# Multi-player Pursuit-Evasion Games with One Superior Evader *

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**Abstract**

Inspired by the hunting and foraging behaviors of group predators, this paper addresses a class of multi-player pursuit-evasion games with one superior evader, who moves faster than the pursuers. We are concerned with the conditions under which the pursuers can capture the evader, involving the minimum number and initial spatial distribution required as well as the cooperative strategies of the pursuers. We present some necessary or sufficient conditions to regularize the encirclement formed by the pursuers to the evader. Then we provide a cooperative scheme for the pursuers to maintain and shrink the encirclement until the evader is captured. Finally, we give some examples to illustrate the theoretical results.

*Key words:* Pursuit-Evasion Game, Superior Evader, Fishing Game, Capture Condition, Cooperative Strategies

## 1 Introduction

There was an interesting Chinese proverb saying: " If the tiger went down to plain land, it would be insulted by dogs", which means that even though the tiger has better explosive force and a faster speed, it might be insulted or hunted by canids (hyenas) who have slower speed but are adept in besieging their preys cooperatively. In the real world, this hunting or foraging behavior by predators for a larger or faster prey is a wide spread phenomenon [18, 23]. For example, the pronghorn's speed is usually $80-100km/h$ while the lion's speed is only $70-80km/h$. But a group of lions are able to capture the pronghorn through effective cooperation.

These hunting phenomena can be naturally generalized to the field of robotics and control, where multiple slower robots (pursuers) try to capture one faster target (evader) who, conversely, attempts to escape. Theoretically, it is known as multi-player pursuit-evasion games with one superior evader [25]. Here, the term " superior" signifies that the evader has comparatively more advantageous control resources than the pursuers.

Pursuit-evasion game, as a common model in differential game theory, has been studied by many researchers during the past decades [4, 7, 11, 20], involving extensive applications such as interception problems of missile and satellite, formation control and jamming confrontation of unmanned aerial vehicles (UAVs), search and rescue operations of robots, and so on. In recent years, the group behaviors (such as team collaboration, distributed control, artificial intelligence, population evolution, etc.) in multi-agent systems have received increasing attention [2, 9, 10, 19, 22]. The conventional approach introduced by [14] is applicable to two-player pursuit-evasion game, which is based on the underlying idea of state reversal: Starting from the terminal manifold, an optimal trajectory of the states is depicted retrograde and the value function of the game can be determined

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by using a formulation of the Hamilton-Jacobi-Isaacs (HJI) equation, so as to attain the equilibrium strategies for the players. However, when it comes to the case of multiple players, this approach encounters tremendous difficulties in the determination of terminal manifold, the characterization of cooperation among multiple pursuers or evaders, and the computational complexity in solving HJI equations.

To cope with the above challenges, researchers generally demonstrate the possibility of capture or escape by exhibiting some particular strategies or policies of the players [1,3,5,6,16,26]. These methods can be collectively referred to as "method of explicit policy" [14,26]. For example, [5] proposed a sweep-pursuit-capture strategy to capture a single evader for multiple pursuers in an unbounded planar environment and determined the minimum probability of capture. In [6] they addressed a cooperative Homicidal Chauffeur game and presented a multi-phase cooperative strategy (align, swerve, encircle and close phases). [16] designed entrapment strategies for the pursuers under distributed information (sensing limitation) that the pursuers will spread out around the evader while approaching it.

However, most of the literature assumes that the pursuers move faster than or equal to the evader to ensure capture feasible. Little has been done for the problems with faster evader in a continuous unbounded planar environment as the hunting phenomena in the animal kingdom mentioned before. Breakwell might be the first one to consider the multi-player pursuit-evasion problems with faster evader in the literature [8, 13], where the evader is required to pass between two pursuers. He obtained a closed-form solution by dividing the optimal trajectories of the players into two successive phases, but it is not scalable for more pursuers. [25] preliminarily studied the multi-pursuer single-superior-evader game on an unbounded plane and provided the minimum number of the pursuers required when the evader moves in a straight line.

Other literature related to faster evader usually imposes some additional constraints on the problem formulation, such as polygonal environment (a closed and bounded set in Euclidean space with polygonal boundary) [21], graph environment (discrete time and discrete space with multiple nodes and edges) [24], sensing limitation [17] and limited turning of the evader [12]. With regard to a general planar environment where the motions of the players are simple, however, some fundamental problems have not been solved completely. For example, how many pursuers would be necessary to capture an evader? What conditions make the capture or escape possible? How to design the cooperative scheme among the pursuers?

The main contribution of this paper is to solve the above problems. We obtain the minimum number of pursuers required to guarantee a capture, which only depends on the speed ratio of the pursuers to the evader. Then we present some necessary or sufficient conditions to regularize the encirclement formed by the pursuers to the evader, and provide a cooperative scheme for the pursuers to maintain and shrink the encirclement until the evader is captured. This cooperative scheme contains three phases: besiege, shrink, and capture. For each phase, the feasible strategies of the players are characterized.

In our previous work [26], a fishing game is introduced, which bears resemblance to the pursuit game in [13], i.e., the superior evader must pass the gap between two pursuers. We obtained a complete solution which can be utilized to induce the analysis of capture conditions for the current multi-player game, because the evader will exert itself to break through the encirclement against some two adjacent pursuers on each instant of time. Thus in Section 2, we present the problem formulation based on the work on fishing game. Then in Section 3 we focus on determining the capture conditions and cooperative strategies for the pursuers. In Section 4, we give some examples to illustrate our findings. Finally, conclusions and future work are summarized in Section 5.

## 2 Preliminaries

### *2.1 A fishing game with two pursuers and one evader*

First, we provide a brief introduction to our previous work on the fishing game [26], which is played in the plane by two pursuers and one superior evader. The evader attempts to pass the gap between the two pursuers (as shown in Fig. 1), while avoiding being captured. On the contrary, the pursuers try to capture the evader while preventing the evader from threading.

In a reduced state space formed by the relative distances $d_1$, $d_2$ and the included angle $\theta$, the dynamics are given as follows:

$$\dot{d}_1 = -v_p \cos\hat{\phi}_1 - v_e \cos(\theta - \alpha), \quad d_1(t_0) = d_1^0 \tag{1}$$

$$\dot{d}_2 = v_p \cos\hat{\phi}_2 - v_e \cos\alpha, \quad d_2(t_0) = d_2^0 \tag{2}$$

$$\begin{aligned}\dot{\theta} &= -\frac{v_p}{d_1}\sin\hat{\phi}_1 + \frac{v_e}{d_1}\sin(\theta-\alpha) - \\ &\quad - \frac{v_p}{d_2}\sin\hat{\phi}_2 + \frac{v_e}{d_2}\sin\alpha, \quad \theta(t_0) = \theta_0\end{aligned} \tag{3}$$

The fishing game with point capture terminates when one of the following situations occurs: a) $\min\{d_1, d_2\} = 0$, the pursuers win; b) $\min\{d_1, d_2\} > 0$ and $\theta = \pi$, the evader wins. Then we obtained a complete solution that consists of a barrier and the optimal strategies of the players. Here, we give some main results.

**Theorem 2.1** *The barrier of fishing game is governed*

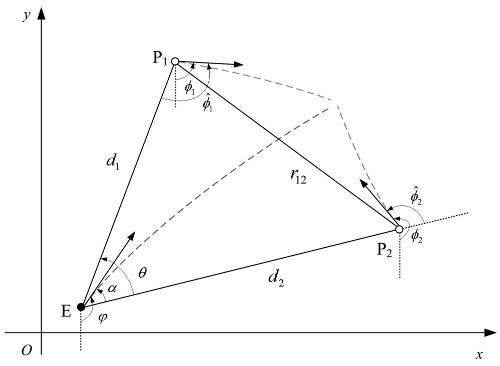

Fig. 1. Fishing game in the fixed reference system.

*by*

$$\mathscr{B} = \{d_1, d_2, \theta \,|\, \cos\theta = \frac{(1-a^2)(d_1+d_2)^2}{2d_1d_2} - 1\}$$
$$= \{d_1, d_2, \theta \,|\, r_{12} = a(d_1+d_2)\} \quad (4)$$

*and the optimal strategies of the players on the barrier are given by*

$$\hat{\phi}_1^* = \frac{\pi}{2}, \qquad \hat{\phi}_2^* = \frac{\pi}{2} \quad (5)$$

$$\sin\alpha^* = \frac{d_1 - d_2\cos\theta}{r_{12}}, \quad \cos\alpha^* = \frac{d_2\sin\theta}{r_{12}} \quad (6)$$

*where* $r_{12} = \sqrt{d_1^2 + d_2^2 - 2d_1d_2\cos\theta}$ *is the distance between the pursuers* $P_1$ *and* $P_2$*, and* $a = v_p/v_e < 1$*.*

The barrier separates the state space into two disjoint regions: capture zone and escape zone. From Theorem 2.1, the capture zone can be described by

$$\mathscr{D}_p = \{d_1, d_2, \theta \,|\, \cos\theta \geq \frac{(1-a^2)(d_1+d_2)^2}{2d_1d_2} - 1\}$$
$$= \{d_1, d_2, \theta \,|\, r_{12} \leq a(d_1+d_2)\} \quad (7)$$

and the escape zone $\mathscr{D}_e$ lies outside of $\mathscr{D}_p$. Further let

$$\Theta = \arccos\Big[\frac{(1-a^2)(d_1+d_2)^2}{2d_1d_2} - 1\Big], \quad (8)$$

we call it as "*non-escape angle*" since the evader will be captured by the pursuers if $\theta \leq \Theta$.

It is convenient to switch the problem to a three-dimensional relative state space, where the state variables are $x$, $y$ and $z$ (see Fig. 2). The coordinates of the pursuers and the evader correspond to $P_1(0, z)$, $P_2(0, -z)$ and $E(x, y)$, respectively. Then for a fixed $z$, the barrier $\mathscr{B}$ is an ellipse in the relative state space, which is given by the following equation

$$x^2 + (1-a^2)y^2 = \frac{(1-a^2)z^2}{a^2}. \quad (9)$$

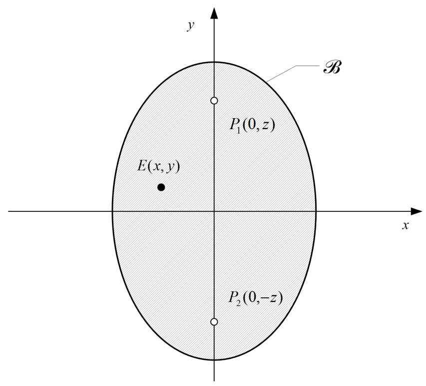

Fig. 2. The barrier of fishing game in the relative state space, where the shaded region (excluding points $P_1$, $P_2$ and the boundary $\mathscr{B}$) corresponds to the escape zone $\mathscr{D}_e$.

## 2.2 Problem formulation

We proceed to present the problem formulation of this paper. Consider a multi-player pursuit-evasion game with $n$ identical pursuers and one superior evader played in an unbounded planar environment. We assume that all the players are of simple motion, i.e., constant speed and unlimited direction, and the speed of the pursuers $v_p$ is strictly less than that of the evader $v_e$, $a := v_p/v_e < 1$. As shown in Fig. 3, from the dynamics of fishing game, we obtain the state equations as follows

$$\dot{d}_i = v_p\cos(\phi_i - \theta_i) - v_e\cos(\varphi - \theta_i) \quad (10)$$
$$\dot{\theta}_i = \frac{v_p}{d_i}\sin(\phi_i - \theta_i) - \frac{v_e}{d_i}\sin(\varphi - \theta_i) \quad (11)$$

where the control variables of $P_i$ and $E$ are $\phi_i$ and $\varphi$, respectively. The included angle between two adjacent pursuers $P_i$ and $P_{i+1}$ with the evader is denoted as $\theta_{i,i+1}$, where $\theta_{n,n+1} = \theta_{n,1}$. Further, the distance between two adjacent pursuers $P_i$ and $P_{i+1}$ is denoted as $r_{i,i+1}$, and the encirclement formed by $n$ pursuers is denoted as the polygon $\mathcal{G} = P_1P_2\cdots P_n$.

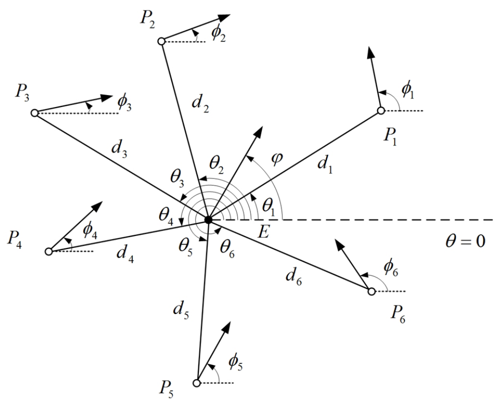

Fig. 3. The multi-player pursuit-evasion game in polar coordinate system.

The multi-player pursuit-evasion game terminates when the pursuers capture the evader, i.e., $\min_{1\le i\le n} d_i \le d_c$, where $d_c \ge 0$ is called as *capture radius* (we assume $d_c = 0$ in the theoretical analysis of this paper). In addition, we assume that the instantaneous position and velocity of the evader is available to all pursuers. Then the purpose of the pursuers is to capture the evader as soon as possible while the evader aims to prevent it infinitely. In terms of game of kind [1], to capture the superior evader, generally, at least three conditions should be satisfied: first, the pursuers are enough to participate in a group cooperatively; second, the initial positions of these pursuers are around of the evader; and third, the encirclement formed by the pursuers is shrinkable to trap the evader in progressively smaller regions. Based on these conditions, the following problems will be considered in this paper:

-1- How many pursuers would be required to guarantee the capture?

-2- What kind of position distribution of the pursuers is an effective encirclement to the evader?

-3- What strategies would be available for the pursuers to shrink the encirclement regardless of the evader's strategy?

## 3 Capture Conditions and Cooperative Strategies

### *3.1 A simple extension from the fishing game*

In fact, the pursuit-evasion problem with $n$ pursuers and one evader discussed in this paper, can be regarded as a combination of $n$ fishing games, because the evader will exert itself to break through the encirclement against some two adjacent pursuers on each instant of time. Thus naturally some results of the fishing game can be extended to the current multi-player game. First, we address a simple problem, assuming that the evader will choose some two adjacent pursuers to play the fishing game while disregarding other pursuers. To distinguish, we call this simple problem as "*multi-player fishing game*", and then we have the following results:

**Definition 3.1 (Surrounding Condition)** *An evader is surrounded by n pursuers if the configuration of the evader and any pair of adjacent pursuers satisfies*

$$\theta_{i,i+1} \le \Theta_{i,i+1} = \arccos\left[\frac{(1-a^2)(d_i+d_{i+1})^2}{2d_i d_{i+1}} - 1\right] \tag{12}$$

$$\textit{or} \qquad r_{i,i+1} \le a(d_i + d_{i+1}), \ \ \forall\, i = 1, 2, \ldots, n. \tag{13}$$

It is obvious that when the evader is surrounded by the pursuers, it will be captured in the multi-player fishing game. For a fixed encirclement $\mathcal{G}$ formed by the pursuers, Fig. 4 shows the possible position of the evader (shaded region) satisfying the conditions (12)∼(13). According to the characteristics of barrier [26], we know that the boundary of shaded region is exactly the barrier of multi-player fishing game, and correspondingly the shaded region represents the capture zone.

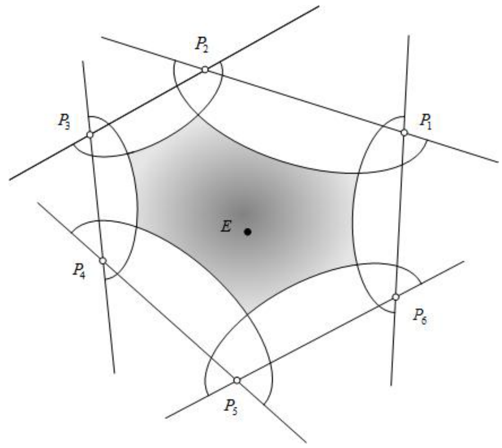


Fig. 4. The initial position distribution of the players when the surrounding condition is satisfied, where the boundary of shaded region is composed of the barriers of $n$ fishing games.

Based on the previous work [26], the optimal strategies of the players are accessible to describe as follows:

I. For the evader, at first, it should find out whether there exists an $i$ subject to $r_{i,i+1}/(d_i + d_{i+1}) > a$. If so, the evader chooses pursuers $P_i$ and $P_{i+1}$ as opponents to play the fishing game, and implements the strategy (6) to escape. If not, to maximize the capture time the evader should choose $P_j$ and $P_{j+1}$ as the opponents, where the barrier associated with line segment $P_jP_{j+1}$ is the closest one to the evader, or more specifically, the value of $r_{j,j+1}/(d_j + d_{j+1})$ is a maximum for all $j = 1, 2, \ldots, n$.

II. For the pursuers, correspondingly, their strategies are to ensure that the evader is in the capture zone first, then $P_k$ and $P_{k+1}$ perform the strategy (5) to capture the evader if the evader moves towards them, while other pursuers only need to follow the evader.

Next, we discuss the minimum number of the pursuers required to satisfy the surrounding condition. It is easy to verify that only if $d_i = d_{i+1}$, the non-escape angle $\Theta_{i,i+1} = \arccos(1-2a^2)$; otherwise, $\Theta_{i,i+1} < \arccos(1-$

[1] Game of kind, introduced by [14] in distinction to one of degree, is a pursuit game in which we are concerned with what conditions make capture possible for the pursuers or escape for the evader, rather than seeking the best procedures in terms of optimizing some continuous payoff.

$2a^2$). Thus, from

$$\sum_{i=1}^{n} \theta_{i,i+1} = 2\pi, \quad \theta_{i,i+1} \le \Theta_{i,i+1}$$

$$\Rightarrow \quad n_{\min} \ge \left[\frac{2\pi}{\arccos(1-2a^2)}\right] = \left[\frac{\pi}{\arcsin a}\right] \qquad (14)$$

where $[\cdot]$ is the symbol of integer conversion, e.g. $[x]$ is the smallest integer greater than $x$. Same result as (14) was given at [15,25] using different approaches, but here we provide a more general argument that the minimum number of the pursuers should be strictly greater than $\pi/\arcsin a$ if $d_i \ne d_{i+1}$ for some $i \in \{1, 2, \cdots, n\}$.

### *3.2 The challenges of capturing an agile evader*

When the superior evader moves freely in the plane (see Fig. 5), it can seek the breakthrough from any two adjacent pursuers at any time. Thus, for the pursuers, it is more difficult to capture such an agile evader. At this point, the surrounding condition (12) no longer works, because it is easy to be destroyed by the evader. When $\theta_{i,i+1}(t) \le \Theta_{i,i+1}(t)$, we cannot ensure that $\theta_{i,i+1}(t+1)$ is still no greater than $\Theta_{i,i+1}(t+1)$. Moreover, even though the surrounding condition could be satisfied for any time, the relative distance $d_i$ would not be guaranteed decreasing with time; because if the evader chooses the direction of vector $\overrightarrow{P_iE}$, the minimum value of $d_i(t+1)$ in one time step will be $d_i(t) + (v_e - v_p)\Delta t > d_i(t)$. Consequently, it is possible that the encirclement $\mathcal{G}$ can only change its shape and position with the evader's moving, but not shrink its size.

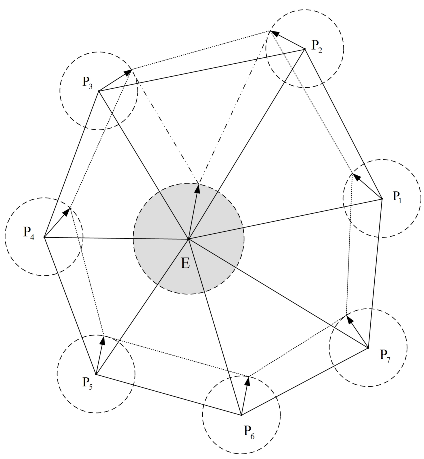


Fig. 5. Multiple pursuers chasing one superior evader who moves freely, where the dotted circles are the accessible position of the players in one time step.

Additionally, it is well recognized that cooperation is a good way to bring the pursuers' advantage in numbers into the game. However, effective cooperation scheme continues to be a significant challenge for the pursuers [3, 5, 6, 16]. In fact, cooperation can be construed as two distinct concepts: *Natural Cooperation* and *Artificial Cooperation*. The former refers to a natural collaboration evolved by intelligent species for the realization of some specific and common objective (such as hunting, foraging and migrating). Theoretically, this evolving cooperation will reach a stable and optimal equilibrium. The latter is a kind of behavior rules applied artificially on the multi-agent systems to achieve some specific objective (such as formation, lead-follow and flocking). It might be able to optimize the objective in the corresponding mathematical model, but not necessarily pure optimization of the original multi-agent systems. Throughout the researches of biological population collaborative behaviors in artificial intelligence, almost all of them focus on constructing artificial cooperation scheme to simulate or approximate the unknown natural cooperation as much as possible.

Inspired by the hunting and foraging behaviors of group predators, we concentrate on dealing with the above challenges in the following, where the artificial cooperation scheme of multi-pursuers will be divided into three phases: besiege, shrink and capture.

### *3.3 Besieged status*

First, for a general multi-player pursuit-evasion game, since the surrounding condition is only necessary to capture the evader rather than sufficient, we define a novel concept named "*besieged status*" in discrete time.

**Definition 3.2 (Besieged Status)** *For $k$ from 0 to $K(\in \mathbb{N})$ and any $i = 1, 2, \ldots, n$, if the inequality*

$$r_{i,i+1}(k) \le a\big[d_i(k) + d_{i+1}(k)\big] \qquad (15)$$

*holds, then we say that the game is in $K$-besieged status, and the evader is besieged by the pursuers without escape angle for $K$ time steps.*

**Theorem 3.1 (Besieging Condition)** *If at the initial time,*

$$r_{i,i+1}(0) \le a\big[d_i(0) + d_{i+1}(0)\big] - K(3-2a)v_p\Delta t \qquad (16)$$

*holds for any $i = 1, 2, \ldots, n$, then the game is in $K$-besieged status, where $\Delta t$ represents the length of one time step.*

**PROOF.** From Fig. 5, for any non-worst strategy of the pursuers (the worst strategy refers to that the pursuers move dispersedly or away from the evader) and any strategy of the evader

$$r_{i,i+1}(k+1) \le r_{i,i+1}(k) + v_p\Delta t,$$

$$d_i(k+1) \geq d_i(k) - (v_e - v_p)\Delta t.$$

Then we have

$$\begin{aligned} r_{i,i+1}(1) &\leq r_{i,i+1}(0) + v_p\Delta t \\ &\leq a\big[d_i(0) + d_{i+1}(0)\big] - K(3-2a)v_p\Delta t + v_p\Delta t \\ &\leq a\big[d_i(1) + d_{i+1}(1) + 2(v_e - v_p)\Delta t\big] - \\ &\quad - K(3-2a)v_p\Delta t + v_p\Delta t \\ &= a\big[d_i(1) + d_{i+1}(1)\big] - (K-1)(3-2a)v_p\Delta t. \end{aligned}$$

Continue to derive the above results, and finally

$$r_{i,i+1}(K) \leq a\big[d_i(K) + d_{i+1}(K)\big] - 0.$$

Thus, the theorem is proved. □

Theorem 3.1 provides a broad sufficient condition to ensure the game entering into $K$-besieged status, which has strict requirements on the number and initial position distribution of the pursuers, but thanks to it, the pursuers can choose some specific strategies to decrease the sum of relative distances $d_1, d_2, \ldots, d_n$, without worrying about the destruction of $K$-besieged status. This property comes from the following theorem:

**Theorem 3.2** *If the besieging condition is satisfied, then there exist pursuit strategies such that*

$$\sum_{i=1}^{n} d_i(k+1) - \sum_{i=1}^{n} d_i(k) < 0, \quad k = 0, 1, \ldots, K-1 \tag{17}$$

*regardless of the evasion strategy.*

**PROOF.** From the state equation (10), we know that

$$d_i(k+1) - d_i(k) = [v_p \cos\hat{\phi}_i - v_e \cos(\varphi - \theta_i)]\Delta t$$

where $\hat{\phi}_i = \phi_i - \theta_i$ is the alternative control of $P_i$. Then

$$\frac{1}{v_e\Delta t}\sum_{i=1}^{n}[d_i(k+1) - d_i(k)] = a\sum_{i=1}^{n}\cos\hat{\phi}_i - \sum_{i=1}^{n}\cos(\varphi - \theta_i)$$

Suppose the evader aims to maximize the above expression, then the best strategy of the evader is given by

$$\cos\varphi^* = -\frac{1}{\rho(k)}\sum_{i=1}^{n}\cos\theta_i, \quad \sin\varphi^* = -\frac{1}{\rho(k)}\sum_{i=1}^{n}\sin\theta_i$$

$$\rho(k) = \sqrt{\left(\sum_{i=1}^{n}\cos\theta_i\right)^2 + \left(\sum_{i=1}^{n}\sin\theta_i\right)^2}.$$

Thus, we have

$$\max_{\varphi}\frac{1}{v_e\Delta t}\sum_{i=1}^{n}[d_i(k+1) - d_i(k)] = a\sum_{i=1}^{n}\cos\hat{\phi}_i + \rho(k)$$

At this point, (17) will be satisfied as long as

$$\sum_{i=1}^{n}\cos\hat{\phi}_i < -\frac{\rho(k)}{a}. \tag{18}$$

We claim that the solutions of (18) are existent based on the following arguments: 1) $\rho(k)$ is bounded for any $\theta_i(k)$. Particularly, when $\theta_{i-1,i}(k) = \theta_{i,i+1}(k)$ for every $i = 2, 3, \ldots, n$, $\rho(k) = 1$; 2) Since the besieging condition is satisfied, the pursuers could choose any non-worst strategy without worrying about the destruction of $K$-besieged status. For example, they can follow the evader to make $\cos\hat{\phi}_i^\circ = -1$; 3) From (14) $n \geq \pi/\arcsin a$, then $\sum\cos\hat{\phi}_i^\circ = -n \leq -\pi/\arcsin a < -1/a$. □

**Remark 3.1** From Definition 3.1, we know that in $K$-besieged status

$$\sum_{i=1}^{n} r_{i,i+1}(k) \leq 2a \cdot \sum_{i=1}^{n} d_i(k) \tag{19}$$

where $\sum r_{i,i+1}(k)$ represents the perimeter of polygonal encirclement $\mathcal{G}$. Thus, according to Theorem 3.2, under the besieging condition, there exist some specific strategies for the pursuers to shrink the encirclement $\mathcal{G}$. ■

### *3.4 Capturing status*

Now, we are going to discuss the condition which can guarantee the pursuers capturing the superior evader.

**Definition 3.3** *We say that the game is in capturing status at the $k$th time step, if*

$$r_{i,i+1}(k) \leq 2v_p\Delta t, \tag{20}$$

*holds for any $i = 1, 2, \ldots, n$ and the evader lies in $\mathcal{G}$.*

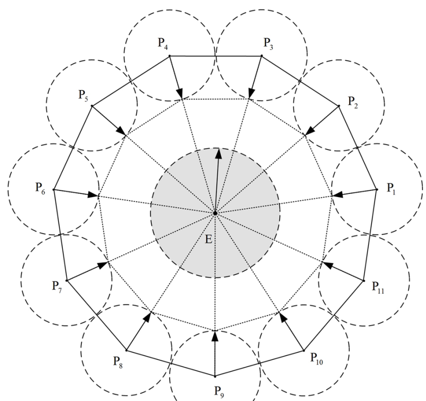


Fig. 6. The capturing status of multi-player pursuit-evasion gamer. Note: the encirclement might not be regular polygon, here is only an extreme example.

As shown in Fig. 6, the game being in capturing status means that the pursuers' accessible regions intersect pairwise in one time step. Thus no matter what strategy the evader chooses, the pursuers can capture it as long as they move towards the barycenter of encirclement $\mathcal{G}$ collectively. Correspondingly, to maximize the capture time, the optimal strategy of the evader will also be moving to this center.

It is well-known that when each side length of a convex polygon is not greater than $2v_p\Delta t$, the radius of whose circumcircle will be no greater than $v_p\Delta t/\sin(\pi/n)$. Thus, if the game is in capturing status, from Fig. 6

$$\frac{v_p\Delta t}{\sin\frac{\pi}{n}} \geq (v_e+v_p)\Delta t \quad\Rightarrow\quad \sin\frac{\pi}{n} \leq \frac{v_p}{v_e+v_p} = \frac{a}{1+a}.$$

Then, the minimum number of the pursuers satisfies

$$n \geq \frac{\pi}{\arcsin\frac{a}{1+a}} > \frac{\pi}{\arcsin a}. \tag{21}$$

Furthermore, if $a=1$,

$$\frac{\pi}{\arcsin\frac{a}{1+a}} = 6, \tag{22}$$

which implies that only if the minimum number is greater than 6, the pursuers capturing one free superior evader is possible (since $a<1$).

### 3.5 Transitions from $K$-besieged status to capturing status

Now, we commence at the possibility of transforming the game from $K$-besieged status into capturing status. We have known that there exist pursuit strategies to decrease $\sum d_i(k)$ under the besieging condition. Besides, when the game is in capturing status, the maximum value of $\sum d_i(k)$ is $nv_p\Delta t/\sin(\pi/n)$. The above arguments inspire us that:

- ♢ Whether there exists some $k<K$ such that when $\sum d_i(k) \leq nv_p\Delta t/\sin(\pi/n)$, the game can reach capturing status from $K$-besieged status in the remaining $K-k$ time steps.

We will tackle this problem in two steps. **First**, if at the $k_1$th time step, every $d_i(k_1) \leq v_p\Delta t/\sin(\pi/n)$, then according to the proofs of Theorem 3.1,

$$r_{i,i+1}(k_1) \leq \frac{2av_p\Delta t}{\sin\frac{\pi}{n}} - (K-k_1)(3-2a)v_p\Delta t. \tag{23}$$

In this sense, as shown in Fig. 7, the pursuers need to adjust every $r_{i,i+1}$ to be equidistant nearly whilst shrinking the encirclement $\mathcal{G}$ until $r_{i,i+1}(K) \leq 2v_p\Delta t$. Since $r_{i,i+1}$ decreases at most $2v_p\Delta t$ at each step,

$$(K-k_1)2v_p\Delta t \geq \frac{2av_p\Delta t}{\sin\frac{\pi}{n}} - (K-k_1)(3-2a)v_p\Delta t - 2v_p\Delta t,$$

$$\Rightarrow\quad K-k_1 \geq \frac{2}{5-2a}\left[\frac{a}{\sin\frac{\pi}{n}} - 1\right]. \tag{24}$$

Considering $\sin(\pi/n) \leq a/(1+a)$, then

$$\frac{2}{5-2a}\left[\frac{a}{\sin\frac{\pi}{n}} - 1\right] \geq \frac{2a}{5-2a} > 0. \tag{25}$$

For example, when $a=0.85$ and $n=9$ we have $K-k_1 \geq 0.9001$, which means that at least at the $(K-1)$th time step every $d_i \leq v_p\Delta t/\sin(\pi/n)$, the game can reach capturing status at the $K$th time step.

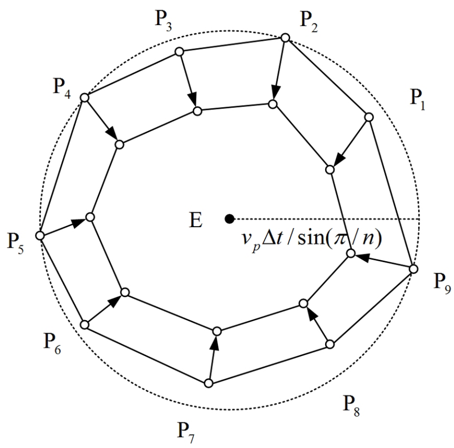


Fig. 7. The strategies of the pursuers when $d_i(k_1) \leq v_p\Delta t/\sin(\pi/n)$, where the radius of dotted circle is $v_p\Delta t/\sin(\pi/n)$.

**Next**, we discuss the other case: when $\sum d_i(k_2) \leq nv_p\Delta t/\sin\frac{\pi}{n}$ at the $k_2$th time step, there exists at least one $i$ subject to $d_i(k_2) > v_p\Delta t/\sin(\pi/n)$. It is worth noting a point that the evader would not let "$d_i(k_2) < (v_e+v_p)\Delta t$" come true, because that is the worst case for the evader. Thus, the minimum value of $d_i(k_2)$ will be $(v_e+v_p)\Delta t$. Besides, since $\sum d_i(k_2)$ is bounded, the maximum value of $d_i(k_2)$ can also be solved

$$\max d_i(k_2) = \frac{nv_p\Delta t}{\sin\frac{\pi}{n}} - (n-1)(v_e+v_p)\Delta t. \tag{26}$$

The strategies of the players are shown in Fig. 8, where all the pursuers move towards the evader to minimize the relative distances. Correspondingly, the evader moves away from the closest pursuer to avoid the distance being less than $(v_e+v_p)\Delta t$.

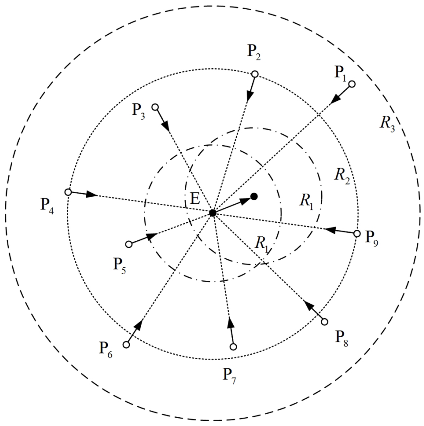


Fig. 8. The strategies of the players when existing $d_i(k_2) > v_p\Delta t/\sin(\pi/n)$, where the radius of circles $R_1$, $R_2$ and $R_3$ are $(v_e+v_p)\Delta t$, $v_p\Delta t/\sin(\pi/n)$ and formula (26), respectively.

Therefore, after one time step the largest $d_i(k_2)$ will decrease at most $(v_e+v_p)\Delta t$, while the smallest one will increase at most $(v_e-v_p)\Delta t$. Then, to ensure every $d_i(k_1) \le v_p\Delta t/\sin(\pi/n)$ at the $k_1$th time step, the following two conditions must be satisfied simultaneously,

$$\max d_i(k_2) - (k_1-k_2)(v_e+v_p)\Delta t \le \frac{v_p\Delta t}{\sin\frac{\pi}{n}}, \tag{27}$$

$$(v_e+v_p)\Delta t + (k_1-k_2)(v_e-v_p)\Delta t \le \frac{v_p\Delta t}{\sin\frac{\pi}{n}}. \tag{28}$$

Simplify the above inequalities, we have

$$k_1 - f \le k_2 \le k_1 - g, \tag{29}$$

where,

$$f = \frac{a}{(1-a)\sin\frac{\pi}{n}} - \frac{1+a}{1-a}, \tag{30}$$

$$g = (n-1)\left[\frac{a}{(1+a)\sin\frac{\pi}{n}} - 1\right]. \tag{31}$$

Since $\sin(\pi/n) \le a/1+a$, then $f \ge 0$ and $g \ge 0$. The comparison of $f$ and $g$ is shown in Fig. 9. It is clear that there exists a pair of $a$ and $n$, such that $f \ge g$ (e.g., when $a = 0.85$ and $n = 9$, $f = 4.2349$, $g = 2.7470$, and we can let $k_1 - k_2 = 3$). Therefore, the existence of $k_2$ is realizable.

In summary, we have the following theorem:

**Theorem 3.3 (Transition Condition)** *If there exist $k_1$ and $k_2$ ($k_2 \le k_1 \le K$) in K-besieged status, such that $\sum d_i(k_2) \le nv_p\Delta t/\sin(\pi/n)$ and the inequalities (24) and (29) hold, then the pursuers can adopt some suitable strategies to make every $d_i(k_1) \le v_p\Delta t/\sin(\pi/n)$ after $k_1 - k_2$ time steps, and further, force the game into capturing status after $K - k_1$ time steps.*

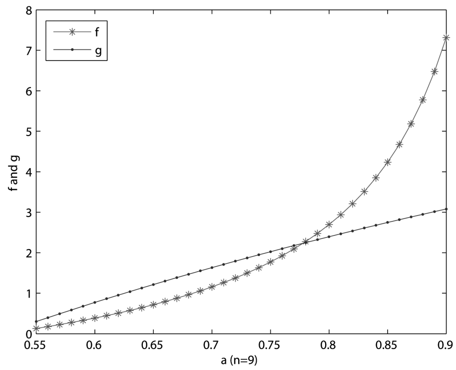


Fig. 9. The comparison of $f$ and $g$.

**Cooperative Pursuit Strategies.** Combine the besieging condition and transition condition, we obtain a cooperative scheme for the pursuers, namely "besiege – shrink – capture". Firstly, the pursuers spread out around the evader to satisfy the besieging condition. Then they shrink the encirclement $\mathcal{G}$ as soon as possible using non-worst strategies to meet the transition condition. At that point, the game can be guaranteed entering into capturing status, as long as the pursuers adopt the corresponding strategies in $k_2 \to k_1$ phase and $k_1 \to K$ phase. Finally, the pursuers can move towards the barycenter of encirclement $\mathcal{G}$ to capture the evader after the game reaches capturing status.

## 4 Examples

In this section, we give some examples to illustrate the theoretical results. We assume that $v_e = 1m/s$, $\Delta t = 1s$ and the capture radius is $v_p\Delta t$ for all the examples. Others such as $v_p$, $n$ and the initial positions of the players will be set distinctively to correspond to different themes.

### *4.1 Examples of multi-player fishing game*

Suppose $v_p = 0.8m/s$, $n = 4$ and the initial positions of the players are $E(20,20)$, $P_1(20,40)$, $P_2(32,18)$, $P_3(18,12)$ and $P_4(5,25)$, respectively. Then the initial surrounding condition is satisfied. As shown in Fig. 10, to maximize the capture time, the evader chose $P_1$ and $P_2$ as the opponents to play fishing game, because the value of $r_{12}/(d_1+d_2)$ was maximum for $1 \le i \le 4$ at the initial time. Then correspondingly the pursuers $P_1$ and $P_2$ adopted strategy (5) to chase the evader, and eventually captured it at $t = 22s$.

Let us take another example, where $v_p = 0.55m/s$, $n = 6$ and the initial positions of the players are $E(20,20)$, $P_1(36,40)$, $P_2(30,20)$, $P_3(25,10)$, $P_4(12,8)$, $P_5(0,22)$ and $P_6(12,40)$, respectively. As shown in Fig. 11, at the initial time, the evader lay inside of the barrier

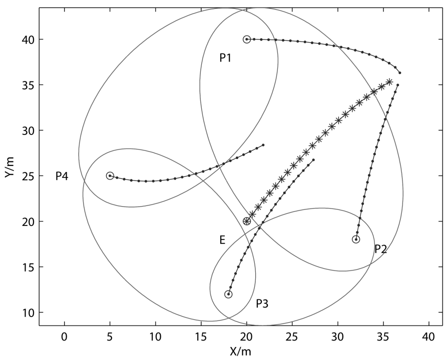


Fig. 10. The evader was captured at $t = 22s$ when the initial surrounding condition held in multi-player fishing game. The ellipses were the barriers of the game, and apparently the evader lay in the capture zone initially.

associated with $P_1$ and $P_2$, thus it chose $P_1$ and $P_2$ as the opponents to play fishing game and finally achieved escape at $t = 20s$.

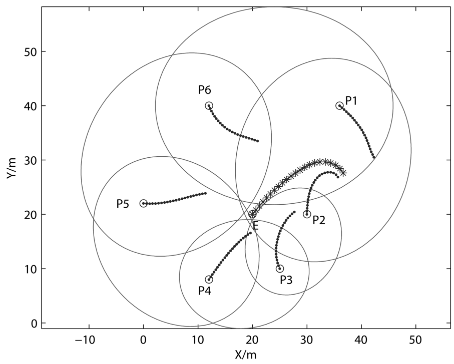


Fig. 11. The evader escaped at $t = 20s$ when its initial position lay in the escape zone, i.e., inside of the barrier associated with $P_1$ and $P_2$.

### 4.2 Examples of general multi-player game

Here, we use the initial settings of Fig. 11 except for $v_p = 0.65m/s$ to compare with the multi-player fishing game. The surrounding condition is satisfied currently, however, the evader can escape when it moves freely. As shown in Fig. 12, the evader changed its directions sharply and continually at the earlier stages, which forced the pursuers to follow its pace back and forth. At the 14th time step, the evader's efforts was rewarded: the surrounded status was destroyed and $r_{56} > a(d_5 + d_6)$ appeared. Then the evader chose $P_5$ and $P_6$ as the opponents of fishing game to escape from the encirclement.

We give the last example to illustrate whether the pursuers can capture the agile evader. Suppose $v_p = 0.85$, $n = 9$ and the initial positions of the players are $E(20, 20)$, $P_1(30, 40)$, $P_2(36, 28)$, $P_3(34, 16)$, $P_4(24, 8)$, $P_5(12, 8)$, $P_6(4, 16)$, $P_7(2, 24)$, $P_8(8, 36)$ and $P_9(18, 42)$,

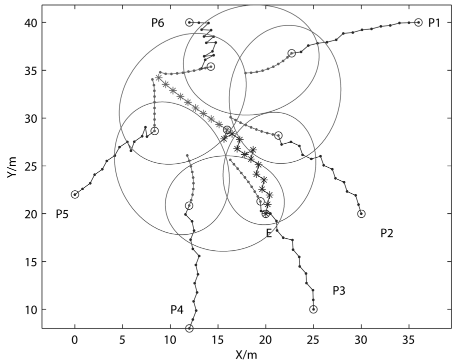


Fig. 12. The evader escaped from 6 pursuers, where the barriers were delineated at the 14th time step. At the moment the evader lay in the escape zone and began to play fishing game with $P_5$ and $P_6$ to escape.

respectively. In this scenario, the number of the pursuers is enough to satisfy the besieging condition, and there exist $k_1 = 21$ and $k_2 = 18$ furnishing the transition condition. It can be seen from Fig. 13 that the game reached capturing status at the 22nd time step. The evader could not destroy the encirclement despite of changing its heading frequently, while the pursuers could advance towards the barycenter of the encirclement gradually. Until the 22nd time step, the evader had nearly nowhere else to go and the pursuers captured it after one time step.

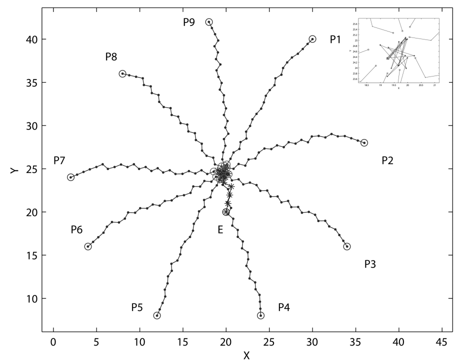


Fig. 13. The evader was captured even though it moved freely, where the game entered into the capturing status at the 22nd time step and terminated at the 23rd time step. The little figure in top right corner amplifies the original figure. We can see that the evader's track is quite chaotic since it can not find any suitable strategy to escape.

## 5 Conclusions and Future Work

Motivated by the hunting and foraging behaviors in group predators, we investigate a class of multi-player pursuit-evasion games with one superior evader in this paper. We are interested in the conditions on the number and initial position distribution of pursuers for which capture can be guaranteed. It shows that for a so-called multi-player fishing game, the minimum number of pursuers $n_{\min} \geq \pi/\arcsin a$; while for a more general case where the superior evader moves freely, the number should be no less than $\pi/\arcsin[a/(1+a)]$.

Further, we provide a surrounding condition and the optimal strategies of the players in the multi-player fishing game. Then for the general case, we define two concepts, besieged status and capturing status, and derive besieging condition and transition condition to ensure the game progressing from $K$-besieged status to capturing status, which comprise a cooperative scheme for the pursuers: besiege − shrink − capture.

In future, the barrier of the game and feasible evasion strategy for the evader still need to be studied in depth. We hope that our findings can be applied for analyzing the interception, tracking, besiegement or collision avoidance of multi-agent systems to some superior targets. To realize these purposes, the future work could focus on the cooperative strategies and more practical motion models of the players under some additional constraints, involving the game environment (map or semi-enclosed polygon), the information structure (limited observation or jamming) and different types of the evader's superiority (more extensive sensing or advantageous position).

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