 \\ 0.5 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}, \\ Q_k &= \text{diag}([0, 1.5, 1]), & \bar{Q}_k &= \text{diag}([1, 1, 0]), \\ R_k &= \text{diag}([1, 1]), & \bar{R}_k &= \text{diag}([1.5, 1]), \\ G_4 &= \text{diag}([0, 1, 1]), & \bar{G}_4 &= \text{diag}([0.5, 1, 0]). \end{aligned}$$

Based on the Riccati equations (3.33), we have Riccati solutions for S_i and T_i for $i = 0, 1, 2, 3$ are given by

$$\begin{aligned} S_0 &= \begin{bmatrix} 0.5227 & 0.3542 & 0.1966 \\ 0.3542 & 1.9655 & 0.3170 \\ 0.1966 & 0.3170 & 1.7009 \end{bmatrix}, & T_0 &= \begin{bmatrix} 4.3329 & 1.7927 & -0.2507 \\ 1.7927 & 4.4213 & 0.4463 \\ -0.2507 & 0.4463 & 3.4720 \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0.5188 & 0.3513 & 0.1951 \\ 0.3513 & 1.9595 & 0.3130 \\ 0.1951 & 0.3130 & 1.6943 \end{bmatrix}, & T_1 &= \begin{bmatrix} 4.2341 & 1.7868 & -0.2366 \\ 1.7868 & 4.4007 & 0.4611 \\ -0.2366 & 0.4611 & 3.4411 \end{bmatrix}, \\ S_2 &= \begin{bmatrix} 0.4862 & 0.3264 & 0.1861 \\ 0.3264 & 1.9219 & 0.2928 \\ 0.1861 & 0.2928 & 1.6660 \end{bmatrix}, & T_2 &= \begin{bmatrix} 3.4908 & 1.6119 & -0.0389 \\ 1.6119 & 4.2394 & 0.4881 \\ -0.0389 & 0.4881 & 3.3283 \end{bmatrix}, \\ S_3 &= \begin{bmatrix} 0.3747 & 0.2421 & 0.1492 \\ 0.2421 & 1.7652 & 0.1849 \\ 0.1492 & 0.1849 & 1.4532 \end{bmatrix}, & T_3 &= \begin{bmatrix} 1.4782 & 0.3777 & 0.3734 \\ 0.3777 & 3.2001 & 0.5548 \\ 0.3734 & 0.5548 & 2.4932 \end{bmatrix}. \end{aligned}$$

Then applying Theorem 3.2, we get the optimal control below

$$u_k^o = M_k^o \mathbb{E}x_k + L_k^o (x_k - \mathbb{E}x_k), \quad k = 0, 1, 2, 3,$$

where

$$\begin{aligned}
M_0^o &= \begin{bmatrix} -0.3286 & -0.4234 & -0.3474 \\ -0.3189 & -0.4351 & -0.7770 \end{bmatrix}, & L_0^o &= \begin{bmatrix} -0.3455 & -0.3271 & -0.4240 \\ -0.2467 & -0.2937 & -0.4941 \end{bmatrix}, \\
M_1^o &= \begin{bmatrix} -0.3436 & -0.4156 & -0.3531 \\ -0.3137 & -0.4381 & -0.7687 \end{bmatrix}, & L_1^o &= \begin{bmatrix} -0.3436 & -0.3235 & -0.4207 \\ -0.2446 & -0.2897 & -0.4885 \end{bmatrix}, \\
M_2^o &= \begin{bmatrix} -0.4029 & -0.3946 & -0.3315 \\ -0.2938 & -0.4160 & -0.7519 \end{bmatrix}, & L_2^o &= \begin{bmatrix} -0.3290 & -0.3009 & -0.4043 \\ -0.2298 & -0.2692 & -0.4650 \end{bmatrix}, \\
M_3^o &= \begin{bmatrix} -0.2418 & -0.2552 & -0.3178 \\ -0.1351 & -0.2213 & -0.5101 \end{bmatrix}, & L_3^o &= \begin{bmatrix} -0.2552 & -0.2084 & -0.2954 \\ -0.1744 & -0.1608 & -0.3028 \end{bmatrix}.
\end{aligned}$$

5 Conclusion

In this paper, we give four methods to deal with the discrete time mean-field LQ problem: the quadratic optimization method in Hilbert space, the operator LQ method, the matrix dynamic optimization method and the method of completing the square. The optimal control is a linear state feedback using two Riccati equation. For future research, we may consider an infinite horizon mean-field LQ problem. In that case, the stability of the system should first be considered. We shall investigate this in the near future.

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