On Potential Equations of Finite Games \star

Xinyun Liu, Jiandong Zhu

School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, P. R. China

Abstract

In this paper, some new criteria for detecting whether a finite game is potential are proposed by solving potential equations. The verification equations with the minimal number for checking a potential game are obtained for the first time. Some connections between the potential equations and the existing characterizations of potential games are established. It is revealed that a finite game is potential if and only if its every bi-matrix sub-game is potential.

Key words: Finite game, Potential game, Potential equation, Semi-tensor product.

1 Introduction

Game theory, the science of strategic decision making pioneered by John von Neumann (see von Neumann, J. & Morgenstern, O., 1953), has wide real-world applications in many fields, including economics, biology, computer science and engineering. The Nash equilibrium, named after John Forbes Nash, Jr, is a fundamental concept in game theory. The existence and computing of Nash equilibria are two central problems in the theory of games. For two-player zero-sum games, von Neumann proved the existence of mixed-strategy equilibria using Brouwer Fixed Point Theorem. Nash proved that if mixed strategies are allowed, then every game with a finite number of players and strategies has at least one Nash equilibrium (Nash, 1951). Although pure strategies are conceptually simpler than mixed strategies, it is usually difficult to guarantee the existence of a pure-strategy equilibrium. However, it is shown that every finite potential game possesses a pure-strategy Nash equilibrium (Monderer & Shapley, 1996). The concept of potential game was proposed by Rosenthal (1973). A game is said to be a potential game if it admits a potential function. The incentive of all players to change their strategy can be expressed by the difference in values of the potential function. For a potential game, the set of pure-strategy Nash equilibria can be found by searching the maximal values of the potential function.

An important problem is how to check whether a game is a potential game. Monderer and Shapley (1996) first proposed necessary and sufficient conditions for potential games. But it is required to verify all the simple closed paths with length 4 for any pair of players. Then Hino (2011) gave an improved condition for detecting potential games, which has a lower complexity than that of Monderer and Shapley (1996) due to that only the adjacent pairs of strategies of two players need to check. In Ui (2000), it is proved that a game is potential if and only if the payoff functions coincide with the Shapley value of a particular class of cooperative games indexed by the set of strategy profiles. Game decomposition is an important method for potential games (Candogan, Menache, Ozdaglar, & Parrilo, 2011; Hwang, & Rey-Bellet 2011; Sandholm, 2010) and some new necessary and sufficient conditions for detecting potential games are obtained. Sandholm (2010) established connections between his results and that in Ui (2000). But the number of the obtained verification equations is not the minimum. In Sandholm (2010), it is proved that a finite game is a potential game if and only if, in each of the component games, all active players have identical payoff functions, and that in this case, the potential function can be constructed.

Recently, Cheng (2014) developed a novel method, based on the semi-tensor product of matrices, to deal with games including potential games, networked games and evolutionary games (Cheng, 2014; Cheng, Xu & Qi, 2014; Cheng, Xu, He, Qi, 2014; Cheng, He, Qi, & Xu, 2015; Guo, Wang, & Li, 2013). In Cheng (2014), a lin-

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Email addresses: liuxinyun1224@163.com (Xinyun Liu), jiandongzhu@njnu.edu.cn (Jiandong Zhu).

ear system, called potential equation, is proposed, and then it is proved that the game is potential if and only if the potential equation is solvable. With a solution of the potential equation, the potential function can be directly calculated.

A natural question is how to establish the connection between the potential equation and the other criteria of potential games. Moreover, an interesting problem is how to get the verification equations with the minimum number. In this paper, we further investigate the solvability of the potential equation. An equivalence transformation is constructed to convert the augmented matrix of the potential equation into the reduced row echelon form. Based on this technique, some new necessary and sufficient conditions for potential games are obtained. For potential games, a new formula to calculate the potential functions is proposed. Based on the obtained results, it is revealed the connection between the potential equation and the results in Hino (2011) and Sandholm (2010).

Throughout the paper, we denote the $k \times k$ identity matrix by I_k , the *i*-th column of I_k by δ_k^i , the *n*-dimensional column vector whose entries are all equal to 1 by $\mathbf{1}_k$, Kronecker product by \otimes and the real number field by \mathbb{R} .

Preliminaries $\mathbf{2}$

Definition 1 (Monderer & Shapley 1996) A finite game is a triple $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$, where

(i) $\mathcal{N} = \{1, 2, \cdots, n\}$ is the set of players; (ii) $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ is the strategy set, where each $\mathcal{S}_i = \{s_1^i, s_2^i, \cdots, s_{k_i}^i\}$ is the strategy set of player *i*; (iii) $C = \{c_1, c_2, \cdots, c_n\}$ is the set of payoff functions, where every $c_i : S \to \mathbb{R}$ is the payoff function of player *i*.

Let $c_{i_1i_2\cdots i_n}^{\mu} = c_{\mu}(s_{i_1}^1, s_{i_2}^2, \cdots, s_{i_n}^n)$ where $1 \leq i_s \leq k_s$ and $s = 1, 2, \cdots, n$. Then the finite game can be described by the arrays

$$C_{\mu} = \{ c^{\mu}_{i_1 i_2 \cdots i_n} | \ 1 \le i_s \le k_s, \ s = 1, 2, \cdots, n \}$$
(1)

with $\mu = 1, 2, \dots, n$. Particularly, for a 2-player game, the $k_1 \times k_2$ matrices $C_1 = (c_{ij}^1)$ and $C_2 = (c_{ij}^2)$ are payoffs of players 1 and 2 respectively. Therefore, a 2player finite game is also called a *bi-matrix game*, which is usually denoted by the simple notation $\mathcal{G} = (C_1, C_2)$.

Definition 2 (Monderer & Shapley 1996) A finite game $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ is said to be *potential* if there exists a function $p : \mathcal{S} \to \mathbb{R}$, called the *potential function*, such that $c_i(x, s^{-i}) - c_i(y, s^{-i}) = p(x, s^{-i}) - p(y, s^{-i})$ for all $x, y \in \mathcal{S}_i, s^{-i} \in \mathcal{S}^{-i}$ and $i = 1, 2, \cdots, n$, where $\mathcal{S}^{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_n$.

Definition 3 (*Cheng, Qi, & Li, 2011*). Assume $A \in$ $\mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}$. Let $\alpha = \operatorname{lcm}(n, p)$ be the least common multiple of n and p. The left semi-tensor product of A and B is defined as $A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{n}})$.

Since the left semi-tensor product is a generalization of the traditional matrix product, the left semi-tensor product $A \ltimes B$ can be directly written as AB. Identifying each strategy s_j^i with the logical vector $\delta_{k_i}^j$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k_i$, Cheng (2014) gave a new expression of the payoff functions using the left semi-tensor product.

Lemma 4 (Cheng, 2014) Let $x_i \in S_i$ be any strategy expressed in the form of logical vectors. Then, for any payoff function c_i of a finite game \mathcal{G} shown in Definition 1, there exists a unique row vector $V_i^c \in \mathbb{R}^n$ such that

$$c_i(x_1, x_2, \cdots, x_n) = V_i^c x_1 x_2 \cdots x_n, \tag{2}$$

where V_i^c is called the structure vector of c_i and $i = 1, 2, \cdots, n$.

Remark 5 It is easy to see that V_i^c is just the row vector composed of the elements of C_i in the lexicographic order (see (1)). Let $C = [(V_1^c)^T, (V_2^c)^T, \cdots, (V_n^c)^T]^T$. Then C is just the *payoff matrix* of \mathcal{G} proposed by Cheng (2014).

Without loss of generality, we assume $k_i = k$ for all $i = 1, 2, \dots, n$. In Cheng (2014), the potential equation is proposed as follows:

$$\Psi \xi = b, \tag{3}$$

where

$$\Psi = \begin{bmatrix} -\Psi_1 & \Psi_2 & & \\ -\Psi_1 & \Psi_3 & & \\ \vdots & & \ddots & \\ -\Psi_1 & & & \Psi_n \end{bmatrix}, \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}, b = \begin{bmatrix} (V_2^c - V_1^c)^T \\ (V_3^c - V_1^c)^T \\ \vdots \\ (V_n^c - V_1^c)^T \end{bmatrix}$$
(4)

and $\Psi_i = I_{k^{i-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-i}}$ for each $i = 1, 2, \cdots, n$.

Lemma 6 (Cheng, 2014) A finite game \mathcal{G} shown in Definition 1 is a potential game if and only if the potential equation (3) has a solution ξ . Moreover, as (3) holds, the potential function p can be calculated by

$$(V^p)^{\mathrm{T}} = (V_1^c)^{\mathrm{T}} - (\mathbf{1}_k \otimes I_{k^{n-1}})\xi_1.$$
 (5)

Bi-matrix games 3

In this section, we consider the 2-player finite game $\mathcal{G} = (C_1, C_2)$, where $C_i \in \mathbb{R}^{k_1 \times k_2}$ for i = 1, 2. In this special case, the coefficients of the potential equation (3) become

$$\Psi = \begin{bmatrix} -\mathbf{1}_{k_1} \otimes I_{k_2} & I_{k_1} \otimes \mathbf{1}_{k_2} \end{bmatrix}, \quad b = (V_2^c - V_1^c)^{\mathrm{T}}.$$
 (6)

Before the main results of this section, we first introduce a basic property on Kronecker product.

Lemma 7 (Horn, 1994) Let $V_r(X)$ denotes the vectorization of the matrix X formed by stacking the rows of Xinto a single column vector. Then

$$\mathbf{V}_{\mathbf{r}}(ABC) = (A \otimes C^{\mathrm{T}})\mathbf{V}_{\mathbf{r}}(B).$$
(7)

In particular, we have

$$V_{\rm r}(AB) = (A \otimes I)V_{\rm r}(B). \tag{8}$$

Theorem 8 Set $B_k = [I_{k-1}, -\mathbf{1}_{k-1}] \in \mathbb{R}^{(k-1) \times k}$. The bi-matrix game $\mathcal{G} = (C_1, C_2)$ is potential if and only if

$$B_{k_1}(C_2 - C_1)B_{k_2}^{\mathrm{T}} = 0, \qquad (9)$$

where $C_1, C_2 \in \mathbb{R}^{k_1 \times k_2}$. Moreover, as (9) holds, the matrix form of each potential function is

$$P = C_1 + [0_{k_1 \times (k_1 - 1)} \ \mathbf{1}_{k_1}](C_2 - C_1) + \lambda \mathbf{1}_{k_1} \mathbf{1}_{k_2}^{\mathrm{T}}, \quad (10)$$

where $\lambda \in \mathbb{R}$ is an arbitrary number.

Proof. Let $D_k = [I_{k-1}, 0] \in \mathbb{R}^{(k-1) \times k}$. Then it is easy to check that

$$B_k D_k^{\mathrm{T}} = I_{k-1}, \quad D_k \delta_k^k = B_k \mathbf{1}_k = 0.$$
 (11)

Let

$$E = \begin{bmatrix} -\delta_{k_1}^{k_1} \otimes I_{k_2}, \ B_{k_1}^{\mathrm{T}} \otimes \delta_{k_2}^{k_2}, \ B_{k_1}^{\mathrm{T}} \otimes B_{k_2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{k_1 k_2 \times k_1 k_2},$$
(12)
$$F = \begin{bmatrix} -\mathbf{1}_{k_1} \otimes I_{k_2}, \ D_{k_1}^{\mathrm{T}} \otimes \mathbf{1}_{k_2}, \ D_{k_1}^{\mathrm{T}} \otimes D_{k_2}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{k_1 k_2 \times k_1 k_2}.$$
(13)

By (11), a straightforward computation shows that

$$EF = I_{k_1k_2}.\tag{14}$$

Eq. (14) shows that E is nonsingular. So the potential equation is equivalent to

$$E\Psi\xi = Eb. \tag{15}$$

With simple calculations, we have

$$\begin{split} E[\Psi \ b] &= \begin{bmatrix} -(\delta_{k_1}^{k_1})^{\mathrm{T} \otimes I_{k_2}} \\ B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}} \\ B_{k_1} \otimes B_{k_2} \end{bmatrix} \begin{bmatrix} -\mathbf{1}_{k_1} \otimes I_{k_2} & I_{k_1} \otimes \mathbf{1}_{k_2} & b \end{bmatrix} \\ &= \begin{bmatrix} I_{k_2} \ -(\delta_{k_1}^{k_1})^{\mathrm{T} \otimes \mathbf{1}_{k_2}} & -((\delta_{k_1}^{k_1})^{\mathrm{T} \otimes I_{k_2}})b \\ 0 & B_{k_1} & (B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}})b \\ 0 & 0 & (B_{k_1} \otimes B_{k_2})b \end{bmatrix} \end{split}$$

$$= \begin{bmatrix} I_{k_2} & 0 & -\mathbf{1}_{k_2} & -((\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2})b \\ 0 & I_{k_1-1} & -\mathbf{1}_{k_1-1} & (B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}})b \\ 0 & 0 & 0 & (B_{k_1} \otimes B_{k_2})b \end{bmatrix} .$$
(16)

From (16), it follows that (15) is solvable if and only if

$$(B_{k_1} \otimes B_{k_2})b = 0, \tag{17}$$

whose matrix form is just (9) by (7). As (9) holds, from (16), we get

$$\xi_1 = -((\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2})(V_2^c - V_1^c)^{\mathrm{T}} - c\mathbf{1}_{k_2}, \qquad (18)$$

where c is an arbitrary constant. Substituting (18) into (5) yields

$$(V^{p})^{\mathrm{T}} = (V_{1}^{c})^{\mathrm{T}} + (\mathbf{1}_{k_{1}}(\delta_{k_{1}}^{k_{1}})^{\mathrm{T}} \otimes I_{k_{2}})(V_{2}^{c} - V_{1}^{c})^{\mathrm{T}} + c\mathbf{1}_{k_{1}k_{2}}.$$
 (19)

Using (8), from (19), we get

$$P = C_1 + \mathbf{1}_{k_1} (\delta_{k_1}^{k_1})^{\mathrm{T}} (C_2 - C_1) + c \mathbf{1}_{k_1} \mathbf{1}_{k_2}^{\mathrm{T}}, \qquad (20)$$

which is just (10). \Box

Corollary 9 Given a bi-matrix game $\mathcal{G} = (C_1, C_2)$, we write the relative payoffs in the matrix form as

$$R = (r_{ij}) = C_2 - C_1 = \begin{bmatrix} R_1 & \eta \\ \zeta^{\rm T} & \eta_{k_1k_2} \end{bmatrix}, \qquad (21)$$

0

where $R_1 \in \mathbb{R}^{(k_1-1)\times(k_2-1)}$. Then \mathcal{G} is potential if and only if

$$R_1 - \mathbf{1}_{k_1 - 1} \zeta^{\mathrm{T}} - \eta \mathbf{1}_{k_2 - 1}^{\mathrm{T}} + \eta_{k_1 k_2} \mathbf{1}_{k_1 - 1} \mathbf{1}_{k_2 - 1}^{\mathrm{T}} = 0, \quad (22)$$

namely,

$$r_{ij} - r_{ik_2} - r_{k_1j} + r_{k_1k_2} = 0$$
(23)
for all $i = 1, 2, \cdots, k_1 - 1$ and $j = 1, 2, \cdots, k_2 - 1$.

Proof. Considering

$$B_{k_1}RB_{k_2}^{\mathrm{T}} = [I_{k_1-1}, -\mathbf{1}_{k_1-1}] \begin{bmatrix} R_1 & \eta \\ \zeta^{\mathrm{T}} & \eta_{k_1k_2} \end{bmatrix} \begin{bmatrix} I_{k_1-1} \\ -\mathbf{1}_{k_1-1}^{\mathrm{T}} \end{bmatrix}$$
$$= R_1 - \mathbf{1}_{k_1-1}\zeta^{\mathrm{T}} - \eta \mathbf{1}_{k_2-1}^{\mathrm{T}} + \eta_{k_1k_2}\mathbf{1}_{k_1-1}\mathbf{1}_{k_2-1}^{\mathrm{T}}, \quad (24)$$

we get the corollary from Theorem 8. \Box

The condition (23) of Corollary 9 is similar to the condition proposed in Theorem 3 of Hino (2011). It should be noted that Theorem 3 of Hino (2011) considers finite weighted potential games. Here, for the convenience of comparing our result with that in Hino (2011), we rewrite Theorem 3 of Hino (2011) for the special case of 2-player potential games in the language of relative payoff matrix as follows:

Proposition 10 (see Theorem 3 of Hino, 2011) The bimatrix game $\mathcal{G} = (C_1, C_2)$ is potential if and only if

$$r_{ij} - r_{i+1,j} - r_{i,j+1} + r_{i+1,j+1} = 0$$
(25)

for all $i = 1, 2, \dots, k_1 - 1$ and $j = 1, 2, \dots, k_2 - 1$.

The original four-cycle condition proposed by Monderer & Shapley (1996) is rewritten as:

Proposition 11 (see Corollary 2.9 of Monderer & Shapley, 1996) The bi-matrix game $\mathcal{G} = (C_1, C_2)$ is potential if and only if

$$r_{ij} - r_{i',j} - r_{i,j'} + r_{i',j'} = 0 (26)$$

for all $i, i' = 1, 2, \dots, k_1$ and $j, j' = 1, 2, \dots, k_2$.

Obviously, our condition (23) is different from Hino's condition (25) and they have the same complexity, but (26) has a larger complexity (Hino, 2011).

Remark 12 From (9) or (17), we see that, given the strategy set for bi-matrix games, the set of all the relative payoff matrices of potential bi-matrix games is a $(k_1 + k_2 - 1)$ -dimensional subspace, which is isomorphic to

$$\mathcal{P} = \{ b \in \mathbb{R}^{k_1 k_2} | (B_{k_1} \otimes B_{k_2}) b = 0 \}.$$
 (27)

Here, we call \mathcal{P} the *potential subspace*. If a bi-matrix game $\mathcal{G} = (C_1, C_2)$ is not a potential game, then one can use the orthogonal projection onto \mathcal{P} to yield corresponding potential games.

The basic result on orthogonal projection is stated as follows:

Lemma 13 (see page 430 of Meyer, 2000) Consider a linear subspace of \mathbb{R}^n as follows:

$$\mathcal{X} = \{ v \in \mathbb{R}^n | Bv = 0 \}.$$

$$(28)$$

If B has a full row rank, then the orthogonal projection of u onto \mathcal{X} is

$$\operatorname{Proj}_{\mathcal{X}} u = (I_n - B^{\mathrm{T}} (BB^{\mathrm{T}})^{-1} B) u.$$
 (29)

Now we consider the orthogonal projection onto the potential subspace.

Lemma 14 Consider a bi-matrix game $\mathcal{G} = (C_1, C_2)$, where $C_1, C_2 \in \mathbb{R}^{k_1 \times k_2}$. Denote the relative payoff matrix by $R = (r_{ij}) = C_2 - C_1$ and let $H_k = I_k - \frac{1}{k} \mathbf{1}_k \mathbf{1}_k^{\mathrm{T}}$. Then

$$\operatorname{Proj}_{\mathcal{P}} \operatorname{V}_{\mathbf{r}}(R) = (I_{k_1 k_2} - H_{k_1} \otimes H_{k_2}) \operatorname{V}_{\mathbf{r}}(R).$$
(30)

Proof. Let $\tilde{B} = B_{k_1} \otimes B_{k_2}$. By Lemma 13, we have

$$\begin{aligned} &\operatorname{Proj}_{\mathcal{P}} \operatorname{V}_{\mathrm{r}}(R) \\ = & (I_{k_{1}k_{2}} - \tilde{B}^{\mathrm{T}}(\tilde{B}\tilde{B}^{\mathrm{T}})^{-1}\tilde{B})\operatorname{V}_{\mathrm{r}}(R) \\ = & (I_{k_{1}k_{2}} - (B_{k_{1}}^{\mathrm{T}}(B_{k_{1}}B_{k_{1}}^{\mathrm{T}})^{-1}B_{k_{1}}) \otimes (B_{k_{2}}^{\mathrm{T}}(B_{k_{2}}B_{k_{2}}^{\mathrm{T}})^{-1}B_{k_{2}}))\operatorname{V}_{\mathrm{r}}(R). (31) \end{aligned}$$

A straightforward computation shows that

$$B_{k}^{\mathrm{T}}(B_{k}B_{k}^{\mathrm{T}})^{-1}B_{k}$$

$$= \begin{bmatrix} I_{k-1} \\ -\mathbf{1}_{k-1}^{\mathrm{T}} \end{bmatrix} (I_{k-1} + \mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}})^{-1}[I_{k-1} - \mathbf{1}_{k-1}]$$

$$= \begin{bmatrix} I_{k-1} \\ -\mathbf{1}_{k-1}^{\mathrm{T}} \end{bmatrix} (I_{k-1} - \frac{1}{k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}})[I_{k-1} - \mathbf{1}_{k-1}]$$

$$= \begin{bmatrix} I_{k-1} - \frac{1}{k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}} - \frac{1}{k}\mathbf{1}_{k-1} \\ -\frac{1}{k}\mathbf{1}_{k-1}^{\mathrm{T}} & \frac{k-1}{k} \end{bmatrix}$$

$$= I_{k} - \frac{1}{k}\mathbf{1}_{k}\mathbf{1}_{k}^{\mathrm{T}} = H_{k}.$$
(32)

From (31) and (32), it follows that (30) holds. \Box

Theorem 15 Consider a bi-matrix game $\mathcal{G} = (C_1, C_2)$, where $C_1, C_2 \in \mathbb{R}^{k_1 \times k_2}$. Let the relative payoff matrix be $R = (r_{ij}) = C_2 - C_1$. Then the following statements are equivalent: (i) \mathcal{G} is a potential game; (ii) $H_{k_1}RH_{k_2} = 0$, where $H_k = I_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}}$;

(iii) $r_{ij} = r_{i-\text{ave}} + r^{j-\text{ave}} - r_{\text{ave}}$ for all $i = 1, 2, \cdots, k_1$ and $j = 1, 2, \cdots, k_2$, where

$$r_{i-\text{ave}} = \frac{1}{k_2} \sum_{\mu=1}^{k_2} r_{i\mu}, \quad r^{j-\text{ave}} = \frac{1}{k_1} \sum_{\lambda=1}^{k_1} r_{\lambda j}, \quad (33)$$

$$r_{\rm ave} = \frac{1}{k_1 k_2} \sum_{\lambda=1}^{k_1} \sum_{\mu=1}^{k_2} r_{\lambda\mu}.$$
 (34)

Proof. Obviously, \mathcal{G} is a potential game if and only if $\operatorname{Proj}_{\mathcal{P}} V_r(R) = V_r(R)$, where \mathcal{P} is the potential subspace. Further by Lemma 14, we have that \mathcal{G} is potential if and only if $(H_{k_1} \otimes H_{k_2})V_r(R) = 0$, i.e. $H_kRH_k = 0$. Moreover, a straightforward calculation shows that

$$H_{k}RH_{k} = (I_{k_{1}} - \frac{1}{k_{1}}\mathbf{1}_{k_{1}}\mathbf{1}_{k_{1}}^{\mathrm{T}})R(I_{k_{2}} - \frac{1}{k_{2}}\mathbf{1}_{k_{2}}\mathbf{1}_{k_{2}}^{\mathrm{T}})$$
$$= R - \frac{1}{k_{1}}\mathbf{1}_{k_{1}}\mathbf{1}_{k_{1}}^{\mathrm{T}}R - \frac{1}{k_{2}}R\mathbf{1}_{k_{2}}\mathbf{1}_{k_{2}}^{\mathrm{T}} + \frac{\mathbf{1}_{k_{1}}^{\mathrm{T}}R\mathbf{1}_{k_{2}}}{k_{1}k_{2}}\mathbf{1}_{k_{1}}\mathbf{1}_{k_{2}}^{\mathrm{T}}.$$
 (35)

From (33)-(35), the equivalence between (ii) and (iii) follows. \Box

Remark 16 For the case of $k_1 = k_2$, Sandholm (2010) obtained the results of Theorem 15 using the method of game decomposition. A similar result can be seen in Proposition 2.14 of Hwang & Rey-Bellet (2011). But here, we get the results from the potential equation. Therefore, we have established a connection between the potential equation and the results obtained by Sandholm (2010). From (33) and (34), we see that $r_{i-\text{ave}}$ is the average relative payoff for given strategy $s_i^1 \in S_1$, $r^{j-\text{ave}}$ is the average relative payoff for given strategy $s_j^2 \in S_2$ and r^{ave} is the average relative payoff of all the strategies. Therefore, Theorem 15 displays an economic meaning of potential games.

4 The General Potential Equation

In this section, we consider the general potential equation for multi-player games and give new detecting conditions for potential games.

Multiplying (3) on the left by

$$\begin{bmatrix} 0 & 1 \\ -I_{n-1} & \mathbf{1}_{n-1} \end{bmatrix} \otimes I_{k^n}, \tag{36}$$

we get the equivalent equation

$$\begin{bmatrix} -\Psi_{1} & \Psi_{n} \\ -\Psi_{2} & \Psi_{n} \\ & \ddots & \vdots \\ & & -\Psi_{n-1} & \Psi_{n} \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \vdots \\ \xi_{n} \end{bmatrix} = \begin{bmatrix} (V_{n}^{c} - V_{1}^{c})^{\mathrm{T}} \\ (V_{n}^{c} - V_{2}^{c})^{\mathrm{T}} \\ \vdots \\ (V_{n}^{c} - V_{n-1}^{c})^{\mathrm{T}} \end{bmatrix}. \quad (37)$$

Construct nonsingular matrix $T = [T_1^T \ T_2^T \ T_3^T]^T$, where

$$T_{i} = \begin{bmatrix} -T_{i1} & & \\ & -T_{i2} & \\ & & \ddots & \\ & & & -T_{i,n-1} \end{bmatrix}$$
(38)

with

$$T_{1j} = I_{k^{j-1}} \otimes (\delta_k^k)^{\mathrm{T}} \otimes I_{k^{n-j}},$$
(39)

$$T_{2j} = I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^k)^{\mathrm{T}}, \tag{40}$$

$$T_{3j} = I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes B_k \tag{41}$$

for all $j = 1, 2, \dots, n$. It is easy to check that

$$T_{1j}\Psi_{j} = (I_{k^{j-1}} \otimes (\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-j}})(I_{k^{j-1}} \otimes \mathbf{1}_{k} \otimes I_{k^{n-j}}) = I_{k^{n-1}}, (42)$$

$$T_{1j}\Psi_{n} = (I_{k^{j-1}} \otimes (\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-j}})(I_{k^{n-1}} \otimes \mathbf{1}_{k})$$

$$=I_{k^{j-1}}\otimes(\delta_k^k)^{\mathrm{T}}\otimes I_{k^{n-j-1}}\otimes\mathbf{1}_k,\tag{43}$$

$$T_{2j}\Psi_{j} = (I_{k,j-1} \otimes B_{k} \otimes I_{k,n-j-1} \otimes (\delta_{k}^{k})^{1})(I_{k,j-1} \otimes \mathbf{1}_{k} \otimes I_{k,n-j}) = 0,(44)$$

$$T_{2j}\Psi_{n} = (I_{k,j-1} \otimes B_{k} \otimes I_{k,n-j-1} \otimes (\delta_{k}^{k})^{T})(I_{k,n-1} \otimes \mathbf{1}_{k})$$

$$= I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}}, \tag{45}$$

$$T_{3j}\Psi_j = (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes B_k)(I_{k^{j-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-j}}) = 0, \quad (46)$$

$$I_{3j}\Psi_n = (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes B_k)(I_{k^{n-1}} \otimes \mathbf{1}_k) = 0.$$
(47)

So, multiplying (37) on the left by T yields

$$\begin{bmatrix} I_{k^{n-1}} & -(\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_{k} \\ & \ddots & \vdots \\ & I_{k^{n-1}} & -I_{k^{n-2}} \otimes (\delta_{k}^{k})^{\mathrm{T}} \otimes \mathbf{1}_{k} \\ 0 & \cdots & 0 & -B_{k} \otimes I_{k^{n-2}} \\ 0 & \cdots & 0 & -I_{k} \otimes B_{k} \otimes I_{k^{n-3}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -I_{k^{n-2}} \otimes B_{k} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \vdots \\ \xi_{n} \end{bmatrix} = T \begin{bmatrix} (V_{n}^{c} - V_{1}^{c})^{\mathrm{T}} \\ (V_{n}^{c} - V_{2}^{c})^{\mathrm{T}} \\ \vdots \\ (V_{n}^{c} - V_{n-1}^{c})^{\mathrm{T}} \end{bmatrix},$$

$$(48)$$

that is,

$$\begin{bmatrix} I_{(n-1)k^{n-1}} & \Gamma \\ 0 & \Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \xi_n \end{bmatrix} = \begin{bmatrix} T_1 \tilde{b} \\ T_2 \tilde{b} \\ T_3 \tilde{b} \end{bmatrix}, \quad (49)$$

where $\tilde{\xi} = [\xi_1^{\mathrm{T}}, \cdots, \xi_{n-1}^{\mathrm{T}}]^{\mathrm{T}}, \tilde{b} = [V_n^c - V_1^c, \cdots, V_n^c - V_{n-1}^c]^{\mathrm{T}}, \Phi = [\Phi_1^{\mathrm{T}}, \Phi_2^{\mathrm{T}}, \cdots, \Phi_{n-1}^{\mathrm{T}}]^{\mathrm{T}}, \Gamma = [\Gamma_1^{\mathrm{T}}, \Gamma_2^{\mathrm{T}}, \cdots, \Gamma_{n-1}^{\mathrm{T}}]^{\mathrm{T}}$ with

$$\Phi_{i} = -I_{k^{i-1}} \otimes B_{k} \otimes I_{k^{n-i-1}}, \ \Gamma_{i} = -I_{k^{i-1}} \otimes (\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-1-i}} \otimes \mathbf{1}_{k} \ (50)$$

for each $i = 1, 2, \dots, n-1$.

By (49), we get the proposition as follows:

Proposition 17 The finite game G is potential if and only if $T_3\tilde{b} = 0$ and the linear equation

$$\Phi \xi_n = T_2 \hat{b} \tag{51}$$

has a solution ξ_n .

In the following, we consider (51). Let

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} & N_{13} & \cdots & N_{1,n-2} & N_{1,n-1} \\ M_{21} & L_{22} & L_{23} & \cdots & L_{2,n-2} & L_{2,n-1} \\ & M_{32} & L_{33} & \cdots & L_{3,n-2} & L_{3,n-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & L_{n-2,n-2} & L_{n-2,n-1} \\ & & & M_{n-1,n-2} & L_{n-1,n-1} \end{bmatrix},$$
(52)

where

$$N_{ij} = D_{k^{n-i}} (I_{k^{j-i}} \otimes D_k^{\mathrm{T}} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}}), \quad (53)$$

$$L_{ij} = -I_{k^{i-2}} \otimes B_k \otimes N_{ij}, \quad (54)$$

$$M_{i,i-1} = I_{k^{i-2} \otimes B_k \otimes N_{ij}},$$
(54)
$$M_{i,i-1} = I_{k^{i-2}(k-1)} \otimes B_{k^{n-i}}$$
(55)

$$m_{i,i-1} \quad m_{k^{i-2}(k-1)} \otimes \mathcal{D}_{k^{n-i}} \tag{3}$$

for all $i = 1, 2, \dots, n-1$ and $j = i, i+1, \dots, n-1$.

It is easy to check that S is a square matrix with order $(n-1)(k-1)k^{n-2}$. In order to prove S is nonsingular, we construct a square matrix as follows:

$$U = \begin{bmatrix} -\Phi_1 D_{k^{n-1}}^{\mathrm{T}} & G_1 & & \\ -\Phi_2 D_{k^{n-1}}^{\mathrm{T}} & G_2 & & \\ \vdots & & \ddots & \\ -\Phi_{n-2} D_{k^{n-1}}^{\mathrm{T}} & & G_{n-2} \\ -\Phi_{n-1} D_{k^{n-1}}^{\mathrm{T}} & & 0 \end{bmatrix}, \quad (56)$$

where $G_i = I_{k^{i-1}(k-1)} \otimes D_{k^{n-i-1}}^{\mathrm{T}}$ for $i = 1, 2, \cdots, n-2$.

Lemma 18 For matrices S and U, we have

$$-\sum_{j=1}^{n-1} N_{1j} \Phi_j = B_{k^{n-1}}, \tag{57}$$

$$M_{i,i-1}\Phi_{i-1} + \sum_{j=i}^{n-1} L_{ij}\Phi_j = 0 \quad (i=2,3,\cdots,n-1), \quad (58)$$

$$SU = I_{(n-1)(k-1)k^{n-2}}.$$
(59)

Proof. See Appendix.

From (59), it follows that S is nonsingular. Multiplying (51) on the left by S, we get an equivalent equation

$$S\Phi\xi_n = ST_2\tilde{b}.\tag{60}$$

Let $S\Phi = [\Upsilon_1^{\mathrm{T}}, \Upsilon_2^{\mathrm{T}}, \cdots, \Upsilon_{n-1}^{\mathrm{T}}]^{\mathrm{T}}$. From (57) and (58), it follows that $\Upsilon_1 = B_{k^{n-1}} = [I_{k^{n-1}-1} - \mathbf{1}_{k^{n-1}-1}]$ and $\Upsilon_i = 0$

for all $i = 2, 3, \dots, n-1$. Thus the linear equation (60) is just

$$[I_{k^{n-1}-1} - \mathbf{1}_{k^{n-1}-1}]\xi_n = S_1 T_2 \tilde{b}, \tag{61}$$

$$0 = S_2 T_2 \tilde{b}. \tag{62}$$

Then we get the following proposition.

Proposition 19 The linear equation (51) has a solution ξ_n if and only if $S_2T_2\tilde{b} = 0$.

Theorem 20 Consider the finite game $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ described by Definition 1 with payoff functions in (2). \mathcal{G} is potential if and only if

$$\begin{bmatrix} S_2 T_2 \\ T_3 \end{bmatrix} \tilde{b} = 0, \tag{63}$$

where $\tilde{b} = [V_n^c - V_1^c, \cdots, V_n^c - V_{n-1}^c]^T$ and the matrices T_2, T_3 and S_2 are shown in (38) and (52). Moreover, as (63) holds, a potential function is described by

$$p(x_1, \cdots, x_n) = V^p x_1 x_2 \cdots x_n, \tag{64}$$

where

$$(V^{p})^{\mathrm{T}} = (V_{1}^{c})^{\mathrm{T}} + (\mathbf{1}_{k}(\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-1}})(V_{n}^{c} - V_{1}^{c})^{\mathrm{T}} - \sum_{j=2}^{n-1} (\mathbf{1}_{k}(\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{j-2}} \otimes D_{k}^{\mathrm{T}} B_{k} \otimes \mathbf{1}_{k^{n-j}} (\delta_{k^{n-j}}^{k^{n-j}})^{\mathrm{T}}) \cdot (V_{n}^{c} - V_{j}^{c})^{\mathrm{T}} + c\mathbf{1}_{k^{n}}.$$
(65)

Proof. From Proposition 17 and Proposition 19, it follows that \mathcal{G} is potential if and only if (63) holds. Now, we compute the potential function. From (61), we get the

$$\xi_n = \begin{bmatrix} S_1 \\ 0 \end{bmatrix} T_2 \tilde{b} + c \mathbf{1}_{k^{n-1}}, \tag{66}$$

where c is an arbitrary constant. From (52) and (53), it follows that

$$S_1 = D_{k^{n-1}} \hat{S}_1, (67)$$

where $\tilde{S}_1 = [\tilde{S}_{11}, \tilde{S}_{12}, \cdots, \tilde{S}_{1,n-1}]$ with

$$\tilde{S}_{1j} = I_{k^{j-1}} \otimes D_k^{\mathrm{T}} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}}.$$
(68)

Considering the last row of \tilde{S}_1 is 0, by (66) and (67), we have

$$\xi_n = \tilde{S}_1 T_2 \tilde{b} + c \mathbf{1}_{k^{n-1}}.$$
(69)

By the first equation of (48), we obtain that

$$\begin{aligned} \xi_1 &= -T_{11} (V_n^c - V_1^c)^{\mathrm{T}} + ((\delta_k^k)^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_k) \xi_n \\ &= -T_{11} (V_n^c - V_1^c)^{\mathrm{T}} + ((\delta_k^k)^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_k) (\tilde{S}_1 T_2 \tilde{b} + c \mathbf{1}_{k^{n-1}}). (70) \end{aligned}$$

Substituting (70) into (5), we have

$$(V^{p})^{\mathrm{T}} = (V_{1}^{c})^{\mathrm{T}} - (\mathbf{1}_{k} \otimes I_{k^{n-1}})\xi_{1}$$

= $(V_{1}^{c})^{\mathrm{T}} + (\mathbf{1}_{k} \otimes I_{k^{n-1}})T_{11}(V_{n}^{c} - V_{1}^{c})^{\mathrm{T}}$
 $- ((\delta_{k}^{k})^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_{k}) \sum_{j=1}^{n-1} \tilde{S}_{1j}T_{2j}(V_{n}^{c} - V_{j}^{c})^{\mathrm{T}} + c\mathbf{1}_{k^{n}}.(71)$

Since $D_k \delta_k^k = 0$, we have

$$((\delta_k^k)^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_k) \tilde{S}_{11}$$

= $((\delta_k^k)^{\mathrm{T}} \otimes I_{k^{n-2}} \otimes \mathbf{1}_k) (D_k^{\mathrm{T}} \otimes \mathbf{1}_{k^{n-2}} (\delta_{k^{n-2}}^{k^{n-2}})^{\mathrm{T}}) = 0.$ (72)

Substituting T_{11} , T_{2j} , \tilde{S}_{1j} and (72) into (71) yields (65).

Remark 21 In Section 4 of Sandholm (2010), it is revealed that the minimal number of linear equations to test potential games is $(n-1)k^n - nk^{n-1} + 1$ for a *n*-player games with k strategies. However, for both the methods of Sandholm (2010) and Hino (2011), the number of equalities to be verified is $\frac{1}{2}n(n-1)k^{n-2}(k-1)^2$ (see the footnote of page 455 of Sandholm (2010)), which is much greater than the minimal number. Fortunately, using the potential equation in Cheng (2014), we get the minimal number of equations described by (63).

Theorem 22 For the finite game $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ described by Definition 1 with payoff functions shown in (2), the following statements are equivalent: (i) \mathcal{G} is potential;

(ii) equalities

$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{n-i-1}} \otimes B_k) (V_n^c - V_i^c)^{\mathrm{T}} = 0,$$
(73)
$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^k)^{\mathrm{T}}) (V_i^c - V_i^c)^{\mathrm{T}} = 0$$
(74)

hold for all $1 \le i < j \le n - 1$; (iii) equalities

$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j}})(V_j^c - V_i^c)^{\mathrm{T}} = 0 \quad (75)$$

hold for all $1 \leq i < j \leq n$.

Proof. (i) \Rightarrow (ii) Assume that \mathcal{G} is potential. Then, from Proposition 17, it follows that $T_3\tilde{b} = 0$ and the linear equation (51) is solvable. It is easy to check that $T_3\tilde{b} = 0$ is just (73). Moreover, (51) implies that

$$(I_{k^{s-1}} \otimes B_k \otimes I_{k^{n-s-1}}) \xi_n = (I_{k^{s-1}} \otimes B_k \otimes I_{k^{n-s-1}} \otimes (\delta_k^k)^{\mathrm{T}}) (V_n^c - V_s^c)^{\mathrm{T}}.$$
(76)
Letting $s = i$ in (76) and multiplying (76) on the left by
 $I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}}$ yield

$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i+1}} \otimes B_k \otimes I_{k^{n-j-1}})\xi_n = (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i+1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^k)^{\mathrm{T}})(V_n^c - V_i^c)^{\mathrm{T}}.$$
 (77)

Similarly, letting s = j in (76) and multiplying (76) on the left by $I_{k^{i-1}} \otimes B_k \otimes I_{k^{n-i-1}}$ yield

$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i+1}} \otimes B_k \otimes I_{k^{n-j-1}})\xi_n = (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i+1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^k)^{\mathrm{T}})(V_n^c - V_j^c)^{\mathrm{T}}.$$
(78)

Subtracting (78) from (77), we get (74). (ii) \Rightarrow (iii) As j = n, (75) is just (73). For the case of $1 \le i < j \le n - 1$, from (73) and (74), it follows that

$$(I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j}}) (V_j^c - V_i^c)^{\mathrm{T}}$$

$$= (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (D_k^{\mathrm{T}} B_k + \mathbf{1}_k (\delta_k^k)^{\mathrm{T}})) (V_j^c - V_i^c)^{\mathrm{T}}$$

$$= (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes D_k^{\mathrm{T}} B_k) (V_n^c - V_i^c)^{\mathrm{T}}$$

$$- (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes D_k^{\mathrm{T}} B_k) (V_n^c - V_j^c)^{\mathrm{T}}$$

$$+ (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} + \mathbf{1}_k (\delta_k^k)^{\mathrm{T}}) (V_j^c - V_i^c)^{\mathrm{T}}$$

$$= (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes D_k^{\mathrm{T}}) (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes B_k) (V_n^c - V_i^c)^{\mathrm{T}}$$

$$- (I_{k^{i-1}} \otimes B_k \otimes I_{k^{n-i-1}} \otimes D_k^{\mathrm{T}}) (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes B_k) (V_n^c - V_j^c)^{\mathrm{T}}$$

$$- (I_{k^{n-2}} \otimes \mathbf{1}_k) (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^k)^{\mathrm{T}}) (V_j^c - V_i^c)^{\mathrm{T}}$$

$$= 0.$$

$$(79)$$

(iii) \Rightarrow (ii) Multiplying (75) on the left by $I_{k^{n-1}} \otimes (\delta_k^k)^T$ yields (74).

(ii) \Rightarrow (i) Since $T_3\tilde{b} = 0$ is just (73), by Theorem 15, we only need to check $S_2T_2 = 0$, i.e.

$$M_{i+1,i}T_{2i}\tilde{b}_i - \sum_{j=i+1}^{n-1} L_{i+1,j}T_{2j}\tilde{b}_j = 0$$
 (80)

for all $i = 1, 2, \dots, n-2$. With simple calculations, we have

$$L_{i+1,j}T_{2j} = (I_{k^{i-1}} \otimes B_k \otimes D_{k^{n-i-1}} (I_{k^{j-i-1}} \otimes D_k^{\mathrm{T}} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}})) \\ \cdot (I_{k^{j-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^{k})^{\mathrm{T}}) = (I_{k^{i-1}(k-1)} \otimes D_{k^{n-i-1}}) (I_{k^{j-2}(k-1)} \otimes D_k^{\mathrm{T}} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}}) \\ \cdot (I_{k^{i-1}} \otimes B_k \otimes I_{k^{j-i-1}} \otimes B_k \otimes I_{k^{n-j-1}} \otimes (\delta_k^{k})^{\mathrm{T}}).$$
(81)

From (74) and (81), it follows that

$$L_{i+1,j}T_{2j}\tilde{b}_j = L_{i+1,j}T_{2j}\tilde{b}_i.$$
 (82)

Therefore, by (82) and (58), we have

$$M_{i+1,i}T_{2i}\tilde{b}_{i} - \sum_{j=i+1}^{n-1} L_{i+1,j}T_{2j}\tilde{b}_{j}$$

= $(M_{i+1,i}T_{2i} - \sum_{j=i+1}^{n-1} L_{i+1,j}T_{2j})\tilde{b}_{i}$
= $((M_{i+1,i}\Phi_{i} - \sum_{j=i+1}^{n-1} L_{i+1,j}\Phi_{j}) \otimes (\delta_{k}^{k})^{\mathrm{T}})\tilde{b}_{i} = 0.$ (83)

Using the concept of multi-indexed matrix proposed by Cheng (2012), we can simplify (75). For details of multiindexed matrix, please refer Definition 1.1 and Definition 1.3 of Cheng (2012). Here, we only give an intuitive example. Given a 4-dimensional data

$$X = \{x_{i_1 i_2 i_3 i_4} | i_1 = 1, 2; i_2 = 1, 2, 3; i_3 = 1, 2; i_4 = 1, 2, 3, 4\},\$$

we arrange X into a matrix

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$$X_{23}^{14} = \begin{bmatrix} x_{11}^{11} & x_{12}^{11} & x_{21}^{11} & x_{22}^{11} & x_{31}^{11} & x_{32}^{11} \\ x_{11}^{12} & x_{12}^{12} & x_{21}^{12} & x_{22}^{12} & x_{31}^{12} & x_{32}^{12} \\ x_{11}^{13} & x_{12}^{13} & x_{13}^{13} & x_{12}^{13} & x_{31}^{13} & x_{32}^{13} \\ x_{11}^{14} & x_{12}^{14} & x_{21}^{14} & x_{22}^{14} & x_{31}^{14} & x_{32}^{14} \\ x_{21}^{11} & x_{21}^{12} & x_{21}^{21} & x_{22}^{21} & x_{31}^{21} & x_{32}^{22} \\ x_{21}^{21} & x_{21}^{22} & x_{21}^{22} & x_{21}^{23} & x_{32}^{22} \\ x_{21}^{22} & x_{22}^{22} & x_{22}^{22} & x_{31}^{22} & x_{32}^{22} \\ x_{21}^{21} & x_{12}^{22} & x_{21}^{22} & x_{22}^{22} & x_{31}^{23} & x_{32}^{22} \\ x_{21}^{21} & x_{21}^{22} & x_{21}^{22} & x_{22}^{22} & x_{31}^{23} & x_{32}^{22} \\ x_{21}^{21} & x_{21}^{22} & x_{21}^{22} & x_{22}^{23} & x_{33}^{23} & x_{32}^{23} \\ x_{21}^{21} & x_{21}^{22} & x_{21}^{22} & x_{22}^{24} & x_{31}^{24} & x_{32}^{24} \end{bmatrix}$$

where $x_{i_2i_3}^{i_1i_4} = x_{i_1i_2i_3i_4}$, the multi-index of the rows of X_{23}^{14} is i_1i_4 and the multi-index of the columns is i_2i_3 . We usually say that X_{23}^{14} is in the order of $id(i_1, i_4; 2, 4) \times id(i_1, i_4; 2, 4)$ $id(i_2, i_3; 3, 2).$

Lemma 23 Suppose $Y = (A_1 \otimes A_2 \otimes \cdots \otimes A_l)X$, where X and Y be column vectors and $A_i \in \mathbb{R}^{m_i \times n_i}$ for each $i = 1, 2, \dots, m$. Assume that the elements of X and Y are in the order of $id(j_1, \dots, j_l; n_1, \dots, n_l)$ and $id(i_1, \dots, i_l; m_1, \dots, m_l)$ respectively. Then

$$Y_{r_{1}\cdots r_{l-t}}^{s_{1}\cdots s_{t}} = (A_{s_{1}}\otimes\cdots\otimes A_{s_{l}})X_{r_{1}\cdots r_{l-t}}^{s_{1}\cdots s_{t}}(A_{r_{1}}\otimes\cdots\otimes A_{r_{l-t}})^{\mathrm{T}},$$

where $\{s_{1},\cdots,s_{t},r_{1},\cdots,r_{l-t}\} = \{1,2,\cdots,l\}.$

By Lemma 23 and (75) of Theorem 22, we get the following corollary.

Corollary 24 Consider finite game $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ described by Definition 1 with payoff functions shown in (2). Let $R^{j \to i}$ be the multi-dimensional data of the relative payoffs with respect to j > i, i.e. $R^{j \to i} = V_j^c - V_i^c$. Then \mathcal{G} is potential if and only if

$$(B_k \otimes B_k)(R^{j \to i})^{ij}_{12 \cdots \hat{i} \cdots \hat{j} \cdots n} = 0$$
(84)

for all $1 \leq i < j \leq n$, where a caret is used to denote missing terms.

Remark 25 Similar to Theorem 15, (84) is equivalent to

$$(H_k \otimes H_k)(R^{j \to i})^{ij}_{12 \cdots \hat{i} \cdots \hat{j} \cdots n} = 0, \qquad (85)$$

which is consistent with the result of Sandholm (2010). But our result is derived from the potential equation. Thus we have established a connection between the result of Cheng (2014) and that of Sandholm (2010).

For players i and j, arbitrarily given the strategies of the other players, the payoffs of i and j admit a bimatrix game, which is called a *bi-matrix sub-game* of \mathcal{G} . Obviously, the relative payoffs of each bi-matrix subgame are just lie in one column of some $(R^{j \to i})^{ij}_{12 \cdots \hat{i} \cdots \hat{j} \cdots n}$. Therefore, we have the following corollary.

Corollary 26 A finite game \mathcal{G} is potential if and only if every bi-matrix sub-game of \mathcal{G} is potential.

5 An example

Example 27 Consider a finite game \mathcal{G} with n = 3, k =2 and payoff matrix $C = (c^{\mu}_{i_1 i_2 i_3})$. Let

$$R = \begin{bmatrix} r_{111}^1 & r_{112}^1 & r_{121}^1 & r_{122}^1 & r_{211}^1 & r_{212}^1 & r_{221}^1 & r_{222}^1 \\ r_{111}^2 & r_{112}^2 & r_{121}^2 & r_{222}^2 & r_{211}^2 & r_{221}^2 & r_{222}^2 \end{bmatrix},$$

where each $r_{ijk}^{\mu} = c_{ijk}^3 - c_{ijk}^{\mu}$ for $\mu = 1, 2$. A computation shows that the coefficient matrix of (63) is

0	1 0	-1	0	-1	0	1 0	-1	$0 \ 1 \ 0$	1	0	-1]
1	-1 0	0	-1	1	0	0 0	0	$0 \ 0 \ 0$	0	0	0
0	$0 \ 1$	-1	0	0	-1	1 0	0	$0 \ 0 \ 0$	0	0	0
0	0 0	0	0	0	0	0 1	-1	-1 1 0	0	0	0
0	0 0	0	0	0	0 (0 0	0	$0 \ 0 \ 1$	-1	-1	1

Thus, by Theorem 20, the game is a potential game if and only if

$$\begin{cases} r_{112}^1 - r_{122}^1 - r_{212}^1 + r_{222}^1 - r_{112}^2 + r_{122}^2 + r_{212}^2 - r_{222}^2 = 0, \\ r_{111}^1 - r_{112}^1 - r_{211}^1 + r_{212}^1 = 0, \\ r_{121}^1 - r_{122}^1 - r_{221}^1 + r_{222}^1 = 0, \\ r_{111}^2 - r_{112}^2 - r_{211}^2 + r_{212}^2 = 0, \\ r_{121}^2 - r_{122}^2 - r_{221}^2 + r_{222}^2 = 0, \end{cases}$$

which has the minimal number of equations for detecting whether \mathcal{G} is potential.

Conclusions 6

For detecting whether a finite game is potential, new necessary and sufficient conditions have been obtained by investigating the potential equations. The number of the obtained verification equalities is minimal. The connections between the potential equations and the existing results on potential games have been revealed. It has been shown that a finite game is potential if and only if its every bi-matrix sub-game is potential. In the future work, we will use the potential equation to investigate near-potential games, networked game and so on.

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Appendix: Proof of Lemma 18

By the basic fact $D_k^{\mathrm{T}} B_k = I_k - \mathbf{1}_k (\delta_k^k)^{\mathrm{T}}$, we have

$$-\sum_{j=1}^{n-1} N_{1j} \Phi_{j}$$

$$=\sum_{j=1}^{n-1} D_{k^{n-1}} (I_{k^{j-1}} \otimes D_{k}^{\mathrm{T}} B_{k} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}})$$

$$=D_{k^{n-1}} \sum_{j=1}^{n-1} I_{k^{j-1}} \otimes (I_{k} - \mathbf{1}_{k} (\delta_{k}^{k})^{\mathrm{T}}) \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}}$$

$$=D_{k^{n-1}} (\sum_{j=1}^{n-1} I_{k^{j}} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{\mathrm{T}} - \sum_{j=1}^{n-1} I_{k^{j-1}} \otimes \mathbf{1}_{k^{n-j}} (\delta_{k^{n-j}}^{k^{n-j}})^{\mathrm{T}})$$

$$=D_{k^{n-1}} (I_{k^{n-1}} - \mathbf{1}_{k^{n-1}} (\delta_{k^{n-1}}^{k^{n-1}})^{\mathrm{T}})$$

$$=D_{k^{n-1}} D_{k^{n-1}}^{\mathrm{T}} B_{k^{n-1}}$$

$$=B_{k^{n-1}} = [I_{k^{n-1}-1} - \mathbf{1}_{k^{n-1}-1}].$$
(86)

For every $i = 2, 3, \dots, n-1$, similarly, we have

$$-M_{i,i-1}\Phi_{i-1} - \sum_{j=i}^{n-1} L_{ij}\Phi_{j}$$

$$= I_{k^{i-2}} \otimes B_{k} \otimes B_{k^{n-i}} - \sum_{j=i}^{n-1} I_{k^{i-2}} \otimes B_{k} \otimes (D_{k^{n-i}})^{T} + (I_{k^{j-i}} \otimes D_{k}^{T}B_{k} \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{T}))$$

$$= I_{k^{i-2}} \otimes B_{k} \otimes B_{k^{n-i}} - \sum_{j=i}^{n-1} I_{k^{i-2}} \otimes B_{k} \otimes (D_{k^{n-i}})^{T} + (I_{k^{j-i}} \otimes (I_{k} - \mathbf{1}_{k} (\delta_{k}^{k})^{T}) \otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{T}))$$

$$= I_{k^{i-2}} \otimes B_{k} \otimes B_{k^{n-i}} - I_{k^{i-2}} \otimes B_{k} \otimes D_{k^{n-i}} (I_{k^{n-i}} - \mathbf{1}_{k^{n-1}} (\delta_{k^{n-i}}^{k^{n-j-1}})^{T}))$$

$$= I_{k^{i-2}} \otimes B_{k} \otimes B_{k^{n-i}} - I_{k^{i-2}} \otimes B_{k} \otimes D_{k^{n-i}} D_{k^{n-i}}^{T} - B_{k^{n-i}} B_{k^{n-i}}$$

$$= 0.$$
(87)

Moreover, by the fact that $B_k D_k^{\mathrm{T}} = I_{k-1}$ we have

$$M_{i,i-1}G_{i-1} = (I_{k^{i-2}(k-1)} \otimes B_{k^{n-i}})(I_{k^{i-2}(k-1)} \otimes D_{k^{n-i}}^{\Gamma})$$

= $I_{k^{i-2}(k-1)(k^{n-i}-1)}.$ (88)

From $D_k \delta_k^k = 0$, it follows that

$$L_{ij}G_{j} = -(I_{k^{i-2}} \otimes B_{k} \otimes N_{ij})(I_{k^{j-1}(k-1)} \otimes D_{k^{n-j-1}}^{T})$$

$$= -I_{k^{i-2}} \otimes B_{k} \otimes N_{ij}(I_{k^{j-i}(k-1)} \otimes D_{k^{n-j-1}}^{T})$$

$$= -I_{k^{i-2}} \otimes B_{k} \otimes D_{k^{n-i}}(I_{k^{j-i}} \otimes D_{k}^{T})$$

$$\otimes \mathbf{1}_{k^{n-j-1}} (\delta_{k^{n-j-1}}^{k^{n-j-1}})^{T})(I_{k^{j-i+1}} \otimes D_{k^{n-j-1}}^{T})$$

$$= -I_{k^{i-2}} \otimes B_{k} \otimes D_{k^{n-i}}(I_{k^{j-i}} \otimes D_{k}^{T})$$

$$\otimes \mathbf{1}_{k^{n-j-1}} (D_{k^{n-j-1}} \delta_{k^{n-j-1}}^{k^{n-j-1}})^{T})$$

$$= 0.$$
(89)

From (86)-(89), we get (59).