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Brief paper

Stabilization of linear system with input saturation and unknown constant delays[☆]

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ABSTRACT

In this paper, we study the stabilization of linear critically unstable systems subject to input saturation and multiple unknown input delays. We find tight upper bounds for delays which are inversely proportional to the maximal magnitude of open-loop eigenvalues on the imaginary axis. For delays satisfying these upper bounds, linear low-gain state and finite dimensional dynamic measurement feedbacks are constructed to solve the semi-global stabilization problem. The effectiveness of the proposed design is illustrated by an example.

1. Introduction

In the last few decades, time-delayed systems have been greeted with great enthusiasm from researchers in recognition of its theoretical and applied importance (see Richard, 2003). Many control problems have been extensively studied, among which stability and stabilization are of particular interest (see for instances Choi & Lim, 2006; Fridman, 2001; Gu, Kharitonov & Chen, 2003; Kharitonov, Niculescu, Moreno & Michiels, 2005; Niculescu, 2001; Niculescu & Michiels, 2004, and references therein). Like time delay, actuator saturation is also ubiquitous in control application and is well known as the bane of closed-loop performance and stability. The study on stabilization subject to actuator saturation has a long history and still receives renewed attention. Numerous results have been reported in the literature. Some earlier work is surveyed in Bernstein and Michel (1995), Hu and Lin (2001), Kapila and Grigoriadis (2002), Saberi and Stoorvogel (1999), Saberi, Stoorvogel and Sannuti (2000), Tarbouriech and Garcia (1997).

When both actuator saturation and input time-delay are present, controller design can be challenging. What is worse, precise knowledge of delay in most circumstances is not available and only an approximation, usually an upper bound, is known. Mazenc, Mondie and Niculescu (2003) studied the global asymptotic stabilization for chains of integrators using nested-saturation type controller originally developed by Teel (1992). This result was later extended to a class of nonlinear feedforward systems by Mazenc, Mondie and Francisco (2004). Chains of integrators were also studied by Michiels and Roose (2001). A linear low-gain state feedback was constructed to achieve the semi-global stabilization for integrator chains with input saturation and unknown input delay that has a known upper bound which can be arbitrarily large. A different low-gain design based on the parametric Lyapunov equation was used by Zhou, Lin and Duan (2010) to prove a similar result for a broader class critically unstable systems with eigenvalues on the imaginary axis being zero. Both state and measurement feedbacks were developed. However, in the measurement feedback case, delays have to be known by the observer.

In this paper, we investigate the stabilization of general linear critically unstable system subject to input saturation and multiple unknown constant input delays. We give upper bounds on the delays which are inversely proportional to the maximal magnitude of the open-loop eigenvalues on the imaginary axis. This makes sense because when the delay is unknown, a system with highly oscillatory behavior is obviously more difficult to stabilize than a system with dynamics that do not change "direction" so frequently. As the eigenvalues on the imaginary axis move towards

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the origin, the upper bounds on delay turn to infinity. For unknown input delays satisfying these bounds, a *linear time invariant* low-gain state or finite-dimensional measurement feedback controller can be designed to achieve semi-global stabilization. The design in this paper only relies on the upper bounds. This paper recovers and expands upon the results in Michiels and Roose (2001) and Zhou et al. (2010).

There is another line of research in the literature which studies the maximal input delay that a feedback controlled linear system can handle (see Middleton & Miller, 2007, and references therein). It is shown that there has to be an upper bound only for linear time invariant controller. A time varying controller may tolerate arbitrarily large delay. For LTI feedback controlled system, similar upper bounds on delay are also proposed in some cases, which are inversely proportional to the eigenvalues on the imaginary axis (Middleton & Miller, 2007). However, this is an inherently different problem from the one that is studied in this paper. Middleton and Miller (2007) studied the largest delay at which the stability of the feedback controlled system can be spoiled and there is no synthesis, while this paper focuses on a *systematic controller design* to achieve certain delay robustness.

The rest of the paper is organized as follows. In Section 2, we formulate the stabilization problems studied in this paper and make necessary assumptions. Main results are presented in Section 3. We illustrate our designs with a numerical example in Section 4. Section 5 is the conclusion. Proofs of some auxiliary lemmas are given in the Appendix.

The following notations will be used. Let $\mathcal{C}_\tau^n := C([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of all continuous functions from $[-\tau, 0] \to \mathbb{R}^n$ with norm

$$\|x\|_C = \sup_{t\in[-\tau,0]} \|x(t)\|.$$

We will denote by $\mathrm{diag}\{A_i\}_{i=1}^m$, the block-diagonal matrix with $A_1, \ldots, A_m$ on the diagonal. A standard saturation function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$\sigma(s) = \begin{cases} 1, & s \geq 1; \\ s, & -1 < s < 1; \\ -1, & s \leq -1. \end{cases}$$

2. Problem formulation

Consider the following system:

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} B_i \sigma[u_i(t-\tau_i)], \\ y = Cx, \\ x(\theta) = \phi(\theta), \quad \theta \in [-\bar{\tau}, 0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y \in \mathbb{R}^p$, $\phi \in \mathcal{C}_{\bar{\tau}}^n$. Each input u_i has a delay $\tau_i \in [0, \bar{\tau}_i]$ and $\bar{\tau} = \max \bar{\tau}_i$.

We formulate two semi-global stabilization problems as follows.

Problem 1. The semi-global asymptotic stabilization via state feedback problem for system (1) is to find, if possible, for any a priori given bounded set of initial conditions $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^n$ with $\bar{\tau} = \max\{\bar{\tau}_i\}$, a linear state feedback controller $u = Fx$ independent of the specific delay such that the zero solution of the closed-loop system is locally asymptotically stable for any $\tau_i \in [0, \bar{\tau}_i]$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \ldots, m$:

(1) $\forall \nu > 0, \exists\, \eta$ such that if $\|\phi\|_C \leq \eta$ then we have $\|x(t)\| \leq \nu$ for all $t \geq 0$;
(2) $\forall \phi \in \mathcal{W}, x(t) \to 0$ as $t \to \infty$.

Problem 2. The semi-global asymptotic stabilization via measurement feedback problem for system (1) is to find, if possible, a positive integer $q > 0$ such that for any a priori given bounded set $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^{n+q}$ with $\bar{\tau} = \max\{\bar{\tau}_i\}$, there exists a linear finite dimensional measurement feedback controller independent of the delay

$$\begin{cases} \dot{\chi} = A_c\chi + B_c y, \quad \chi \in \mathbb{R}^q \\ u = C_c\chi + D_c y, \end{cases} \tag{2}$$

for which the zero solution of the closed-loop system is locally asymptotically stable for all $\tau_i \in [0, \bar{\tau}_i]$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\tau_i \in [0, \bar{\tau}_i]$:

(1) $\forall \nu > 0, \exists\, \eta$ such that if $\|(\phi; \psi)\|_C \leq \eta$ then we have $\|x(t)\| \leq \nu$ for all $t \geq 0$;
(2) $\forall (\phi; \psi) \in \mathcal{W}, (x(t), \chi(t)) \to 0$ as $t \to \infty$.

If $\tau_i = 0, i = 1, \ldots, m$, it is well known that the semi-global stabilization problem is solvable only if system (1) is asymptotically null controllable with bounded control, i.e. the following assumption holds.

Assumption 1. (A, B) is stabilizable with $B = (B_1 \quad \cdots \quad B_m)$ and A has all its eigenvalues in the closed left half plane.

Moreover, for stabilization via measurement feedback, the next assumption is also necessary.

Assumption 2. (A, C) is detectable.

3. Main result

We start by designing the state and measurement feedback controllers that will solve the stabilization problems studied in this paper. The methodology we use here is the classical H_2-ARE based low-gain feedback design (see Wang, Stoorvogel, Saberi, Grip & Sannuti, 2011) which was originally developed by Lin, Stoorvogel and Saberi (1996) in the context of semi-global stabilization of linear systems subject to input saturation.

Assume (A, B) is stabilizable and A has all its eigenvalues in the closed left half plane. For $\varepsilon \in (0, 1]$, let P_ε be the solution of Algebraic Riccati Equation

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon I = 0. \tag{3}$$

The low-gain state feedback can be constructed as

$$u = \alpha F_\varepsilon x = -\alpha B'P_\varepsilon x \tag{4}$$

for suitably chosen α and ε. For the system (1) we denote:

$$u = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}'$$

and hence $u_i = \alpha F_i x$ where $F_i = B_i'P_\varepsilon$ for $i = 1, \ldots, m$.

The low-gain state feedback (4) can be implemented as a dynamic compensator, which we refer to as a low-gain compensator

$$\begin{cases} \dot{\chi} = A\chi + BF_\varepsilon\chi - K(y - C\chi) \\ u = \alpha F_\varepsilon\chi, \end{cases} \tag{5}$$

where K is chosen such that $A + KC$ is Hurwitz stable.

In the design of (4) and (5), ε is called a low-gain parameter. The role of α and ε is reminiscent of the low-and-high-gain design. However, in our case α can be chosen independent of ε and independent of the set of initial conditions but only dependent on the delay bounds. Note that αF_ε will still be arbitrarily small as $\varepsilon \to 0$ and hence we still have a low-gain design.

With a properly chosen ε, the low-gain feedback (4) and low-gain compensator (5) solve Problems 1 and 2 respectively for suitably chosen $\bar{\tau}_i$. To prove this, we will proceed in two steps: first, we will show that our controllers globally asymptotically stabilize (1) without saturation and provide us with the required input-delay tolerance. Then, we will extend the result to the case where saturation is present by selecting the low-gain parameter differently.

3.1. Global stabilization of linear systems with input delay

Ignoring saturation, we can write (1) as follows:

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} B_i u_i(t-\tau_i) \\ y = Cx \\ x(\theta) = \phi(\theta), \quad \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{6}$$

Since the system is linear, it is possible to solve the global asymptotic stabilization problems for (6) using the low-gain feedback (4) and compensator (5), which means in Problems 1 and 2, that the bounded set of initial condition $\mathcal{W}$ is actually the entire Banach space $\mathcal{C}^n_{\bar{\tau}}$ and $\mathcal{C}^{n+q}_{\bar{\tau}}$, respectively.

In order to present our result, we need the following notation. For each input u_i, $i = 1, \ldots, m$, define the maximal controllable frequency as

$$\omega^i_{\max} := \max\{\omega \in \mathbb{R} \mid \exists v \in \mathbb{C}^n, \text{s.t.}\, A'v = j\omega v \text{ and } B_i'v \neq 0\}. \tag{7}$$

Note that we set $\omega^i_{\max}$ equal to zero if there exist no $\omega \in \mathbb{R}$ and $v \in \mathbb{C}^n$ for which $A'v = j\omega v$ and $B_i'v \neq 0$. We state that an eigenvalue $j\omega_k$ of the matrix A is controllable via the input u_i if there exists a $v \in \mathbb{C}^n$ for which $A'v = j\omega v$ and $B_i'v \neq 0$.

It is clear that $j\omega^i_{\max}$ is the eigenvalue of A on the imaginary axis with the maximal magnitude which is (at least partially) controllable via input channel u_i. Now, we are ready to present the following theorem.

Theorem 1. *If*

$$\omega^i_{\max}\bar{\tau}_i < \frac{\pi}{2}, \quad i = 1, \ldots, m, \tag{8}$$

then with any $\alpha > \max\left\{\frac{1}{2\cos(\omega^i_{\max}\bar{\tau}_i)}, 1\right\}$, *there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$, *the closed-loop of* (6) *and the low-gain feedback* (4) *is globally asymptotically stable for any* $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \ldots, m$.

In order to prove the above theorem we need two lemmas. The first one is adapted from Zhang, Knospe and Tsiotras (2000).

Lemma 1. *Consider a linear time-delay system*

$$\dot{x} = Ax + \sum_{i=1}^{m} A_i x(t-\tau_i). \tag{9}$$

Assume

$$A + \sum_{i=1}^{m} A_i$$

is Hurwitz. In that case, (9) *is globally asymptotically stable for any* $\tau_i \in [0, \bar{\tau}_i]$ *where* $i = 1, \ldots, m$ *if*

$$\det\left[j\omega I - A - \sum_{i=1}^{m} e^{-j\omega\tau_i} A_i\right] \neq 0$$

for all $\omega \in \mathbb{R}$ *and* $\tau_i \in [0, \bar{\tau}_i]$.

Assume A has r eigenvalues on the imaginary axis which are denoted by $j\omega_k$, $k = 1, \ldots, r$. Suppose $\omega^i_{\max}\bar{\tau}_i < \frac{\pi}{2}$ for $i = 1, \ldots, m$ and we choose $\alpha > \max\left\{\frac{1}{2\cos(\omega^i_{\max}\bar{\tau}_i)}, 1\right\}$. There exists a $\zeta > 0$ such that

(1) The neighborhoods $\mathcal{E}_k := [\omega_k - \zeta, \omega_k + \zeta]$, $k = 1, \ldots, r$ around these eigenfrequencies, are mutually disjoint;
(2) If $j\omega_k$ is controllable via input u_i for some i then $\omega\bar{\tau}_i < \frac{\pi}{2}$ for $\omega \in \mathcal{E}_k$.
(3) We have
$$2\alpha\cos(\omega\bar{\tau}_i) > 1, \quad \forall i, \tag{10}$$
for all $\omega \in \mathcal{E}_k$ and those $k \in \{1, \ldots, m\}$ for which $j\omega_k$ is controllable via input i.

Lemma 2. *The following properties hold:*

(1) *If* $j\omega_k$ *is not controllable via input* u_i *for some* i, *then*
$$\lim_{\varepsilon \downarrow 0} F_\varepsilon (j\omega I - A - BF_\varepsilon)^{-1} B_i = 0,$$
uniformly in ω *for* $\omega \in \mathcal{E}_k$ *where* F_ε *is given by* (4).
(2) *For any* $\rho > 0$, *there exists* ε^* *such that for* $\varepsilon \in (0, \varepsilon^*]$,
$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}B\| \leq \rho, \quad \forall \omega \in \Omega := \mathbb{R} \setminus \cup_{k=1}^{r} \mathcal{E}_k.$$

Proof. See Appendix ■

With the help of the above two lemmas we can prove Theorem 1.

Proof of Theorem 1. Consider the closed-loop system

$$\dot{x} = Ax + \sum_{i=1}^{m} \alpha B_i F_i x(t-\tau_i), \tag{11}$$

where $\bar{\tau}_i$ satisfies condition (8) and we choose some $\alpha > \max\left\{\frac{1}{2\cos(\omega^i_{\max})\bar{\tau}_i}, 1\right\}$. Let α be fixed and let sets $\mathcal{E}_k$ be as defined before Lemma 2.

Since

$$A + \sum_{i=1}^{m} \alpha B_i F_i = A - \alpha BB'P_\varepsilon$$

is Hurwitz stable for $\alpha > 1$, it follows from Lemma 1 that system (11) is asymptotically stable if

$$\det\left[j\omega I - A - \sum_{i=1}^{m} \alpha e^{-j\omega\tau_i} B_i F_i\right] \neq 0, \tag{12}$$

for all $\omega \in \mathbb{R}$ and all $\tau_i \in [0, \bar{\tau}_i]$ where $i = 1, \ldots, m$. We define:

$$G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B \tag{13}$$
$$\Delta(s) = \operatorname{diag}\left\{\alpha e^{-\tau_i s} - 1\right\}_{i=1}^{m}. \tag{14}$$

We have:

$$\det\left[j\omega I - A - \sum_{i=1}^{m} \alpha e^{-j\omega\tau_i} B_i F_i\right] = \det\left[j\omega I - A + BB'P_\varepsilon\right] \det\left[I - G_\varepsilon(j\omega)\Delta(j\omega)\right].$$

Since $A - BB'P_\varepsilon$ is Hurwitz we know that the first determinant on the right is nonzero. Therefore we only need to show that

$$\det\left[I - G_\varepsilon(j\omega)\Delta(j\omega)\right] \neq 0 \tag{15}$$

for all $\omega \in \mathbb{R}$ and all $\tau_i \in [0, \bar{\tau}_i]$ where $i = 1, \ldots, m$.

We check (15) first for $\omega \in \mathbb{R} \setminus \cap_{k=1}^{r} \mathcal{E}_k$. By Lemma 2 there exists for $\rho = (1+\alpha)^{-1}$ an ε_1 such that for $\varepsilon < \varepsilon_1$ we have that $\|G_\varepsilon(j\omega)\| < (1+\alpha)^{-1}$ while $\|\Delta(j\omega)\| < 1 + \alpha$. This implies (15) is satisfied.

Next, we need to consider $\omega \in \mathcal{E}_k$. By Lemma 2, there exists ε_2 such that

$$\|F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B_i\| \leq \mu$$

for $\varepsilon < \varepsilon_2$ for all i for which $j\omega_k$ is not controllable via input u_i.

We first establish that

$$\det\left[I - G_\varepsilon(j\omega)\tilde{\Delta}_k(j\omega)\right]$$

is bounded away from zero where $\tilde{\Delta}_k$ is obtained from Δ by setting $\tau_i = 0$ for any i for which $j\omega_k$ is not controllable via input u_i.

We have:

$$I - G_\varepsilon(j\omega)\tilde{\Delta}_k(j\omega) = I + \tilde{\Delta}_k(j\omega) - (I + G_\varepsilon(j\omega))\tilde{\Delta}_k(j\omega). \tag{16}$$

Note that

$$I + \tilde{\Delta}_k(s) = \text{diag}\left\{ \alpha e^{-\tau_i s} \right\}_{i=1}^m$$

is invertible and

$$\tilde{\Delta}_k(j\omega)(I + \tilde{\Delta}_k(j\omega))^{-1} = \text{diag}\left\{ 1 - \alpha^{-1} e^{j\omega\tau_i} \right\}_{i=1}^m .$$

Note that for any i for which $j\omega_k$ is not controllable via input u_i, we have set $\tau_i = 0$ and hence

$$|1 - \alpha^{-1} e^{j\omega\tau_i}| = |1 - \alpha^{-1}| < 1.$$

Otherwise,

$$|1 - \alpha^{-1} e^{j\omega\tau_i}|^2 = 1 - 2\alpha^{-1}\cos(\omega\tau_i) + \alpha^{-2} < 1$$

since $2\alpha\cos(\omega\tau_i) > 1$. The above implies there exists $\beta > 0$ such that

$$\|\tilde{\Delta}_k(j\omega)(I + \tilde{\Delta}_k(j\omega))^{-1}\| < 1 - \beta. \tag{17}$$

Next, we know that for all $\varepsilon > 0$

$$\underline{\sigma}[I - F_\varepsilon(j\omega I - A)^{-1}B] \geq 1, \quad \forall\omega \tag{18}$$

(see Anderson & Moore, 1971, Section 7.1, p.122), and this implies that

$$\|I + G_\varepsilon(j\omega)\| \leq 1, \quad \forall\omega. \tag{19}$$

But (16) together with (17) and (19) imply that

$$\det(I - G_\varepsilon(j\omega)\tilde{\Delta}_k(j\omega))$$

is bounded away from zero. Using item (1) of Lemma 2 we conclude that for ε small enough

$$\det(I - G_\varepsilon(j\omega)\Delta(j\omega))$$

is bounded away from zero. Since this is valid for $\omega \in \mathcal{E}_k$ for any $k \in \{1, \ldots, m\}$, it shows that there exists an ε_3 such that (15) is satisfied for $\varepsilon < \varepsilon_3$, which completes the proof of Theorem 1. ■

In the special case where A has all its eigenvalues at the origin, the low-gain feedback can tolerate arbitrary large but bounded delays.

Corollary 1. *Suppose A has only zero eigenvalues. For any $\bar{\tau}_i > 0$, $i = 1, \ldots, m$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (6) and (4) is asymptotically stable for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \ldots, m$.*

Remark 1. We feel the following comments would be helpful to the readers. Essentially, Theorem 1 is built upon two fundamental results.

The first result is the robustness of LQR. As we know from the Nyquist criterion, the stabilizability of a single input linear system with time delay is closely related to the notion of phase margin. It is also known that LQR has a guaranteed phase margin of $\frac{\pi}{3}$. In this paper, we actually prove that for critically unstable linear systems, a special class of LQR which is the low-gain feedback has a guaranteed phase margin of $\frac{\pi}{2}$ which gives the right hand side of (8).

The second result is the low-gain properties. It should be noted that the $\frac{\pi}{2}$ phase margin only provides an upper bound on the phase uncertainty and does not say how much delay can be tolerated. It is the properties of low-gain feedback that translate the "phase margin" to the amount of tolerable delay in each channel, which is explicitly related to the maximal magnitude of eigenvalues on the imaginary axis.

This is more transparent for single input systems, in which case according to Nyquist, we are mainly concerned with "gain crossover frequencies". It can be immediately seen from the transfer function $F_\varepsilon(j\omega I - A)^{-1}B$ that as the low-gain parameter approaches zero, the "gain crossover frequencies" converge to those that correspond to some open loop eigenvalues on imaginary axis. Then condition (8) is natural given a $\frac{\pi}{2}$ phase margin.

For multi-input systems, the problem is not so obvious. We use a simple frequency domain stability criterion (Lemma 1) and the proof is quite involved. However the thoughts behind our proof are similar to the single input case, that is, in concerning with stability in the frequency domain (Lemma 1), not all ω matter but only those close to the eigenfrequencies (this is proved in Lemma 2).

The next theorem concerns stabilization of (6) via measurement feedback.

Theorem 2. *If*

$$\omega^i_{\max}\bar{\tau}_i < \frac{\pi}{2}, \tag{20}$$

then with any $\alpha > \max\left\{\frac{1}{2\cos(\omega^i_{\max}\bar{\tau}_i)}, 1\right\}$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (6) and low-gain compensator (5) is asymptotically stable for $\tau_i \in [0, \bar{\tau}_i]$.

We first present a lemma needed in the proof.

Lemma 3. *Let $G_\varepsilon(s)$ be given by (13). Then*

$$\lim_{\varepsilon\downarrow 0} G^m_\varepsilon(j\omega) = G_\varepsilon(j\omega)$$

uniformly in ω, where

$$G^m_\varepsilon(s) = -F_\varepsilon(sI - A - BF_\varepsilon)^{-1}KC(sI - A - KC)^{-1}B. \tag{21}$$

Proof. See Appendix. ■

Proof of Theorem 2. The closed-loop system is given by

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^m \alpha B_i F_i \chi(t - \tau_i) \\ \dot{\chi} = (A + BF_\varepsilon + KC)\chi - KCx \\ x(\theta) = \phi(\theta), \quad \forall\theta \in [-\bar{\tau}, 0] \\ \chi(\theta) = \psi(\theta), \quad \forall\theta \in [-\bar{\tau}, 0]. \end{cases} \tag{22}$$

It follows from Lemma 1 that (22) is globally asymptotically stable if and only if

$$\det\left[j\omega I - \begin{pmatrix} A & 0 \\ -KC & A + BF_\varepsilon + KC \end{pmatrix} - \sum_{i=1}^m \begin{pmatrix} 0 & \alpha B_i F_i \\ 0 & 0 \end{pmatrix} e^{-j\omega\tau_i} \right] \neq 0$$

for all $\omega \in \mathbb{R}$ and for all $\tau_i \in [0, \bar{\tau}_i]$. This is equivalent to:

$$\det[I - G^m_\varepsilon(j\omega)\Delta(j\omega)] \neq 0, \quad \forall\omega \in \mathbb{R}, \forall\tau_i \in [0, \bar{\tau}_i], \tag{23}$$

since $A + BF_\varepsilon$ and $A + KC$ are Hurwitz where $G^m_\varepsilon(s)$ and $\Delta(s)$ are defined by (21) and (14).

From the proof of Theorem 1, we note that there exists an ε_3 such that for all $\varepsilon \in (0, \varepsilon_3]$ we have that (15) is satisfied. It is then easily checked using Lemma 3 that we can find an $\varepsilon_4 \leq \varepsilon_3$ such that (23) holds for all $\varepsilon \in (0, \varepsilon_4]$. ■

3.2. Semi-global stabilization subject to input saturation

In this subsection, we shall extend the results for linear systems to the case where input saturation is considered and solve the semi-global stabilization problems as formulated in Problems 1 and 2.

Theorem 3. *Consider the system (1). The semi-global asymptotic stabilization via state feedback problem can be solved by the low-gain feedback (4). Specifically, for a set of positive real numbers* $\bar{\tau}_i < \frac{\pi}{2\omega^i_{\max}}$, $i = 1, \dots, m$ *and any a priori given compact set of initial conditions* $\mathcal{W} \subset \mathcal{C}^n_{\bar{\tau}}$*, there exists an* ε^* *such that for any* $\varepsilon \in (0, \varepsilon^*]$*, the low-gain feedback (4) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing* $\mathcal{W}$ *for any* $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \dots, m$.

Proof. The closed-loop system can be written as

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} B_i \sigma(\alpha F_i x(t-\tau_i)) \\ x(\theta) = \phi(\theta), \quad \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{24}$$

Suppose $\bar{\tau}_i$'s satisfy the bound $\omega^i_{\max}\bar{\tau}_i < \frac{\pi}{2}$. Let ε_1 be such that the closed-loop system (11) in the absence of saturation, is asymptotically stable for $\varepsilon \le \varepsilon_1$. Then the local stability of (24) as required in part (1) of Problem 1 follows with $\varepsilon \le \varepsilon_1$.

It remains to show the attractivity. It is sufficient to prove that for system (24), given $\mathcal{W}$, there exists an $\varepsilon_2 \le \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, we have

$$\|\alpha F_i x(t-\tau_i)\| \le 1, \quad \forall t \ge 0, \tag{25}$$

for all $\tau_i \le \bar{\tau}_i$ and for $i = 1, \dots, m$. Then we can avoid saturation for all $t \ge 0$. The closed-loop system becomes linear and the attractivity of the zero solution is therefore guaranteed with $\varepsilon \le \varepsilon_2$.

We define the linear time-invariant operator $g_\varepsilon : v_\varepsilon \to w_\varepsilon$ with state space representation:

$$\begin{cases} \dot{\xi} = (A + BF_\varepsilon)\xi + Bv_\varepsilon, \qquad \xi(0) = 0 \\ w_\varepsilon = F_\varepsilon \xi. \end{cases} \tag{26}$$

Next, we define the linear time-invariant operator δ by:

$$g(t) = \delta(f)(t)$$

with $g(t), f(t) \in \mathbb{R}^m$ where g is defined componentwise by:

$$g_i(t) = \begin{cases} \alpha f_i(t-\tau_i) - f_i(t) & \text{if } t > \tau_i \\ -f_i(t) & \text{otherwise} \end{cases}$$

for $t > 0$. The Laplace transform of these two operators is given by the transfer matrices (13) and (14). From the proof of Theorem 1, we know there exists ε_3 such that for all $\varepsilon < \varepsilon_3$ we have that (15) is satisfied which guarantees that there exists a μ such that

$$\underline{\sigma}(I - G_\varepsilon(j\omega)\Delta(j\omega)) > \mu, \forall \omega \in \mathbb{R}, \quad \forall \tau_i \in [0, \bar{\tau}_i]$$

for all $\varepsilon \le \varepsilon_3$ and this μ only depends on $\bar{\tau}_i$ provided that $\varepsilon \le \varepsilon_3$. This implies that

$$\|(I - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \le \frac{1}{\mu}.$$

Moreover, we already have in (19)

$$\bar{\sigma}(I + G_\varepsilon(j\omega)) \le I, \quad \forall \omega \in \mathbb{R}$$

which implies $\|G_\varepsilon(s)\|_\infty \le 2$.

Note that for $t \ge 0$, (24) implies that

$$\dot{x} = (A + BF_\varepsilon)x + B\delta(F_\varepsilon x) + Bv_\varepsilon,$$

where

$$v_\varepsilon(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_m(t) \end{bmatrix}, \qquad v_i(t) = \begin{cases} \alpha F_i \phi(t-\tau_i), & t < \tau_i, \\ 0, & t \ge \tau_i. \end{cases}$$

Since $v_\varepsilon(t)$ vanishes for $t \ge \bar{\tau}$, $\phi \in \mathcal{W}$ and $F_\varepsilon \to 0$, we have for any $\phi \in \mathcal{W}$, $\|v_\varepsilon\|_\infty \to 0$ and $\|v_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$.

We have

$$F_\varepsilon x(t) = F_\varepsilon e^{(A+BF_\varepsilon)t}x(0) + (g_\varepsilon \circ \delta)(F_\varepsilon x)(t) + g_\varepsilon(v_\varepsilon)(t)$$

and hence

$$F_\varepsilon x(t) = (1 - g_\varepsilon \circ \delta)^{-1}\left[F_\varepsilon e^{(A+BF_\varepsilon)t}x(0) + g_\varepsilon(v_\varepsilon)(t)\right]. \tag{27}$$

Let $w_\varepsilon(t) = g_\varepsilon(v_\varepsilon)(t)$. By the definition of g_ε, we have (26). Clearly, $\|w_\varepsilon\|_2 \le \|G_\varepsilon(s)\|_\infty \|v_\varepsilon\|_2 \le 2\|v_\varepsilon\|_2$. Hence for any given initial condition ϕ, $\|w_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$. For $t \in [0, \bar{\tau}]$,

$$\begin{aligned} \dot{w}_\varepsilon(t) &= F_\varepsilon(A + BF_\varepsilon)\xi(t) + F_\varepsilon B v_\varepsilon(t) \\ &= F_\varepsilon(A + BF_\varepsilon)\int_0^t e^{(A+BF_\varepsilon)(t-r)}Bv_\varepsilon(s)\mathrm{d}r + F_\varepsilon B v_\varepsilon(t). \end{aligned}$$

Since $A + BF_\varepsilon$ is bounded for all $\varepsilon \in [0, 1]$ and $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$, we will have

$$\sup_{t\in[0,\bar{\tau}]} \|\dot{w}_\varepsilon(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{28}$$

This also implies

$$\int_0^{\bar{\tau}} \|\dot{w}(t)\|^2 \mathrm{d}t \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{29}$$

From $\bar{\tau}$ onward, $v_\varepsilon(t)$ vanishes and

$$\dot{w}(t) = F_\varepsilon e^{(A+BF_\varepsilon)t}(A + BF_\varepsilon)\xi(\bar{\tau}).$$

It is shown by Wang et al. (2011) that

$$\int_{\bar{\tau}}^{\infty} \|\dot{w}(t)\|^2 \mathrm{d}t \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{30}$$

provided that $\xi(\bar{\tau})$ is bounded which is obvious by noticing that

$$\xi(\bar{\tau}) = \int_0^{\bar{\tau}} e^{(A+BF)(\bar{\tau}-t)}Bv_\varepsilon(t)\mathrm{d}t$$

and $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$. Combining (29) and (30), we have shown that for any given $\phi \in \mathcal{W}$, $\|\dot{w}\|_2 \to 0$ as $\varepsilon \to 0$.

Now let us go back to (27). We get

$$\begin{aligned} \|F_\varepsilon x\|_2 &\le \|(1 - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x(0)\|_2 \\ &\quad + \|(1 - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \|w_\varepsilon\|_2 \\ &\le \frac{1}{\mu}\|F_\varepsilon e^{(A+BF_\varepsilon)t}x(0)\|_2 + \frac{1}{\mu}\|w_\varepsilon\|_2. \end{aligned}$$

Since for any ϕ, $\|F_\varepsilon e^{(A+BF_\varepsilon)t}x(0)\|_2 \to 0$ (see Wang et al., 2011) and $v_\varepsilon \to 0$ as $\varepsilon \to 0$ and μ is independent of ε (provided ε is smaller than ε_3), there exists an ε_4 such that for $\varepsilon \in (0, \varepsilon_4]$, we get

$$\|F_\varepsilon x\|_2 \le \frac{1}{2\alpha}, \quad \forall \phi \in \mathcal{W}. \tag{31}$$

Note that (27) also yields

$$F\dot{x}(t) = (1 - g_\varepsilon \circ \delta)^{-1}\left[F_\varepsilon e^{(A+BF_\varepsilon)t}(A + BF_\varepsilon)x(0) + \dot{w}_\varepsilon(t)\right],$$

and thus

$$\begin{aligned} \|F_\varepsilon \dot{x}\|_2 &\le \|(1 - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}\tilde{x}\|_2 \\ &\quad + \|(1 - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \|\dot{w}_\varepsilon\|_2 \\ &\le \frac{1}{\mu}\|F_\varepsilon e^{(A+BF_\varepsilon)t}\tilde{x}\| + \frac{1}{\mu}\|\dot{w}_\varepsilon\|_2 \end{aligned}$$

with $\tilde{x} = (A + BF_\varepsilon)x(0)$. There exists an ε_5 such that for $\varepsilon \in (0, \varepsilon_5]$, we have

$$\|F_\varepsilon \dot{x}\|_2 \le \frac{1}{2\alpha}, \quad \forall \phi \in \mathcal{W}. \tag{32}$$

Applying the Cauchy–Schwarz inequality, we can prove that for any $t \ge 0$,

$$\left| \|F_\varepsilon x(t)\|^2 - \|F_\varepsilon x(0)\|^2 \right| \le 2\|F_\varepsilon \dot{x}\|_2 \|F_\varepsilon x\|_2,$$

and

$$\|F_\varepsilon x(t)\|^2 \le \|F_\varepsilon x(0)\|^2 + 2\|F_\varepsilon \dot{x}\|_2 \|F_\varepsilon x\|_2. \tag{33}$$

Next, we note that there exists an ε_6 such that for $\varepsilon \in (0, \varepsilon_6]$

$$\|F_\varepsilon x(0)\|^2 \le \|F_\varepsilon \phi\|_C^2 \le \frac{1}{2\alpha}, \quad \phi \in \mathcal{W}. \tag{34}$$

Finally, there exists ε_7 such that for $\varepsilon < \varepsilon_7$ we have:

$$\begin{aligned} \|\alpha F_i x(t - \tau_i)\| &\le \|\alpha F_\varepsilon x(t - \tau_i)\| \\ &\le \|\alpha F_\varepsilon \phi\|_C \le 1, \quad \forall t \in [0, \tau_i]. \end{aligned} \tag{35}$$

Let $\varepsilon^* = \min\{\varepsilon_1, \dots, \varepsilon_7\}$. We conclude from (31)–(34) that for $\varepsilon \in (0, \varepsilon^*]$,

$$\|\alpha F_i x(t - \tau_i)\| \le \|\alpha F_\varepsilon x(t - \tau_i)\| \le 1, \quad \forall t \ge \tau_i.$$

Together with (35), this implies (25) is satisfied. Hence the system avoids saturation which implies the required attractivity. ■

The next theorem solves Problem 2.

Theorem 4. *Consider the system (1). The semi-global asymptotic stabilization via measurement feedback problem can be solved by the low-gain compensator (5). Specifically, for any a priori given compact set of initial conditions $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^{2n}$ and a set of positive real numbers $\omega_{\max}^i \bar{\tau}_i < \frac{\pi}{2}$, $i = 1, \dots, m$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the low-gain feedback (5) achieves local asymptotic stability of the closed-loop system for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \dots, m$ with the domain of attraction containing $\mathcal{W}$.*

In order to derive our main result for Problem 2 we need a preliminary lemma.

Lemma 4. *For any $\xi \in \mathbb{R}^{2n}$,*

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty \|\mathcal{F}_\varepsilon e^{(\mathcal{A} + \mathcal{B}\mathcal{F}_\varepsilon)t} \xi\|^2 \mathrm{d}t = 0,$$

where

$$\mathcal{A} = \begin{bmatrix} A & BF_\varepsilon \\ -KC & A + BF_\varepsilon + KC \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad \mathcal{F} = \begin{bmatrix} 0 & F_\varepsilon \end{bmatrix}.$$

Proof. See Appendix. ■

Proof of Theorem 4. The closed-loop system can be written as

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^m B_i \sigma(\alpha F_i \chi(t - \tau_i)) \\ \dot{\chi} = (A + BF_\varepsilon + KC)\chi - KCx \\ x(\theta) = \phi(\theta), \quad \forall \theta \in [-\bar{\tau}, 0] \\ \chi(\theta) = \psi(\theta), \quad \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{36}$$

Suppose $\bar{\tau}_i$'s satisfy the bound $\omega_{\max}^i \bar{\tau}_i < \frac{\pi}{2}$. Let ε^* be given by Theorem 2 such that the closed-loop system without saturation is asymptotically stable. Then the local stability of (36) as required in part (1) of Problem 2 follows with $\varepsilon \le \varepsilon^*$.

Define two linear time invariant operators g_ε^m and δ with Laplace transform

$$G_\varepsilon^m(s) = -F_\varepsilon(sI - A - BF_\varepsilon)^{-1} KC (sI - A - KC)^{-1} B$$
$$\Delta(s) = \mathrm{diag}\{\alpha e^{-\tau_i s} - 1\}_{i=1}^m.$$

From the proof of Theorem 2, we know that there exists ε_4 such that (23) holds for $\varepsilon \le \varepsilon_4$. There exists a $\mu > 0$ such that

$$\underline{\sigma}(I - G_\varepsilon^m(j\omega)\Delta(j\omega)) > \mu, \quad \forall \omega \in \mathbb{R}, \forall \tau_i \in [0, \bar{\tau}_i], \tag{37}$$

where μ is independent of ε provided that $\varepsilon \le \varepsilon_4$. It follows from Lemma 3 that $G_\varepsilon^m(j\omega) \to G_\varepsilon(j\omega)$ uniformly in ω where $G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B$. Hence given $\bar{\sigma}(G_\varepsilon(j\omega)) \le 2$ for any $\varepsilon > 0$ and $\omega \in \mathbb{R}$, there exists an ε_5 such that

$$\bar{\sigma}(G_\varepsilon^m(j\omega)) \le 3, \quad \forall \omega \in \mathbb{R}. \tag{38}$$

Given (37), (38) and Lemma 4 hold, we can use exactly the same argument as in the proof of Theorem 3 to prove that there exists an $\varepsilon_6 \le \min\{\varepsilon^*, \varepsilon_4, \varepsilon_5\}$ such that for $\varepsilon \in (0, \varepsilon_6]$,

$$\|\alpha F_\varepsilon \chi(t - \bar{\tau})\| \le 1, \quad \forall t \ge 0, \ (\phi, \psi) \in \mathcal{W}. \quad ■$$

4. Example

Let us consider the following system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t - \tau)$$

where $\tau \in [0, \bar{\tau}]$.

First, in this example we have $\omega_{\max} = 1$. According to Theorem 1, the maximal tolerable delay is $\bar{\tau} < \frac{\pi}{2}$.

Second, we shall examine four delay bounds: $\bar{\tau} = 0.5$, $\bar{\tau} = 1$, $\bar{\tau} = 1.3$ and $\bar{\tau} = 1.4$. For each delay, we perform a simulation with four different initial conditions: [2; 2], [−2; −2], [2; −2] and [−2; 2].

4.1. Parameters

In our simulation, we choose

$$\alpha = 1 + \frac{1}{\cos(\omega_{\max}\bar{\tau})},$$

which obviously satisfies Theorem 1.

The resulting ε^* (and approximate settling time) are shown in the following table.

$\bar{\tau}$	α	ε^*	Settling time (s)
0.5	2.1395	0.7	50
1.0	2.8508	0.04	150
1.3	4.7383	0.002	400
1.4	6.8835	0.0002	1000

To save space, we only show the simulation data for two cases (see Figs. 1 and 2).

Remark 2. Though the dependence of ε^* on $\bar{\tau}$ is implicit and the way we choose ε^* is through experiments, it appears that as the maximal delay $\bar{\tau}$ approaches the bound in Theorem 1, i.e. $\frac{\pi}{2}$ in this case, the ε^* shrinks to zero with an increasing rate. Also, the performance deteriorates with larger delay and smaller ε^*.

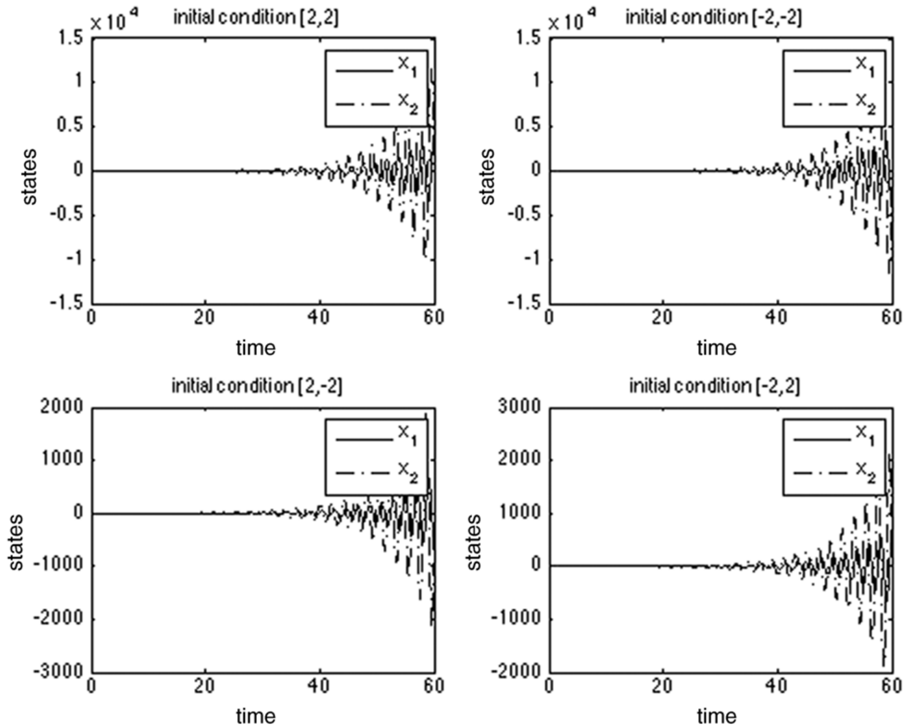

Fig. 1. $\bar{\tau} = 0.5$ and $\varepsilon^* = 1$. The system is unstable.

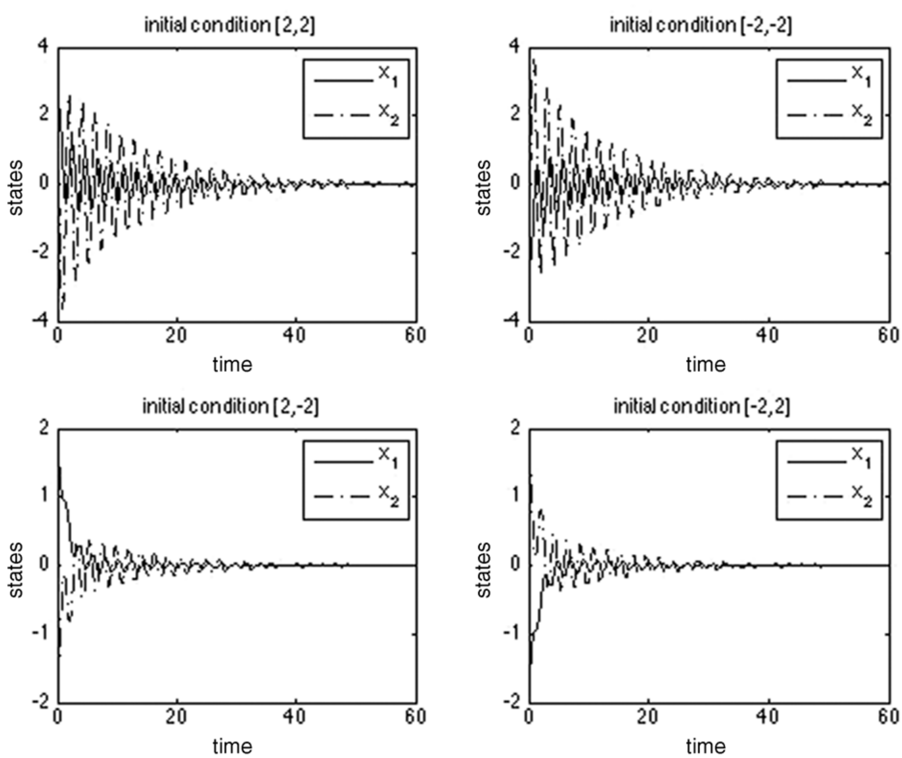

Fig. 2. $\bar{\tau} = 0.5$ and $\varepsilon^* = 0.7$. It takes about 50 s to stabilize the system.

5. Conclusion

In this paper, the semi-global stabilization problems for general uncritically unstable systems subject to input saturation and multiple unknown input delays are solved. Upper bounds on delays are found for which a low-gain state feedback or a low-gain compensator can be constructed to achieve the semi-global stabilization.

Appendix

Proof of Lemma 2. To prove item (1), we first note that

$$\begin{aligned}&F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i\\&\quad = F_\varepsilon(I - (j\omega I - A)^{-1}BF_\varepsilon)^{-1}(j\omega I - A)^{-1}Be_i\\&\quad = (I - F_\varepsilon(j\omega I - A)^{-1}B)^{-1}F_\varepsilon(j\omega I - A)^{-1}Be_i,\end{aligned}$$

where e_i is the standard basis (indicator vector) in $\mathbb{R}^m$. Next we note that:

$$\bar{\sigma}(I - F_\varepsilon(j\omega I - A)^{-1}B)^{-1} \leq 1, \quad \forall \omega \in \mathbb{R}$$

(see Anderson & Moore, 1971). Moreover, $\forall \omega \in \mathcal{E}_k$, $(j\omega I - A)^{-1}Be_i$ has no pole and therefore

$$\|(j\omega I - A)^{-1}Be_i\| \leq M, \quad \forall \omega \in \mathcal{E}_k,$$

for $M > 0$ independent of ω. But then

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i\| \leq M\|F_\varepsilon\|, \quad \forall \omega \in \mathcal{E}_k,$$

and since F_ε converges to zero we get

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i\| \to 0$$

as $\varepsilon \to 0$ uniformly in $\mathcal{E}_k$.

It remains to show item (2). By definition, $\det(j\omega I - A) \neq 0$ for all $\omega \in \Omega$. There exists a μ such that

$$\underline{\sigma}(j\omega I - A) > \mu, \quad \forall \omega \in \Omega.$$

After all assume this is not the case. Then there exists a sequence $\omega^i \in \Omega$ such that

$$\underline{\sigma}(j\omega^i I - A) \to 0$$

as $i \to \infty$. We can ensure that this sequence ω^i is bounded since for ω satisfying $|\omega| > \|A\| + 1$ we have:

$$\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1.$$

But a bounded sequence ω^i has a convergent subsequence whose limit, denoted by $\bar{\omega}$, is in Ω (since Ω is closed). The limit $\bar{\omega}$ would have the property

$$\underline{\sigma}(j\bar{\omega}I - A) = 0.$$

This implies $\bar{\omega}$ is an eigenvalue of A which is in contradiction with the definition of Ω.

Choose ε^* such that $\|F_\varepsilon\| \leq \rho\frac{\mu}{2}\|B\|^{-1}$ for $\varepsilon \leq \varepsilon^*$. In that case:

$$\underline{\sigma}(j\omega I - A - BF) > \mu - \|B\|\|F_\varepsilon\| > \frac{\mu}{2}, \quad \forall \omega \in \Omega,$$

where we assume, without loss of generality that $\rho < 1$. Hence

$$\|(j\omega I - A - BF_\varepsilon)^{-1}\| < \frac{2}{\mu}, \quad \forall \omega \in \Omega,$$

but then

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}B\| \leq \|F_\varepsilon\|\|(j\omega I - A - BF_\varepsilon)^{-1}\|\|B\| \leq \rho$$

for all $\omega \in \Omega$. ■

Proof of Lemma 3. The error between $G_\varepsilon^m(s)$, given by (13), and $G_\varepsilon(s)$, given by (21), is

$$\begin{aligned} G_\varepsilon(s) - G_\varepsilon^m(s) &= \left[I + F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B\right]F_\varepsilon(sI - A - KC)^{-1}B \\ &= [I + G_\varepsilon(s)]\,F_\varepsilon(sI - A - KC)^{-1}B. \end{aligned}$$

From (19) we obtain

$$\bar{\sigma}(I + G_\varepsilon(j\omega)) \leq 1, \quad \forall \varepsilon > 0,\ \omega \in \mathbb{R}.$$

Moreover,

$$\|F_\varepsilon(sI - A - KC)^{-1}B\|_\infty \leq \|F_\varepsilon\|\|(sI - A - KC)^{-1}B\|_\infty.$$

Since $F_\varepsilon \to 0$ as $\varepsilon \to 0$, we immediately have that

$$\lim_{\varepsilon \downarrow 0} G_\varepsilon^m(j\omega) - G_\varepsilon(j\omega) = 0,$$

uniformly in ω. ■

Proof of Lemma 4. Define a system as

$$\begin{cases} \dot{x}_1 = Ax_1 + BF_\varepsilon x_2 \\ \dot{x}_2 = (A + BF_\varepsilon + KC)x_2 - KCx_1\,, \\ z = F_\varepsilon x_2. \end{cases} \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \xi.$$

It is obvious that for any ξ

$$\|z\|_2 = \int_0^\infty \|\mathcal{F}_\varepsilon e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}\xi\|^2 \mathrm{d}t.$$

Let $e = x_1 - x_2$. In the new coordinates of (x_1, e), the above system can be written as

$$\begin{cases} \dot{x}_1 = (A + BF_\varepsilon)x_1 - BF_\varepsilon e \\ \dot{e} = (A + KC)e \\ \dot{z} = F_\varepsilon(x_1 - e), \end{cases}$$

with $e_1(0) = x_1(0) - x_2(0)$. We get $\|z\|_2 \leq \|F_\varepsilon e\|_2 + \|F_\varepsilon x_1\|_2$.

Since $A + KC$ is Hurwitz, there exists a γ such that $\|e\|_2 \leq \gamma\|e(0)\|$ for any $e(0) \in \mathbb{R}^n$. Then

$$\|F_\varepsilon e\|_2 \leq \gamma\|F_\varepsilon\|\|e(0)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

But for x_1, we have

$$\begin{aligned} \|F_\varepsilon x_1\|_2 &\leq \|G_\varepsilon(s)\|_\infty\|F_\varepsilon e\|_2 + \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t \\ &\leq 2\gamma\|F_\varepsilon\|\|e(0)\| + \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t \end{aligned}$$

where $G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B$. It was shown in Wang et al. (2011) that

$$\lim_{\varepsilon \downarrow 0} \|F_\varepsilon x_1\|_2 = \lim_{\varepsilon \downarrow 0} \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t = 0,$$

and thus

$$\lim_{\varepsilon \downarrow 0} \|z\|_2 = 0. \quad ■$$

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