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On the ISS properties of a class of parabolic DPS' with discontinuous control using sampled-in-space sensing and actuation ⋆

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Abstract

A reaction-diffusion-advection equation with uncertain parameters, and additionally subject to disturbances of both matched and mismatched nature, is considered. It is assumed that only a finite number of point-wise sensing and actuation devices, suitably located in an equi-spaced manner along the 1-dimensional spatial domain of interest, is available. The variable structure control approach is adopted to design the underlying discontinuous feedback control laws. The existence of the resulting closed-loop trajectories is addressed in depth. The state of the closed-loop system is shown to feature an exponential ISS property with respect to mismatched disturbances thereby constituting a new result in the DPS' setting, capturing point-wise sensing and actuation. Apart from this, it is established that an arbitrary level of the mismatched disturbance attenuation can be achieved by employing sufficiently many sensing and actuation devices. It is also established that the matched disturbances entering the control channels can be fully rejected by the proposed design. Tuning rules of the controller parameters are constructively derived by means of the Lyapunov approach, and simulation results are brought into play to support the theoretical development.

Key words: Reaction-diffusion-advection equation; Collocated sensing; Point-wise actuation. Distributed-parameter systems; Sliding mode control.

1 Introduction

Variable structure control has long been recognized as a powerful control method to counteract non-vanishing external disturbances affecting dynamical systems of finite and infinite dimension (see [Utkin, 1992]). As many important systems and industrial processes (such as, e.g., flexible manipulators and chemical reactors) are governed by partial differential equations (PDEs) with uncertain parameters and external disturbances, significant interest has emerged in extending the discontinuous control methods to the infinite-dimensional distributed-parameter systems. Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented (see, e.g., [Levaggi, 2002], [Orlov *et al.*, 1987], [Orlov, 2009]) and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart.

State feedback distributed controllers possess powerful stabilizing features but they are hardly implementable in many practical situations as they require infinitely many sensors and actuators. Due to this, control design methods for distributed-parameter systems (DPS') employing only a finite number of collocated in-domain sensing and actuating devices have been studied [Fridman *et al.*, 2012], [Demetriou *et al.*, 2005], [Demetriou, 2009].

In [Pisano *et al.*, 2012] a boundary second-order sliding-mode controller, providing rejection of matched boundary disturbances, has been proposed. In the presence of mismatched disturbances, however, their rejection becomes in general unfeasible and the notion of input-to-state stability (ISS) has been introduced (see, e.g., the survey paper [Sontag, 2008] and the references therein) to characterize the response of the closed-loop system against them. Roughly speaking, ISS means that once the underlying system is initialized at the origin, the state norm is upper bounded by a continuous disturbance-dependent function, escaping to zero when the disturbance magnitude is nullified whereas the effect of the arbitrary initial conditions is captured by an additional term, being a function of class $\mathcal{KL}$ (see [Khalil, 2002]) depending both on the magnitude of the initial condition and on time, which asymptotically decays as time goes to infinity. Further generalizations of the ISS concept to the finite-time stability setting were given in [Hong *et al.*, 2010], where

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such a $\mathcal{KL}$-class function was required to vanish in finite-time rather than asymptotically as in [Sontag, 2008]. ISS is the subject to be addressed in the present work in the PDEs setting, incorporating point-wise sensing and actuation.

The ISS of DPS' has been addressed in the literature, e.g., in [Prieur *et al.*, 2012], [Dashkovskiy *et al.*, 2013] [Karafyllis *et al.*, 2016] to name a few (see also references therein). The integral ISS (iISS) of DPS' has been tackled as well [Mironchenko *et al.*, 2015], and particularly, within the linear $\mathcal{H}_\infty$ framework (see, e.g., [Morris, (2001),Kasinathan *et al.*, 2013]). A constructive output-feedback design was proposed in [Morris, (2001)] whereas in [Kasinathan *et al.*, 2013] a methodology was given to optimally locate a pre-specified number of actuators in the spatial domain. In this paper, a relevant problem is attacked to determine how many actuator/sensor pairs are needed to achieve a pre-specified arbitrary level of attenuation of in-domain distributed mismatched disturbances.

More precisely, the present work investigates the practical stabilization problem for a class of uncertain parabolic linear PDEs, confined to the finite spatial domain $x \in [0, 1]$, and perturbed by matched and mismatched disturbances. State measurements are assumed to only be available in a finite number of fixed and equi-spaced spatial points $x_i = (i-1)h$ with $i = 1, 2, ..., N$ and $h = 1/(N-1)$, and actuation is applied at these N points x_i only. Therefore, a finite array of stationary sensors and actuators is needed to implement the suggested control scheme.

The underlying control process is uncertain as it possesses unknown diffusivity, advection and reaction coefficients, and it is perturbed by two types of disturbances. These disturbances are formed by a distributed disturbance term $f(x, t)$, acting along the entire spatial domain, and by a number of collocated disturbances $\psi_i(t),\ i = 1, 2, ..., N$, matching the actuation channels. It is worth noticing that within the proposed point-wise actuation framework, the distributed disturbance turns out to be a mismatched one.

The design to be developed is based on the sliding mode control approach, and along with the complete rejection of the matched disturbances it proves to be capable of ensuring an arbitrary level of attenuation of the mismatched disturbances provided that the number N of actuation and sensing devices is taken large enough. The proposed design is accompanied with a detailed ISS Lyapunov analysis to support the expected robustness properties.

The present work makes a step beyond the conference paper [Orlov, *et al.* 2014] where a more narrow class of PDEs was dealt with no advection and reaction terms. Besides the higher complexity of the underlying PDE, the presence of the advection and reaction poses a new challenge that the open loop system is admitted to be unstable unlike that of [Orlov, *et al.* 2014]. Due to this, the closed-loop stabilization and disturbance attenuation are deeper analyzed to yield the minimal number of required actuator-sensor pairs. Additionally, the appropriate ISS framework is introduced in the present work to better describe its contribution.

The structure of the paper is as follows. The present section is concluded by the description of the adopted notation and by an instrumental lemma to be used in the sequel. The ISS problem of interest is formulated in Section 2 for scalar reaction-diffusion-advection processes with point-wise collocated sensing and actuation and with in-domain distributed disturbances. The stabilizing synthesis is then developed in Section 3, where the existence of the closed-loop trajectories is addressed and the main stability results are presented. In Section 4, the effectiveness of the proposed synthesis is supported by numerical simulations. Finally, Section 5 presents some conclusions and discusses perspectives of the future research.

1.1 Notation and instrumental lemma

The symbol $\mathcal{K}$ stands for the class of continuous and strictly increasing functions $\gamma : \Re^+ \to \Re^+$ such that $\gamma(0) = 0$ and $\gamma(t) > 0$ for $t > 0$. $H^\ell(a, b)$ with $a \le b$ and $\ell = 0, 1, 2, \ldots$ denotes the Sobolev space of absolutely continuous scalar functions $z(x)$ on (a, b) with square integrable derivatives $z^{(i)}(x)$ up to the order ℓ and the H^ℓ-norm

$$\|z(\cdot)\|_{H^\ell(a,b)} = \sqrt{\int_a^b \Sigma_{i=0}^{\ell}[z^{(i)}(x)]^2 dx}. \tag{1}$$

Throughout the paper, the standard notations $H^0(a, b) = L_2(a, b)$ and $L_\infty(a, b)$ are used as well. The symbol $L_\infty(0, T; L_2(a, b))$ is reserved for the set of functions $f(x, t)$ such that $f(\cdot, t) \in L_2(a, b)$ for almost all $t \in (0, T)$, $\int_a^b f(x, t)\phi(x)dx$ is Lebesgue measurable in t for all $\phi(\cdot) \in L_2(a, b)$, and $ess\sup_{t\in(0,T)} \int_a^b f^2(x, t)dx < \infty$. For ease of reference, the notations $L_\infty^{loc} = \bigcap_{T>0} L_\infty(0, T)$ and $L_\infty^{loc}(L_2(a, b)) = \bigcap_{T>0} L_\infty(0, T; L_2(a, b))$ are in play.

Lemma 1.1 *Let $z(x) \in H^1(a, b)$ and $x_1, x_2 \in \Re$ be such that $a \le x_1 \le x_2 = x_1 + h \le b$ for some $h \ge 0$. Then, the following inequality*

$$\|z(\cdot)\|^2_{L_2(x_1,x_2)} \le 2h\Big[z^2(x_i) + h\|z_x(\cdot)\|^2_{L_2(x_1,x_2)}\Big] \tag{2}$$

holds for $i = 1, 2$ and $z_x(\cdot) = dz/dx$.

Proof of Lemma 1.1. Given $z(\cdot) \in H^1(a, b)$, it is absolutely continuous and therefore,

$$z(\eta) = z(x_1) + \int_{x_1}^{\eta} z_x(x)dx \quad \forall\, \eta \in [x_1, x_2]. \tag{3}$$

Now squaring both sides of (3), exploiting the well-known inequality $2ab \le a^2 + b^2$, successively applying the Hölder inequality, and taking into account that $\eta \in [x_1, x_1 + h]$ by construction, one arrives at the next chain of inequalities

$$
\begin{aligned}
z^2(\eta) &\le 2\left[z^2(x_1) + \left(\int_{x_1}^{\eta} z_x(x)dx\right)^2\right] \\
&\le 2\left[z^2(x_1) + (\eta - x_1)\int_{x_1}^{\eta} z_x^2(x)dx\right] \\
&\le 2\left[z^2(x_1) + h\|z_x(\cdot)\|^2_{L_2(x_1,x_2)}\right]. \qquad (4)
\end{aligned}
$$

Integrating both sides of (4) with respect to η from x_1 to x_2 yields (2) with $i = 1$. The proof of (2) with $i = 2$ becomes identical under the change of coordinate $\zeta = x_2 - x$. $\square$

2 Problem formulation

Consider the space- and time-varying scalar field $z(x, t)$, evolving in the space $L_2(0, 1)$, with the spatial variable $x \in [0, 1]$ and time variable $t \ge 0$. Let it be governed by the next perturbed parabolic boundary-value problem (BVP)

$$
\begin{aligned}
z_t(x,t) &= \theta z_{xx}(x,t) + dz_x(x,t) + \lambda z(x,t) + f(x,t) \\
&\quad + \sum_{i=2}^{N-1} b_i(x)\left[u_i(t) + \psi_i(t)\right], \qquad (5)\\
z_x(0,t) &= -[u_1(t) + \psi_1(t)], \qquad (6)\\
z_x(1,t) &= u_N(t) + \psi_N(t) \qquad (7)
\end{aligned}
$$

of Neumann type where θ, d, λ are uncertain diffusion, advection and reaction coefficients, respectively, and $f(x, t)$ is an uncertain distributed disturbance of class $L_\infty^{loc}(L_2(0, 1))$. The control signals $u_i(t),\ i = 2, ..., N-1$ and matched disturbances $\psi_i(t) \in L_\infty^{loc}$ enter the in-domain control channels, which are characterized by their spatial localizations $b_i(x)$ whereas $u_1(t)$ and $u_N(t)$ are manipulable boundary control inputs, and $\psi_1(t), \psi_N(t) \in L_\infty^{loc}$ are boundary disturbances. Along with this, the BVP is equipped with N collocated sensors at the boundaries and in the interior of the considered spatial domain.

The associated initial condition (IC) is

$$
z(x, 0) = z^0(x) \in L_2(0, 1). \qquad (8)
$$

Due to the simultaneous presence of the boundary control inputs and perturbations, invoking the usual compatibility conditions $z^0(0) = u_1(0) + \psi_1(0)$ and $z^0(1) = u_N(1) + \psi_N(1)$ in the closed-loop setting appears to be rather restrictive. Instead, the meaning of the BVP (5)-(7) is subsequently viewed in the mild sense. For later use, recall [Butkovskiy, 1982] that the mild solutions of (5)-(7) coincide with the corresponding weak solutions of the so-called standardizing PDE in distributions

$$
\begin{aligned}
z_t(x,t) &= \theta z_{xx}(x,t) + dz_x(x,t) + \lambda z(x,t) + f(x,t) \\
&\quad + \sum_{i=2}^{N-1} b_i(x)\left[u_i(t) + \psi_i(t)\right] + \theta[u_1(t) + \psi_1(t)]\delta(x) \\
&\quad + \theta[u_N(t) + \psi_N(t)]\delta(x-1), \qquad (9)
\end{aligned}
$$

subject to the homogeneous BCs

$$
z_x(0, t) = 0, \qquad z_x(1, t) = 0, \qquad (10)
$$

and to the same IC (8). Since the right-hand side of (9) contains the Dirac distributions $\delta(x)$ and $\delta(x-1)$, the meaning of the BVP (9), (10) is defined indirectly according to the weak solution concept (see, e.g., [Pazy, 2002]).

Definition 1 *A continuous function $z(\cdot, t) \in H^1(0, 1)$, satisfying the BCs (10), is said to be a weak solution of the BVP (9)-(10) on $[0, \tau)$ if for every $\varphi(\xi) \in H^1(0, 1)$, the function $\int_0^1 z(\xi, t)\varphi(\xi)d\xi$ is absolutely continuous on $[0, \tau)$ and relation*

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 z(\xi,t)\varphi(\xi)d\xi = -\theta\int_0^1 z_\xi(\xi,t)\varphi_\xi(\xi)d\xi \\
&+d\int_0^1 z_\xi(\xi,t)\varphi(\xi)d\xi + \int_0^1 [\lambda z(\xi,t) + f(\xi,t)]\varphi(\xi)d\xi \\
&+\sum_{i=2}^{N-1}\int_0^1 b_i(\xi)\left[u_i(t) + \psi_i(t)\right]\varphi(\xi)d\xi \\
&+\theta\big[u_1(t) + \psi_1(t)\big]\varphi(0) + \theta\big[u_N(t) + \psi_N(t)\big]\varphi(1) \qquad (11)
\end{aligned}
$$

holds for almost all $t \in [0, \tau)$.

The weak solution concept (11) relies on the well-defined action $\int_0^1 \varphi(\xi)\delta(\xi - \zeta)d\xi = \varphi(\zeta)$ of the shifted Dirac distribution $\delta(\xi - \zeta),\ \zeta \in [0, 1]$ on an arbitrary test function $\varphi(\xi) \in H^1(0, 1)$ and it is based on the integration-by-parts property

$$
\int_0^1 z_{\xi\xi}(\xi,t)\varphi(\xi)d\xi = -\int_0^1 z_\xi(\xi,t)\varphi_\xi(\xi)d\xi \qquad (12)
$$

of the Sobolev derivatives of the $H^1(0, 1)$-valued functions under the BCs (10).

In the sequel, the spatial distribution functions of the in-domain actuators are specified as follows. Let the points $0 = x_1 < x_2 < \ldots < x_N = 1$ be taken equi-spaced in the spatial domain $[0, 1]$, i.e.,

$$
x_i = (i-1)h, \quad i = 1, 2, ..., N, \quad h = \frac{1}{N-1}. \qquad (13)
$$

The points x_i correspond to the fixed location of the collocated in-domain and boundary sensing and actuation devices. The present investigation is confined to point-wise in-domain actuators, which are located at (13) and contribute to the state PDE (9) in a similar manner as those located at the boundaries.

Thus, the spatial distribution function of the i-th in domain actuator is assumed to be of the form

$$b_i(x) = \theta\delta(x - x_i) \tag{14}$$

of a Dirac distribution, located at the corresponding point x_i and pre-multiplied by the diffusivity parameter θ. Then substituting (14) into (9) for the spatial localization functions $b_i(x),\ i = 2, ..., N-1$ yields the next standardizing plant PDE in distributions

$$\begin{aligned} z_t(x,t) &= \theta z_{xx}(x,t) + dz_x(x,t) + \lambda z(x,t) + f(x,t) \\ &\quad + \theta \sum_{i=1}^{N} \delta(x - x_i)\left[u_i(t) + \psi_i(t)\right], \end{aligned} \tag{15}$$

coupled to the homogenous BCs (10) and the IC (8).

The control objective is to properly design collocated control laws $u_i(t)$, rejecting the matched disturbances $\psi_i(t)$ while also attenuating the mismatched disturbance $f(x,t)$. Potential solutions of the closed-loop BVP (10), (15) should then satisfy the exponential ISS inequality (see, e.g., [Dashkovskiy *et al.*, 2013] for details)

$$\begin{aligned} &\|z(\cdot,t)\|^2_{L_2(0,1)} \le e^{-\beta t}\|z(\cdot,0)\|^2_{L_2(0,1)} \\ &+\gamma_0\left(\|f\|_{L_\infty(0,t;L_2(0,1))}\right) + \sum_{i=1}^{N}\gamma_i\left(\|\psi_i\|_{L_\infty(0,t)}\right) \end{aligned} \tag{16}$$

for any IC (8), $\forall t \ge 0$, $\forall f \in L^{loc}_\infty(L_2(0,1)), \forall \psi_i \in L^{loc}_\infty$, for some constant $\beta > 0$, and for some functions $\gamma_j,\ j = 0, 1, \ldots, N$, of class $\mathcal{K}$.

In addition, *a question is addressed* on how many collocated actuator-sensor pairs are needed to guarantee the exponential ISS property (16) with a pre-specified arbitrarily large decay rate β.

The above ISS issues are subsequently treated under the following assumption on the available information on the uncertain system parameters.

Assumption 1 *There exist a-priori known constants $\theta_0 > 0$, $D \ge 0$, and Λ such that*

$$0 < \theta_0 \le \theta, \quad \lambda \le \Lambda, \quad |d| \le D. \tag{17}$$

□

Apart from this, an extra assumption on admissible magnitudes of the external disturbances is involved.

Assumption 2 *There exist a nonnegative constant F and a-priori known constants Ψ_i, $i = 1, 2, ..., N$ such that*

$$\|f(\cdot,t)\|_{L_2(0,1)} \le F, \qquad |\psi_i(t)| \le \Psi_i \quad \forall\, t > 0. \tag{18}$$

□

Since under Assumption 2 the exponential ISS property (16) is enforced, the guaranteed closed-loop accuracy

$$\|z(\cdot,t)\|^2_{L_2(0,1)} \le \sigma_0\gamma_0(F) + \sigma\Sigma_{i=1}^{n}\gamma_i(\Psi_i), \quad t \ge T \tag{19}$$

can be achieved in a finite transient time $T > 0$ with arbitrary $\sigma_0 > 1$ and $\sigma > 1$. Moreover, by employing sliding mode control components, it becomes possible to read the closed-loop accuracy (19) with $\sigma = 0$.

3 Control synthesis

The collocated feedback inputs

$$u_i(t) = -k_i z(x_i,t) - M_i \mathrm{sign}(z(x_i,t)),\ i = 1, \ldots, N \tag{20}$$

are involved to attain the stated control objective by properly tuning the proportional and switching gains $k_i \ge 0$ and $M_i \ge 0$.

The meaning of the closed-loop system (10), (15), driven by the proportional-discontinuous feedback (20), is adopted in the generalized sense [Orlov, 2000] as a limiting result obtained through the regularization procedure, similar to that proposed for finite-dimensional systems [Utkin, 1992]. According to this procedure, the weak solutions of the BVP (10), (15), (20) are only considered whenever they are beyond any of the discontinuity manifolds $z(x_i,t) = 0,\ i = 1, \ldots, N$ whereas in a vicinity of these manifolds the original system is replaced by a related system, which takes into account all possible imperfections in the new input functions $u^\delta_i(t)$ (e.g., delay, hysteresis, saturation, etc.) and for which there exists a weak solution. *A generalized solution* of the system in question is then obtained by making the characteristics of the new system approach those of the original one. As established in [Levaggi, 2002], such a generalized solution is nothing else than a weak solution of the multi-valued closed-loop system (10), (15) with the switched feedback (20), where the sign function is defined in the *Filippov* sense

$$\mathrm{sign}(z) \in \begin{cases} 1 & \text{if } z > 0, \\ [-1, 1] & \text{if } z = 0, \\ -1 & \text{if } z < 0. \end{cases} \tag{21}$$

The ISS analysis of the closed-loop system (10), (15) with the switched multi-valued inputs (20), (21) is preceded by establishing specific solutions of the resulting BVP to globally exist. The uniqueness of such a solution is actually questionable for potential sliding modes even in the finite-dimensional setting with multi inputs [Utkin, 1992, Chapter 3] what is however irrelevant within the ISS analysis to be conducted where all (possibly, non-unique) plant trajectories are required to exponentially decay according to (16). The interested reader may refer to [Guo *et al.*, 2015] for uniquely determined Filippov solutions in the PDE setting under a single boundary control input.

3.1 Existence of closed-loop trajectories

The aim of this section is to demonstrate that generalized solutions of the BVP (10), (15) with the discontinuous input (20) globally exist. To facilitate the exposition, a particular case $N = 2$ is addressed in depth. The general treatment follows the same line of arguing by separately viewing the coupled plant PDEs over the plant localizations $(x_i, x_{i+1}),\ i = 1, \dots, N-1$.

Passing back to the original plant representation, the closed-loop system (10), (15), (20) with $N = 2$ is given by

$$\begin{aligned} z_t(x,t) = {}& \theta z_{xx}(x,t) + dz_x(x,t) + \lambda z(x,t) + f(x,t) \\ & -\theta\delta(x)\left[M_1 \text{sign}(z(0,t)) - \psi_1(t)\right] \\ & -\theta\delta(x-1)\left[M_2 \text{sign}(z(1,t)) - \psi_2(t)\right], \qquad (22) \\ z_x(0,t) = {}& k_1 z(0,t), \quad z_x(1,t) = -k_2 z(1,t). \qquad (23) \end{aligned}$$

In the sequel, potential dynamics of the BVP (22), (23) are separately investigated out of the discontinuity surface and along it. In the former case, the functions sign$(z(0,t))$ and sign$(z(1,t))$ are single-valued whereas in the latter case of the multi-valued sign function (21), potential sliding modes along $z(0,t) = 0$ and/or $z(1,t) = 0$ are specified according to the equivalent control method [Orlov, 2000], resulting in the Neumann/Robin (mixed) BCs (23) with $k_1 = 0$ and/or $k_2 = 0$ (dependent on whether the sliding mode is at the left boundary and/or at the right boundary) and respectively yielding $M_1\text{sign}(z(0,t)) = \psi_1(t)$ and/or $M_2\text{sign}(z(0,t)) = \psi_2(t)$.

For demonstrating a generalized solution of the multi-valued BVP (22), (23) to globally exist, the invertible state transformation

$$Q(x,t) = e^{\frac{d}{2\theta}x} z(x,t) \qquad (24)$$

is applied to the BVP (10), (15) (properly specified outside and along the discontinuity surface) to simplify it to the one

$$\begin{aligned} Q_t(x,t) = {}& \theta Q_{xx}(x,t) + (\lambda - \frac{d^2}{4\theta})Q(x,t) + e^{\frac{d}{2\theta}x} f(x,t) \\ & -\theta\delta(x)\left[M_1 \text{sign}(z(0,t)) - \psi_1(t)\right] \\ & -\theta\delta(x-1)e^{\frac{d}{2\theta}}\left[M_2 \text{sign}(z(1,t)) - \psi_2(t)\right] \qquad (25) \end{aligned}$$

with no advection term, with the associated IC

$$Q(x,0) = e^{\frac{d}{2\theta}x} z^0(x) \in L_2(0,1), \qquad (26)$$

and with Robin (mixed) boundary conditions

$$\begin{aligned} Q_x(0,t) &= (\frac{d}{2\theta} + k_1)Q(0,t), \\ Q_x(1,t) &= (\frac{d}{2\theta} - k_2)Q(1,t). \qquad (27) \end{aligned}$$

It is worth recalling that the sliding modes along the discontinuity surface(s) $z(0,t) = 0$ and/or $z(1,t) = 0$ are governed by the PDE (25) with nullified terms $M_1\text{sign}(z(0,t)) - \psi_1(t) = 0$ and/or $M_2\text{sign}(z(1,t)) - \psi_2(t) = 0$, and with BCs (27), specified with $k_1 = 0$ and/or $k_2 = 0$. In any case, a generalized solution of such a BVP, is subsequently shown to admit the Fourier expansion

$$Q(x,t) = \sum_{m=1}^{\infty} q_m(t) r_m(x),\ x \in [0,1], t \geq 0 \qquad (28)$$

in terms of the eigenfunctions $r_m(x) \in L_2(0,1),\ k = 1, \dots$ of the Sturm-Liouville problem (cf. [Boyce *et al.*, 1997], [Karafyllis *et al.*, 2016])

$$\begin{aligned} & \theta r_{xx}(x) = -\mu r(x), \\ & r_x(0) = (\frac{d}{2\theta} + k_1)r(0),\ r_x(1) = (\frac{d}{2\theta} - k_2)r(1). \qquad (29) \end{aligned}$$

In $L_2(0,1)$, the Sturm-Liouville problem (29) is actually well-recognized [Butkovskiy, 1982] to possess an orthonormal basis of the uniformly bounded eigenfunctions

$$r_m(\cdot): \max_{x\in[0,1]} |r_m(x)| \leq R,\ m = 1, 2, \dots \qquad (30)$$

with some positive constant R and with corresponding real eigenvalues $0 < \mu_1 < \mu_2 < \dots < \mu_m < \dots$ such that

$$\sum_{m=1}^{\infty} \mu_m^{-1} < \infty. \qquad (31)$$

By substituting (28) into (25) and taking into account the Fourier expansion of the shifted Dirac distribution [1]

$$\delta(x - \xi) = \sum_{m=1}^{\infty} r_m(\xi) r_m(x), \qquad (32)$$

[1] In order to reproduce (32) it suffices to verify that the action $\int_0^1 \sum_{m=1}^{\infty} r_m(\xi) r_m(x)\varphi(x)dx = \sum_{k=1}^{\infty} \varphi_m r_m(\xi) = \varphi(\xi)$ of the right-hand side of (32) on an arbitrary test function $\varphi(x) = \sum_{m=1}^{\infty} \varphi_m r_m(x)$, (as a matter of fact, admitting the standard Fourier representation) coincides with that of the shifted Dirac distribution $\delta(x - \xi)$.

the solution Fourier modes $q_m(t),\ m = 1, 2, \ldots$ prove to be governed by

$$
\begin{aligned}
&\dot{q}_m(t) = \left(-\mu_m + \lambda - \frac{d^2}{4\theta}\right) q_m(t) + f_m^e(t) \\
&-\theta r_m(0)\Big[M_1 \mathrm{sign}(< q(t), r(0) >) - \psi_1(t)\Big] \\
&-\theta r_m(1) e^{\frac{d}{2\theta}}\Big[M_2 \mathrm{sign}(< q(t), r(1) >) - \psi_2(t)\Big]
\end{aligned} \tag{33}
$$

where

$$
f_m^e(t) = \int_0^1 e^{\frac{d}{2\theta}x} f(x,t) r_m(x) dx \tag{34}
$$

is the corresponding Fourier coefficient of the external disturbance $e^{\frac{d}{2\theta}x} f(x,t)$ and

$$
< q(\cdot), r(\cdot) >= \sum_{m=1}^{\infty} q_m(\cdot) r_m(\cdot). \tag{35}
$$

It follows that a potential solution (28) of the BVP (25)-(27) can be represented in the integral form

$$
\begin{aligned}
&Q(x,t) = \sum_{m=1}^{\infty} e^{\left(-\mu_m + \lambda - \frac{d^2}{4\theta}\right)t} q_m^0 r_m(x) \\
&+ \sum_{m=1}^{\infty} r_m(x) \int_0^t e^{\left(-\mu_m + \lambda - \frac{d^2}{4\theta}\right)(t-\tau)} \Big\{ f_m^e(\tau) \\
&-\theta r_m(0)\Big[M_1 \mathrm{sign}(Q(0,\tau)) - \psi_1(\tau)\Big] \\
&-\theta r_m(1) e^{\frac{d}{2\theta}}\Big[M_2 \mathrm{sign}(Q(1,\tau)) - \psi_2(\tau)\Big]\Big\} d\tau
\end{aligned} \tag{36}
$$

where

$$
q_m^0 = \int_0^1 Q(x,0) r_m(x) dx,\ m = 1, 2, \ldots \tag{37}
$$

are the Fourier coefficients of the initial distribution (26). In order to conclude that such a solution, formally satisfying the BVP (25)-(27), does globally exist it suffices to note that the right hand side of (36) is straightforwardly verified to absolutely converge in $L_2(0,1)$ for any $t \geq 0$, $\forall f \in L_\infty^{loc}(L_2(0,1))$, $\forall \psi_1, \psi_2 \in L_\infty^{loc}$, and for any admissible value of the sign function (21). Indeed, taking into account that due to (30), (31), the series $\sum_{m=1}^{\infty} \left(\mu_m - \lambda + \frac{d^2}{4\theta}\right)^{-1} r_m(x)$ is absolutely convergent for any $x \in [0,1]$, one derives

$$
\begin{aligned}
&\|Q(x,t)\|_{L_2(0,1)} \leq \|Q(0,t)\|_{L_2(0,1)} e^{\left(-\mu_1 + \lambda - \frac{d^2}{4\theta}\right)t} \\
&+ \|f\|_{L_\infty(0,t;L_2(0,1))} \sum_{m=1}^{\infty} \left(\mu_m - \lambda + \frac{d^2}{4\theta}\right)^{-1} \\
&+ \theta\Big(M_1 + \|\psi_1\|_{L_\infty(0,t)}\Big)\Big[\sum_{m=1}^{\infty} \left(\mu_m - \lambda + \frac{d^2}{4\theta}\right)^{-1} r_m(0)\Big] \\
&+ \theta e^{\frac{d}{2\theta}} \Big(M_2 + \|\psi_2\|_{L_\infty(0,t)}\Big) \\
&\times \Big[\sum_{m=1}^{\infty} \left(\mu_m - \lambda + \frac{d^2}{4\theta}\right)^{-1} r_m(1)\Big] < \infty
\end{aligned} \tag{38}
$$

for all $t \geq 0$.

Summarizing and extrapolating from $N = 2$ to an arbitrary N, the following result is obtained.

Theorem 1 *The BVP* (10)*, (15) with the proportional-discontinuous feedback (20) globally possesses a generalized solution for any IC* (8)*, for any plant parameters* (17)*, for any controller gains $k_i, M_i \geq 0,\ i = 1, \ldots, N$, and for any $f \in L_\infty^{loc}(L_2(0,1))$, $\psi_1, \psi_2 \in L_\infty^{loc}$.*

Proof of Theorem 1. In a particular case where only $N = 2$ boundary actuators were in play, it was explicitly shown that by means of the invertible transformation (24), a generalized solution of the closed-loop BVP (10), (15), (20) can be expressed in terms of a generalized solution of the auxiliary system (25), (27), which admits the integral representation (36) of the absolutely and uniformly convergent Fourier series (28). The extension of the proof to an arbitrary number N of the available actuators is straightforward. Such a technical extension is however rather lengthy and its details are left to the interested reader. □

3.2 ISS analysis

Once the closed-loop BVP (10), (15), (20) is established to possess generalized solutions, its Lyapunov ISS analysis becomes eligible. Before applying such an analysis, suppose that the collocated proportional-discontinuous output feedback (20) is tuned with non-negative switching gains M_i and with proportional gains such that

$$
k_1 > 1 + \frac{1}{2}D,\ k_i > 1,\ i = 2, ..., N-1,\ k_N \geq \frac{1}{2}D. \tag{39}
$$

Provided that the number N of actuator-sensor pairs is large enough to meet the condition

$$
N \geq \max\left\{1 + \frac{\beta + 2\Lambda}{\theta_0}, 0\right\} \tag{40}
$$

with β being the decay rate from (16), the next result proves to be in force.

Theorem 2 *Under Assumption 1 on the plant parameters, consider the perturbed reaction-diffusion-advection process (15) with the BCs* (10) *and arbitrary ICs* (8). *Let it be controlled by the output feedback (20) with proportional gains, tuned according to* (39)*, with arbitrary switching magnitudes* $M_i, i = 1, \dots, N$*, and with sufficiently large actuators number* N*, satisfying* (40). *Then the closed-loop system is exponentially ISS so that* (16) *holds with*

$$\gamma_0(r) := \frac{\sqrt{2}}{\varepsilon\beta} r, \quad \gamma_j(r) := \frac{\theta_0}{\varepsilon\beta} r, \quad j = 1, \dots, N, \tag{41}$$

and sufficiently small $\varepsilon > 0$ *such that*

$$\frac{\theta_0}{h} - 2\Lambda - \frac{\sqrt{2}\varepsilon}{2} > 0 \tag{42}$$

$$\varepsilon < 2 \min\left\{k_1 - \frac{D}{2} - 1, k_i - 1, k_N - \frac{D}{2}\right\} \tag{43}$$

for $i = 2, \dots, N-1$.

Proof of Theorem 2. Consider the Lyapunov functional candidate

$$V(z) = \frac{1}{2}\int_0^1 z^2(x,t)dx = \frac{1}{2}\|z(\cdot,t)\|^2_{L_2(0,1)}. \tag{44}$$

By applying the differentiation rule (11) (specified with $\varphi(\xi) = z(\xi,t)$, the spatial variable $\xi = x$, and the frozen time instant t) to the generalized (weak) solutions of the BVP (10), (15), the time derivative of $V(z) = V(z(\cdot,t))$ along these solutions, which is for simplicity referred to as $\dot{V}(t)$, is evaluated as follows

$$\begin{aligned}\dot{V}(t) &= \int_0^1 z(x,t)z_t(x,t)dx = -\theta\int_0^1 z_x^2(x,t)dx \\ &+ \frac{1}{2}d\left[z^2(1,t) - z^2(0,t)\right] \\ &+ \lambda\int_0^1 z^2(x,t)dx + \int_0^1 z(x,t)f(x,t)dx \\ &+ \theta\sum_{i=1}^N z(x_i,t)\left[u_i(t) + \psi_i(t)\right]dx.\end{aligned} \tag{45}$$

Estimating the right-hand side of the above relation (45) by applying the well-known Hölder integral inequality (see, e.g., [Abramowitz *et al.*, 1974]) to its fourth term, then substituting the control law (20) into its fifth term, and finally employing the plant parameter estimates (17), one arrives at

$$\begin{aligned}\dot{V}(t) &\le -\theta_0\|z_x(\cdot,t)\|^2_{L_2(0,1)} + \Lambda\|z(\cdot,t)\|^2_{L_2(0,1)} \\ &+ \|f(\cdot,t)\|_{L_2(0,1)}\|z(\cdot,t)\|_{L_2(0,1)} - \theta_0\sum_{i=1}^N \tilde{k}_i z^2(x_i,t) \\ &- \theta_0\sum_{i=1}^N\left[M_i - |\psi_i(t)|\right]|z(x_i,t)|\end{aligned} \tag{46}$$

where

$$\begin{aligned}&\tilde{k}_1 = k_1 - \frac{1}{2}D, \quad \tilde{k}_N = k_N - \frac{1}{2}D, \\ &\tilde{k}_i = k_i, \quad i = 2, \dots, N-1.\end{aligned} \tag{47}$$

By applying the straightforward norm decomposition

$$\|z(\cdot)\|^2_{L_2(0,1)} = \sum_{i=1}^{N-1}\|z(\cdot)\|^2_{L_2(x_i,x_{i+1})} \tag{48}$$

to the solution derivative $z_x(\cdot)$, it follows that

$$\begin{aligned}\dot{V}(t) &\le -\theta_0\sum_{i=1}^{N-1}\left[\|z_x(\cdot)\|^2_{L_2(x_i,x_{i+1})} + \tilde{k}_i z^2(x_i,t)\right] \\ &+ \Lambda\|z(\cdot,t)\|^2_{L_(0,1)} + \|f(\cdot,t)\|_{L_2(0,1)}\|z(\cdot,t)\|_{L_2(0,1)} \\ &- \theta_0\tilde{k}_N z^2(x_N,t) - \theta_0\sum_{i=1}^N\left[M_i - |\psi_i(t)|\right]|z(x_i,t)|.\end{aligned} \tag{49}$$

Now taking into account that relation (2) under sampling (13) yields

$$\begin{aligned}&\|z_x(\cdot)\|^2_{L_2(x_i,x_{i+1})} + \tilde{k}_i z^2(x_i,t) \ge h\|z_x(\cdot)\|^2_{L_2(x_i,x_{i+1})} \\ &+ z^2(x_i,t) + \left(\tilde{k}_i - 1\right)z^2(x_i,t) \\ &\ge \frac{1}{2h}\|z(\cdot)\|^2_{L_2(x_i,x_{i+1})} + \left(\tilde{k}_i - 1\right)z^2(x_i,t)\end{aligned} \tag{50}$$

for $i = 1, 2, ..N-1$, for $h = 1/(N-1) < 1$, and for the controller gains (39), inequality (49) is further manipulated to

$$\begin{aligned}\dot{V}(t) &\le -\frac{\theta_0}{2h}\sum_{i=1}^{N-1}\|z(\cdot)\|^2_{L_2(x_i,x_{i+1})} + \Lambda\|z(\cdot,t)\|^2_{L_2(0,1)} \\ &+ \|f(\cdot,t)\|_{L_2(0,1)}\|z(\cdot,t)\|_{L_2(0,1)} - \theta_0\sum_{i=1}^N k_i^* z^2(x_i,t) \\ &- \theta_0\sum_{i=1}^N\left[M_i - |\psi_i(t)|\right]|z(x_i,t)|\end{aligned} \tag{51}$$

where

$$k_i^* = \tilde{k}_i - 1, \quad i = 1, \ldots, N-1, \text{ and } k_N^* = \tilde{k}_N. \tag{52}$$

Since (48) ensures that the Lyapunov functional candidate (44) satisfies the relation

$$\sum_{i=1}^{N-1} \|z(\cdot)\|^2_{L_2(x_i,x_{i+1})} = 2V(z), \tag{53}$$

inequality (51) is simplified to

$$\begin{aligned}\dot{V}(t) \leq & -\left[\frac{\theta_0}{h} - 2\Lambda\right] V(z) \\ & + \|f(\cdot,t)\|_{L_2(0,1)} \sqrt{2V(z)} - \theta_0 \sum_{i=1}^{N} k_i^* z^2(x_i,t) \\ & - \theta_0 \sum_{i=1}^{N} \left[M_i - |\psi_i(t)|\right] |z(x_i,t)|. \end{aligned} \tag{54}$$

The latter inequality verifies that relation (44) determines an exponential ISS Lyapunov functional [Prieur *et al.*, 2012]. Indeed, by taking into account the well-known inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, which is valid for arbitrary real numbers a and b and for an arbitrary $\varepsilon > 0$, (54) results in

$$\begin{aligned}\dot{V}(t) \leq & -\left[\frac{\theta_0}{h} - 2\Lambda - \frac{\varepsilon}{\sqrt{2}}\right] V(z) \\ & -\theta_0 \sum_{i=1}^{N} \left[k_i^* - \frac{\varepsilon}{2}\right] z^2(x_i,t) + \frac{\sqrt{2}}{2\varepsilon} \|f(\cdot,t)\|^2_{L_2(0,1)} \\ & + \frac{\theta_0}{2\varepsilon} \sum_{i=1}^{N} \psi_i^2(t). \end{aligned} \tag{55}$$

To complete the proof it remains to note that under conditions (39), (40), (43), coupled to notations (47), (52), the factors $\frac{\theta_0}{h} - 2\Lambda - \frac{\sqrt{2}\varepsilon}{2}$ and $k_i^* - \frac{\varepsilon}{2}$, $i = 1, \ldots, N$ are positive, and with this in mind, applying [Khalil, 2002, Comparison Lemma 3.4] to (55) yields the ISS property (16), specified with (41). □

Theorem 2 presents the proportional gains tuning rules (39), (40) for the closed-loop BVP (10), (15), (20) to be ISS in the presence of mismatched disturbances, distributed over the plant domain. Properly tuning the magnitudes of the discontinuous components of the proposed feedback (20) allows one to additionally reject the matched disturbances, collocated with the available point-wise actuators. Such a strengthened version of Theorem 2 is as follows.

Theorem 3 *Let along with the conditions of Theorem 2, Assumption 2 on the disturbance magnitudes be additionally in force and let the proportional gain tuning rules* (39)*,* (40) *be accompanied with their counterpart*

$$M_i > \Psi_i, \quad i = 1, 2, \ldots N \tag{56}$$

on the discontinuous control components. Then starting from a finite time instant

$$T = \frac{\|z^0(\cdot)\|_{L_2(0,1)}}{(\sigma_0 - 1)F}, \tag{57}$$

dependent on the plant IC (8)*, relation*

$$\|z(\cdot,t)\|_{L_2(0,1)} \leq \gamma^2 F, \quad t \geq T \tag{58}$$

holds true with

$$\gamma^2 = \frac{2h}{\theta_0 - 2h\Lambda}\sigma_0, \tag{59}$$

and an arbitrary parameter $\sigma_0 > 1$ for all generalized solutions $z(\cdot,t)$ of the closed-loop BVP (10)*, (15), (20).*

Proof of Theorem 3. Under Assumption 2, the differential inequality (54) on the Lyapunov functional (44) is simplified to

$$\dot{V}(t) \leq -\left[\left(\frac{\theta_0}{h} - 2\Lambda\right)\sqrt{V(z)} - \sqrt{2}F\right]\sqrt{V(z)}. \tag{60}$$

The latter inequality ensures that the domain

$$\sqrt{V(z)} \leq \sigma_0 \frac{\sqrt{2}hF}{\theta_0 - 2h\Lambda} \tag{61}$$

is invariant and it is reached in finite time with any $\sigma_0 > 1$. Indeed, (60) guarantees that the differential inequality

$$\dot{V}(t) \leq -\sqrt{2}F(\sigma_0 - 1)\sqrt{V(z)} \tag{62}$$

holds true outside of domain (61) whereas (62) ensures the finite-time attractiveness of this domain with a transient time T, estimated as follows

$$T \leq \frac{\sqrt{2V(z^0)}}{(\sigma_0 - 1)F}. \tag{63}$$

To complete the proof it suffices to note that by taking into account the explicit Lyapunov functional representation (44), the transient time estimate (63) and the attraction domain (61) itself are readily reproduced in terms of the state $z(\cdot,t)$ of the plant in the form of (57) and (58). respectively. □

4 Simulation results

Consider the BVP (15), (10), with parameters $\theta = 10$, $d = 1$ and $\lambda = 2$, distributed disturbance $f(x,t) = 20sin(2\pi x)sin(10\pi t))$ and collocated matched disturbances $\psi_i(t) = sin(2\pi t)$ ($i = 1, 2, ..., N$). Due to the chosen disturbance profiles, the bounds in (18) take the value $F = 20/\sqrt{2} \approx 14.14$ and $\Psi_i = 1$. The IC is set as $z(\xi, 0) = 1$ and the limiting values for the uncertain parameters, reported in (17), are taken as $\theta_0 = 5$, $D = 2$, $\Lambda = 3$. In accordance with (39) and (56), the controller parameters are set to $k_i = M_i = 2,\ i = 1, 2, .., N$.

For solving the closed-loop PDE, a standard finite-difference approximation method is used by discretizing the spatial solution domain $x \in [0, 1]$ into a finite number of 200 uniformly spaced solution nodes. The resulting 200-th order discretized system is then solved by the fixed-step forward Euler method with step $T_s = 10^{-5}$.

First, the performance of the open-loop system is numerically analyzed. Figure 1 depicts the unstable spatiotemporal profile of the solution $z(x, t)$ with $u_i(t) = 0,\ i = 1, 2, .., N$.

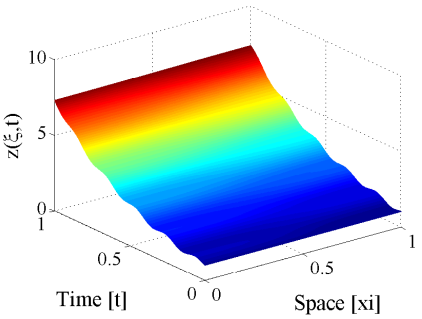

Fig. 1. Spatiotemporal solution profile in the open loop test

The disturbance attenuation problem is then addressed under the pre-specified attenuation level $\gamma^2 = 0.1$ for the attenuation factor (59) appearing in (58) and the decay rate $\beta = 10$. By applying relations (59), (13), (40), it turns out that at least $N = 32$ actuator-sensor pairs are needed to guarantee the desired disturbance attenuation level and and decay rate. The corresponding closed-loop behaviour is shown in Fig. 2, which reports the long-term time evolution of the L_2 norm $\|z(\cdot, t)\|_{0,0,1}$ (left-plot) and a zoom on the steady state profile (right plot). The left plot shows clearly the finite-time convergence towards the invariant domain (58). The zoom in the right plot shows that the steady-state accuracy remains within the guaranteed bound $\|z(\cdot, t)\|_{0,0,1} \leq \gamma^2 F \approx 1.4$, specified according to (58), (59). As typical in the variable structure control design, the actual accuracy appears to be much higher than that predicted by the theoretical computations, what is due to the worst-case nature of the underlying analysis.

To investigate how the number of actuator-sensor pairs varies with the desired level of attenuation γ^2, the computation of the minimal number N_{min} of required actuators has been made by considering relation (59) with different values of the desired attenuation coefficient γ^2. The resulting diagram is shown in Fig. 3 which highlights the inverse dependence of N_{min} on γ^2.

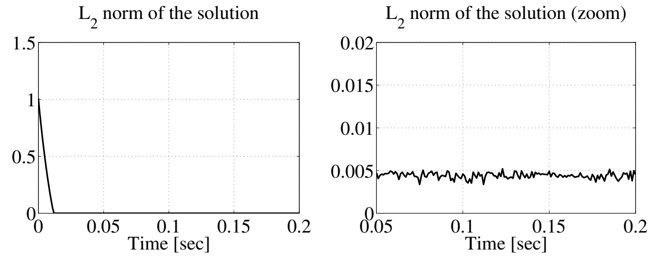

Fig. 2. Solution L_2 norm $\|z(\cdot, t)\|_{0,0,1}$ in the closed loop test with $N = 32$

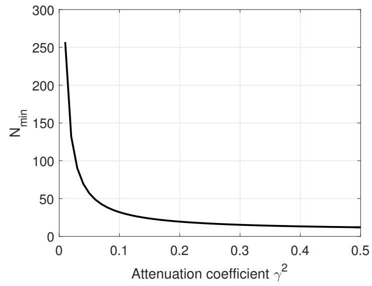

Fig. 3. Minimal number N_{min} of actuator-sensor pairs vs. desired attenuation coefficient γ^2

Particularly, in order to attain the less restrictive disturbance attenuation level $\gamma^2 = 0.2$, compared to the value $\gamma^2 = 0.1$ used in the previous test, one needs to utilize at least $N = 19$ actuator-sensor pairs. While the left plot of Fig. 4 illustrates that using 19 actuator-sensor pairs, the transient time remains almost the same, the right plot of the figure shows that the actual accuracy decreases by nearly two times as compared with the previous test when 32 actuator-sensor pairs were used.

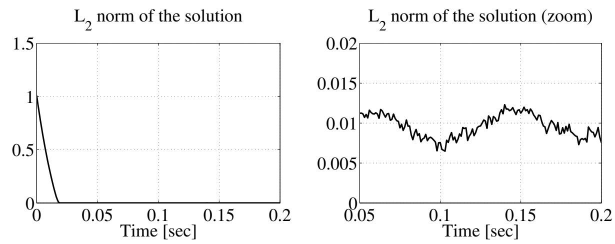

Fig. 4. Solution L_2 norm $\|z(\cdot, t)\|_{0,0,1}$ in the closed loop test with $N = 19$

5 Concluding remarks

A variable-structure control system, using a finite number of actuation and sensing devices, is suggested to address the ISS of an uncertain reaction-diffusion-advection process, affected by matched and mismatched disturbances. It is shown that the level of the mismatched disturbance attenuation can be set arbitrarily, and the complete rejection of the distributed disturbance is theoretically achievable when the number of devices tends to infinity. Among the relevant problems to be attacked within the proposed framework, it is of great interest to implement sampled-in-space second-order sliding mode distributed controllers recently developed in [Orlov *et al.*, 2011a,Orlov *et al.*, 2011b]. Such controllers are expected to attenuate the chattering phenomenon which, due to the discontinuous nature of the control algorithm, suggested in the present work, is a source of the potential drawback in applications.

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