

Analysis of Quantum Linear Systems' Response to Multi-photon States

Guofeng Zhang^a

^a*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong*

Abstract

The purpose of this paper is to present a mathematical framework for analyzing the response of quantum linear systems driven by multi-photon states. Both the factorizable (namely, no correlation among the photons in the channel) and unfactorizable multi-photon states are treated. Pulse information of multi-photon input state is encoded in terms of tensor, and response of quantum linear systems to multi-photon input states is characterized by tensor operations. Analytic forms of output correlation functions and output states are derived. The proposed framework is applicable no matter whether the underlying quantum dynamic system is passive or active. The results presented here generalize those in the single-photon setting studied in (Milburn, 2008) and (Zhang & James, 2013). Moreover, interesting multi-photon interference phenomena studied in (Sanaka, Resch & Zeilinger, 2006), (Ou, 2007), and (Bartley, et al., 2012) can be reproduced in the proposed framework.

Key words: quantum linear systems, multi-photon states, tensors.

1 Introduction

Analysis of system response to various types of input signals is fundamental to control systems engineering. Step response enables a control engineer to visualize system transient behavior such as rise time, overshoot and settling time; frequency response design methods are among the most powerful methods in classical control theory; response analysis of linear systems initialized in Gaussian states driven by Gaussian input signals is the basis of Kalman filtering and linear quadratic Gaussian (LQG) control (see, e.g., Anderson & Moore, 1971; Kwakernaak & Sivan, 1972; Anderson & Moore, 1979; Zhou, Doyle & Glover, 1996; Qiu & Zhou, 2009).

Over the last two decades, there has been rapid advance in experimental demonstration and theoretical investigation of quantum (namely, non-classical) control systems due to their promising applications in a wide range of areas such as quantum communication, quantum computation, quantum metrology, laser-induced chemical reaction, and nano electronics (Gardiner & Zoller, 2000; Loudon, 2000; Nielsen & Chuang, 2000; D'Alessandro, 2007; Walls & Milburn, 2008; Wiseman & Milburn, 2010; Belavkin, 1983; Huang, Tarn & Clark, 1983; Yurke & Denker, 1984; Gardiner, 1993; Doherty & Jacobs, 1999; Khaneja, Brockett & Glaser, 2001; Albertini & D'Alessandro, 2003; Yanagisawa & Kimura, 2003; Stockton, van Handel & Mabuchi, 2004; Mabuchi & Khaneja, 2005; van Handel, Stockton & Mabuchi, 2005; Altafini, 2007; Mirrahimi & van Handel, 2007; James, Nurdin & Petersen, 2008; Rouchon, 2008; Bonnard, Chyba & Sugny, 2009; Gough & James, 2009; Li & Khaneja, 2009; Mirrahimi & Rouchon, 2009; Nurdin, James & Doherty, 2009; Yamamoto & Bouten, 2009; Bloch, Brockett & Rangan, 2010; Bolognani & Ticozzi, 2010; Brif, Chakrabarti & Rabitz, 2010; Dong & Petersen, 2010; Gough, James & Nurdin, 2010; Munro, Nemoto & Milburn, 2010; Wang & Schirmer, 2010; Maalouf & Petersen, 2011; Zhang & James, 2011; Altafini & Ticozzi, 2012; Amini, Mirrahimi & Rouchon, 2012; Zhang, et al., 2012; Qi, 2013). Within this program quantum linear systems play a prominent role. Quantum linear systems are characterized by linear quantum stochastic differential equations (linear QSDEs). In quantum optics, linear systems are widely used because they are easy to manipulate and, more importantly, linear dynamics often serve well as good approximation of more general dynamics (Gardiner & Zoller, 2000; Loudon,

Email address: Guofeng.Zhang@polyu.edu.hk (Guofeng Zhang).

2000; Walls & Milburn, 2008; Wiseman & Milburn, 2010). Besides their broad applications in quantum optics, linear systems have also found applications in many other quantum-mechanical systems such as opto-mechanical systems (Massel, et al., 2011, Eqs. (15)-(18)), circuit quantum electrodynamics (circuit QED) systems (Matyas, et al., 2011, Eqs. (18)-(21)), atomic ensembles (Stockton, van Handel & Mabuchi, 2004, Eqs. (A1),(A4)), quantum memory (Hush, Carvalho, Hedges & James, 2013, Eqs. (12)-13). From a signals and systems point of view, quantum linear systems driven by Gaussian input states have been studied extensively, and results like quantum filtering and measurement-based feedback control have been well established (Wiseman & Milburn, 2010).

In addition to Gaussian states there are other types of non-classical states, for example single-photon states and multi-photon states. Such states describe electromagnetic fields with a definite number of photons. Due to their highly non-classical nature and recent hardware advance, there is rapidly growing interest in the generation and engineering (e.g., pulse shaping) of photon states, and it is generally perceived that these photon states hold promising applications in quantum communication, quantum computing, quantum metrology and quantum simulations (Cheung, Migdall & Rastello, 2009; Gheri, Ellinger, Pellizzari & Zoller, 1998; Sanaka, Resch & Zeilinger, 2006; Ou, 2007; Bartley, et al., 2012; Milburn, 2008; Gough, James & Nurdin, 2013; Hush, Carvalho, Hedges & James, 2013). Thus, a new and important problem in the field of quantum control engineering is: How to characterize and engineer interaction between quantum linear systems and photon states? The interaction of quantum linear systems with continuous-mode photon states has recently been studied in the literature, primarily in the physics community. For example, interference phenomena of photons passing through beamsplitters have been studied, see, e.g., Sanaka, Resch & Zeilinger, 2006; Ou, 2007; Bartley, et al., 2012. Milburn discussed how to use an optical cavity to manipulate the pulse shape of a single-photon light field (Milburn, 2008). Quantum filtering for systems driven by single-photon fields has been investigated in Gough, James & Nurdin, 2013, based on which nonlinear phase shift of coherent signal induced by single-photon field has been studied in Carvalho, Hush & James, 2012. Intensities of output fields of quantum systems driven by continuous-mode multi-photon light fields have been studied in Baragiola, Cook, Brańczyk & Combes, 2012. In Zhang & James, 2013 the response of quantum linear systems to single-photon states has been studied. Formulas for intensities of output fields have been derived. In particular, a new class of optical states, photon-Gaussian states, has been proposed.

In the analysis of the response of quantum linear systems to single-photon states, matrix presentation is sufficient because two indices are adequate: one for input channels, and the other for output channels. However, this is not the case in the multi-photon setting. In addition to indices for input and output channels, we need another index to count photon numbers in channels. As a result, tensor representation and operation are essential in the multi-photon setting. To be specific, multi-photon state processing by quantum linear systems can be mathematically represented in terms of tensor processing by transfer functions. The key ingredient for such an operation is the following (for the passive case). Let $E(t) = (E^{jk}(t)) \in \mathbb{C}^{m \times m}$ be the transfer function of a quantum linear passive system with m input channels. For each $j = 1, \dots, m$, let $\mathcal{V}_j(t_1, \dots, t_{\ell_j})$ be an ℓ_j -way m -dimensional tensor function that encodes the pulse information of the j -th input channel containing ℓ_j photons. Denote the entries of $\mathcal{V}_j(t_1, \dots, t_{\ell_j})$ by $\mathcal{V}_{j,k_1, \dots, k_{\ell_j}}(t_1, \dots, t_{\ell_j})$. For all given $1 \leq r_1, \dots, r_{\ell_j} \leq m$, define an ℓ_j -way m -dimensional tensor \mathcal{W}_j with entries given by the following multiple convolution

$$\begin{aligned} & \mathcal{W}_{j,r_1, \dots, r_{\ell_j}}(t_1, \dots, t_{\ell_j}) \\ = & \sum_{k_1, \dots, k_{\ell_j}=1}^m \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} E^{r_1 k_1}(t_1 - \iota_1) \dots E^{r_{\ell_j} k_{\ell_j}}(t_{\ell_j} - \iota_{\ell_j}) \mathcal{V}_{j,k_1, \dots, k_{\ell_j}}(t_1, \dots, t_{\ell_j}) d\iota_1 \dots d\iota_{\ell_j}. \end{aligned}$$

It turns out that the tensors \mathcal{W}_j ($j = 1, \dots, m$) encode the pulse information of the output field. That is, an ℓ_j -way input tensor is mapped to an ℓ_j -way output tensor by the quantum linear passive system.

The contributions of this paper are three-fold. First, the analytic form of the steady-state output state of a quantum linear system driven by a multi-photon input state is derived. When the quantum linear system is a beamsplitter (a static passive device), interesting multi-photon interference phenomena studied in (Sanaka, Resch & Zeilinger, 2006), (Ou, 2007), and (Bartley, et al., 2012) are re-produced by means of our approach, see Examples 1,2,3. Second, when the underlying quantum linear system is not passive (e.g., a degenerate parametric amplifier), the steady-state output state with respect to a multi-photon input state is not a multi-photon state. In terms of tensor representation, a more general class of states is defined. Such rigorous mathematical description paves the way for multi-photon state engineering. Third, both the factorizable and unfactorizable multi-photon states are treated in this paper. Here a factorizable multi-photon state is a state for which the photons in a given channel are not correlated, while for an



Fig. 1. Quantum linear system G with input b and output b_{out}

unfactorizable multi-photon state there exists correlation among the photons. This difference cannot occur in the single-photon state case. Thus, the mathematical framework presented here is more general.

The rest of the paper is organized as follows. Preliminary results are presented in Section 2. Specifically, quantum linear systems are briefly reviewed in Subsection 2.1 with focus on stable inversion and covariance function transfer, in Subsection 2.2 several types of tensors and their associated operations are introduced. The multi-photon state processing when input states are factorizable in terms of pulse shapes is studied in Section 3. (Here the word “factorizable” means there is no correlation among photons in each specific channel.) Specifically, single-channel and multi-channel multi-photon states are presented in Subsections 3.1 and 3.2 respectively, covariance functions and intensities of output fields are studied in Subsection 3.3, while an analytic form of steady-state output states is derived in Subsection 3.4. The unfactorizable case is investigated in Section 4. Specifically, unfactorizable multi-channel multi-photon states are defined in Subsection 4.1, the analytic form of the steady-state output state is presented in Subsection 4.2 where the underlying system is passive, the active case is studied in Subsection 4.3. Some concluding remarks are given in Section 5.

Notations. m is the number of input channels, and n is the number of degrees of freedom of a given quantum linear stochastic system. $|\phi\rangle$ denotes the initial state of the system which is always assumed to be vacuum, $|0\rangle$ denotes the vacuum state of free fields. Given a column vector of complex numbers or operators $x = [x_1 \dots x_k]^T$ where k is a positive integer, define $x^\# = [x_1^* \dots x_k^*]^T$, where the asterisk $*$ indicates complex conjugation or Hilbert space adjoint. Denote $x^\dagger = (x^\#)^T$. Furthermore, define the doubled-up column vector to be $\check{x} = [x^T \ (x^\#)^T]^T$. Let I_k be an identity matrix and 0_k a zero square matrix, both of dimension k . Define $J_k = \text{diag}(I_k, -I_k)$ and $\Theta_k = [0 \ I_k; -I_k \ 0]$ (The subscript “ k ” is often omitted.) Then for a matrix $X \in \mathbb{C}^{2j \times 2k}$, define $X^\flat := J_k X^\dagger J_j$. \otimes_c denotes the Kronecker product. Given a function $f(t)$ in the time domain, define its two-sided Laplace transform (Sogo, 2010, (13)) to be $F[s] = \mathcal{L}_b\{f(t)\}(s) := \int_{-\infty}^{\infty} e^{-st} f(t) dt$. Given two constant matrices $U, V \in \mathbb{C}^{r \times k}$, define $\Delta(U, V) = [U \ V; V^\# \ U^\#]$. Similarly, given time-domain matrix functions $E^-(t)$ and $E^+(t)$ of compatible dimensions, define $\Delta(E^-(t), E^+(t)) = [E^-(t) \ E^+(t); E^+(t)^\# \ E^-(t)^\#]$. Given two operators A and B , their commutator is defined to be $[A, B] := AB - BA$. For any integer $r > 1$, we write \int_r for integration in the space \mathbb{R}^r . We also write $dt_{1 \rightarrow r}$ for $dt_1 \dots dt_r$. Finally, given a column vector a , we use a_j to denote its entries. Given a matrix A , we use A^{jk} to denote its entries. Given a 3-way tensor \mathcal{A} (also called a tensor of order 3), we use \mathcal{A}_{ijk} to denote its entries; we do the similar thing for higher order tensors.

2 Quantum linear systems and tensors

This section records preliminary results necessary for the development of the paper. Quantum linear systems are briefly discussed in Subsection 2.1. Tensors and their associated operations, the appropriate mathematical language to describe the interaction of a quantum linear system with multi-photon channels, is introduced in Subsection 2.2.

2.1 Quantum linear systems

In this subsection quantum linear systems are described in the input-output language, which makes it natural to present transfer of covariance function of input fields. Moreover, the input-output framework also enables the definition of the stable inversion of quantum linear systems.

2.1.1 Fields and systems

In this part we set up the model which is a quantum linear system driven by boson fields, (Gardiner & Zoller, 2000; Walls & Milburn, 2008; Wiseman & Milburn, 2010).

The triple (S, L, H) provides a compact way for the description of open quantum systems (Gough & James, 2009; Gough, James & Nurdin, 2010; Zhang & James, 2012). Here the self-adjoint operator H is the initial system Hamiltonian, S is a unitary scattering operator, and L is a coupling operator that describes how the system is coupled to its environment. The environment is an m -channel electromagnetic field in free space, represented by a column vector of annihilation operators $b(t) = [b_1(t), \dots, b_m(t)]^T$. Let t_0 be the initial time, namely, the time when the quantum system starts interacting with its environment. Define a gauge process $\Lambda(t)$ by $\Lambda(t) = \int_{t_0}^t b^\#(\tau) b^T(\tau) d\tau = (\Lambda^{jk}(t))_{j,k=1,\dots,m}$ with operator entries $\Lambda^{jk}(t)$ on the Fock space \mathfrak{F} for the free field (Gardiner & Zoller, 2000, Walls & Milburn, 2008). In this paper it is assumed that these quantum stochastic processes are *canonical*, that is, they have the following non-zero Ito products

$$\begin{aligned} dB_j(t)dB_k^*(t) &= \delta_{jk}dt, \quad d\Lambda^{jk}dB_l^*(t) = \delta_{kl}dB_j^*(t), \\ dB_j(t)d\Lambda^{kl}(t) &= \delta_{jk}dB_l(t), \quad d\Lambda^{jk}(t)d\Lambda^{lm}(t) = \delta_{kl}d\Lambda^{jm}(t), \quad j, k, l = 1, \dots, m, \end{aligned}$$

where $B(t) = [B_1(t), \dots, B_m(t)]^T$ is a column vector of the integrated field operators defined via $B(t) := \int_{t_0}^t b(r)dr$. In the *interaction picture* the stochastic Schrodinger's equation for the open quantum system driven by the free field $b(t)$ is, in Ito form (Gardiner & Zoller, 2000, Chapter 11),

$$dU(t, t_0) = \left\{ \text{Tr}[(S - I_m)d\Lambda(t)^T] + dB^\dagger(t)L - L^\dagger S dB(t) - \left(\frac{1}{2}L^\dagger L + iH\right)dt \right\} U(t, t_0), \quad t \geq t_0, \quad (1)$$

with $U(t, t_0) = I$ being an identity operator for all $t \leq t_0$.

Specific to the linear case, the open quantum linear system G shown in Fig. 1 represents a collection of n interacting quantum harmonic oscillators $a(t) = [a_1(t), \dots, a_n(t)]^T$ (defined on a Hilbert space \mathfrak{H}_G) coupled to m boson fields $b(t)$ described above (Gardiner & Zoller, 2000, Wiseman & Milburn, 2010, Zhang & James, 2011, Zhang & James, 2012). Here, a_j ($j = 1, \dots, n$) is the annihilation operator of the j th oscillator satisfying the canonical commutation relations $[a_j, a_k^*] = \delta_{jk}$. Denote $\check{a}(t_0) = \check{a}$. The vector operator $L \in \mathfrak{H}_G$ is defined as $L = C_- a + C_+ a^\#$ with $C_-, C_+ \in \mathbb{C}^{m \times n}$. The initial Hamiltonian $H \in \mathfrak{H}_G$ is $H = \frac{1}{2}\check{a}^\dagger \Delta(\Omega_-, \Omega_+) \check{a}$ with $\Omega_-, \Omega_+ \in \mathbb{C}^{n \times n}$ satisfying $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^T$. By (1), the dynamic model for the system G is

$$\dot{\check{a}}(t) = A\check{a}(t) + B\check{b}(t), \quad \check{a}(t_0) = \check{a}, \quad (2)$$

$$\check{b}_{\text{out}}(t) = C\check{a}(t) + D\check{b}(t), \quad (3)$$

in which system matrices are given in terms of the physical parameters S, L, H , specifically,

$$D = \Delta(S, 0), \quad C = \Delta(C_-, C_+), \quad B = -C^b \Delta(S, 0), \quad A = -\frac{1}{2}C^b C - iJ_n \Delta(\Omega_-, \Omega_+).$$

The *transfer function* (impulse response function) for the system G is

$$g_G(t) := \begin{cases} \delta(t)D + Ce^{At}B, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (4)$$

This, together with (2) and (3), yields

$$\check{b}_{\text{out}}(t) = Ce^{A(t-t_0)}\check{a} + \int_{t_0}^t g_G(t-r)\check{b}(r)dr. \quad (5)$$

The system G is said to be *passive* if both $C^+ = 0$ and $\Omega^+ = 0$. The system G is said to be *asymptotically stable* if the matrix A is Hurwitz (Zhang & James, 2011, Section III-A).

Define matrix functions

$$g_{G^-}(t) := \begin{cases} \delta(t)S - [C_- \ C_+]e^{At} \begin{bmatrix} C_-^\dagger \\ -C_+^\dagger \end{bmatrix} S, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$g_{G^+}(t) := \begin{cases} -[C_- \ C_+]e^{At} \begin{bmatrix} -C_+^T \\ C_-^T \end{bmatrix} S^\#, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

(Note that when G is passive, $g_{G^+}(t) \equiv 0$.) With these functions, the transfer function in (4) can be re-written as

$$g_G(t) = \Delta(g_{G^-}(t), g_{G^+}(t)).$$

Finally, assume that the system (2)-(3) is asymptotically stable. Letting $t_0 \rightarrow -\infty$ and noticing (4), equation (5) becomes

$$\check{b}_{\text{out}}(t) = \int_{-\infty}^{\infty} g_G(t-r)\check{b}(r)dr, \quad (6)$$

which characterizes the steady-state relation between the input and the output.

2.1.2 Stable inversion

In this par some results for the stable inversion of quantum linear systems G are recorded, which are used in the derivation of output states of quantum linear systems driven by multi-photon states, cf. Sections 3 and 4.

For the transfer function $g_G(t)$ defined in (4), let $\Xi_G[s]$ be its two-sided Laplace transform (see the *Notations* part and Sogo, 2010, Eq. (13)). Define a matrix function $g_{G^{-1}}(t)$ to be

$$g_{G^{-1}}(t) := \mathcal{L}_b^{-1}\{\Xi_G[s]^{-1}\}(t). \quad (7)$$

The following result is proved in Zhang & James, 2013, Lemma 1.

Lemma 1 *Assume that the system G is asymptotically stable. Then*

$$g_{G^{-1}}(t) = \Delta(g_{G^-}(-t)^\dagger, -g_{G^+}(-t)^T). \quad (8)$$

Remark 1. Because the system G is asymptotically stable, it has no zeros on the imaginary axis, $\Xi_G[s]^{-1}$ is well defined. It is worth pointing out that the matrix function $g_{G^{-1}}(-t)$ turns out to be the transfer function of the inverse system defined in Gough, James & Nurdin, 2010, (71).

Lemma 2 *Assume the quantum linear system G is asymptotically stable. Define an operator*

$$\check{b}^-(t, t_0) := U(t, t_0)\check{b}(t)U(t, t_0)^*, \quad t \geq t_0.$$

Then

$$\check{b}^-(t, -\infty) = \int_{-\infty}^{\infty} g_{G^{-1}}(t-r)\check{b}(r)dr. \quad (9)$$

Proof. Because $\check{b}_{\text{out}}(t) = U(t, t_0)^*\check{b}(t)U(t, t_0)$, equation (5) gives

$$U(t, t_0)^*\check{b}(t)U(t, t_0) = Ce^{A(t-t_0)}\check{a} + \int_{t_0}^t g_G(t-r)\check{b}(r)dr.$$

That is,

$$\check{b}(t) = Ce^{A(t-t_0)}U(t, t_0)\check{a}U(t, t_0)^* + \int_{t_0}^t g_G(t-r)\check{b}^-(t, t_0)dr.$$

Letting $t_0 \rightarrow -\infty$ and noticing (4) we have

$$\check{b}(t) = \int_{-\infty}^{\infty} g_G(t-r)\check{b}^-(r, -\infty)dr. \quad (10)$$

However, by Eq. (7), we have $g_{G^{-1}} * g_G(t) = \delta(t)$. Substituting it into (10) yields (9).

Remark 2. Operators $\check{b}^-(r, t_0)$ and $\check{b}^-(r, -\infty)$ are formally defined mathematically, they may not correspond to physical variables. However, they do help in the derivation of the steady-state state of output fields.

2.1.3 Steady-state covariance transfer

Here we record results concerning covariance function transfer by the quantum linear system G defined in (2)-(3).

Assume the quantum linear system G is in the vacuum state $|\phi\rangle$. Assume further that the input field is in a zero-mean state ρ_f . (specific types of ρ_f will be studied in the sequel.) Denote the covariance functions of the input field $b(t)$ and the output field $b_{\text{out}}(t)$ by $R(t, r)$ and $R_{\text{out}}(t, r)$ respectively, that is,

$$R(t, r) = \text{Tr}[\rho_f \check{b}(t)\check{b}^\dagger(r)], \quad R_{\text{out}}(t, r) = \text{Tr}[|\phi\rangle\langle\phi| \otimes \rho_f \check{b}_{\text{out}}(t)\check{b}_{\text{out}}^\dagger(r)]. \quad (11)$$

According to (6) and (11) we have

Lemma 3 *Assume that the system (2)-(3) is asymptotically stable. Let the input field have covariance $R(t, r)$ defined in (11). The steady-state (namely $t_0 \rightarrow -\infty$) output covariance function $R_{\text{out}}(t, r)$ is*

$$R_{\text{out}}(t, r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_G(t-\tau_1)R(\tau_1, \tau_2)g_G(r-\tau_2)^\dagger d\tau_1 d\tau_2. \quad (12)$$

In the frequency domain, we have

Theorem 4 *Assume that the system (2)-(3) is asymptotically stable. If the input field is stationary with spectral density matrix $R[i\omega]$ (namely, the Fourier transform of $R(t, t)$), the output spectral density matrix is given by*

$$R_{\text{out}}[i\omega] = \Xi_G[i\omega]R[i\omega]\Xi_G[i\omega]^\dagger. \quad (13)$$

In particular, if the input field is in the vacuum state $|0\rangle$, that is, $R[i\omega] = \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix}$, then the output state is a Gaussian state with output spectral density matrix

$$R_{\text{out}}[i\omega] = \Xi_G[i\omega] \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix} \Xi_G[i\omega]^\dagger. \quad (14)$$

In what follows we focus on the Gaussian input field states. Denote the initial joint system-field Gaussian state by $\rho_{0g} = |\phi\rangle\langle\phi| \otimes \rho_f$ where ρ_f is a Gaussian field state. Define the steady-state joint state

$$\rho_{\infty g} := \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} U(t, t_0)\rho_{0g}U(t, t_0)^*, \quad (15)$$

and the steady-state output field state

$$\rho_{\text{field},g} := \text{Tr}_{\text{sys}}[\rho_{\infty g}],$$

where the subscript ‘‘sys’’ indicates that the trace operation is performed with respect to the system. Moreover, if the input state ρ_f is stationary with spectral density $R[i\omega]$, according to Theorem 4, $\rho_{\text{field},g}$ is the steady-state output field density with covariance function $R_{\text{out}}[i\omega]$ given in (13). Finally, if $\rho_f = |0\rangle\langle 0|$, then $\rho_{\text{field},g}$ is stationary zero-mean Gaussian with $R_{\text{out}}[i\omega]$ given in (14).

Remark 3. Because $\rho_{\text{field},g}$ is obtained by tracing out the system, it is in general a mixed state. Moreover, $\rho_{\infty g}$ is in general not the vacuum state even if $\rho_f = |0\rangle\langle 0|$. However, if the system G is *passive*, then by (14),

$$R_{\text{out}}[i\omega] = \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix} = R_{\text{in}}[i\omega]. \quad (16)$$

That is, in the passive case the steady-state output state $\rho_{\text{field},g}$ is again the vacuum state.

2.2 Tensors

In this subsection several types of tensors and their associated operations are introduced. Because different channels may have different numbers of photons, fibers of the tensors may thus have different lengths, see e.g., (17). Nonetheless, with slight abuse of notation, we still call these objects tensors.

Given positive integers m and ℓ_1, \dots, ℓ_m , let $\mathbb{C}^{m \times (\ell_1, \dots, \ell_m)}$ be a space of matrix-like objects, whose element ξ is of the form

$$\xi = \begin{bmatrix} \xi^{11} & \dots & \xi^{1\ell_1} \\ \vdots & \ddots & \vdots \\ \xi^{m1} & \dots & \xi^{m\ell_m} \end{bmatrix}.$$

In this paper ξ is used to represent m -channel multi-photon input states with ℓ_j denoting the photon number in the j -th channel, $j = 1, \dots, m$. Because channels may have different numbers of photons, ℓ_1, \dots, ℓ_m may not equal each other. Nonetheless in the paper we still call ξ a matrix. Next we define a tensor space $\mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$, whose elements \mathcal{S} are defined in the following way: For each $i, j = 1, \dots, m$, the model-3 fiber is

$$\mathcal{S}_{ij} = \begin{bmatrix} \mathcal{S}_{ij1} \\ \vdots \\ \mathcal{S}_{ij\ell_j} \end{bmatrix} \in \mathbb{C}^{\ell_j}. \quad (17)$$

That is, when the first two indices i, j are fixed, we have a vector of dimension ℓ_j . \mathcal{S} looks like a 3-way tensor (Kolda & Bader, 2009), but its mode-3 fibers may have different dimensions for different j . Nevertheless, in this paper we still call \mathcal{S} a 3-way tensor and $\mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ a space of 3-way tensors over the field of complex numbers. Given a matrix $\xi \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m)}$, we may represent it as a 3-way tensor $\xi^\dagger \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$, by defining horizontal slices to be

$$\xi_{i::}^\dagger = \begin{bmatrix} \left. \begin{array}{c} 0 \cdots 0 \\ \vdots \cdots \vdots \\ 0 \cdots 0 \end{array} \right\}_{i-1} \\ \xi^{i1} \dots \xi^{i\ell_i} \\ \left. \begin{array}{c} 0 \cdots 0 \\ \vdots \cdots \vdots \\ 0 \cdots 0 \end{array} \right\}_{m-i} \end{bmatrix} \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m)}, \quad \forall i = 1, \dots, m. \quad (18)$$

This update turns out to be very useful because the output state of a quantum passive linear system driven by an m -channel multi-photon state encoded by a matrix $\xi \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m)}$ can be characterized by a tensor in $\mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$, see Sec. 3.4.

We adopt notations in Kolda & Bader, 2009. For each $j = 1, \dots, m$ and $k = 1, \dots, \ell_j$,

$$\mathcal{S}_{:jk} = \begin{bmatrix} \mathcal{S}_{1jk} \\ \vdots \\ \mathcal{S}_{mjk} \end{bmatrix} \in \mathbb{C}^m$$

is mode-1 (column) fiber. $\mathcal{S}_{i::}$ and $\mathcal{S}_{:j}$ are respectively horizontal and lateral slices (matrices) of the form

$$\mathcal{S}_{i::} = \begin{bmatrix} \mathcal{S}_{i1}^T \\ \vdots \\ \mathcal{S}_{im}^T \end{bmatrix} \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m)}, \quad \mathcal{S}_{:j} = \begin{bmatrix} \mathcal{S}_{:j1} & \cdots & \mathcal{S}_{:j\ell_j} \end{bmatrix} \in \mathbb{C}^{m \times \ell_j}, \quad \forall i, j = 1, \dots, m.$$

Finally, let $\mathcal{C} \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m) \times (\ell_1, \dots, \ell_m)}$ be a 3-way tensor. We say that \mathcal{C} is partially Hermitian in modes 2 and 3 if all the horizontal slices are Hermitian matrices. That is, for all $i = 1, \dots, m$, the horizontal slices $\mathcal{C}_{i::} \in \mathbb{C}^{\ell_i \times \ell_i}$ satisfy $\mathcal{C}_{i::}^\dagger = \mathcal{C}_{i::}$. This is a natural extension of the concept *partially symmetric* discussed in (Kolda & Bader, 2009) to the complex domain.

In what follows we define operations associated to these tensors. Given 3-way tensors $\mathcal{S}(t), \mathcal{T}(r) \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ and partially Hermitian tensor $\mathcal{C} \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m) \times (\ell_1, \dots, \ell_m)}$, we define a matrix $\mathcal{S}(t) \otimes \mathcal{T}(r) \in \mathbb{C}^{m \times m}$ whose (i,k)-th entry is

$$(\mathcal{S}(t) \otimes \mathcal{T}(r))_{ik} := \sum_{j=1}^m \frac{1}{N_{\ell_j}} \sum_{\beta=1}^{\ell_j} \sum_{\alpha=1}^{\ell_j} \mathcal{C}_{j\alpha\beta} \mathcal{S}_{ij\alpha}(t) \mathcal{T}_{kj\beta}(r), \quad \forall i, k = 1, \dots, m, \quad (19)$$

where N_{ℓ_j} ($j = 1, \dots, m$) are positive scalars. (The physical interpretation of N_{ℓ_j} will be given in Sec. 3.) It can be verified that

$$(\mathcal{S}(t) \otimes \mathcal{T}(r))^\dagger = \mathcal{T}(r)^\# \otimes \mathcal{S}(t)^\#. \quad (20)$$

In this paper, we call \mathcal{C} a ‘‘core tensor’’ for the operation \otimes . According to (19) and the definition of ξ^\dagger in (18), we have

$$\text{diag}_{j=1, \dots, m} \left(\frac{1}{N_{\ell_j}} \sum_{i,k=1}^{\ell_j} \mathcal{C}_{jik} \xi^{ji}(r)^* \xi^{jk}(t) \right) = \xi^\dagger(r)^\# \otimes \xi^\dagger(t), \quad (21)$$

Given a matrix function $E(t) \in \mathbb{C}^{m \times m}$ and a 3-way tensor $\mathcal{S}(t) \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$, define $\mathcal{T} \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ whose (i, j, k) -th element is

$$\mathcal{T}_{ijk}(t) := \sum_{r=1}^m \int_{-\infty}^{\infty} E^{ir}(t - \tau) \mathcal{S}_{rjk}(\tau) d\tau.$$

In compact form we write

$$\mathcal{T} = \mathcal{S} \times_1 E,$$

where \times_1 is called *1-mode matrix product* (Kolda & Bader, 2009, Sec. 2.5).

Given two matrices $E(t), F(t) \in \mathbb{C}^{m \times m}$ and two tensors $\mathcal{S}(t), \mathcal{T}(t) \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$, define

$$\Delta(\mathcal{S}, \mathcal{T}) \times_1 \Delta(E, F) := \Delta(\mathcal{S} \times_1 E + \mathcal{T}^\# \times_1 F, \mathcal{T} \times_1 E + \mathcal{S}^\# \times_1 F). \quad (22)$$

That is, the operation \times_1 is performed block-wise. This operation is useful in studying the output state of a quantum linear system driven by a multi-channel multi-photon input state.

Finally, we define another type of operations between matrices and tensors. Let $E(t) = (E^{jk}(t)) \in \mathbb{C}^{m \times m}$ be the transfer function of the underlying quantum linear system with m input channels. For each $j = 1, \dots, m$, let $\mathcal{V}_j(t_1, \dots, t_{\ell_j})$ be an ℓ_j -way m -dimensional tensor function that encodes the pulse information of the j th input field containing ℓ_j photons. Denote the entries of $\mathcal{V}_j(t_1, \dots, t_{\ell_j})$ by $\mathcal{V}_{j,k_1, \dots, k_{\ell_j}}(t_1, \dots, t_{\ell_j})$. Define an ℓ_j -way m -dimensional tensor \mathcal{W}_j with entries given by the following multiple convolution

$$\begin{aligned} & \mathcal{W}_{j,r_1, \dots, r_{\ell_j}}(t_1, \dots, t_{\ell_j}) \\ = & \sum_{k_1, \dots, k_{\ell_j}=1}^m \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} E^{r_1 k_1}(t_1 - \iota_1) \dots E^{r_{\ell_j} k_{\ell_j}}(t_{\ell_j} - \iota_{\ell_j}) \mathcal{V}_{j,k_1, \dots, k_{\ell_j}}(\iota_1, \dots, \iota_{\ell_j}) d\iota_1 \dots d\iota_{\ell_j} \end{aligned}$$

for all $1 \leq r_1, \dots, r_{\ell_j} \leq m$. In compact form we write

$$\mathcal{W}_j = \mathcal{V}_j \times_1 E \times_2 \dots \times_{\ell_j} E, \quad \forall j = 1, \dots, m, \quad (23)$$

cf. Kolda & Bader, 2009, Sec. 2.5. More discussions on tensors will be given in Section 4.3.

3 The factorizable case

In this section we study how a quantum linear system responds to a factorizable multi-photon input state, here the word ‘‘factorizable’’ means that photons in each input channel are not statistically correlated. The single-channel and multi-channel multi-photon input states are defined in Subsections 3.1 and 3.2 respectively, output covariance functions and intensities are presented in Subsection 3.3, while the output states are derived in Subsection 3.4.

3.1 Single-channel multi-photon states

In this subsection single-channel ℓ -photon states are defined and their statistical properties are discussed.

For any given positive integer ℓ and real numbers t_1, \dots, t_ℓ , let $P(t_1, \dots, t_\ell)$ be a permutation of the numbers t_1, \dots, t_ℓ . Denote the set of all such permutations by S_ℓ . For arbitrary functions $\xi_1(t), \dots, \xi_\ell(t)$ defined on the real line, define

$$N_\ell := \sum_{P \in S_\ell} \int_{-\infty}^{\infty} \xi_\ell(t_\ell)^* \dots \xi_1(t_1)^* \xi_1(P(t_1)) \dots \xi_\ell(P(t_\ell)) dt_{1 \rightarrow \ell}, \quad (24)$$

provided the above multiple integral converges (this is always assumed in the paper). The subscript ‘‘ ℓ ’’ in N_ℓ indicates the number of photons. It can be shown that $N_\ell > 0$. A single-channel *continuous-mode* ℓ -photon state $|\psi_\ell\rangle$ is defined via

$$|\psi_\ell\rangle := \frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} B^*(\xi_k)|0\rangle, \quad (25)$$

where $B^*(\xi_k) := \int_{-\infty}^{\infty} b^*(t)\xi_k(t)dt$, ($k = 1, \dots, m$.) Because $|\psi_\ell\rangle$ is a product of single integrals, there is no correlation among the photons. This type of multi-photon states is therefore called *factorizable* photon states. It can be shown that

$$\langle 0 | \prod_{i=1}^{\ell} B(\xi_i) \prod_{k=1}^{\ell} B^*(\xi_k) | 0 \rangle = \sum_{P \in S_\ell} \int_{-\infty}^{\infty} \xi_\ell(t_\ell)^* \dots \xi_1(t_1)^* \xi_1(P(t_1)) \dots \xi_\ell(P(t_\ell)) dt_{1 \rightarrow \ell} = N_\ell.$$

Thus $\langle \psi_\ell | \psi_\ell \rangle = 1$. That is, $|\psi_\ell\rangle$ is normalized.

When $\ell = 1$, $N_1 = \int_{-\infty}^{\infty} |\xi_1(t)|^2 dt$, $|\psi_1\rangle$ is a single-photon state, (Loudon, 2000, (6.3.4); Milburn, 2008, (9)). On the other hand, when $\xi_1(t) \equiv \dots \equiv \xi_\ell(t) \equiv \xi(t)$ and $\int_{-\infty}^{\infty} |\xi(t)|^2 dt = 1$, the input light field contains ℓ indistinguishable photons; such states are called *continuous-mode* Fock states which have been intensely studied, in e.g., Gheri, Ellinger, Pellizzari & Zoller, 1998, (3); Baragiola, Cook, Brańczyk & Combes, 2012, (13).

For convenience, define a matrix $\mathcal{C} \in \mathbb{C}^{\ell \times \ell}$ whose entries are

$$\mathcal{C}^{ik} = \langle 0 | \prod_{\alpha=1, \alpha \neq i}^{\ell} B(\xi_\alpha) \prod_{\beta=1, \beta \neq k}^{\ell} B^*(\xi_\beta) | 0 \rangle, \quad \ell \geq 2. \quad (26)$$

Clearly, $\mathcal{C} = \mathcal{C}^\dagger$.

Lemma 5 \mathcal{C}^{ik} defined in (26) satisfies

$$\sum_{k=1}^{\ell} \mathcal{C}^{ik} \int_{-\infty}^{\infty} \xi_i(t)^* \xi_k(t) dt = N_\ell, \quad \forall i = 1, \dots, \ell.$$

In what follows we study statistical properties of the ℓ -photon state $|\psi_\ell\rangle$. It is easy to show that for all $t \geq t_0$, $\langle \psi_\ell | b(t) | \psi_\ell \rangle = 0$. That is, the field has zero average field amplitude. The following result summarizes the second-order statistical information of the ℓ -photon state $|\psi_\ell\rangle$.

Lemma 6 Let $\bar{n}(t)$ denote the field intensity with respect to the state $|\psi_\ell\rangle$, namely,

$$\bar{n}(t) = \langle \psi_\ell | b^*(t) b(t) | \psi_\ell \rangle.$$

(In quantum optics, the second-order moment $\bar{n}(t)$ is the count rate (Gardiner & Zoller, 2000).) Moreover, let the field covariance function be

$$R(t, r) = \langle \psi_\ell | \check{b}(t) \check{b}^\dagger(r) | \psi_\ell \rangle,$$

as given by (11). Then we have

$$R(t, r) = \delta(t - r) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{N_\ell} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \Delta(\mathcal{C}^{ik} \xi_k(t) \xi_i(r)^*, 0), \quad (27)$$

$$\bar{n}(t) = \frac{1}{N_\ell} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \mathcal{C}^{ik} \xi_i(t)^* \xi_k(t). \quad (28)$$

Proof. Clearly,

$$R(t, r) = \delta(t - r) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \Delta(\langle \psi_\ell | b^*(r) b(t) | \psi_\ell \rangle, \langle \psi_\ell | b(t) b(r) | \psi_\ell \rangle). \quad (29)$$

Observing that

$$b(t) | \psi_\ell \rangle = \frac{1}{\sqrt{N_\ell}} \sum_{k=1}^{\ell} \xi_k(t) \prod_{r=1, r \neq k}^{\ell} B^*(\xi_r) | 0 \rangle, \quad (30)$$

we have

$$\langle \psi_\ell | b^*(r) b(t) | \psi_\ell \rangle = \frac{1}{N_\ell} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \mathcal{C}^{ik} \xi_i(r)^* \xi_k(t), \quad \langle \psi_\ell | b(t) b(r) | \psi_\ell \rangle = 0. \quad (31)$$

Substituting (31) into (29) establishes (27). Finally, because $\bar{n}(t)$ is the 2-by-2 entry of $R(t, t)$, (28) follows (27).

In particular, for the single-photon case, the field covariance function is

$$R(t, r) = \delta(t - r) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \Delta(\xi_1(t) \xi_1(r)^*, 0), \quad (32)$$

which is the same as Zhang & James, 2012, (35).

Remark 4. According to Lemma 6, the ℓ -photon state $|\psi_\ell\rangle$ is not Gaussian; it may not be stationary either. So, its first and second order moments cannot provide all statistical information of the input field.

3.2 Multi-channel multi-photon states

In this subsection multi-channel multi-photon states are defined.

Let there be m input field channels. For the j -th field channel, let ℓ_j be the number of photons ($j = 1, \dots, m$). Similar to (25), define the j -th channel ℓ_j -photon state by

$$|\Psi_j\rangle := \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} B_j^*(\xi^{jk})|0\rangle, \quad (33)$$

where the subscript “ j ” indicates the j -th channel, and ℓ_j indicates that there are ℓ_j photons in this channel. In analog to (24), for each $j = 1, \dots, m$, the normalization coefficient N_{ℓ_j} is defined to be

$$N_{\ell_j} := \sum_{P \in S_{\ell_j}} \int_{\ell_j} \xi^{j\ell_j}(t_{\ell_j})^* \cdots \xi^{j1}(t_1)^* \xi^{j1}(P(t_1)) \cdots \xi^{j\ell_j}(P(t_{\ell_j})) dt_{1 \rightarrow \ell_j}.$$

We define an m -channel multi-photon state as

$$|\Psi\rangle := |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_m\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} B_j^*(\xi^{jk})|0^{\otimes m}\rangle. \quad (34)$$

In particular, for each $j = 1, \dots, m$, if $\xi^{j1}(t) \equiv \cdots \equiv \xi^{j\ell_j}(t)$, then (34) defines a multi-channel continuous-mode Fock state, see eg., Baragiola, Cook, Brańczyk & Combes, 2012, (D1).

3.3 Output covariance functions and intensities

In this subsection analytical forms of output covariance functions $R_{\text{out}}(t, r)$ and intensities $\bar{n}_{\text{out}}(t)$ are presented.

3.3.1 Steady-state output covariance function

In this part we derive an explicit expression of $R_{\text{out}}(t, r)$ when the input is in the multi-channel multi-photon state $|\Psi\rangle$ defined in (34).

We first introduce some notation. Define a 3-way tensor $\mathcal{C} \in \mathbb{C}^{m \times (\ell_1, \dots, \ell_m) \times (\ell_1, \dots, \ell_m)}$, whose elements are

$$\mathcal{C}_{jik} := \langle 0 | \prod_{\alpha=1, \alpha \neq i}^{\ell_j} B_j(\xi^{j\alpha}) \prod_{\beta=1, \beta \neq k}^{\ell_j} B_j^*(\xi^{j\beta}) | 0 \rangle, \quad \forall j = 1, \dots, m, \text{ and } i, k = 1, \dots, \ell_j. \quad (35)$$

Clearly, \mathcal{C} is partially Hermitian, that is, $\mathcal{C}_{j::} = \mathcal{C}_{j::}^\dagger \in \mathbb{C}^{\ell_j \times \ell_j}$, ($j = 1, \dots, m$). Similar to (31), for each $j = 1, \dots, m$,

$$\langle \Psi | b_j(t) b_j^*(r) | \Psi \rangle = \delta(t - r) + \frac{1}{N_{\ell_j}} \sum_{i=1}^{\ell_j} \sum_{k=1}^{\ell_j} \mathcal{C}_{jik} \xi^{ji}(r)^* \xi^{jk}(t).$$

Consequently, the input covariance function is

$$R(t, r) = \delta(t - r) \begin{bmatrix} I_m & \\ & 0_m \end{bmatrix} + \begin{bmatrix} \xi^\dagger(r)^\# \otimes \xi^\dagger(t) & \\ & (\xi^\dagger(r)^\# \otimes \xi^\dagger(t))^\dagger \end{bmatrix}. \quad (36)$$

For state $|\Psi\rangle$ defined in (34), let ξ^\dagger be the 3-way tensor defined via (18). Then we can define 3-way tensors $\eta^-, \eta^+ \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ by

$$\Delta(\eta^-, \eta^+) := \Delta(\xi^\dagger, \mathbf{0}) \times_1 g_G, \quad (37)$$

where the operation \times_1 has been defined in (22). For example, for a single-channel ℓ -photon input state defined in (25), equation (37) yields

$$\eta_k^-(t) := \int_{-\infty}^{\infty} g_{G^-}(t-\tau)\xi_k(\tau)d\tau, \quad \eta_k^+(t) := \int_{-\infty}^{\infty} g_{G^+}(t-\tau)\xi_k(\tau)^*d\tau, \quad k = 1, 2, \dots, \ell. \quad (38)$$

Theorem 7 *Assume that the quantum linear system G is asymptotically stable. Let the input field have covariance $R(t, r)$ be that given in (36). Then the steady-state output covariance function is*

$$R_{\text{out}}(t, r) = \int_{-\infty}^{\infty} g_G(t-\tau) \begin{bmatrix} I_m & \\ & 0_m \end{bmatrix} g_G(r-\tau)^\dagger d\tau \quad (39)$$

$$+ \Delta((\eta^-(r)^\# \otimes \eta^-(t)), (\eta^+(r) \otimes \eta^-(t)))^T + \Delta(\eta^+(t) \otimes \eta^+(r)^\#, \eta^+(t) \otimes \eta^-(r)),$$

where the tensors $\eta^-(t)$ and $\eta^+(t)$ are given by (37), and the core tensor for the operation \otimes is the tensor \mathcal{C} given in (35).

Proof. (39) can be derived by substituting (36) into (12) and with the aid of (20)-(21) and (37).

3.3.2 Steady-state output intensity

For the multi-channel multi-photon input state $|\Psi\rangle$ defined in (34), the steady-state ($t_0 \rightarrow -\infty$) intensity of the output field is

$$\bar{n}_{\text{out}}(t) = \langle \phi\Psi | b_{\text{out}}^\#(t) b_{\text{out}}^T(t) | \phi\Psi \rangle. \quad (40)$$

Because $\bar{n}_{\text{out}}(t)$ is the 2-by-2 block of $R_{\text{out}}(t, t)$, the following result is an immediate consequence of Theorem 7.

Corollary 8 *Assume the quantum linear system G is asymptotically stable. The steady-state ($t_0 \rightarrow -\infty$) intensity $\bar{n}_{\text{out}}(t)$ of the output field defined in (40), of the system G driven by the m -channel multi-photon input field $|\Psi\rangle$ defined in (34), is given by*

$$\bar{n}_{\text{out}}(t) = \int_{-\infty}^{\infty} g_{G^+}(t)^\# g_{G^+}(t)^T dt + (\eta^+(t) \otimes \eta^+(t)^\#)^T + \eta^-(t)^\# \otimes \eta^-(t), \quad (41)$$

where $\eta^-(t)$ and $\eta^+(t)$ are given by (37), and the core tensor for the operation \otimes is given in (35).

3.4 Steady-state output state

The preceding subsections studied the first and second order moments of output fields of quantum linear systems driven by multi-photon states. Because the output states are in general not Gaussian, these moments cannot provide the complete information of output fields. In this subsection we derive the analytic form of output states.

A multi-channel continuous-mode multi-photon state $|\Psi\rangle$ defined in (34) is parameterized by the functions $\xi^{jk}(t)$, each of which has two indices j and k . The index j (from 1 to m) indicates the j -th input channel, while the index k (from 1 to ℓ_j for each given j) is used to count the number of photons in each channel. On the other hand, according to (37), the steady-state output covariance function (in (39)) and intensity $\bar{n}_{\text{out}}(t)$ (in (41)) of the linear quantum system G driven by $|\Psi\rangle$ are parameterized by tensor functions $\eta_{ijk}^-(t)$ and $\eta_{ijk}^+(t)$, each of which has three indices i, j, k . Formally, the index i indicates that each output channel is a linear combination of the input channels. Interestingly, let $\xi^- = \xi^\dagger$ (c.f. (18)), that is,

$$\xi_{ijk}^-(t) := \begin{cases} 0, & i \neq j, \\ \xi^{jk}(t), & i = j. \end{cases}$$

Define further $\xi^+ \in \mathcal{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ to be a zero tensor. Then the m -channel multi-photon state $|\Psi\rangle$ can be re-written as

$$|\Psi\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) |0^{\otimes m}\rangle. \quad (42)$$

Accordingly, (37) can be re-written as

$$\Delta(\eta^-, \eta^+) = \Delta(\xi^-, \xi^+) \times_1 g_G. \quad (43)$$

In light of the above discussion, we derive the steady-state output state of the quantum linear system G driven by an input state of the form

$$\rho_{\xi, R} = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \rho_R \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right)^*,$$

where

$$\xi(t) = \Delta(\xi^-(t), \xi^+(t)), \quad (44)$$

with $\xi^-, \xi^+ \in \mathbb{C}^{m \times m \times (\ell_1, \dots, \ell_m)}$ and ρ_R is a stationary zero-mean Gaussian state with covariance function $R[i\omega]$.

In order for $\rho_{\xi, R}$ to be a valid state, ξ^-, ξ^+ and ρ_R have to satisfy certain conditions. We first introduce some notation. Given a tensor $\varphi \in \mathbb{C}^{2m \times m \times (\ell_1, \dots, \ell_m)}$, define tensor products consisting of ℓ_j vectors, each of dimension $2m$:

$$\begin{aligned} M_{\varphi:j}(t_{1 \rightarrow \ell_j}) &:= \varphi_{:j1}(t_1) \otimes_c \cdots \otimes_c \varphi_{:j\ell_j}(t_{\ell_j}), \quad j = 1, \dots, m, \\ M_{\varphi:j}^+(t_{1 \rightarrow \ell_j}) &:= \varphi_{:j\ell_j}(t_1) \otimes_c \cdots \otimes_c \varphi_{:j1}(t_j), \quad j = 1, \dots, m, \end{aligned}$$

where \otimes_c is the Kronecker product as introduced in the *Notations* part. Then define tensor products of the form

$$\begin{aligned} M_{\varphi}(t_{1 \rightarrow \ell_1 + \dots + \ell_m}) &:= \frac{1}{\sqrt{N_{\ell_1}}} M_{\varphi:1}(t_{1 \rightarrow \ell_1}) \otimes_c \cdots \otimes_c \frac{1}{\sqrt{N_{\ell_m}}} M_{\varphi:m}(t_{\ell_1 + \dots + \ell_{m-1} + 1 \rightarrow \ell_1 + \dots + \ell_m}), \\ M_{\varphi}^+(t_{1 \rightarrow \ell_1 + \dots + \ell_m}) &:= \frac{1}{\sqrt{N_{\ell_m}}} M_{\varphi:m}^+(t_{1 \rightarrow \ell_m}) \otimes_c \cdots \otimes_c \frac{1}{\sqrt{N_{\ell_1}}} M_{\varphi:1}^+(t_{\ell_2 + \dots + \ell_{m-1} + 1 \rightarrow \ell_1 + \dots + \ell_m}). \end{aligned}$$

Similarly, for the operators $\check{b}(t)$, define

$$M_{\check{b}}(t_{1 \rightarrow k}) := \check{b}(t_1) \otimes_c \cdots \otimes_c \check{b}(t_k), \quad \forall k \geq 1.$$

Finally for a matrix A , let $A^{\otimes_c k} := A \otimes_c \cdots \otimes_c A$ be an k -way Kronecker tensor product.

The following equation will be used in Definition 9.

$$\begin{aligned} \int_2 \sum_{j=1}^m \int_{\ell_j} (M_{\xi}^+(t_{1 \rightarrow \ell_1 + \dots + \ell_m})^{\#} \otimes_c M_{\xi}(t_{\ell_1 + \dots + \ell_{m-1} + 1 \rightarrow 2(\ell_1 + \dots + \ell_m)}))^T J^{\otimes_c^{\ell_1 + \dots + \ell_m}} \\ \otimes_c \Theta^{\otimes_c^{\ell_1 + \dots + \ell_m}} \text{Tr}[\rho_R M_{\check{b}}(t_{1 \rightarrow 2(\ell_1 + \dots + \ell_m)})] dt_{1 \rightarrow 2(\ell_1 + \dots + \ell_m)} = 1, \end{aligned} \quad (45)$$

where ξ is given in (44) and $\Theta = [0 \ I; -I \ 0]$ as introduced in the *Notations* part.

Definition 9 A state $\rho_{\xi,R}$ is said to be a photon-Gaussian state if it belongs to the set

$$\mathcal{F}_0 := \left\{ \rho_{\xi,R} = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \rho_R \right. \\ \left. \times \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right)^* : \xi \text{ and } \rho_R \text{ satisfy (45)} \right\}. \quad (46)$$

Remark 5. Clearly, the m -channel multi-photon state $|\Psi\rangle$ defined in (42) belongs to \mathcal{F}_0 .

Proposition 10 The photon-Gaussian states $\rho_{\xi,R} \in \mathcal{F}_0$ are normalized, that is $\text{Tr}[\rho_{\xi,R}] = 1$.

Proof. For each $j = 1, \dots, m$, define $\xi_{:,j}(t) \in \mathbb{C}^{2m \times 2\ell_j}$ by

$$\xi_{:,j}(t) := \Delta(\xi_{:,j}^-(t), \xi_{:,j}^+(t)) = [\xi_{:,j1}(t) \cdots \xi_{:,j(2\ell_j)}(t)],$$

where

$$\xi_{:,jk}(t) = \begin{bmatrix} \xi_{:,jk}^-(t) \\ \xi_{:,jk}^+(t)^\# \end{bmatrix} \in \mathbb{C}^{2m}, \quad \forall k = 1, \dots, 2\ell_j.$$

It can be shown that

$$\begin{aligned} & \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \\ &= \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \int_{-\infty}^{\infty} [-\xi_{:,jk}^+(t)^\dagger \xi_{:,jk}^-(t)^T] \check{b}(t) dt \\ &= \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \int_{-\infty}^{\infty} \xi_{:,jk}(t)^T (\Theta \check{b}(t)) dt \\ &= \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \int_{\ell_j} (\xi_{:,j1}(t_1) \otimes_c \xi_{:,j2}(t_2) \otimes_c \cdots \otimes_c \xi_{:,j\ell_j}(t_{\ell_j}))^T (\Theta \check{b}(t_1)) \otimes (\Theta \check{b}(t_2)) \cdots \otimes (\Theta \check{b}(t_{\ell_j})) dt_{1 \rightarrow \ell_j} \\ &= \int_{\ell_1 + \cdots + \ell_m} (M_{\xi,1}(t_{1 \rightarrow \ell_1}) \otimes_c \cdots \otimes_c M_{\xi,m}(t_{\ell_1 + \cdots + \ell_{m-1} + 1 \rightarrow \ell_1 + \cdots + \ell_m}))^T (\Theta \check{b}(t_1)) \otimes \cdots \otimes (\Theta \check{b}(t_{\ell_1 + \cdots + \ell_m})) dt_{1 \rightarrow \ell_1 + \cdots + \ell_m} \\ &= \int_{\ell_1 + \cdots + \ell_m} M_{\xi}(t_{1 \rightarrow \ell_1 + \cdots + \ell_m})^T \Theta^{\otimes_{c^{\ell_1 + \cdots + \ell_m}}} M_{\check{b}}(t_{1 \rightarrow \ell_1 + \cdots + \ell_m}) dt_{1 \rightarrow \ell_1 + \cdots + \ell_m}, \end{aligned}$$

where $\Theta = [0 \ I; -I \ 0]$ as introduced in the *Notations* part. Thus, that $\text{Tr}[\rho_{\xi,R}] = 1$ is equivalent to that (45) holds. The proof is completed.

The following result is the main result of this subsection.

Theorem 11 Suppose that the linear quantum system G is asymptotically stable. Then the steady-state output state of G driven by a state $\rho_{\xi,R} \in \mathcal{F}_0$ is

$$\rho_{\eta,R,\text{out}} = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \rho_{\text{field,g}} \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \right)^*, \quad (47)$$

where the 3-way tensors η^- and η^+ are given by (43), and $\rho_{\text{field,g}}$ is a stationary zero-mean Gaussian field whose covariance function is

$$R_{\text{out}}[i\omega] = G[i\omega]R[i\omega]G[i\omega]^\dagger.$$

given by the Gaussian transfer (13) in Theorem 4. Clearly, $\rho_{\eta,R_{\text{out}}} \in \mathcal{F}_0$.

Proof. Let $\rho(t, t_0)$ be the density operator of the composite system. Then

$$\rho(t, t_0) = U(t, t_0)|\phi\rangle\langle\phi| \otimes \rho_{\xi,R}U(t, t_0)^*.$$

We study the steady-state behavior of the state, that is, we assume that the interaction starts in the distant past ($t_0 \rightarrow -\infty$), and also let $t \rightarrow \infty$.

$$\begin{aligned} & \rho_\infty \\ := & \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} \rho(t, t_0) \\ = & \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} U(t, t_0)|\phi\rangle\langle\phi| \otimes \rho_{\xi,R}U(t, t_0)^* \\ = & \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t, t_0) \left(I_{\text{sys}} \otimes (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right) U(t, t_0)^* \rho_{\infty,g} \\ & \times \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} \left(\prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t, t_0) \left(I_{\text{sys}} \otimes (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right) U(t, t_0)^* \right)^*, \end{aligned} \quad (48)$$

where $\rho_{\infty,g}$ is given in (15). According to Lemmas 1 and 2, we have

$$\begin{bmatrix} b_j^-(t, -\infty) \\ b_j^{*-}(t, -\infty) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^m \int_{-\infty}^{\infty} g_{G^-}^{kj}(r-t)^* b_k(r) - g_{G^+}^{kj}(r-t) b_k^*(r) dr \\ \sum_{k=1}^m \int_{-\infty}^{\infty} -g_{G^+}^{kj}(r-t)^* b_k(r) + g_{G^-}^{kj}(r-t) b_k^*(r) dr \end{bmatrix}. \quad (49)$$

As a result,

$$\begin{aligned} & \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m U(t, t_0) \left(I_{\text{sys}} \otimes (B_i^*(\xi_{ijk}^-) - B_i(\xi_{ijk}^+)) \right) U(t, t_0)^* \\ = & \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m \int_{-\infty}^{\infty} \left(\xi_{ijk}^-(t_k) b_i^{*-}(t_k, -\infty) dt_k - \xi_{ijk}^+(t_k)^* b_i^-(t_k, -\infty) dt_k \right) \\ = & \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m \int_{-\infty}^{\infty} \left(- \int_{-\infty}^{\infty} \sum_{r=1}^m g_{G^+}^{ri}(\iota - t_k)^* \xi_{ijk}^-(t_k) dt_k b_r(\iota) d\iota + \int_{-\infty}^{\infty} \sum_{r=1}^m g_{G^-}^{ri}(\iota - t_k) \xi_{ijk}^-(t_k) dt_k b_r^*(\iota) d\iota \right. \\ & \left. - \int_{-\infty}^{\infty} \sum_{r=1}^m g_{G^-}^{ri}(\iota - t_k)^* \xi_{ijk}^+(t_k)^* dt_k b_r(\iota) d\iota + \int_{-\infty}^{\infty} \sum_{r=1}^m g_{G^+}^{ri}(\iota - t_k) \xi_{ijk}^+(t_k)^* dt_k b_r^*(\iota) d\iota \right) \\ = & \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m \left(B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+) \right), \end{aligned} \quad (50)$$

where 3-way tensors η^- and η^+ are given by (43). Substituting (50) into (48) we have

$$\rho_\infty = \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \rho_{\infty,g} \left(\prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m (B_i^*(\eta_{ijk}^-) - B_i(\eta_{ijk}^+)) \right)^*.$$

Tracing out the system part, (47) is obtained. The proof is completed.

Restricted to the single-channel case, we have

Corollary 12 *Assume that the quantum linear system G is asymptotically stable. If the single-channel ℓ -photon input state is $|\psi_\ell\rangle$ defined in (25), the steady-state output state is*

$$\rho_{out} = \frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} (B^*(\eta_k^-) - B(\eta_k^+)) \rho_{field,g} \left(\frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} B^*(\eta_k^-) - B(\eta_k^+) \right)^*, \quad (51)$$

where $\eta_k^-(t)$ and $\eta_k^+(t)$ are given by (38) and $\rho_{field,g}$ is given in (2.1.3).

Specific to the passive case, the steady-state output state is a multi-photon state, as given by the following result.

Corollary 13 *Assume that the quantum linear system G is asymptotically stable and passive. The steady-state output state of G driven by the multi-photon state $|\Psi\rangle$ is a pure state*

$$|\Psi_{out}\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \prod_{k=1}^{\ell_j} \sum_{i=1}^m B_i^*(\eta_{ijk}^-) |0^{\otimes m}\rangle.$$

In particular, in the single-channel case, the steady-state output state of G driven by the single-photon state $|\psi_1\rangle$ is

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} B^*(\eta_k^-) |0\rangle, \quad (52)$$

Example 1: Beamsplitter. Consider a beamsplitter with parameters $L = 0$, $H = 0$, and

$$S = \begin{bmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ -\sqrt{1-\eta} & \sqrt{\eta} \end{bmatrix}, \quad (0 < \eta < 1).$$

Let each input channel have two photons. As with (34), define an input state to be $|\Psi\rangle = \prod_{j=1}^2 \frac{1}{\sqrt{N_{2_j}}} \prod_{k=1}^2 B_j^*(\xi^{jk}) |0^{\otimes 2}\rangle$,

where $N_{2_j} = \int_{-\infty}^{\infty} |\xi^{j1}(t)|^2 dt \int_{-\infty}^{\infty} |\xi^{j2}(t)|^2 dt + \left| \int_{-\infty}^{\infty} \xi^{j2}(t)^* \xi^{j1}(t) dt \right|^2$, ($j = 1, 2$). According to (43), we have $\eta^- \in \mathcal{C}^{2 \times 2 \times 2}$ with elements

$$\begin{aligned} \eta_{111}^- &= \sqrt{\eta} \xi^{11}(t), & \eta_{112}^- &= \sqrt{\eta} \xi^{12}(t), & \eta_{121}^- &= \sqrt{1-\eta} \xi^{21}(t), & \eta_{122}^- &= \sqrt{1-\eta} \xi^{22}(t), \\ \eta_{211}^- &= -\sqrt{1-\eta} \xi^{11}(t), & \eta_{212}^- &= -\sqrt{1-\eta} \xi^{12}(t), & \eta_{221}^- &= \sqrt{\eta} \xi^{21}(t), & \eta_{222}^- &= \sqrt{\eta} \xi^{22}(t). \end{aligned}$$

According to Corollary 13,

$$\begin{aligned}
|\Psi_{\text{out}}\rangle &= \frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11})B_1^*(\xi^{12})B_1^*(\xi^{21})B_1^*(\xi^{22}) |0^{\otimes 4}\rangle \\
&+ \frac{\eta\sqrt{\eta(1-\eta)}}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11})B_1^*(\xi^{12}) (B_1^*(\xi^{22})B_2^*(\xi^{21}) + B_1^*(\xi^{21})B_2^*(\xi^{22})) |0^{\otimes 4}\rangle \\
&- \frac{\sqrt{\eta(1-\eta)}(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{21})B_1^*(\xi^{22}) (B_1^*(\xi^{12})B_2^*(\xi^{11}) + B_1^*(\xi^{11})B_2^*(\xi^{12})) |0^{\otimes 4}\rangle \\
&+ \frac{\eta^2}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{11})B_1^*(\xi^{12})B_2^*(\xi^{21})B_2^*(\xi^{22}) |0^{\otimes 4}\rangle \\
&+ \frac{(1-\eta)^2}{\sqrt{N_{2_1}N_{2_2}}} B_1^*(\xi^{21})B_1^*(\xi^{22})B_2^*(\xi^{11})B_2^*(\xi^{12}) |0^{\otimes 4}\rangle \\
&- \frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} (B_1^*(\xi^{11})B_2^*(\xi^{12}) + B_1^*(\xi^{12})B_2^*(\xi^{11})) (B_1^*(\xi^{21})B_2^*(\xi^{22}) + B_1^*(\xi^{22})B_2^*(\xi^{21})) |0^{\otimes 4}\rangle \\
&- \frac{\eta\sqrt{\eta(1-\eta)}}{\sqrt{N_{2_1}N_{2_2}}} (B_1^*(\xi^{11})B_2^*(\xi^{12}) + B_1^*(\xi^{12})B_2^*(\xi^{11})) B_2^*(\xi^{21})B_2^*(\xi^{22}) |0^{\otimes 4}\rangle \\
&+ \frac{\sqrt{\eta(1-\eta)}(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} (B_1^*(\xi^{21})B_2^*(\xi^{22}) + B_1^*(\xi^{22})B_2^*(\xi^{21})) B_2^*(\xi^{11})B_2^*(\xi^{12}) |0^{\otimes 4}\rangle \\
&+ \frac{\eta(1-\eta)}{\sqrt{N_{2_1}N_{2_2}}} B_2^*(\xi^{11})B_2^*(\xi^{12})B_2^*(\xi^{21})B_2^*(\xi^{22}) |0^{\otimes 4}\rangle. \tag{53}
\end{aligned}$$

Assume $\eta = \frac{1}{2}$, that is the system is a balanced beamsplitter. If $\xi^{11}(t) \equiv \xi^{12}(t) \equiv \xi^{21}(t) \equiv \xi^{22}(t)$ and $\int_{-\infty}^{\infty} |\xi^{11}(t)|^2 dt = 1$, then $N_{2_1} = N_{2_2} = 2$. Let $\frac{1}{\sqrt{i!k!}}|i, k\rangle$ be the state with i photons in the first channel and k photons in the second channel respectively, ($i = 0, \dots, 4$). (53) reduces to

$$|\Psi_{\text{out}}\rangle = \sqrt{\frac{3}{8}}|4, 0\rangle - \frac{1}{2}|2, 0\rangle|0, 2\rangle + \sqrt{\frac{3}{8}}|0, 4\rangle. \tag{54}$$

(54) is the same as (15) in (Ou, 2007). In a similar way, (17) in (Ou, 2007) can also be re-produced.

4 The unfactorizable case

The factorizable multi-photon states studied in Secion 3 form a subclass of more general multi-photon states, e.g., Gheri, Ellinger, Pellizzari & Zoller, 1998, (58) and Baragiola, Cook, Brańczyk & Combes, 2012, Section 2. In this section, we study the response of quantum linear systems to general multi-channel multi-photon states where there may exist correlation among photons in channels.

4.1 More general multi-photon states

The unfactorizable multi-photon states are defined in this subsection.

If a single channel has ℓ photons, a general form of continuous-mode ℓ -photon state is

$$|\psi_\ell\rangle = \frac{1}{\sqrt{N_\ell}} \int_\ell \psi(t_1, \dots, t_\ell) b^*(t_1) \cdots b^*(t_\ell) dt_{1 \rightarrow \ell} |0\rangle, \tag{55}$$

where $\psi(t_1, \dots, t_\ell)$ is a multi-variable function, and

$$N_\ell = \sum_{P \in S_\ell} \int_\ell \psi(t_1, \dots, t_\ell) \psi(P(t_1), \dots, P(t_\ell)) dt_{1 \rightarrow \ell}$$

is a normalization parameter, with $P(t_1, \dots, t_\ell)$ and S_ℓ as those defined in subsection 3.1. In general, for the m -channel case, assume the j -th channel has ℓ_j photons, and the state for this channel is

$$|\Psi_j\rangle = \frac{1}{\sqrt{N_{\ell_j}}} \int_{\ell_j} \Psi_j(t_1, \dots, t_{\ell_j}) b_j^*(t_1) \cdots b_j^*(t_{\ell_j}) dt_{1 \rightarrow \ell_j} |0\rangle. \quad (56)$$

Then the state for the m -channel input field can be defined as

$$|\Psi\rangle = \prod_{j=1}^m |\Psi_j\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \int_{\ell_j} \Psi_j(t_1, \dots, t_{\ell_j}) b_j^*(t_1) \cdots b_j^*(t_{\ell_j}) dt_{1 \rightarrow \ell_j} |0^{\otimes m}\rangle. \quad (57)$$

Remark 6. In particular, when $\Psi_j(t_1, \dots, t_{\ell_j}) = \prod_{k=1}^{\ell_j} \Psi^{jk}(t_k)$, ($j = 1, \dots, m$), (56) reduces to (33), and correspondingly (57) reduces to (34), the factorizable case.

4.2 The passive case

In this subsection we study the response of the quantum linear passive system G to an m -channel input field in the state $|\Psi\rangle$ defined in (57).

We first rewrite the m -channel multi-photon state $|\Psi\rangle$ into an alternative form; this will enable us to present the input and output states in a unified form. For $j = 1, \dots, m$, $i = 1, \dots, \ell_j$, and $k_i = 1, \dots, m$, define

$$\Psi_{j, k_1, \dots, k_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) := \begin{cases} \Psi_j(\tau_1, \dots, \tau_{\ell_j}), & k_1 = \dots = k_{\ell_j} = j, \\ 0, & \text{otherwise.} \end{cases} \quad (58)$$

Thus for each $j = 1, \dots, m$ we have an ℓ_j -way m -dimensional tensor, denoted Ψ_j . The multi-channel multi-photon state $|\Psi\rangle$ in (57) can be re-written as

$$|\Psi\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \int_{\ell_j} \Psi_{j, k_1, \dots, k_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) b_{k_1}^*(\tau_1) \cdots b_{k_{\ell_j}}^*(\tau_{\ell_j}) d\tau_{1 \rightarrow \ell_j} |0^{\otimes m}\rangle.$$

We define a class of pure states

$$\mathcal{F}_1 = \left\{ |\Psi\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \int_{\ell_j} \Psi_{j, k_1, \dots, k_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) \prod_{i=1}^{\ell_j} b_{k_i}^*(\tau_i) d\tau_{1 \rightarrow \ell_j} |0^{\otimes m}\rangle : \langle \Psi | \Psi \rangle = 1 \right\}. \quad (59)$$

Theorem 14 *Suppose that the quantum linear system G is asymptotically stable and passive. The steady-state output state of G driven by a state $|\Psi_{\text{in}}\rangle \in \mathcal{F}_1$ is another state $|\Psi_{\text{out}}\rangle \in \mathcal{F}_1$ with wave packet transfer*

$$\Psi_{\text{out}, j} = \Psi_{\text{in}, j} \times_1 g_{G^-} \times_2 \cdots \times_{\ell_j} g_{G^-}, \quad \forall j = 1, \dots, m,$$

where the operation between the matrix and tensor is defined in (23).

Because Theorem 14 is a special case of Theorem 17 for the active case, cf. Remark 9, its proof is omitted.

In particular, for the single-channel case, we have

Corollary 15 *The steady-state output state of a quantum linear passive system G driven by the ℓ -photon state $|\psi_\ell\rangle$ in (55) is an ℓ -photon state*

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{N_\ell}} \int_{\ell} \psi_{\text{out}}^-(t_1, \dots, t_\ell) b^*(t_1) b^*(t_2) \cdots b^*(t_\ell) dt_{1 \rightarrow \ell} |0\rangle,$$

where the multi-variable function ψ_{out}^- is

$$\psi_{\text{out}}^-(t_1, \dots, t_\ell) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_{G^-}(t_1 - \tau_1) \cdots g_{G^-}(t_\ell - \tau_\ell) \psi_\ell(\tau_1, \dots, \tau_\ell) d\tau_1 \cdots d\tau_\ell.$$

4.3 The active case

In this subsection we study the response of the quantum linear system G to the m -channel input field in the state $|\Psi\rangle$ defined in (57). Here G is not necessarily passive. In this case, as shown in Sec. 3.4, g_{G^+} contributes to the output states. So the active case is more mathematically involved.

We first introduce some more notations in order to derive the steady-state output state. Define

$$\text{sgn}(d_i) := \begin{cases} 1, & d_i = 1, \\ 0, & d_i = -1, \end{cases} \quad \forall i = 1, \dots, \max\{\ell_1, \dots, \ell_m\}.$$

For each $j = 1, \dots, m$, $i = 1, \dots, \ell_j$ and $k_i = 1, \dots, m$, define

$$\Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) := \begin{cases} \Psi_j(\tau_1, \dots, \tau_{\ell_j}), & k_1 = \dots = k_{\ell_j} = j, \quad d_1 = \dots = d_{\ell_j} = -1, \\ 0, & \text{otherwise,} \end{cases} \quad (60)$$

where the multi-variable function $\Psi_j(\tau_1, \dots, \tau_{\ell_j})$ is defined in (56). $\Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j})$ can be regarded as a $2\ell_j$ -way

tensor in the tensor space $\mathbb{C}^{\overbrace{m \times \dots \times m}^{\ell_j} \times \overbrace{2 \times \dots \times 2}^{\ell_j}}$. Accordingly, for each $j = 1, \dots, m$, and $i = 1, \dots, \ell_j$ define operators

$$b_j^{d_i}(t) := \begin{cases} b_j^*(t), & d_i = -1, \\ b_j(t), & d_i = 1. \end{cases}$$

Moreover, for each $j, k = 1, \dots, m$, define

$$g_{G^d}^{kj}(t) := \begin{cases} g_{G^-}^{kj}(t), & d = -1, \\ g_{G^+}^{kj}(t)^*, & d = 1. \end{cases}$$

With the above notations, for each $j = 1, \dots, m$, $|\Psi_j\rangle$ defined in (56) can be encoded by a $2\ell_j$ -way tensor in the

tensor space $\mathbb{C}^{\overbrace{m \times \dots \times m}^{\ell_j} \times \overbrace{2 \times \dots \times 2}^{\ell_j}}$. Specifically,

$$|\Psi_j\rangle = \frac{1}{\sqrt{N_{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \int_{\ell_j} \Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) b_j^{d_1}(\tau_1) \cdots b_j^{d_{\ell_j}}(\tau_{\ell_j}) d\tau_{1 \rightarrow \ell_j} |0\rangle. \quad (61)$$

Moreover, for each $j = 1, \dots, m$, $i = 1, \dots, \ell_j$ and $k_i = 1, \dots, m$, define operators

$$b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j) := \Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(t_1, \dots, t_{\ell_j}) b_j^{d_1}(t_1) \cdots b_j^{d_{\ell_j}}(t_{\ell_j}), \quad (62)$$

where the $2\ell_j$ -way tensor $\Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j})$ is that defined in (60). Then (61) becomes

$$|\Psi_j\rangle = \frac{1}{\sqrt{N_{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \int_{\ell_j} b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j) dt_1 \cdots dt_{\ell_j} |0\rangle.$$

Accordingly, the multi-channel state $|\Psi\rangle$ in (57) can be re-written as

$$|\Psi\rangle = \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \int_{\ell_j} b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j) dt_1 \dots dt_{\ell_j} |0^{\otimes m}\rangle. \quad (63)$$

The above motivates us to define a class of states.

Definition 16 Let $\Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j})$ be a $2\ell_j$ -way tensor in the tensor space

$\mathbb{C}^{\overbrace{m \times \dots \times m}^{\ell_j} \times \overbrace{2 \times \dots \times 2}^{\ell_j}}$. A state $\rho_{\Psi, R}$ is said to be a photon-Gaussian state if it belongs to the set

$$\mathcal{F}_2 := \left\{ \rho_{\Psi, R} = \prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \int_{\ell_j} b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j) dt_1 \dots dt_{\ell_j} |0^{\otimes m}\rangle \rho_R \right. \\ \left. \times \left(\prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \int_{\ell_j} b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j) dt_1 \dots dt_{\ell_j} |0^{\otimes m}\rangle \right)^* \right\}, \quad (64)$$

where the operator $b_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\Psi_j)$ is defined in (62), and ρ_R is a zero-mean Gaussian field state with covariance function R . It is assumed that $\text{Tr}[\rho_{\Psi, R}] = 1$.

Remark 7. Clearly, the m -channel multi-photon state $|\Psi\rangle$ defined in (57) belongs to \mathcal{F}_2 . Moreover, when G is passive, $\mathcal{F}_1 = \mathcal{F}_2$.

Next we study how the input state in \mathcal{F}_2 is transformed by the quantum linear system G .

Theorem 17 Suppose that the quantum linear system G is asymptotically stable. The density function $\rho_{\Psi_{\text{out}}, R_{\text{out}}}$ of the steady-state output field of G driven by the density operator $\rho_{\Psi, R} \in \mathcal{F}_2$ is

$$\rho_{\Psi_{\text{out}}, R_{\text{out}}} = \left(\prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \sum_{r_1, \dots, r_{\ell_j}=1}^m \sum_{f_1, \dots, f_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(f_i)} \int_{\ell_j} b_{r_1, \dots, r_{\ell_j}}^{f_1, \dots, f_{\ell_j}}(\Psi_{\text{out}, j}) d\tau_{1 \rightarrow \ell_j} |0^{\otimes m}\rangle \right) \rho_{R_{\text{out}}} \\ \times \left(\prod_{j=1}^m \frac{1}{\sqrt{N^{\ell_j}}} \sum_{r_1, \dots, r_{\ell_j}=1}^m \sum_{f_1, \dots, f_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(f_i)} \int_{\ell_j} b_{r_1, \dots, r_{\ell_j}}^{f_1, \dots, f_{\ell_j}}(\Psi_{\text{out}, j}) d\tau_{1 \rightarrow \ell_j} |0^{\otimes m}\rangle \right), \quad (65)$$

where

$$g_{G^{d_i, f_i}}^{kj}(t) := \begin{cases} g_{G^{-d_i}}^{kj}(t), & f_i = -1, \\ g_{G^{d_i}}^{kj}(t), & f_i = 1, \end{cases} \quad \forall j, k = 1, \dots, m, \quad i = 1, \dots, \ell_j, \quad d_i = \pm 1, \\ b_i^{d_k, f_k}(t) := \begin{cases} b_i^{-d_k}(t), & f_i = -1, \\ b_i^{d_k}(t), & f_i = 1, \end{cases} \quad \forall i = 1, \dots, m, \quad \forall k = 1, \dots, \max\{\ell_1, \dots, \ell_m\},$$

$$\Psi_{k_1 \rightarrow \ell_j, r_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j, f_1 \rightarrow \ell_j}(t_1, \dots, t_{\ell_j}) := \int_{\ell_j} \prod_{i=1}^{\ell_j} g_{G^{d_i, f_i}}^{r_i k_i}(t_i - \tau_i) \Psi_{k_1, \dots, k_{\ell_j}}^{d_1, \dots, d_{\ell_j}}(\tau_1, \dots, \tau_{\ell_j}) d\tau_{1 \rightarrow \ell_j}, \quad (66)$$

$$b_{r_1, \dots, r_{\ell_j}}^{f_1, \dots, f_{\ell_j}}(\Psi_{out, j}) := \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \Psi_{k_1 \rightarrow \ell_j, r_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j, f_1 \rightarrow \ell_j}(t_1, \dots, t_{\ell_j}) \prod_{i=1}^{\ell_j} b_{r_i}^{d_i, f_i}(t_i), \quad (67)$$

and $\rho_{R_{out}}$ is a Gaussian state whose covariance function is obtained by the Gaussian transfer

$$R_{out}[i\omega] = G[i\omega]R[i\omega]G[i\omega]^\dagger.$$

Proof. It is easy to show that (49) can be re-written as

$$b_j^{-d}(t, -\infty) = \sum_{k=1}^m \int_{-\infty}^{\infty} \left(-g_{G^{-d}}^{kj}(r-t)b_k^{-d}(r) + g_{G^d}^{kj}(r-t)b_k^d(r) \right) dr, \quad d = \pm 1.$$

Note that

$$\begin{aligned} & \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} U(t, t_0) \int_{\ell_j} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \\ & \quad \times \Psi_{k_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j}(\tau_1, \dots, \tau_{\ell_j}) b_{k_1}^{d_1}(\tau_1) \cdots b_{k_{\ell_j}}^{d_{\ell_j}}(\tau_{\ell_j}) d\tau_{1 \rightarrow \ell_j} U(t, t_0)^* \\ &= \int_{\ell_j} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \Psi_{k_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j}(\tau_1, \dots, \tau_{\ell_j}) \\ & \quad \times \sum_{r_1=1}^m \int_{-\infty}^{\infty} \left(-g_{G^{-d_1}}^{r_1 k_1}(t_1 - \tau_1) b_{r_1}^{-d_1}(t_1) dt_1 + g_{G^{d_1}}^{r_1 k_1}(t_1 - \tau_1) b_{r_1}^{d_1}(t_1) dt_1 \right) \\ & \quad \cdots \sum_{r_{\ell_j}=1}^m \int_{-\infty}^{\infty} \left(-g_{G^{-d_{\ell_j}}}^{r_{\ell_j} k_{\ell_j}}(t_{\ell_j} - \tau_{\ell_j}) b_{r_{\ell_j}}^{-d_{\ell_j}}(t_{\ell_j}) dt_{\ell_j} + g_{G^{d_{\ell_j}}}^{r_{\ell_j} k_{\ell_j}}(t_{\ell_j} - \tau_{\ell_j}) b_{r_{\ell_j}}^{d_{\ell_j}}(t_{\ell_j}) dt_{\ell_j} \right) d\tau_{1 \rightarrow \ell_j} \\ &= \int_{\ell_j} \sum_{r_1, \dots, r_{\ell_j}=1}^m \sum_{f_1, \dots, f_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(f_i)} \sum_{k_1, \dots, k_{\ell_j}=1}^m \sum_{d_1, \dots, d_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(d_i)} \\ & \quad \times \Psi_{k_1 \rightarrow \ell_j, r_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j, f_1 \rightarrow \ell_j}(t_1, \dots, t_{\ell_j}) \prod_{i=1}^{\ell_j} b_{r_i}^{d_i, f_i}(t_i) dt_{1 \rightarrow \ell_j} \\ &= \sum_{r_1, \dots, r_{\ell_j}=1}^m \sum_{f_1, \dots, f_{\ell_j}=\pm 1} (-1)^{\sum_{i=1}^{\ell_j} \text{sgn}(f_i)} \int_{\ell_j} b_{r_1, \dots, r_{\ell_j}}^{f_1, \dots, f_{\ell_j}}(\Psi_{out}) dt_1 \cdots dt_{\ell_j} \end{aligned}$$

This, together with the Gaussian transfer theorem (Theorem 4), establishes Theorem 17.

Remark 8. It can be verified that the *factorizable* m -channel multi-photon state $|\Psi\rangle$ defined in (34) (equivalently (42)) can be re-written as

$$|\Psi\rangle\langle\Psi| = \prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \sum_{i=1}^m \prod_{k=1}^{\ell_j} (B_j^*(\xi_{ijk}^-) - B_j(\xi_{ijk}^+)) |0^{\otimes m}\rangle\langle 0^{\otimes m}| \left(\prod_{j=1}^m \frac{1}{\sqrt{N_{\ell_j}}} \sum_{i=1}^m \prod_{k=1}^{\ell_j} (B_j^*(\xi_{ijk}^-) - B_j(\xi_{ijk}^+)) \right)^*. \quad (68)$$

There is clear similarity between (34) and (63), or equivalently, between (68) and (64). The implication of such similarity is that all the results for the unfactorizable case can be reduced to those for the factorizable case.

Remark 9. When the quantum linear system G is passive and $\rho_R = |\phi\rangle\langle\phi|$, $\rho_{\Psi, R}$ in (64) becomes a pure state. Moreover, for the case case, $\text{sgn}(d_i) = 0$ for all i . Therefore, in the passive case $\rho_{\Psi, R}$ is a pure state in the class \mathcal{F}_1 defined in (59). As a result, in the passive case Theorem 17 reduces to Theorem 14.

Remark 10. From (66) it can be seen that $\Psi_{k_1 \rightarrow \ell_j, r_1 \rightarrow \ell_j}^{d_1 \rightarrow \ell_j, f_1 \rightarrow \ell_j}(t_1, \dots, t_{\ell_j})$ is a $4\ell_j$ way tensor, not a $2\ell_j$ way tensor in

the space $\mathbb{C}^{\overbrace{m \times \dots \times m}^{\ell_j} \times \overbrace{2 \times \dots \times 2}^{\ell_j}}$. As a result, $\rho_{\Psi_{\text{out}}, R_{\text{out}}}$ in (65) is not an element in the class \mathcal{F}_2 . That is, the class \mathcal{F}_2 is not an invariant set under the steady-state action of the quantum linear system G . However, using a procedure similar to that presented in Theorem 17, it is not hard to derive the steady-state output state when a quantum linear system G is driven by an input state $\rho_{\Psi_{\text{out}}, R_{\text{out}}}$. Clearly, the tensor representation plays a key role in this study.

Next we use three examples to illustrate the results for the unfactorizable photon states.

Example 2: The $(1 + \ell)$ -photon case. Consider a beamsplitter with parameter

$$S = \begin{bmatrix} \sqrt{1-R} & \sqrt{R} \\ \sqrt{R} & -\sqrt{1-R} \end{bmatrix}, \quad (0 < R < 1).$$

Let the input state be

$$|\Psi_{\text{in}}\rangle = B_1^*(\xi) \otimes \frac{1}{\sqrt{N_\ell}} \prod_{k=1}^{\ell} B_2^*(\xi_k) |00\rangle.$$

As with Example 1, the output state can be derived by means of Corollary 13. Alternatively, it can be derived via Theorem 14. Clearly, $m = 2$, $\ell_1 = 1$, and $\ell_2 = \ell$. By Theorem 14, the output state is

$$|\Psi_{\text{out}}\rangle = (\sqrt{1-R}B_1^*(\xi) + \sqrt{R}B_2^*(\xi)) \frac{1}{\sqrt{N_\ell}} \sum_{k_1, \dots, k_\ell=1}^2 B_{k_1}^*(S^{k_1 2} \xi_{k_1}) \cdots B_{k_\ell}^*(S^{k_\ell 2} \xi_{k_\ell}) |00\rangle. \quad (69)$$

In particular, assume $\xi_1(t) \equiv \dots \equiv \xi_\ell(t) \equiv \xi(t)$ and $\int_{-\infty}^{\infty} |\xi(t)|^2 dt = 1$. Then (69) becomes

$$|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{\ell!}} (\sqrt{1-R}B_1^*(\xi) + \sqrt{R}B_2^*(\xi)) (\sqrt{R}B_1^*(\xi) - \sqrt{1-R}B_2^*(\xi))^\ell |00\rangle.$$

The coefficient for the component $\frac{1}{\sqrt{\ell!}} B_1^*(\xi)^\ell B_2^*(\xi) |00\rangle = \frac{1}{\sqrt{\ell!}} |\ell, 1\rangle$ is $\sqrt{R^{\ell-1}}(R - \ell(1-R))$, whose squared value is exactly (in) in Sanaka, Resch & Zeilinger, 2006.

Example 3: The photon-catalyzed optical coherent (PCOC) case. Consider a beamsplitter with parameter

$$S = \begin{bmatrix} T & -R \\ R & T \end{bmatrix}, \quad (R, T > 0, \quad R^2 + T^2 = 1).$$

Let the input be $|\psi_\ell\rangle \otimes |\alpha\rangle$, where $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ is a coherent state. The input stat can be re-written as

$$|\Psi_{\text{in}}\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\ell\rangle \otimes |n\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \prod_{j=1}^2 \prod_{k=1}^{\ell_j} B_j^*(\xi) |0^{\otimes 2}\rangle,$$

where $\ell_1 = \ell$ and $\ell_2 = n$. That is, here it is assumed that all the photons are identical. By (43),

$$\eta_{11k}^- = T\xi, \quad \eta_{12k}^- = -R\xi, \quad \eta_{21r}^- = R\xi, \quad \eta_{22r}^- = T\xi, \quad \forall k = 1, \dots, \ell, \quad \forall r = 1, \dots, n.$$

By Theorem 14,

$$\begin{aligned} |\Psi_{\text{out}}\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \prod_{j=1}^2 \prod_{k=1}^{\ell_j} \sum_{i=1}^2 B_i^*(\eta_{ijk}^-) |0^{\otimes 2}\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sum_{i=0}^{\ell} \sum_{j=0}^n \binom{n}{n-j} \binom{\ell}{i} (-1)^j T^{n+\ell-i-j} R^{i+j} |\ell+j-i\rangle \otimes |n+i-j\rangle. \end{aligned}$$

When the first output channel is measured by means of the state $|\ell\rangle$, the state at the second output channel becomes

$$|\Psi_{\text{out,conditioned}}\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sum_{j=0}^{\min\{\ell,n\}} \binom{n}{n-j} \binom{\ell}{j} (-1)^j T^{n+\ell-2j} R^{2j} |n\rangle,$$

which reproduces the key formula (1) in Bartley, et al., 2012.

Remark 11. Examples 1, 2, and 3 illustrate that the proposed research is also applicable to the discrete-variable multi-photon case.

Example 4: Multi-photon pulse shaping via optical cavity. In this example, we study how an optical cavity responds to an unfactorizable 2-photon input state. The optical cavity has parameters $\Omega_- = \Omega_+ = 0$, $C_- = \sqrt{\kappa}$, $C_+ = 0$, $S = 1$. Thus, $A = -\frac{\kappa}{2}I_2$, $B = -\sqrt{\kappa}I_2$, $C = \sqrt{\kappa}I_2$, $D = I_2$. Let an unfactorizable 2-photon input state be

$$|\psi_2\rangle = \frac{1}{\sqrt{N_2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t_1, t_2) b^*(t_1) b^*(t_2) dt_1 dt_2 |0\rangle,$$

where

$$\psi(t_1, t_2) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} \begin{bmatrix} t_1 - \tau_1 & t_2 - \tau_2 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} t_1 - \tau_1 \\ t_2 - \tau_2 \end{bmatrix}\right),$$

with

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1.$$

That is, the input state has a 2-dimensional Gaussian pulse shape centered at (τ_1, τ_2) and with covariance matrix Σ . When the correlation parameter $\rho = 0$, $|\psi_2\rangle$ reduces to a factorizable state. According to Corollary 15, the steady-state output state $|\psi_{\text{out}}^-\rangle$ is given by

$$\psi_{\text{out}}^-(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{G^-}(t_1 - r_1) g_{G^-}(t_2 - r_2) \psi(r_1, r_2) dr_1 dr_2,$$

where

$$g_{G^-}(t) = \begin{cases} \delta(t) - \kappa e^{-\frac{\kappa}{2}t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

In the following we fix $\tau_1 = \tau_2 = \sigma_1 = \sigma_2 = 1$, and study the pulse shape $\psi_{\text{out}}^-(t_1, t_2)$ of the output state for several pairs of the correlation coefficient ρ and the cavity decay rate κ . Fig. 2 summarizes pulse shaping of multi-photon states by the optical cavity in different scenarios. Fig. 2(a)-(f) are for the case of $\rho = 0.5$. Fig. 2(a) is the shape $\psi(t_1, t_2)$ of the input state, while Fig. 2(b)-(f) are the shapes $\psi_{\text{out}}^-(t_1, t_2)$ for the output state for different decaying rates. Fig. 2(g)-(k) are for the case of $\rho = -0.5$. Fig. 2(g) is the shape $\psi(t_1, t_2)$ of the input state, while Fig. 2(h)-(k) are the shapes $\psi_{\text{out}}^-(t_1, t_2)$ for the output state for different decaying rates.

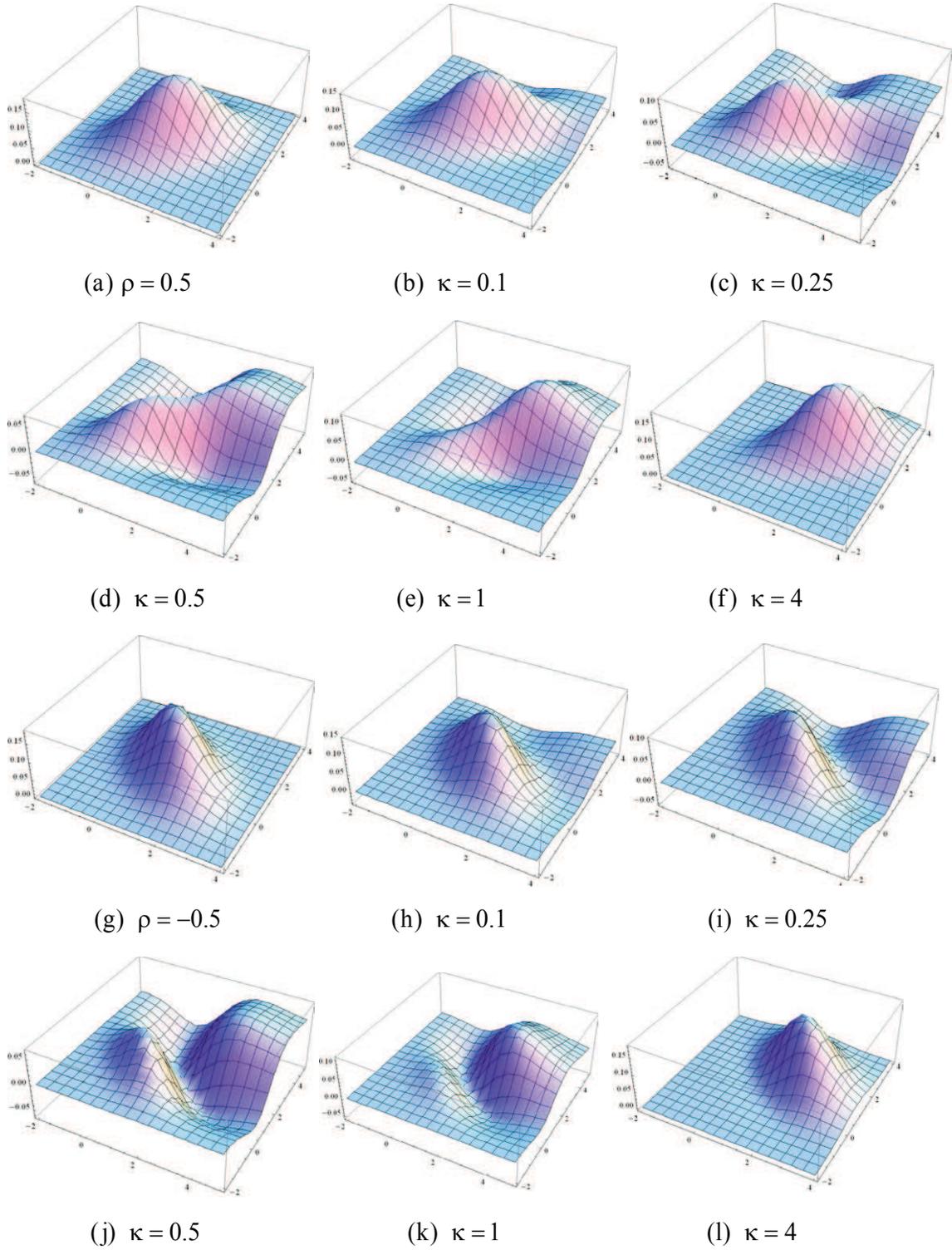


Fig. 2. Multi-photon pulse shaping via an optical cavity. (a) and (g) are pulse shapes for the input 2-photon state with different correlations ρ . When the decay rate is small ((b) and (h)), the shape of ψ_{out}^- of the output 2-photon state is similar to that of ψ of the input 2-photon state; As decay rate increases, the pulse shapes deform ((c)-(e) and (i)-(k)); When the decay rate is large ((f) and (l)), the shape of ψ_{out}^- is similar with that of ψ , however their mean values are significantly different.

5 Conclusion

In this paper we have studied the response of quantum linear systems to multi-channel multi-photon states. New types of tensors are defined to encode pulse information of multi-photon states, for both the factorizable case and the unfactorizable case. The steady-state action of quantum linear systems on multi-photon states are characterized in terms of tensor processing by transfer functions. Explicit forms of output states, output covariance functions and output intensities have been derived. In contrast to the discrete-variable (single-mode) treatments in most discussions on quantum information, we have presented a continuous-variable (multi-mode) treatment of multi-photon processing. As can be seen from Examples 1-3, the continuous-variable treatment is also applicable to many discrete-variable treatments. Moreover, the continuous-variable treatment is closer to a real experimental environment in optical quantum information processing. As demonstrated by Example 4 for pulse shaping by optical cavities, one immediate future research is: How to design desired pulse shapes (which encode time or frequency correlation among photons) by means of quantum linear systems, as has been investigated in Milburn, 2008 and Zhang & James, 2013 in the single-photon setting for the passive case. Another future research is to study how multi-photon pulses can be stored and read out by gradient echo memories (Hush, Carvalho, Hedges & James, 2013), which are indispensable components of complex quantum optical networks for quantum communication and computing.

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