Hysteresis loop of the LuGre Model ⋆

Mohammad Fuad Mohammad Naser [a], Fayçal Ikhouane [b]

[a] *mohammad.fuad@upc.edu*

[b] *faycal.ikhouane@upc.edu*

Universitat Politècnica de Catalunya, Escola Universitària d´Enginyeria, Tècnica Industrial de Barcelona. Departament de Matemàtica Aplicada III, Comte d´Urgell, 187, 08036, Barcelona, Spain.

Abstract

The LuGre friction model is used in the current literature to describe the friction phenomenon for mechanical systems. In this paper, we focus on the hysteresis behaviour of the model. More precisely, we describe analytically the hysteresis loop of the model through the concepts of consistency and strong consistency. The description is illustrated by numerical simulations.

Key words: LuGre model; hysteresis; (strong) consistency.

1 Introduction

Friction is a nonlinear phenomenon that originates from the contact of two bodies. It has two types of characteristics, static and dynamic. The static characteristics of friction include the stiction friction, the kinetic force (the Coulomb force), the viscous force, and the Stribeck effect which are functions of steady state velocity. The static friction models give the friction force as a function of velocity and only describe the steady-state behaviour between velocity and friction force. Static friction models are discontinuous at zero velocity with a dependence on the sign of velocity [1].
This discontinuity does not reflect accurately the real friction behaviour and may cause errors in numerical simulations, or even instability in the algorithms designed to compensate friction [1].
Dynamic friction models capture properties that cannot be captured by typical static friction models; for instance, presliding displacement related to the elastic and plastic deformations of asperities, frictional lag, that is the delay in the change of friction force as a function of a change of velocity, and stick-slip motion. These models do not present a discontinuity at zero velocity which makes them more suitable for numerical simulations and friction compensation [1].

⋆ Supported by grant DPI2011-25822 of the Spanish Ministry of Economy and Competitiveness and by FI-DGR-grant IUE/2365/2009 of the Agency for Management of University and Research Grants (AGAUR).

Dahl friction model is a dynamic model whose steady–state is the Coulomb friction [5]. The main contribution of the model is that it takes into account the existence of hysteresis between the presliding friction force input and the displacement output that is observed experimentally [1]. However, Dahl model does not capture the Stribeck effect. An improvement of this model is implemented in the LuGre model [4] which captures some essential properties of friction such as hysteresis and Stribeck effect and thus can describe stick-slip motion [2]. Therefore, it has been used to describe the friction phenomenon for mechanical systems [10,2]. Necessary and sufficient conditions for the dissipativity of the LuGre model are given in [3]. Also, the model has been used for friction compensation [7,12,13].
In this paper, we focus on the hysteresis behaviour of the LuGre model. More precisely, we investigate the analytical expression of the hysteresis loop of the model through the concepts of consistency and strong consistency [8]. These concepts are particularly useful when dealing with *rate–dependent* hysteresis as is the case of the LuGre model. The reader is referred to [8] for a more detailed explanation and motivation of the concepts of consistency and strong consistency.
The paper is organized as follows. Section 2 presents the needed background from [8]. The problem statement is formulated in Section 3. The main results of this paper are presented in Section 4. These results are commented upon is Section 5, and a simulation example is provided in Section 6. The conclusion is given in Section 7.

2 Background Results

This section summarizes the results obtained in [8].

2.1 Class of inputs

The Lebesgue measure on $\mathbb{R}$ is denoted μ. For a measurable function $p: I \subset \mathbb{R}_+ \to \mathbb{R}^m$, $\|p\|_{\infty,I}$ denotes the essential supremum of $|p|$ on I where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. When $I = \mathbb{R}_+$, it is denoted $\|p\|_\infty$.
Consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u: \mathbb{R}_+ \to \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative $\dot{u}$ is defined a.e, and we have $\|u\|_\infty < \infty$, $\|\dot{u}\|_\infty < \infty$.
For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)| \,\mathrm{d}\tau \in \mathbb{R}_+$. The function ρ_u is well defined, nondecreasing and absolutely continuous. Observe that ρ_u may not be invertible. Let I_u be the range of ρ_u.

Lemma 1 *[8] Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant so that I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^m)$ that satisfies $\psi_u \circ \rho_u = u$. The function ψ_u satisfies $\|\dot{\psi}_u\|_{\infty,I_u} = 1$ and $\mu\left[\left\{\varrho \in I_u / \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right] = 0$.*

Lemma 2 *[8] Define $s_\gamma(t) = t/\gamma, \forall \gamma > 0, t \geq 0$. Then $\forall \gamma > 0, I_{u \circ s_\gamma} = I_u$ and $\psi_{u \circ s_\gamma} = \psi_u$.*

2.2 Class of operators

Let Ξ be a set of initial conditions and consider the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. The operator $\mathcal{H}$ is said to be causal if [14, p.60]: $\forall \left(u_1, \xi^0\right), \left(u_2, \xi^0\right) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, $\forall \tau > 0$ if $u_1 = u_2$ on $[0, \tau]$ then $\mathcal{H}\left(u_1, \xi^0\right) = \mathcal{H}\left(u_2, \xi^0\right)$ on $[0, \tau]$.

Assumption 3 *[8] Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ and $y = \mathcal{H}\left(u, \xi^0\right) \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$; if $\exists \theta \in \mathbb{R}_+$ such that u is constant on $[\theta, \infty)$, then y is constant on $[\theta, \infty)$.*

Lemma 4 *[8] Assume that $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is causal and satisfies Assumption 3. Then, $\exists! \varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ that satisfies $\varphi_u \circ \rho_u = y$. Moreover $\|\varphi_u\|_{\infty,I_u} \leq \|y\|_\infty$. If y is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty,I_u} = \|y\|_\infty$.*

2.3 Definition of consistency and strong consistency

Definition 5 *[8] Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. The operator $\mathcal{H}$ is said to be consistent with respect to (u, ξ^0) if the sequence of functions $\{\varphi_{u \circ s_\gamma}\}_{\gamma > 0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$ as $\gamma \to \infty$. Denote $L^\infty(I_u, \mathbb{R}^m) \ni \varphi_u^\star = \lim_{\gamma \to \infty} \varphi_{u \circ s_\gamma}$.*

Observe that, in Definition 5 of consistency, the input u needs not be periodic. Now, to characterize the hysteresis loop of the operator $\mathcal{H}$ we introduce the concept of strong consistency.

Definition 6 *[8] Let $T > 0$. A T-periodic function $w : \mathbb{R}_+ \to \mathbb{R}$ is said to be wave periodic if there exists some $T^+ \in (0, T)$ such that*

- *The function w is continuous on $\mathbb{R}_+$*
- *The function w is continuously differentiable on $(0, T^+)$ and on (T^+, T)*
- *The function w is increasing on $(0, T^+)$ and is decreasing on (T^+, T)*

Lemma 7 *[8] If the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is non-constant and T-periodic, then $I_u = \mathbb{R}_+$ and $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is $\rho_u(T)$-periodic. Furthermore, if $n = 1$ and u is wave periodic, then a more precise result can be stated. The function ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$.*

$\forall k \in \mathbb{N}$, let $\varphi_{u,k}^\star \in L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ be defined as $\varphi_{u,k}^\star(\varrho) = \varphi_u^\star(\rho_u(T)k + \varrho), \forall \varrho \in [0, \rho_u(T)]$.

Definition 8 *[8] Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. The operator $\mathcal{H}$ is said to be strongly consistent with respect to (u, ξ^0) if it is consistent with respect to (u, ξ^0), and the sequence of functions $\varphi_{u,k}^\star$ converges in $L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ as $k \to \infty$. Define $L^\infty([0, \rho_u(T)], \mathbb{R}^m) \ni \varphi_u^\circ = \lim_{k \to \infty} \varphi_{u,k}^\star$.*

If the operator $\mathcal{H}$ is strongly consistent with respect to (u, ξ^0), then the graph $\{(\psi_u(\varrho), \varphi_u^\circ(\varrho)), \varrho \in [0, \rho_u(T)]\}$ represents the so-called hysteresis loop.

3 Problem Statement

The LuGre model is given by [2]:

$$\dot{x}(t) = -\sigma_0 \frac{|\dot{u}(t)|}{g(\dot{u}(t))} x(t) + \dot{u}(t), \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$F(t) = \sigma_0 x(t) + \sigma_1 \dot{x}(t) + f(\dot{u}(t)), \tag{3}$$

where $t \geq 0$ denotes time; the parameters $\sigma_0 > 0$ and $\sigma_1 > 0$ are respectively the stiffness and the microscopic damping friction coefficients; the function $g \in C^0(\mathbb{R}, \mathbb{R})$ represents the macrodamping friction with $g(\vartheta) > 0, \forall \vartheta \in \mathbb{R}$; $x(t) \in \mathbb{R}$ is the average deflection of the bristles; $x_0 \in \mathbb{R}$ is the initial state; $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ is the relative displacement and is the input of the system; $F(t)$ is the friction force and is

the output of the system; and $f \in C^0(\mathbb{R},\mathbb{R})$.
In Equation (1), the function $g(\dot{u})$ is measurable, thus, the differential equation (1) can be seen as a linear time-varying system that satisfies all assumptions of [6, Theorem 3]. This implies that a unique absolutely continuous solution of (1) exists on $\mathbb{R}_+$.
On the other hand, define $M_u = \sup_{\alpha\in[-\|\dot{u}\|_\infty,\|\dot{u}\|_\infty]} g(\alpha)$. Then $0 < M_u < \infty$ since g is continuous and $g(\vartheta) > 0, \forall\vartheta \in \mathbb{R}$. We also have $0 < g(\dot{u}(t)) \leq M_u$ for almost all $t \geq 0$. Thus it follows from [2] that $|x(t)| \leq \frac{M_u}{\sigma_0}, \forall t \geq 0$ if $|x_0| \leq \frac{M_u}{\sigma_0}$. If $|x_0| > \frac{M_u}{\sigma_0}$ then $|x(t)| \leq |x_0|, \forall t \geq 0$. Thus, $\forall x_0 \in \mathbb{R}$ we have $x \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$.
Now, in equations (1)-(3), consider the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}(u,x_0) = F$. Then it can be shown that $\mathcal{H}$ is causal and satisfies Assumption 3. This implies that the concepts introduced in Section 2 apply.
$\forall\gamma > 0$, when the input $u \circ s_\gamma$ is used instead of u, system (1)-(3) becomes

$$\dot{x}_\gamma(t) = -\sigma_0 \frac{\left|\frac{\dot{u}\circ s_\gamma(t)}{\gamma}\right|}{g\left(\frac{\dot{u}\circ s_\gamma(t)}{\gamma}\right)} x_\gamma(t) + \frac{\dot{u}\circ s_\gamma(t)}{\gamma}, \tag{4}$$
$$x_\gamma(0) = x_0, \tag{5}$$
$$F_\gamma(t) = \sigma_0 x_\gamma(t) + \sigma_1 \dot{x}_\gamma(t) + f\left(\frac{\dot{u}\circ s_\gamma(t)}{\gamma}\right). \tag{6}$$

Lemma 4 shows that there exists a unique function $x_{u\circ s_\gamma} \in L^\infty(I_u,\mathbb{R}) \cap C^0(I_u,\mathbb{R})$ such that $x_{u\circ s_\gamma} \circ \rho_{u\circ s_\gamma} = x_\gamma$, a unique function $v_u \in L^\infty(I_u,\mathbb{R})$ such that $v_u \circ \rho_u = \dot{u}$, and a unique function $\varphi_{u\circ s_\gamma} \in L^\infty(I_u,\mathbb{R})$ such that $\varphi_{u\circ s_\gamma} \circ \rho_{u\circ s_\gamma} = F_\gamma$. Using the change of variables $\varrho = \rho_{u\circ s_\gamma}(t)$, it follows that

$$\dot{x}_{u\circ s_\gamma}(\varrho) = -\frac{\sigma_0}{g\left(\frac{v_u(\varrho)}{\gamma}\right)} x_{u\circ s_\gamma}(\varrho) + \dot{\psi}_u(\varrho), \tag{7}$$
$$x_{u\circ s_\gamma}(0) = x_0, \tag{8}$$
$$\varphi_{u\circ s_\gamma}(\varrho) = \sigma_0 x_{u\circ s_\gamma}(\varrho) + \frac{\sigma_1}{\gamma}|v_u(\varrho)|\,\dot{x}_{u\circ s_\gamma}(\varrho) + f\left(\frac{v_u(\varrho)}{\gamma}\right), \tag{9}$$

for all $\gamma > 0$ and for almost all $\varrho \in I_u$.

Problem statement: The aim of this paper is to analyze the convergence properties of the sequence of functions $\varphi_{u\circ s_\gamma}$ in order to study the consistency and strong consistency of the operator $\mathcal{H}$ and find the analytical expression of the hysteresis loop of the operator $\mathcal{H}$.

4 Main Results

This section presents the main results of the paper, which are given in Theorem 9.

Theorem 9 *Consider the LuGre model given by equations (7)-(9) in which $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ is non–constant. Then there exist $E, \gamma_1 > 0$ such that $\|F_\gamma\|_\infty \leq E$, $\forall\gamma > \gamma_1$. The operator $\mathcal{H}$ is consistent with respect to (u, x_0) and $\forall\varrho \in I_u$ we have*

$$\varphi^\star_u(\varrho) = \sigma_0 e^{\frac{-\sigma_0\varrho}{g(0)}}\Big[x_0 + \int_0^\varrho e^{\frac{\sigma_0\tau}{g(0)}} \cdot \dot{\psi}_u(\tau)\, d\tau\Big] + f(0). \tag{10}$$

Moreover, if u is T-periodic, then the operator $\mathcal{H}$ is strongly consistent with respect to (u, x_0) and

$$\varphi^\circ_u(\varrho) = \sigma_0 e^{\frac{-\sigma_0\varrho}{g(0)}}\Big[\frac{\varphi^\circ_u(0)}{\sigma_0} + \int_0^\varrho e^{\frac{\sigma_0\tau}{g(0)}} \cdot \dot{\psi}_u(\tau)\, d\tau\Big] + f(0)$$
$$\forall\varrho \in [0, \rho_u(T)]. \tag{11}$$

Suppose now that the input u is wave periodic. Denote $u_{\max} = u(T^+)$ and $u_{\min} = u(0)$. Then, $\rho_u(T^+) = u_{\max} - u_{\min}$, $\rho_u(T) = 2(u_{\max} - u_{\min})$. Define $c = \frac{\sigma_0(u_{\max}-u_{\min})}{g(0)}$, $\alpha = \frac{2}{1+e^{-c}}$. Then

$$\varphi^\circ_u(\varrho) = g(0)\big[1 - \alpha e^{-\frac{\sigma_0\varrho}{g(0)}}\big] + f(0)$$
$$\forall\varrho \in \big[0, \rho_u(T^+)\big] \tag{12}$$
$$\varphi^\circ_u(\varrho) = g(0)\big[-1 + \alpha e^{\frac{-\sigma_0\left(\varrho-\rho_u(T^+)\right)}{g(0)}}\big] + f(0)$$
$$\forall\varrho \in \big[\rho_u(T^+), \rho_u(T)\big]. \tag{13}$$

PROOF. Consider the linear system with input $\dot{\psi}_u$ and output $h \in W^{1,\infty}(I_u,\mathbb{R})$.

$$\dot{h}(\varrho) = -\frac{\sigma_0}{g(0)}h(\varrho) + \dot{\psi}_u(\varrho), \forall\varrho \in I_u \tag{14}$$
$$h(0) = x_0, \tag{15}$$

The solution of (14)-(15) is

$$h(\varrho) = e^{\frac{-\sigma_0\varrho}{g(0)}}\Big[x_0 + \int_0^\varrho e^{\frac{\sigma_0\tau}{g(0)}}\dot{\psi}_u(\tau)\, d\tau\Big], \forall\varrho \in I_u. \tag{16}$$

Define $W = h^2$. From (14)-(15) it follows that $W(0) = x_0^2$ and $\dot{W} = -\frac{2\sigma_0}{g(0)}W + 2\,\dot{\psi}_u h$. By Lemma 1 we have $\|\dot{\psi}_u\|_{\infty,I_u} = 1$ so that $\dot{W} \leq -\frac{2\sigma_0}{g(0)}W + 2\sqrt{W}$. This leads to $\dot{W} \leq 0$, whenever $W > \big(\frac{g(0)}{\sigma_0}\big)^2$. Using [9, Lemma 17] it comes that $W \leq \max\big(x_0^2, \big(\frac{g(0)}{\sigma_0}\big)^2\big)$ so that $|h(\varrho)| \leq \max\big(|x_0|, \frac{g(0)}{\sigma_0}\big), \forall\varrho \in I_u$. We now claim that $\lim_{\gamma\to\infty}\|\chi_\gamma\|_{\infty,I_u} = 0$, where $\chi_\gamma : I_u \to \mathbb{R}$ is defined a.e. as $\chi_\gamma(\varrho) = \frac{1}{g(\frac{v_u(\varrho)}{\gamma})} - \frac{1}{g(0)}$. Indeed, let $\varepsilon > 0$. Since g is continuous and non-zero, we have $\lim_{\vartheta\to 0}\frac{1}{g(\vartheta)} = \frac{1}{g(0)}$.

Hence there exists some $\delta_\varepsilon > 0$ that depend solely on ε, such that $|\frac{1}{g(\vartheta)} - \frac{1}{g(0)}| < \varepsilon$, whenever $\vartheta \in (-\delta_\varepsilon, \delta_\varepsilon)$. By Lemma 4 we have $\|v_u\|_{\infty,I_u} \leq \|\dot{u}\|_\infty$. Thus there exists $\gamma_\star > 0$ such that $\frac{\|v_u\|_{\infty,I_u}}{\gamma} < \delta_\varepsilon, \forall\gamma > \gamma_\star$. Thus $|\frac{1}{g\left(\frac{v_u(\varrho)}{\gamma}\right)} - \frac{1}{g(0)}| < \varepsilon$, for almost all $\varrho \in I_u, \forall\gamma > \gamma_\star$. Our claim is thus proved.

For any $\gamma \geq 1$, define $y_\gamma : I_u \to \mathbb{R}$ as $y_\gamma = x_{u\circ s_\gamma} - h$. Since $x_{u\circ s_\gamma}(0) = h(0) = x_0$, we have for all $\gamma \geq 1$ and for almost all $\varrho \in I_u$ that

$$\dot{y}_\gamma(\varrho) = \frac{-\sigma_0}{g(\frac{v_u(\varrho)}{\gamma})} y_\gamma(\varrho) - \sigma_0\, \chi_\gamma(\varrho)\, h(\varrho), \tag{17}$$

$$y_\gamma(0) = 0. \tag{18}$$

where we have used equations (7) and (14). Consider the Lyapunov function $V_\gamma : I_u \to \mathbb{R}_+$ defined by $V_\gamma(\varrho) = y_\gamma^2(\varrho), \forall\varrho \in I_u$. By (17) we have for all $\gamma \geq 1$ and for almost all $\varrho \in I_u$ that

$$\dot{V}_\gamma(\varrho) \leq -2\frac{\sigma_0}{g(\frac{v_u(\varrho)}{\gamma})} V_\gamma(\varrho) + D_1 \|\chi_\gamma\|_{\infty,I_u} \sqrt{V_\gamma(\varrho)}, \tag{19}$$

where $D_1 = 2\sigma_0 \|h\|_{\infty,I_u} > 0$. On the other hand, since v_u is essentially bounded and the function g is continuous and positive, there exists $M > 0$ such that

$$g(v_u(\varrho)/\gamma) > M, \text{ for almost all } \varrho \in I_u, \forall\gamma \geq 1. \tag{20}$$

Thus, we obtain from (19) that

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq -\frac{2\sigma_0}{M} V_\gamma(\varrho) + D_1 \|\chi_\gamma\|_{\infty,I_u} \sqrt{V_\gamma(\varrho)} \\ \forall\gamma \geq 1, \text{ for almost all } \varrho \in I_u, \end{cases}$$

so that

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq 0, \text{ for almost all } \varrho \in I_u, \forall\gamma \geq 1 \\ \text{that satisfy } V_\gamma(\varrho) > \left(\frac{D_1 M \|\chi_\gamma\|_{\infty,I_u}}{2\sigma_0}\right)^2. \end{cases} \tag{21}$$

The fact that $V_\gamma(0) = 0$ along with [9, Lemma 17] imply that $V_\gamma(\varrho) \leq \left(\frac{D_1 M \|\chi_\gamma\|_{\infty,I_u}}{2\sigma_0}\right)^2$ for all $\gamma \geq 1$ and all $\varrho \in I_u$. Hence $|y_\gamma(\varrho)| = \left|x_{u\circ s_\gamma}(\varrho) - h(\varrho)\right| \leq \frac{MD_1}{2\sigma_0} \|\chi_\gamma\|_{\infty,I_u}$. Thus we conclude from the previous claim that

$$\lim_{\gamma\to\infty} \|y_\gamma\|_{\infty,I_u} = \lim_{\gamma\to\infty} \left\|x_{u\circ s_\gamma} - h\right\|_{\infty,I_u} = 0. \tag{22}$$

On the other hand, it follows from the continuity of x_γ that $\left\|x_{u\circ s_\gamma}\right\|_{\infty,I_u} = \|x_\gamma\|_\infty, \forall\gamma \geq 1$ (see Lemma 4). Thus (22) implies that there exists $\gamma_1 \geq 1$ such that $\|x_\gamma\|_\infty = \left\|x_{u\circ s_\gamma}\right\|_{\infty,I_u} < 1 + \|h\|_{\infty,I_u} = E_1, \forall\gamma > \gamma_1$. Hence, we get from equation (4), inequality (20) and the essential boundedness of $\dot{u}$, that, for all $\gamma > \gamma_1$, the function $\dot{x}_\gamma$ is essentially bounded by a number that does not depend on γ. Therefore, by (6) and the continuity of f, there exists some $E > 0$ such that $\|F_\gamma\|_\infty \leq E, \forall\gamma > \gamma_1$. On the other hand, by (17), inequality (20) and the boundedness of h, there exist $D_2 > 0$ and $D_3 > 0$ such that for all $\gamma > \gamma_1$ we have $\|\dot{y}_\gamma\|_{\infty,I_u} \leq D_2 \|y_\gamma\|_{\infty,I_u} + D_3 \|\chi_\gamma\|_{\infty,I_u}$. Thus (22) implies that $\lim_{\gamma\to\infty} \|\dot{y}_\gamma\|_{\infty,I_u} = \lim_{\gamma\to\infty} \|\dot{x}_{u\circ s_\gamma} - \dot{h}\|_{\infty,I_u} = 0$. This means that $x_{u\circ s_\gamma}$ converges to h in $W^{1,\infty}(I_u, \mathbb{R})$ as $\gamma \to \infty$ because of (22). Hence, (9), the essential boundedness of v_u and the continuity of f imply that $\lim_{\gamma\to\infty} \left\|\varphi_{u\circ s_\gamma} - \varphi_u^\star\right\|_{\infty,I_u} = 0$, where $\varphi_u^\star = \sigma_0\, h + f(0) \in W^{1,\infty}(I_u, \mathbb{R})$ since $h \in W^{1,\infty}(I_u, \mathbb{R})$. This fact proves the first part of Theorem 9.

Now, assume that the input u is T-periodic, then ψ_u and $\dot{\psi}_u$ are $\rho_u(T)$-periodic (see Lemma 7). For any integer $k \geq 0$, let $h_k(\varrho) = h(\rho_u(T)k + \varrho), \forall\varrho \in [0, \rho_u(T)]$. The periodicity of $\dot{\psi}_u$ implies that the function h_k satisfies (14) for each k, with initial condition $h_k(0) = h(\rho_u(T)k)$. Let $k_1, k_2 \geq 0$ be integers. Let $V_{k_1,k_2} = (h_{k_1} - h_{k_2})^2$, then we have $\dot{V}_{k_1,k_2} = \frac{-2\sigma_0}{g(0)} V_{k_1,k_2}$, so that

$$V_{k_1,k_2}(\varrho) = V_{k_1,k_2}(0)\, e^{\frac{-2\sigma_0}{g(0)}\varrho}, \forall\varrho \in [0, \rho_u(T)]. \tag{23}$$

Therefore, we get $V_{k_1,k_2}(\rho_u(T)) = V_{k_1,k_2}(0)\,\beta$, where $\beta = e^{\frac{-2\rho_u(T)\sigma_0}{g(0)}} \in (0,1)$. Observe that $V_{k_1,k_2}(\rho_u(T)) = V_{k_1+1,k_2+1}(0)$ so that $V_{k_1+1,k_2+1}(0) = \beta V_{k_1,k_2}(0)$. Therefore, it can be verified by induction that $V_{k_1,k_2}(0) \leq \beta^{\min\{k_1,k_2\}} D_4$, where D_4 is a positive constant that depends on $\|h\|_\infty$ and x_0. This means that $V_{k_1,k_2}(0)$ converges to 0 as k_1 and k_2 go to ∞. Thus, we obtain from (23) that $\|V_{k_1,k_2}\|_{\infty,[0,\rho_u(T)]} \to 0$, as $k_1, k_2 \to \infty$, which means that the sequence of functions h_k is a Cauchy sequence in the Banach space $C^0([0, \rho_u(T)], \mathbb{R})$. This implies that the sequence h_k converges with respect to the norm $\|\cdot\|_{\infty,[0,\rho_u(T)]}$ to a continuous function h_∞. By applying the Dominated Lebesgue Convergence Theorem in equation (14) written in the integral form, we deduce that h_∞ satisfies the same equation (14). This fact implies that $h_\infty \in W^{1,\infty}([0, \rho_u(T)], \mathbb{R})$. Observe that $h_\infty(0)$ may be different than x_0.

Since $\varphi_u^\star = \sigma_0\, h + f(0)$ it follows that $\varphi_{u,k}^\star = \sigma_0\, h_k + f(0)$, thus defining $\varphi_u^\circ \in W^{1,\infty}([0, \rho_u(T)], \mathbb{R})$ by the relation $\varphi_u^\circ(\varrho) = \sigma_0\, h_\infty(\varrho) + f(0), \forall\varrho \in [0, \rho_u(T)]$ we obtain $\lim_{k\to\infty} \|\varphi_{u,k}^\star - \varphi_u^\circ\|_{\infty,[0,\rho_u(T)]} = 0$, which ends the proof of the second part of Theorem 9.

If the input u is wave periodic, Lemma 7 states that ψ_u is also wave periodic, $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$. Note that $\rho_u(T^+) = u(T^+) - u(0)$ and $\rho_u(T) = 2(u(T^+) - u(0))$. Denote $u_{\max} = u(T^+)$ the largest value of u and $u_{\min} = u(0)$ the smallest value of u. Then $\rho_u(T^+) = u_{\max} - u_{\min}$ and

$\rho_u(T) = 2(u_{\max} - u_{\min})$. Define $c = \frac{\sigma_0(u_{\max}-u_{\min})}{g(0)} > 0$ and $a = \frac{g(0)}{\sigma_0}\left(2e^c - 1 - e^{2c}\right) = -\frac{g(0)}{\sigma_0}\left(e^c - 1\right)^2$. Then, from (16) we get $h(\rho_u(T)k) = e^{-2ck}[x_0 + R_k], \forall k \in \mathbb{N}$ where

$$
\begin{aligned}
R_k &= \sum_{j=1}^{k}\left[\int_{\rho_u(T)(j-1)}^{\rho_u(T)(j-1)+\rho_u(T^+)} e^{\frac{\sigma_0\tau}{g(0)}}d\tau - \int_{\rho_u(T)(j-1)+\rho_u(T^+)}^{\rho_u(T)j} e^{\frac{\sigma_0\tau}{g(0)}}d\tau\right] \\
&= a\sum_{j=1}^{k} e^{2c(j-1)} = a\frac{1-e^{2ck}}{1-e^{2c}} \qquad (24)
\end{aligned}
$$

From (24) we obtain $h(\rho_u(T)k) = e^{-2ck}x_0 + a\frac{e^{-2ck}-1}{1-e^{2c}}$. Therefore $h_\infty(0) = \lim_{k\to\infty} h(\rho_u(T)k) = \frac{a}{(e^c-1)(e^c+1)}$ We have seen that h_∞ satisfies equation (14), thus $\forall\varrho \in [0, \rho_u(T^+)]$ we have $h_\infty(\varrho) = e^{\frac{-\sigma_0\varrho}{g(0)}}\left[h_\infty(0) + \int_0^\varrho e^{\frac{\sigma_0\tau}{g(0)}}d\tau\right]$ and $\forall\varrho \in [\rho_u(T^+), \rho_u(T)]$ we have $h_\infty(\varrho) = e^{\frac{-\sigma_0\varrho}{g(0)}}\left[h_\infty(0) + \int_0^{\rho_u(T^+)} e^{\frac{\sigma_0\tau}{g(0)}}d\tau - \int_{\rho_u(T^+)}^{\varrho} e^{\frac{\sigma_0\tau}{g(0)}}d\tau\right]$. The last part of Theorem 9 follows from the relation $\varphi_u^\circ(\varrho) = \sigma_0 h_\infty(\varrho) + f(0), \forall\varrho \in [0, \rho_u(T)]$.

5 Comments on Theorem 9

Comment 1. When the input u is wave periodic, Lemma 7 states that ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$. Thus we get $\psi_u(\varrho) = \varrho + u_{\min}$ for all $\varrho \in [0, u_{\max} - u_{\min}]$ and $\psi_u(\varrho) = -\varrho + 2u_{\max} - u_{\min}$ for all $\varrho \in [u_{\max} - u_{\min}, 2(u_{\max} - u_{\min})]$. To obtain the equations of the hysteresis loop we have to write $\varphi_u^\circ(\varrho)$ as a function of $\psi_u(\varrho)$. From equations (12)-(13) we get

$$
\begin{aligned}
&\varphi_u^\circ(\varrho) = g(0)\left[1 - \alpha e^{-\frac{\sigma_0}{g(0)}(\psi_u(\varrho)-u_{\min})}\right] + f(0) \\
&\forall\varrho \in [0, u_{\max} - u_{\min}] \qquad (25) \\
&\varphi_u^\circ(\varrho) = g(0)\left[-1 + \alpha e^{-\frac{\sigma_0}{g(0)}(u_{\max}-\psi_u(\varrho))}\right] + f(0) \\
&\forall\varrho \in [u_{\max} - u_{\min}, 2(u_{\max} - u_{\min})] \qquad (26)
\end{aligned}
$$

Since u is wave periodic, $\forall\tau \in [0, T^+], \rho_u(\tau) = u(\tau) - u_{\min}$ and $\forall\tau \in [T^+, T], \rho_u(\tau) = -u(\tau) + 2u_{\max} - u_{\min}$. Observe that ρ_u is increasing. Consider the change of variable $\varrho = \rho_u(\tau)$, then we get

$$
\begin{aligned}
&F^\circ(\tau) = g(0)\left[1 - \alpha e^{-\frac{\sigma_0}{g(0)}(u(\tau)-u_{\min})}\right] + f(0) \\
&\forall\tau \in [0, T^+] \qquad (27) \\
&F^\circ(\tau) = g(0)\left[-1 + \alpha e^{-\frac{\sigma_0}{g(0)}(u_{\max}-u(\tau))}\right] + f(0) \\
&\forall\tau \in [T^+, T] \qquad (28)
\end{aligned}
$$

where $F^\circ \in W^{1,\infty}([0,T],\mathbb{R})$ is defined by the relation $F^\circ = \varphi_u^\circ \circ \rho_u$. Equation (27) corresponds to the so-called loading (u increasing). By an abuse of notation, it is written as

$$
F_\uparrow^\circ(u) = g(0)\left[1 - \alpha e^{-\frac{\sigma_0}{g(0)}(u-u_{\min})}\right] + f(0) \qquad (29)
$$

in which the subscript of the output force $F_\uparrow^\circ$ refers to loading and the superscript refers to the fact that it is obtained in "steady-state". In equation (29) the variable u is a real number that belongs to the interval $[u_{\min}, u_{\max}]$ (this is the abuse of notation), and $F_\uparrow^\circ(u)$ is the value given by the right-hand side of equation (29). This means that $F_\uparrow^\circ$ is a function defined from the interval $[u_{\min}, u_{\max}]$ to $\mathbb{R}$.
Equation (28) corresponds to the so-called unloading (u decreasing). Similarly, it is written as

$$
F_\downarrow^\circ(u) = g(0)\left[-1 + \alpha e^{-\frac{\sigma_0}{g(0)}(u_{\max}-u)}\right] + f(0) \qquad (30)
$$

(29)-(30) constitute the equations of the hysteresis loop of the LuGre model.
Comment 2. Let $u_1 \in [\frac{u_{\max}+u_{\min}}{2}, u_{\max}]$, $u_2 \in [u_{\min}, \frac{u_{\max}+u_{\min}}{2}]$ be real numbers such that $u_1 - \frac{u_{\max}+u_{\min}}{2} = \frac{u_{\max}+u_{\min}}{2} - u_2$. Then it can be checked that $F_\uparrow^\circ(u_1) - f(0) = -(F_\downarrow^\circ(u_2) - f(0))$. This means that the hysteresis loop is symmetric with respect to the point $(\frac{u_{\max}+u_{\min}}{2}, f(0))$.

6 Numerical simulations

Consider the LuGre model (1)-(3) with the values that are taken from [2, p. 105]. We choose $f(\vartheta) = \sigma_2\vartheta, \forall\vartheta \in \mathbb{R}$, where the parameter σ_2 is the viscous friction coefficient. A possible choice for $g(\vartheta)$ is $g(\vartheta) = F_c + (F_s - F_c)e^{-|\vartheta/v_s|^\beta}, \forall\vartheta \in \mathbb{R}$, where $F_c > 0$ is the Coulomb friction force, $F_s > 0$ is the stiction force, $v_s > 0$ is the Stribeck velocity, and β is a positive constant. The values of the differents constants are: $\sigma_0 = 1.47 \cdot 10^6$ N/m, $\sigma_1 = 2.42 \cdot 10^3$ kg/s, $\sigma_2 = 0$ kg/s, $F_c = 2.94$ N, $F_s = 5.88$ N, $v_s = 0.001$ m/s, $u_{\min} = 0$ m, $u_{\max} = 6 \cdot 10^{-6}$ m, $\beta = 1$. We take $x(0) = x_0 = 0$ m.
Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be the wave periodic function of period $T = 2\pi$ (seconds), with $T^+ = \pi$ (seconds), such that $u(t) = U_o - U\cos(t)$ (in meters), $\forall t \in [0, T]$, where $U = \frac{u_{\max}-u_{\min}}{2}$, $U_o = u_{\min} + U$ and time $t \geq 0$. The displacement input u is shown in Figure 1 for $t \in [0, 3T]$. Theorem 9 shows that $\mathcal{H}$ is consistent with respect to (u, x_0); that is $\lim_{\gamma\to\infty}\left\|\varphi_{u\circ s_\gamma} - \varphi_u^\star\right\|_\infty = 0$, where $\varphi_u^\star$ is given by equation (10). Figure 2 shows $\varphi_{u\circ s_\gamma}(\varrho)$ versus ϱ for $\gamma = 0.005$, $\gamma = 0.01$ and $\gamma = 0.1$ (in grey). These plots have been obtained by solving the differential equation (7)-(8) using Matlab solver ode23s and substituting in equation (9). It can be seen that the plot that corresponds to $\gamma = 0.1$ is close to that of $\varphi_u^\star(\varrho)$ versus ϱ (in black). The latter plot has been calculated using equation (10). Theorem 9 also shows that $\mathcal{H}$ is strongly consistent with respect to (u, x_0);

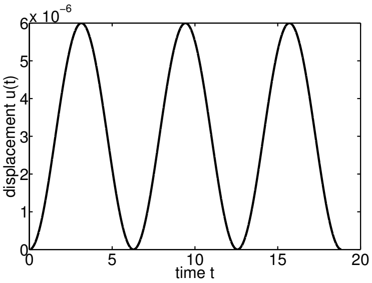

Figure 1. $u(t)$ versus t.

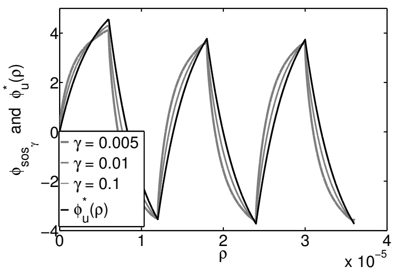

Figure 2. Grey: $\varphi_{u\circ s_\gamma}(\varrho)$ versus ϱ. Black: $\varphi_u^\star(\varrho)$ versus ϱ.

that is $\lim_{k\to\infty}\|\varphi_{u,k}^\star - \varphi_u^\circ\|_{\infty,[0,\rho_u(T)]} = 0$, where φ_u° is given by equations (12)-(13). Figure 3 gives $\varphi_{u,k}^\star(\varrho)$ versus ϱ for $k = 0$, $k = 1$ and $k = 2$ (in grey). These plots have been calculated from equation (10). Observe that the plot that corresponds to $k = 2$ is close to that of $\varphi_u^\circ(\varrho)$ versus ϱ (in black). The latter plot has been calculated using equations (12)-(13). Figure 4 gives the

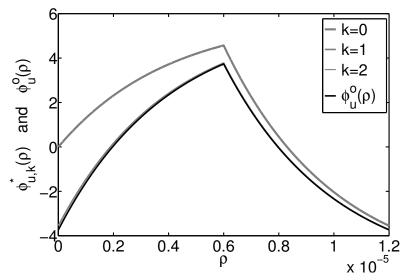

Figure 3. Grey: $\varphi_{u,k}^\star(\varrho)$ versus ϱ. Black: $\varphi_u^\circ(\varrho)$ versus ϱ.

hysteresis loop $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$. This same loop can be obtained in its loading part as $F_\uparrow^\circ(u)$ versus u (Figure 4 grey plot), and in its unloading part as $F_\downarrow^\circ(u)$ versus u (Figure 4 black plot). The symmetry property of this hysteresis loop is illustrated in Figure 4 where the marker in the center of the figure corresponds to the point $\left(\frac{u_{\max}+u_{\min}}{2}, f(0)\right)$.

7 Conclusion

An explicit expression for the hysteresis loop of the LuGre model is provided through the concepts of consistency and strong consistency.

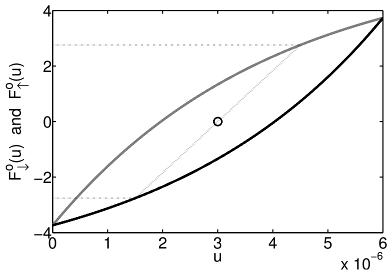

Figure 4. Grey: $F_\uparrow^\circ(u)$ versus u. Black: $F_\downarrow^\circ(u)$ versus u.

Acknowledgment

Supported by grant DPI2011-25822 of the Spanish Ministry of Economy and Competitiveness and by FI-DGR-grant 2012FI-B2 00257 of Agency for Management of University and Research Grants (AGAUR).

References

[1] Armstrong-Hélouvry, B., Dupont, P., & Canudas De Wit, C. (1994). A survey of models, analysis tools and compensation methods for the control of machines with friction. *Automatica*, 30(7), 1083-1138.

[2] Åström, K., Canudas-de-Wit, C. (2008). Revisiting the LuGre friction model. *IEEE Control Systems Magazine*, 28(6), 101–114.

[3] Barahanov, N., & Ortega, R. (2000). Necessary and sufficient conditions for passivity of the LuGre friction model. *IEEE Transactions on Automatic Control*, 45(4), 830–832.

[4] Canudas de Wit, C., Olsson, H., Åström, K., & Lischinsky, P. (1995). A new model for control of systems with friction. *IEEE Transactions on Automatic Control*, 40(3), 419-425.

[5] Dahl, P. (1976). Solid friction damping of mechanical vibrations. *AIAA J.*, 14(2), 1675–1682.

[6] Filippov, A. F. (1988). *Differential equations with discontinuous right-hand sides.* Kluwer.

[7] Freidovich, L., Robertsson, A., Shiriaev, A., & Johansson, R. (2010). LuGre-model-based friction compensation. *IEEE Transactions on Control Systems Technology*, 18(1), 194-200.

[8] Ikhouane, F. (2013). Characterization of hysteresis processes. *Mathematics of Control, Signals, and Systems.* 25, 294-310.

[9] Naser, M. F. M., & Ikhouane, F. (2013). Consistency of the Duhem model with hysteresis. *Mathematical Problems in Engeering*, 2013, Article ID 586130.

[10] Padthe, A. K., Drincic, B., Oh, J., Rizos, D. D., Fassois, S. D., & Berstein, D. S. (2008). Duhem modeling of friction-induced hysteresis. *IEEE Control Systems Magazine*, 28(5), 90-107.

[11] San, P. P., Ren, B., Ge, S. S., Lee, T. H., & Liu, J.-K. (2011). Adaptive neural network control of hard disk drives with hysteresis friction nonlinearity. *IEEE Transactions on Control Systems Technology*, 19(2), 351-358.

[12] Swevers, J., Al-Bender, F., Ganseman, C. G., & Projogo, T. (2000). An integrated friction model structure with improved presliding behavior for accurate friction compensation. *IEEE Transactions on Automatic Control*, 45(4), 675-686.

[13] Visintin, A. (1994). *Differential models of hysteresis.* Springer-Verlag, Berlin, Heidelberg.