Controllability of networked MIMO systems [*]

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Abstract

In this paper, we consider the state controllability of networked systems, where the network topology is directed and weighted and the nodes are higher-dimensional linear time-invariant (LTI) dynamical systems. We investigate how the network topology, the node-system dynamics, the external control inputs, and the inner interactions affect the controllability of a networked system, and show that for a general networked multi-input/multi-output (MIMO) system: 1) the controllability of the overall network is an integrated result of the aforementioned relevant factors, which cannot be decoupled into the controllability of individual node-systems and the properties solely determined by the network topology, quite different from the familiar notion of consensus or formation controllability; 2) if the network topology is uncontrollable by external inputs, then the networked system with identical nodes will be uncontrollable, even if it is structurally controllable; 3) with a controllable network topology, controllability and observability of the nodes together are necessary for the controllability of the networked systems under some mild conditions, but nevertheless they are not sufficient. For a networked system with single-input/single-output (SISO) LTI nodes, we present precise necessary and sufficient conditions for the controllability of a general network topology.

Key words: Networked system; state controllability; structural controllability; directed network; MIMO LTI system.

1 Introduction

Complex networks of dynamical systems are ubiquitous in nature and science, as well as in engineering and technology. When control to a network is taken into consideration, the controllability of the network is essential, which is a classical concept [Kalman, 1962] applicable to multi-variable control systems [Gilbert, 1963, Shields & Pearson, 1976], composite systems [Davison & Wang, 1975] and decentralized control systems [Kobayashi, Hanafusa, & Yoshikawa, 1978, Tarokh, 1992], etc.

The subject of system controllability has been extensively studied over the last half a century. To date, various criteria have been well developed, including different kinds of matrix rank conditions and substantial graphic properties [Gilbert, 1963, Hautus, 1969, Lin, 1974, Davison & Wang, 1975, Glover & Silverman, 1976, Shields & Pearson, 1976, Kobayashi, Hanafusa, & Yoshikawa,

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1978, Tarokh, 1992, Jarczyk, Svaricek, & Alt, 2011, Liu, Slotine, & Barabási, 2011, Ruths, & Ruths, 2014]. One closely-related subject is the controllability of multi-agent systems, including the controllability of consensus systems [Liu *et al.*, 2008, Rahmani *et al.*, 2009, Lou & Hong, 2012, Ji, Lin, & Yu, 2012, Zhang, Camlibel, & Cao, 2012, Nabi-Abdolyousefi & Mesbahi, 2013, Ni, Wang, & Xiong, 2013, Xiang *et al.*, 2013], formation [Cai & Zhong, 2010], and pinning strategies [Chen, 2014].

Noticeably, many existing results on controllability are derived under the assumption that the dimension of the state of each node is one [Liu, Slotine, & Barabási, 2011, Lou & Hong, 2012, Zhang, Camlibel, & Cao, 2012, Nabi-Abdolyousefi & Mesbahi, 2013, Wang *et al.*, 2014]. However, most real-world networks of dynamical systems have higher-dimensional node states, and many multi-input/multi-output (MIMO) nodes are interconnected via multi-dimensional channels. In this paper we study the controllability of networked higher-dimensional systems with higher-dimensional nodes mainly for the MIMO setting.

In the literature, few results are developed for the controllability of networked higher-dimensional systems. Controllability and observability of specified cartesian

product networks are investigated in [Chapman, Nabi-Abdolyousefi, & Mesbahi, 2014], while a general networked system is considered in [Zhou, 2015] with every subsystem subject to external control input. It should be noted that some recent studies have addressed consensus controllability of networks with MIMO nodes [Cai & Zhong, 2010, Ni, Wang, & Xiong, 2013, Xiang *et al.*, 2013], where the controllability usually can be decoupled into two independent parts: one is about the controllability of each individual node and the other is solely determined by the network topology. A general setting of complex dynamical networks *cannot* be decoupled, however, as will be seen from the discussions below.

This paper addresses networked MIMO LTI dynamical node-systems in a directed and weighted topology, where there is no requirement for every subsystem to have an external control input. Some controllability conditions on the network topology, node dynamics, external control inputs and inner interactions are developed, so that effective criteria can be obtained for determining the large-scale networked system controllability. On one hand, some necessary and sufficient conditions on the controllability of a networked system with SISO higher-dimensional nodes are derived. On the other hand, some interesting results on the controllability of a networked system with MIMO nodes are obtained: 1) Under some mild conditions, controllability and observability of the nodes together are necessary for the controllability of the networked systems but, nevertheless, they are not sufficient; 2) the controllability of the network topology is necessary for the controllability of the generated networked system; 3) the controllability of individual node is necessary for the controllability of chain-networks, but not necessary for cycle-networks; 4) interactions among the states of different nodes play an important role in determining the controllability of a general networked system. Surprisingly, for the same network topology with the same node systems, the interactions among the states of nodes not only can lead controllable nodes to form an uncontrollable networked system, but also can assemble uncontrollable nodes into a controllable networked system.

The rest of the paper is organized as follows: In Section 2, some preliminaries and the general model of networked MIMO LTI systems are presented. Controllability conditions on various networked systems are investigated in Section 3. Finally, conclusions are drawn with some discussions in Section 4.

2 Preliminaries and the networked system

2.1 Preliminaries

Throughout, let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively, $\mathbb{R}^n$ ($\mathbb{C}^n$) the vector space of real (complex) n-vectors, $\mathbb{R}^{n \times m}$ ($\mathbb{C}^{n \times m}$) the set of $n \times m$ real (complex) matrices, I_N the $N \times N$ identity matrix, and $diag(a_1, \cdots, a_N)$ the $N \times N$ diagonal matrix with diagonal elements $a_1, \cdots, a_N$. Denote by $\sigma(A)$ the set of all the eigenvalues of matrix A and by $\otimes$ the Kronecker product.

In a directed graph, an edge (i, j) is directed from i to j, where i is the tail and j is the head of the edge. As reviewed for a directed graph in [Liu, Slotine, & Barabási, 2011], a matching is a set of edges that do not share any common tail or head, and a node being the head of an edge in the matching is called a matched node; otherwise, it is an unmatched node. A maximum matching is a matching that contains the largest possible number of edges in the graph. A perfect matching is a matching which matches all nodes in the graph. A graph formed by a sequence of edges $\{(v_i, v_{i+1}) \,|\, i = 1, \cdots, \ell - 1\}$ with no repeated node is called a path, denoted as $v_1, \cdots, v_\ell$, where v_1 is the beginning and v_ℓ is the end of the path, and v_ℓ is said to be reachable from v_1. If $v_1, \cdots, v_\ell$ is a path, then the graph formed by adding the edge (v_ℓ, v_1) is a cycle. A graph without cycles is called a tree. The node in a tree which can reach every other node is called the root of the tree. A leaf in a rooted tree is a node of degree 1 that is not the root.

Specifically, the notion of system controllability includes state controllability and structural controllability. For an n-dimensional system $\dot{x} = Ax + Bu$, it is said to be *state controllable*, if it can be driven from any initial state to the origin in finite time by a piecewise continuous control input. (A, B) is state controllable if and only if the controllability matrix $(B, AB, A^2B, \cdots, A^{n-1}B)$ has a full row rank [Kalman, 1962, Chui & Chen, 1998]. A parameterized system (A, B) (i.e., all of their nonzero elements are parameters) is said to be *structurally controllable*, if it is possible to choose a set of nonzero parameter values such that the resulting system (A, B) is state controllable [Lin, 1974]. In this paper, for brevity, *controllability* always means state controllability unless otherwise specified, e.g., structural controllability.

2.2 The networked system model

Consider a general directed and weighted network consisting of MIMO LTI node-systems in the following form:

$$\begin{cases} \dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} H y_j, \\ y_i = C x_i, \qquad i = 1, 2, \cdots, N, \end{cases} \tag{1}$$

in which $x_i \in \mathbb{R}^n$ is the state vector and $y_i \in \mathbb{R}^m$ the output vector of node i, $H \in \mathbb{R}^{n \times m}$ denotes the inner coupling matrix, and $\beta_{ij} \in \mathbb{R}$ represent the communication channels between different nodes. As usual, assume that $\beta_{ii} = 0$ and $\beta_{ij} \neq 0$ if there is an edge from node j to node i, otherwise $\beta_{ij} = 0$, for all $i, j = 1, 2, \cdots, N$.

When subjected to control inputs, the above networked system becomes

$$\dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} HC x_j + \delta_i Bu_i, \quad i = 1, 2, \cdots, N, \quad (2)$$

where $u_i \in \mathbb{R}^p$ is the external control input to node i, $B \in \mathbb{R}^{n \times p}$, with $\delta_i = 1$ if node i is under control, but otherwise $\delta_i = 0$, for all $i = 1, 2, \cdots, N$. To avoid trivial situations, always assume that $N \geq 2$.

Here and throughout, for statement simplicity a network consisting of more than one dynamical node, with or without control inputs (e.g., (1) and (2)), will be called a *networked system*.

Denote

$$L = [\beta_{ij}] \in \mathbb{R}^{N \times N} \quad \text{and} \quad \Delta = diag(\delta_1, \cdots, \delta_N), \quad (3)$$

which represent the network topology and the external input channels of the networked system (2), respectively. Let $X = [x_1^T, x_2^T, \cdots, x_N^T]^T$ be the whole state of the networked system, and $U = [u_1^T, u_2^T, \cdots, u_N^T]^T$ the total external control input. Then, this networked system can be rewritten in a compact form as

$$\dot{X} = \Phi X + \Psi U, \quad (4)$$

with

$$\Phi = I_N \otimes A + L \otimes HC, \quad \Psi = \Delta \otimes B. \quad (5)$$

In this paper, the focus is on how the network topology (described by the matrix L), the node-system (A, B, C), the external control input (determined by the matrix Δ), and the inner interactions specified by H affect the controllability of the whole networked system.

2.3 Some counter-intuitive examples

In [Liu, Slotine, & Barabási, 2011], it is shown that a network is structurally controllable if and only if there is an external input on each unmatched node and there are directed paths from controlled nodes with input signals to all matched nodes. For the networked system (4)-(5) formed by nodes with higher-dimensional state vectors, however, its controllability can be much more complicated, as demonstrated by the following example.

Example 2.1 *Consider a network of three identical nodes, with $\beta_{21} = \beta_{31} = 1$, and $\delta_1 = 1$, $\delta_2 = \delta_3 = 0$. It is not structurally controllable with one external input if each node has a one-dimensional state, since nodes 2*

and 3 can not be matched simultaneously. If each node has a higher-dimensional state with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $a_{ij} \neq 0$ and $h_i \neq 0$, $i, j = 1, 2$, then obviously (A, B) is controllable and (A, C) is observable. However, based on the results from Subsection 3.2 below, one knows that the networked system is uncontrollable for any matrix A, although it is structurally controllable due to the existence of self-matched cycle in every MIMO node.

The following three examples show that, even the network is a cycle having a perfect matching, the controllability of (A, B) is neither necessary nor sufficient for the controllability of the whole networked system.

Example 2.2 *Consider a network of two mutually connected identical nodes, with $\beta_{12} = \beta_{21} = 1$. Suppose that both nodes have external control inputs, i.e. $\delta_1 = \delta_2 = 1$, and*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \ 0].$$

It is easy to check that (A, B) is controllable. However, the networked system (2) is uncontrollable, although each node has an independent external input.

Example 2.3 *Consider a simple network of two mutually connected identical nodes, with $\beta_{12} = \beta_{21} = 1$, $\delta_1 = 1$, $\delta_2 = 0$, and*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0 \ 1].$$

Then, (A, B) is controllable, (A, C) is observable. However, the networked system (2) is uncontrollable.

Example 2.4 *Consider a network of two mutually connected identical nodes, with $\beta_{12} = \beta_{21} = 1$, $\delta_1 = 1$, $\delta_2 = 0$, and*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = [0 \ 1].$$

Then, (A, B) is uncontrollable. However, the networked system (2) is controllable, although there is only one node under external control.

Comparing the above three examples, their network topologies are the same and their node-system matrices A are identical. However, these networked systems have

very different controllabilities. The interactions among the states of nodes not only can lead controllable nodes to form an uncontrollable network, but also can assemble uncontrollable nodes into a controllable network.

3 Main Results

The controllability of the networked system (4)-(5) is considered in this section, where results on general network topologies are obtained, with specific and precise conditions obtained for some typical networks including tress and cycles.

First, recall the Popov-Belevitch-Hautus (PBH) rank condition [Hautus, 1969]: the networked system (4)-(5) is controllable if and only if

$$\text{rank}(sI_{N\cdot n} - \Phi, \Psi) = N \cdot n \tag{6}$$

is satisfied for any complex number s.

3.1 A general network topology

Based on the PBH rank condition (6), one can prove the following results.

Theorem 1 *If there exists one node without incoming edges, then to reach controllability of the networked system (4)-(5), it is necessary that (A, B) is controllable and moreover an external control input is applied onto this node which has no incoming edges.*

Proof: Assume that node i does not have any incoming edge. Then, the ith block row of Φ in (5) becomes $[0, \cdots, 0, A, 0, \cdots, 0]$. If there is no external control input onto node i, that is, $\delta_i = 0$, then for any $s_0 \in \sigma(A)$, the row rank of $[s_0 I - \Phi, \Psi]$ will be reduced at least by 1. If (A, B) is uncontrollable, then there exists an $s_0 \in \sigma(A)$ such that $\text{rank}(s_0 I - A, B) \leq n - 1$, which will also result in the reduction of the rank of $[s_0 I - \Phi, \Psi]$. ∎

Theorem 2 *If there exists one node without external control inputs, then for networked system (4)-(5) to be controllable, it is necessary that (A, HC) is controllable.*

Proof: Assume that node i does not have any external input, that is, $\delta_i = 0$. If (A, HC) is uncontrollable, then there exist an $s_0 \in \sigma(A)$ and a nonzero vector $\xi \in \mathbb{C}^{1\times n}$, such that $\xi(s_0 I - A) = 0$ and $\xi HC = 0$. Let $\alpha = [0, \cdots, 0, \xi, 0 \cdots, 0]$ with ξ located at the ith block. Then, it is easy to verify that $\alpha(s_0 I - \Phi) = 0$ and $\alpha\Psi = 0$. ∎

Theorem 3 *If the number of nodes with external control inputs is m, and $N > m \cdot rank(B)$, then for the networked system (4)-(5) to be controllable, it is necessary that (A, C) is observable.*

Proof: Suppose that (A, C) is unobservable. Then, there exist an $s_0 \in \sigma(A)$ and a nonzero vector $\xi \in \mathbb{C}^n$ such that

$$C\xi = 0 \quad \text{and} \quad (s_0 I - A)\xi = 0\,. \tag{7}$$

Consider the matrix $\Phi_{s_0} \triangleq s_0 I - \Phi$, and partition it into N column blocks, $[\Phi_{s_0}^1, \Phi_{s_0}^2, \cdots, \Phi_{s_0}^N]$, with $\Phi_{s_0}^i = [\beta_{1i}(HC)^T, \cdots, \beta_{i-1,i}(HC)^T, (sI - A)^T, \beta_{i+1,i}(HC)^T, \cdots, \beta_{Ni}(HC)^T]^T$, which corresponds to node i.

Based on formula (7), one has

$$\Phi_{s_0}^i \xi = 0, \quad i = 1, \cdots, N, \tag{8}$$

which implies that $\text{rank}(\Phi_{s_0}^i) \leq n - 1$. Therefore, $\text{rank}(\Phi_{s_0}) \leq N \cdot (n - 1)$. In view of $\text{rank}(\Psi) \leq m \cdot \text{rank}(B) < N$, one has $\text{rank}(s_0 I - \Phi, \Psi) < N \cdot n$, showing that the networked system (4)-(5) is uncontrollable. ∎

Theorem 4 *If (L, Δ) is uncontrollable, then the networked system (4)-(5) is uncontrollable.*

Proof: If (L, Δ) is uncontrollable, then there exist an $s_0 \in \sigma(L)$ and a nonzero vector $\xi \in \mathbb{C}^{1\times N}$ such that

$$\xi(s_0 I - L) = 0\,, \quad \xi\Delta = 0\,. \tag{9}$$

Therefore,

$$\begin{aligned}
(\xi \otimes I)\dot{X} &= (\xi \otimes I)\big((I \otimes A + L \otimes HC)X + (\Delta \otimes B)U\big)\\
&= (\xi \otimes A + s_0\xi \otimes HC)X\\
&= \big(\xi \otimes (A + s_0 HC)\big)X\,,
\end{aligned}$$

that is,

$$\left(\sum_{i=1}^N \xi_i x_i\right)' = (A + s_0 HC)\sum_{i=1}^N \xi_i x_i\,. \tag{10}$$

This implies that the variable $\sum_{i=1}^N \xi_i x_i$ is unaffected by the external control input U. For the zero initial state $x_i(t_0) = 0$, $i = 1, \cdots, N$, one has $\sum_{i=1}^N \xi_i x_i(t_0) = 0$. Moreover, $\sum_{i=1}^N \xi_i x_i(t) = 0$ for all $t > t_0$, because of the uniqueness of the solution to the linear equation (10). Consequently, for any state $\widetilde{X} \triangleq [\widetilde{x}_1^T, \cdots, \widetilde{x}_N^T]^T$ with $\sum_{i=1}^N \xi_i \widetilde{x}_i \neq 0$, there is no external control input U that can drive the networked system (4)-(5) to traverse from state 0 to $\widetilde{X}$. Thus, it is uncontrollable. ∎

If the network is not structurally controllable by external inputs, then (L, Δ) is uncontrollable, and thus the networked system will be uncontrollable. Since a network having more leaf nodes than nodes with external control input is not structurally controllable [Ruths, & Ruths, 2014], the following result comes accordingly.

Corollary 3.1 *If there are more leaf nodes than the nodes with external control inputs, then the networked system (4)-(5) is uncontrollable.*

If (L, Δ) is controllable, Examples 3.2, 3.4, and 3.6 discussed bellow for specific network topologies show that, for the networked system (4)-(5) to be controllable, it is not sufficient to ensure (A, B) and (A, HC) be both controllable and (A, C) be observable. Moreover, Example 3.4 shows that even every node has an external input and (A, B) is controllable, the networked system may still be uncontrollable.

Next, some necessary and sufficient conditions for the controllability of networked systems with SISO nodes are developed. First, a lemma is given.

Lemma 3.1 *Assume that $C \in \mathbb{R}^{1 \times n}$ is nonzero. Then, (A, HC) is controllable if and only if (A, H) is controllable.*

Proof: Since $C \in \mathbb{R}^{1 \times n}$, one has $H \in \mathbb{R}^{n \times 1}$ and $\operatorname{rank}(HC) = 1$. Therefore, $\operatorname{rank}(sI - A, HC) = \operatorname{rank}(sI - A, H)$, leading to the conclusion. ∎

Before moving on to the theorem, some new notations are needed. Denote the set of nodes with external control inputs by

$$\mathfrak{U} = \{i \mid \delta_i \neq 0, \, i = 1, \cdots, N\}. \tag{11}$$

For any $s \in \sigma(A)$, define a matrix set

$$\Gamma(s) = \left\{ [\alpha_1^T, \cdots, \alpha_N^T] \, \middle| \, \begin{array}{l} \alpha_i \in \Gamma^1(s) \text{ for } i \notin \mathfrak{U} \\ \alpha_i \in \Gamma^2(s) \text{ for } i \in \mathfrak{U} \end{array} \right\}, \tag{12}$$

where

$$\Gamma^1(s) = \{\xi \in \mathbb{C}^{1 \times n} \mid \xi(sI - A) = 0\},$$
$$\Gamma^2(s) = \{\xi \in \mathbb{C}^{1 \times n} \mid \xi B = 0, \, \xi \in \Gamma^1(s)\}.$$

Theorem 5 *Suppose that $|\mathfrak{U}| < N$, $B \in \mathbb{R}^{n \times 1}$, and $C \in \mathbb{R}^{1 \times n}$. Then, the networked system (4)-(5) is controllable if and only if the following hold:*

(i) (A, H) is controllable;
(ii) (A, C) is observable;
(iii) for any $s \in \sigma(A)$ and $\kappa \in \Gamma(s)$, $\kappa L \neq 0$ if $\kappa \neq 0$;
(iv) for any $s \notin \sigma(A)$, $\operatorname{rank}(I - L\gamma, \Delta\eta) = N$, with $\gamma = C(sI - A)^{-1}H$ and $\eta = C(sI - A)^{-1}B$.

Proof: Necessity. From Theorem 2 and Lemma 3.1, it follows that condition *(i)* is necessary. From Theorem 3, it follows that condition *(ii)* is also necessary.

Now, suppose that condition *(iii)* is not necessary. Then, there exist an $s_0 \in \sigma(A)$ and a nonzero matrix $\kappa \in \Gamma(s_0)$ such that

$$\kappa L = 0.$$

For matrix $M \in \mathbb{C}^{p \times q}$, denote by $vec(M) \in \mathbb{C}^{pq \times 1}$ the vectorization of matrix M formed by stacking the columns of M into a single column vector. Furthermore, let $\alpha = vec(\kappa)^T$. Since $\kappa \in \Gamma(s_0)$, it is easy to verify that $\alpha\Psi = 0$ and

$$\begin{aligned} \alpha(s_0 I - \Phi) &= \alpha(I_N \otimes (s_0 I - A) - L \otimes HC) \\ &= -\alpha(L \otimes HC) \\ &= -vec(C^T H^T \kappa L)^T = 0, \end{aligned}$$

which contradicts the network controllability.

Finally, suppose that condition *(iv)* is not necessary. Then, there exists an $s_0 \notin \sigma(A)$ satisfying

$$\operatorname{rank}(I - L\gamma_0, \Delta\eta_0) < N,$$

with $\gamma_0 = C(s_0 I - A)^{-1}H$ and $\eta_0 = C(s_0 I - A)^{-1}B$. Thus, there exists a nonzero vector $\zeta = [\zeta_1, \cdots, \zeta_N] \in \mathbb{C}^{1 \times N}$, such that

$$\zeta(I - L\gamma_0) = 0 \quad \text{and} \quad \zeta\Delta\eta_0 = 0.$$

Let $\alpha = [\alpha_1, \cdots, \alpha_N]$ with $\alpha_i = \zeta_i C(s_o I - A)^{-1}$. Then, since $\zeta \neq 0$, one has $\alpha \neq 0$. Moreover,

$$\begin{aligned} \alpha\Psi &= (\zeta \otimes C(s_0 I - A)^{-1}) \cdot (\Delta \otimes B) \\ &= (\zeta\Delta) \otimes (C(s_o I - A)^{-1}B) \\ &= \zeta\Delta\eta_0 = 0, \end{aligned}$$

and

$$\begin{aligned} &\alpha(s_0 I - \Phi) \\ &= (\zeta \otimes C(s_o I - A)^{-1}) \cdot (I_N \otimes (s_0 I - A) - L \otimes HC) \\ &= \zeta \otimes C - \zeta L \otimes (C(s_o I - A)^{-1}H)C \\ &= (\zeta - \zeta L\gamma_0) \otimes C = 0. \end{aligned}$$

This is also in conflict with the controllability of the networked system.

Sufficiency. For $s \in \mathbb{C}$, suppose that there exists a vector $\alpha = [\alpha_1, \cdots, \alpha_N]$, with $\alpha_i \in \mathbb{C}^{1 \times n}$, such that $\alpha(sI - \Phi) = 0$ and $\alpha\Psi = 0$. That is,

$$\alpha_i(sI - A) - \sum_{j \neq i} \beta_{ji}\alpha_j HC = 0, \quad i = 1 \cdots, N, \tag{13}$$

and

$$\alpha_i B = 0, \quad i \in \mathfrak{U}. \tag{14}$$

If $s \in \sigma(A)$, then $\text{rank}(sI - A) < n$. From (13), it follows that, for all $i = 1, \cdots, N$,

$$\sum_{j \neq i} \beta_{ji} \alpha_j H = 0. \tag{15}$$

If not, then $\text{rank}\left(\begin{bmatrix} C \\ sI - A \end{bmatrix}\right) = \text{rank}(sI - A) < n$,

which contradicts with the observability of (A, C). Moreover, based on (13), one has

$$\alpha_i(sI - A) = 0, \quad i = 1 \cdots, N. \tag{16}$$

Therefore, for all $i = 1 \cdots, N$, one has

$$\sum_{j \neq i} \beta_{ji} \alpha_j (sI - A) = 0. \tag{17}$$

Combining it with (15) and the controllability of (A, H), one obtains

$$\sum_{j \neq i} \beta_{ji} \alpha_j = 0, \quad i = 1 \cdots, N. \tag{18}$$

Next, let $\kappa = [\alpha_1^T, \cdots, \alpha_N^T]$. In view of (14), (16) and (18), it is easy to verify that $\kappa L = 0$ with $\alpha_i(sI - A) = 0$ for $i = 1, \cdots, N$, and $\alpha_i B = 0$ for $i \in \mathfrak{U}$. Therefore, by condition *(iii)*, one has $\alpha = 0$.

If $s \notin \sigma(A)$, then $sI - A$ is invertible. From (13), one has

$$\alpha_i = \sum_{j \neq i} \beta_{ji} \alpha_j H C (sI - A)^{-1}, \quad i = 1, \cdots, N. \tag{19}$$

Let $\zeta_i = \sum_{j \neq i} \beta_{ji} \alpha_j H$. Then, for $i = 1, \cdots, N$,

$$\alpha_i = \zeta_i C (sI - A)^{-1}, \tag{20}$$

and

$$\begin{aligned} \zeta_i &= \sum_{j \neq i} \beta_{ji} \alpha_j H = \sum_{j \neq i} \beta_{ji} \zeta_j C (sI - A)^{-1} H \\ &= \sum_{j \neq i} \beta_{ji} \zeta_j \gamma. \end{aligned} \tag{21}$$

Let $\zeta = [\zeta_1, \cdots, \zeta_N]$, and rewrite (21) as

$$\zeta(I - L\gamma) = 0. \tag{22}$$

Then, from (14) and (20), it follows that $\zeta_i C(sI - A)^{-1} B = 0$ for $i \in \mathfrak{U}$, which is equivalent to

$$\zeta \Delta \eta = 0. \tag{23}$$

Consequently, by combining it with (22) and condition *(iv)*, one has $\zeta = 0$, which together with (20) imply that $\alpha = 0$.

It follows from the above analysis that, for any $s \in \mathbb{C}$, the row vectors of matrix $[sI - \Phi, \Psi]$ are linearly independent, hence $\text{rank}(sI - \Phi, \Psi) = N \cdot n$. Thus, the networked system (4)-(5) is controllable. ∎

Next, some typical network structures, i.e., trees and circles are discussed in details.

3.2 Trees

In view of Corollary 3.1, the following result comes easily.

Corollary 3.2 *Consider a tree-network, in which every node is reachable from the root, and only the root has an external control input. If there is more than one leaf node in the tree, then the networked system is uncontrollable. Consequently, a star networked system with $N > 2$ is uncontrollable.*

A tree with only one leaf is a chain, which could be described by a path $1 \to 2 \to \cdots \to n$. Based on Theorem 1, node 1 should be under external control.

This chain-networked system (4)-(5) has

$$\Phi = \begin{bmatrix} A & 0 & \cdots & 0 \\ \beta_{21}HC & A & & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \beta_{N,N-1}HC & A \end{bmatrix}, \quad \Psi = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{24}$$

where $\beta_{i,i-1} \neq 0$ for $i = 2, \cdots, N$, and $\beta_{ij} = 0$ for $j \neq i - 1$, $i = 1, \cdots, N$.

From Theorems 1 and 2, one obtains the following result.

Corollary 3.3 *A necessary condition for the controllability of the chain networked system (4)-(24) is that (A, B) and (A, HC) are both controllable.*

Remark 3.1 *The observability of (A, C) is not necessary for the controllability of the chain networked system (4)-(24), as shown by the following example.*

Example 3.1 *Consider a chain-network of two identical nodes, with $\beta_{21} = 1$ and*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C = [1 \; 0].$$

It is easy to check that (A, B) and (A, HC) are both controllable and (A, C) is unobservable. The coupled networked system (24) has $\mathrm{rank}(\Psi, \Phi\Psi, \Phi^2\Psi, \Phi^3\Psi) = 4$, indicating that it is controllable. Therefore, the observability of (A, C) is indeed not necessary.

Remark 3.2 *Suppose that (A, B) and (A, HC) are both controllable and (A, C) is observable. However, these are not sufficient to guarantee the controllability of the chain networked system (4)-(24), as shown by the following example.*

Example 3.2 *Consider a chain-network of two nodes, with $\beta_{21} = 1$ and*

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \; H = \begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that (A, B) and (A, HC) are both controllable and (A, C) is observable. However, the coupled networked system (24) has $\mathrm{rank}(6 \cdot I - \Phi, \Psi) = 3 < 4$, indicating that it is uncontrollable.

If the input and output channels are all one-dimensional, namely, if all the nodes are SISO, then Theorem 5 can be restated as follows.

Corollary 3.4 *Assume that $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$. The chain networked system (4)-(24) is controllable if and only if (A, B) and (A, H) are both controllable and (A, C) is observable.*

Proof: Construct $\kappa = [\alpha_1^T, \cdots, \alpha_N^T]$ according to condition *(iii)* of Theorem 5, such that $\alpha_1 \in \Gamma^2$ satisfies $\alpha_1(sI - A) = 0$ and $\alpha_1 B = 0$, and moreover $\alpha_i \in \Gamma^1$ satisfies $\alpha_i(sI - A) = 0$ for $i = 2, \cdots, N$. In view of $\kappa L = [\beta_{21}\alpha_2^T, \beta_{32}\alpha_3^T, \cdots, \beta_{N,N-1}\alpha_N^T, 0]$, the condition $\kappa L \neq 0$ for $\kappa \neq 0$ is equivalent to $\alpha_1 = 0$, which implies the equivalence with the controllability of (A, B). Therefore, condition *(iii)* in Theorem 5 is equivalent to the controllability of (A, B).

Condition *(iv)* in Theorem 5 is automatically satisfied for the chain-network, since

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \beta_{21} & 0 & & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \beta_{N,N-1} & 0 \end{bmatrix}, \qquad (25)$$

and correspondingly for $s \notin \sigma(A)$, $\mathrm{rank}(I - L\gamma) = N$ with $\gamma = C(sI - A)^{-1}H$. ∎

3.3 Cycles

Now, assume that the network topology is a cycle. Since the cycle has a perfect matching, one external input is enough for the structural controllability, which can be added to any node in the cycle. Without loss of generality, assume that node 1 is under external control.

The cycle networked system has

$$\Phi = \begin{bmatrix} A & 0 & \cdots & \beta_{1N}HC \\ \beta_{21}HC & A & & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \beta_{N,N-1}HC & A \end{bmatrix},$$
$$\Psi = \begin{bmatrix} B^T & 0 & \cdots & 0 \end{bmatrix}^T, \qquad (26)$$

where $\beta_{1N} \neq 0$, $\beta_{i,i-1} \neq 0$ for $i = 2, \cdots, N$, and $\beta_{ij} = 0$ otherwise.

From Theorem 2, the controllability of (A, HC) is necessary for the controllability of the networked system (4)-(26).

Remark 3.3 *The controllability of (A, B) and the observability of (A, C) are not necessary for the controllability of the cycle networked system (4)-(26), as can be seen from the following example.*

Example 3.3 *Consider a cycle-network of three identical nodes, with $\beta_{13} = \beta_{21} = \beta_{32} = 1$ and*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check that (A, B) is uncontrollable and (A, C) is unobservable. However, the coupled system (26) has $rank(\Psi, \Phi\Psi, \Phi^2\Psi, \cdots, \Phi^{11}\Psi) = 12$, indicating that the networked system is controllable.

Remark 3.4 *Conditions that (A, B) and (A, HC) are both controllable and (A, C) is observable together are not sufficient to guarantee the controllability of the cycle networked system (4)-(26), as shown by the following example.*

Example 3.4 *Consider a cycle-network of three identical nodes, with $\beta_{13} = \beta_{21} = \beta_{32} = 1$ and*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that (A, B) and (A, HC) are both controllable and (A, C) is observable. However, although every node is driven by a control input, the whole networked system with

$$\Phi = \begin{bmatrix} A & 0 & HC \\ HC & A & 0 \\ 0 & HC & A \end{bmatrix}, \quad \Psi = \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix} \quad (27)$$

has $rank(\Psi, \Phi\Psi, \Phi^2\Psi, \cdots, \Phi^5\Psi) = 5 < 6$, implying that the networked system is uncontrollable.

Remark 3.5 *Even every node is SISO, the controllability of (A, B) is not necessary for the controllability of the networked system (4)-(26), as shown by the following example.*

Example 3.5 *Consider a cycle-network of three identical nodes, with $\beta_{13} = \beta_{21} = \beta_{32} = 1$ and*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

It is easy to check that (A, H) is controllable, (A, C) is observable, and (A, B) is uncontrollable. However, the coupled system (26) has $rank(\Psi, \Phi\Psi, \Phi^2\Psi, \cdots, \Phi^5\Psi) = 6$, indicating that the networked system is controllable.

Remark 3.6 *Assume that every node is SISO. The conditions that (A, B) and (A, H) are controllable and (A, C) is observable together are not sufficient to guarantee the controllability of the networked system (4)-(26), as shown by the following example.*

Example 3.6 *Consider a cycle-network of three identi-*

cal nodes, with $\beta_{13} = -1$, $\beta_{21} = \beta_{32} = 1$, and

$$A = \begin{bmatrix} 1 & 8 & 7 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 4 & 3 & 6 \end{bmatrix}.$$

It is easy to check that (A, B) and (A, H) are both controllable and (A, C) is observable. However, the coupled system (26) has $rank(\Psi, \Phi\Psi, \Phi^2\Psi, \cdots, \Phi^8\Psi) = 8 < 9$, showing that the networked system is uncontrollable.

For circle network with SISO nodes, a new criterion for the controllability is given as follows.

Theorem 6 *Assume that $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$. The cycle networked system (4)-(26) is controllable if and only if (A, H) is controllable, (A, C) is observable, and moreover*

$$rank\left(I - bHC(sI - A)^{-1}, B\right) = n, \quad \forall s \notin \sigma(A), \quad (28)$$

where $b = \beta_{1N} \prod_{i=1}^{N-1} \beta_{i+1,i} \gamma^{N-1}$, with $\gamma = C(sI - A)^{-1}H$.

Proof: For the cycle-network,

$$L = \begin{bmatrix} 0 & 0 & \cdots & \beta_{1,N} \\ \beta_{21} & 0 & & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \beta_{N,N-1} & 0 \end{bmatrix}, \quad (29)$$

which is invertible, therefore condition *(iii)* in Theorem 5 is automatically satisfied.

In the following, it will be proved that condition *(iv)* in Theorem 5 is equivalent to the above rank condition. Note that the two conditions are both given in terms of matrix ranks, yet one is about the network topology which is N-dimensional while the other is about a subsystem which is only n-dimensional.

If $\gamma = 0$, then the two matrices both have full ranks. In the following, assume that $\gamma \neq 0$.

If $rank(I - L\gamma, \Delta\eta) < N$, then there exists a nonzero vector $\mathbf{k} = [k_1, \cdots, k_N] \in \mathbb{C}^{1 \times N}$ such that

$$\mathbf{k} = \mathbf{k}L\gamma \quad \text{and} \quad \mathbf{k}\Delta\eta = 0,$$

that is,

$$k_i = k_{i+1}\beta_{i+1,i}\gamma, \quad i = 1, \cdots, N-1,$$
$$k_N = k_1\beta_{1,N}\gamma, \quad (30)$$
$$k_1\eta = 0.$$

From the recursion formula (30), it follows that $k_1 \neq 0$ since $\mathbf{k} \neq 0$. Moreover, $k_1 = k_2\beta_{21}\gamma = \cdots = k_N \prod\limits_{i=1}^{N-1} \beta_{i+1,i}\gamma^{N-1} = k_1 b\gamma$, which implies that $b\gamma = 1$. Choose $\xi = k_1 C(sI-A)^{-1}$. Then, $\xi \neq 0$, $\xi B = k_1 C(sI-A)^{-1}B = k_1\eta = 0$, and $\xi(I - bHC(sI-A)^{-1}) = k_1 C(sI-A)^{-1} - k_1 b\gamma C(sI-A)^{-1} = 0$, which implies that $\mathrm{rank}(I - bHC(sI-A)^{-1}, B) < n$.

If $\mathrm{rank}(I - bHC(sI-A)^{-1}, B) < n$, then there exists a nonzero vector $\xi \in \mathbb{C}^{1 \times n}$, satisfying

$$\xi = b\xi HC(sI-A)^{-1}, \quad \xi B = 0.$$

Since $\xi \neq 0$, one has $b \neq 0$ and $\xi H \neq 0$. Moreover, $\xi H = b\xi H\gamma$, which implies that $b\gamma = 1$. Now, define

$$
\begin{aligned}
k_1 &= b\xi H, \\
k_N &= \beta_{1,N}k_1\gamma, \\
k_i &= k_{i+1}\beta_{i+1,i}\gamma, \quad i = 2, \cdots, N-1.
\end{aligned}
\tag{31}
$$

One can easily verify that

$$k_1\eta = b\xi HC(sI-A)^{-1}B = \xi B = 0,$$

$$k_2\beta_{21}\gamma = k_N \prod_{i=1}^{N-1} \beta_{i+1,i}\gamma^{N-1} = k_1 b\gamma = k_1.$$

Therefore, $\mathbf{k} = \mathbf{k}L\gamma$ and $\mathbf{k}\Delta\eta = 0$ with $\mathbf{k} = [k_1, \cdots, k_N]$, which implies that $\mathrm{rank}(I - L\gamma, \Delta\eta) < N$. ∎

Looking back to Example 3.5, it can be seen that $\sigma(A) = \{0,0\}$. And, for any $s \neq 0$, one has $b = s^{-4}$ and

$$
\begin{aligned}
&\mathrm{rank}\left(I - bHC(sI-A)^{-1}, B\right) \\
&= \mathrm{rank}\left(\begin{bmatrix} 1 & 0 \\ -s^{-5} & 1 - s^{-6} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2.
\end{aligned}
$$

Moreover, (A, H) is controllable and (A, C) is observable. Therefore, from Theorem 6, it follows that the networked system in Example 3.5 is controllable.

Looking back to Example 3.6, it can be seen that for $s = 2 \notin \sigma(A)$, one has $C(2I-A)^{-1}H = -1$, $b = -1$, and

$$
\begin{aligned}
&\mathrm{rank}(I - bHC(2I-A)^{-1}, B) \\
&= \mathrm{rank}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 2 < 3.
\end{aligned}
$$

Therefore, from Theorem 6, it follows that the networked system in Example 3.6 is uncontrollable.

4 Conclusions

We have investigated a network consisting of MIMO LTI node-systems (A, B, C), in a topology described by matrix L with inner interactions described by matrix H, with or without control inputs determined by matrix Δ. We have studied the integrated effects of the network topology L, node-system (A, B, C) and inner interactions H on the controllability of the networked system.

We have shown that a networked system in the MIMO setting is uncontrollable if the network topology L is uncontrollable by external inputs through Δ, e.g., a non-trivial star-network with a single input to its root. For a networked system to be controllable, the controllability of (A, B) and (A, HC), as well as the observability of (A, C), are necessary under some conditions; but they are not sufficient in general, even for the cycle-network which has a perfect matching.

For SISO nodes with higher-dimensional state vectors, we have presented necessary and sufficient conditions for the controllability of some networked systems, including trees, cycles as well as a general network topology. These results not only provide precise and efficient criteria for determining the controllability of large-scale networked systems, by means of verifying some properties of a few matrices of lower dimensions, but also provide some general guidelines on how to assemble uncontrollable nodes to form a controllable networked system, which is deemed useful in engineering practice.

If each node-system (described by higher-dimensional matrices (A, B, H, C)) is viewed as a sub-network, then the networked system studied in this paper can also be considered as an interdependent network (or interconnected network, multi-layer network, network of networks, multiplex network, etc. [Boccaletti et al., 2014]); therefore, the results obtained in this paper should shed lights onto studying the controllability of such complex networks.

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