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Brief paper

On robustness analysis of linear vibrational control systems[☆]

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ABSTRACT

By injecting high frequency dither signals, it is possible to stabilize an inverted pendulum without any feedback. The concept of the vibrational control system is thus proposed to provide extra design freedom in stabilization or other performance indexes. Although various vibrational control algorithms have been proposed and implemented in literature, little work has been done to show their robustness with respect to disturbances and uncertainties. This paper focuses on the robustness analysis of linear vibrational control systems with additive disturbances. By applying perturbation techniques, the linear vibrational control system is shown to be input-to-state stable with respect to disturbances. When disturbances are periodic, frequency analysis technique obtains a less conservative estimate of the ultimate bound of the system, indicating that disturbances with high frequencies lead to relatively small ultimate bounds. When additive state-dependent disturbances are considered, weak averaging techniques can be used to show the robustness of the system when bounded disturbances are slow time-varying. Numerical results support the theoretic analysis.

1. Introduction

The vibrational control method was proposed to stabilize systems such as inverted pendulums in an open-loop fashion by inserting a high frequency dither instead of using feedback (Meerkov, 1980). It has been shown that high-frequency dithers can introduce extra design freedom in stabilization and performance improvement (Meerkov, 1980), making it attractive to many engineering applications, see, for example, chemical reactors (Cinar, Deng, Meerkov, & Shu, 1987), gas lasers (Meerkov & Shapiro, 1976) and under-actuated robotics (Hong, 2002; Yabuno, Matsuda, & Aoshima, 2005) and references therein.

We adapt the example of vibrational control system (Khalil, 1996 Example 8.10) to illustrate this idea. By vertically oscillating the suspension point using a sine wave dither with a small amplitude but high frequency (Kapitsa, 1951; Stephenson, 1908), an inverted pendulum can be locally stabilized.[1] The dynamics model of this system is presented as:

$$ml\ddot{\theta} + (mg - ma\omega\cos\omega t + ka\sin\omega t)\sin\theta + kl\dot{\theta} = 0, \quad (1)$$

where θ is the angular displacement, m is the mass, l is the length of pendulum, k is the viscous friction coefficient, a and ω are the amplitude and frequency of oscillating dither respectively. By linearizing it around its equilibrium position at $(\pi, 0)$, the linearized model in state-space becomes:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ \frac{g}{l} & -\frac{k}{m}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ \frac{a\omega}{l}\cos\omega t + \frac{ka}{ml}\sin\omega t & 0\end{bmatrix}}_{B(\omega t)}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \quad (2)$$

It consists of two parts. One is the unstable A matrix and the other is the periodic matrix $B(\omega t)$ coming from the sinusoidal dither. It has been proved that if the dither frequency ω is sufficiently large, the inverted pendulum is locally stable (Bogoliubov & Mitropolski, 1961). This example shows that even though the equilibrium is unstable, an open-loop controller using a high frequency dither can locally stabilize the system. In this example, though dither is inserted without using feedback, the system (2) has a "feedback-like" structure. This feedback-like behavior in vibrational control design was described as a natural interaction between the system dynamics and vibrated component as pointed out in Shapiro and Zinn (1997).

A thorough analysis of linear vibrational control systems in the form of $\dot{x} = Ax + B(\omega t)x$ was introduced by Meerkov (1980). In his seminal work, it was assumed that A has a controllable canonical form. Under such a scenario, a necessary and sufficient condition to ensure stabilization is that the trace of A is negative.

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[1] The amplitude of dither is selected as $\frac{a}{\omega}$ to simplify the presentation.

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Moreau and Aeyels (2004) used the idea of vibrational control to enlarge the domain of attraction by designing a periodic output feedback for linear-time-invariant (LTI) systems. Similarly, pole assignment capabilities of vibrational control method were discussed in Kabamba, Meerkov, and Poh (1998). Recently, Berg proposed a design tool using the concept of stability maps for a class of second-order linear periodic systems (Berg & Wickramasinghe, 2015).

Subsequently, the framework of nonlinear vibrational control systems was established by R. Bellman and J. Bentsman in Bellman, Bentsman, and Meerkov (1985, 1986a, b). The criteria of stabilization, controllability and transient behavior for different types of vibrational control systems were addressed. Shapiro and Zinn (1997) showed that a class of dynamic system can be locally stabilized by a nonlinear vibrational controller even if its Jacobian matrix has a positive trace. This result shows that the vibrational control method can be applied to a large class of engineering systems.

Although various stability results of vibrational control systems have been published, there is little work addressing the robustness with respect to disturbances or uncertainties, which is one of the most important performance requirements for engineering applications. This work focuses on linear vibrational control systems and explores its robustness in the presence of two types additive disturbances: one is state-independent and the other is state-dependent.

In the motivating example, there are always external forces/moments that can perturb the inverted pendulum. This leads to a linear vibrational control system in the presence of state-independent disturbances:

$$\dot{x} = Ax + B_1(\omega t)x + B_2 w(t),\ x(t_0) = x_0 \in \mathbb{R}^n, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}, B_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}, B_2 \in \mathbb{R}^{n\times m}, w : [t_0, \infty) \to \mathbb{R}^m$. Similarly, state-dependent disturbances such as variation of friction coefficient can also appear thus robustness analysis is important.

One of the key techniques in stability analysis of vibrational control systems is averaging (Bellman et al., 1986b; Shapiro & Zinn, 1997). The existence of disturbances would perturb the averaged systems, leading to undesirable performance. When state-independent disturbances are considered, the perturbation technique (Khalil, 1996) can be applied to show the robustness. When disturbances are bounded and periodic, by using frequency analysis, our result shows that the ultimate bound of the system is inversely proportional to the frequency of the disturbances.

When state-dependent disturbances are considered, neither perturbation method nor frequency analysis can be directly applied. Recently, strong average and weak average techniques have been developed to analyze the robustness of nonlinear time-varying systems when taking disturbances into consideration (Nešić & Teel, 2001). It is shown that the strong average of the vibrational control system does not exist while the weak average exists. By exploring the stability of the weak averaged system, we show that the linear vibrational control system is robust to bounded but slow time-varying disturbances.

The remainder of this paper is organized as follows. In Section 2, preliminaries are stated. Problem formulation and main results are presented in Section 3 for state-independent disturbances. Section 4 discusses the robustness of the linear vibrational control systems with respect to state-dependent disturbances, followed by simulation examples in Section 5. Section 6 concludes the paper. All proofs are provided in Appendix.

2. Preliminaries

The set of real numbers is denoted as $\mathbb{R}$. A continuous function $\gamma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is of class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ belongs to class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Define the infinity norm as $\|w\|_\infty := \text{ess}\ \sup_{t\geq 0}|w(t)|$. If $\|w\|_\infty < \infty$, it can be called that $w \in \mathcal{L}_\infty$.

2.1. Vibrational stabilization

In literature, a generic form of vibrational control systems is (Bellman et al., 1985; Bullo, 2002; Meerkov, 1980):

$$\dot{x} = f(x) + g\left(\frac{t}{\varepsilon}, x\right), x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with an equilibrium point x_e such that $f(x_e) = 0$. The nonlinear mapping $g : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in both arguments and T-periodic in time.

Definition 1 (*Bellman et al., 1985*). The equilibrium point x_e of $f(x)$ is said to be vibrationally stabilizable (v-stabilizable) if for any $\upsilon > 0$ there exists almost periodic and zero-mean $g(\frac{t}{\varepsilon}, x)$ in the first argument such that system (4) has an almost periodic asymptotically stable solution $x^s(t)$ characterized by

$$|\bar{x}^s - x_e| < \upsilon,$$

where $\bar{x}^s = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^s(\tau)d\tau$.

As a special case, linear multiplicative vibrational systems in Bellman et al. (1985) characterize a class of linear systems stabilized by a linear vibrational control input:

$$\dot{x} = Ax + \frac{1}{\varepsilon}B_1\left(\frac{t}{\varepsilon}\right)x,\ x(t_0) = x_0 \in \mathbb{R}^n, \tag{5}$$

where the matrix $A \in \mathbb{R}^{n\times n}$ and the vibrational matrix $B_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is continuous and periodic in t with zero mean value (see inverted pendulum (2) for example). The positive number ε serves as the design parameter that is related to the dither frequency.

Lemma 1 provides a necessary and sufficient condition of vibrational stabilization for linear multiplicative systems.

Lemma 1 (*Meerkov, 1980*). *Suppose the matrix A in system (5) has a controllable canonical form, then the system is v-stabilizable if and only if the trace of the matrix A is negative.*

Remark 1. As the trace of a square matrix equals the summation of all its eigenvalues, Lemma 1 implies that there may exist positive eigenvalues such that the system (5) is open-loop unstable . By introducing a high frequency vibration, it is possible to shift unstable eigenvalues to stable ones. ∘

2.2. Input-to-state stability

While there exist disturbances in a dynamic system, input-to-state stability (ISS) (Khalil, 1996) is used to address the robustness. We consider the following time-varying system:

$$\dot{x} = f\ (t, x, w)\ ,\ x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \tag{6}$$

where $f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous differentiable in t and locally Lipschitz in x and w. The disturbance $w : [t_0, \infty) \to \mathbb{R}^m$ is time-varying. Without losing generality, let us assume $f(t, 0, 0) = 0$.

Definition 2 (*Khalil, 1996*; *Nešić & Teel, 2001*)**.** The system (6) is said to be input-to-state stable with gain $\gamma \in \mathcal{K}$ if there exists $\beta \in \mathcal{KL}$ such that for each $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^n$, the solution starting at (x_0, t_0) exists and satisfies:

$$|x(t)| \le \max\{\beta(|x_0|, t - t_0), \gamma(\|w\|_\infty)\}, \forall t \ge t_0 \ge 0.$$

Definition 2 infers that in the case of no disturbance ($w = 0$), the system (6) is uniformly globally asymptotically stable. As pointed out in Khalil (1996), the system (6) is ISS if it is Lyapunov ISS. The definition of Lyapunov ISS is provided in Definition 3.

Definition 3 (*Nešić and Teel, 2001*)**.** The system (6) is called Lyapunov-ISS with gain γ if there exists a continuous differentiable function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(|x|) \le V(t, x) \le \alpha_2(|x|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, w) \le -\alpha_3(|x|), \forall |x| > \rho(\|w\|_\infty),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$ and $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

3. Main results

Linear vibrational control systems with additive state-independent disturbances have the following form:

$$\dot{x} = Ax + \frac{1}{\varepsilon} B_1\left(\frac{t}{\varepsilon}\right) x + B_2 w(t),\ x(t_0) = x_0 \in \mathbb{R}^n, \tag{7}$$

where A and $B_1(\cdot)$ come from the system (5). The matrix $B_2 \in \mathbb{R}^{n\times m}$ and $w \in \mathbb{R}^m$ is the time-varying disturbance satisfying $w \in \mathcal{L}_\infty$.

As discussed in Lemma 1, it is assumed that matrices A and $B_1(t)$ satisfy the following assumptions:

Assumption 1. The matrix A has a controllable canonical form with a negative trace.

Assumption 2. The time-varying matrix $B_1(t)$ is continuous and periodic with zero mean value, i.e. $B_1(t) = B_1(t + T)$ and $\frac{1}{T}\int_0^T B_1(\tau)d\tau = 0$.

In order to analyze the robustness of the system (7), the first step is to transform it to a new coordinate in the fast time-scale as done in Bellman et al. (1986b).

In the new time $\tau = \frac{t}{\varepsilon}$, it becomes

$$\frac{dx}{d\tau} = \varepsilon Ax + B_1(\tau)x + \varepsilon B_2 w(\varepsilon\tau), x(\tau_0) = x_0 \in \mathbb{R}^n. \tag{8}$$

An auxiliary system is then introduced to simplify the stability analysis of the system (8):

$$\frac{dz}{d\tau} = B_1(\tau)z, \qquad z(\tau_0) \in \mathbb{R}^n, \forall \tau \ge \tau_0 \ge 0. \tag{9}$$

Suppose that system (9) has a state transition matrix $\Phi(\tau, \tau_0)$, the following coordinate transformation is applied:

$$x(\tau) = \Phi(\tau, \tau_0)y(\tau). \tag{10}$$

After taking derivative with respect to τ in (10), the system (8) could be re-written in the new coordinate y as

$$\begin{aligned}\frac{dy}{d\tau} &= \varepsilon(\Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)y(\tau) \\ &\quad + \Phi^{-1}(\tau, \tau_0)B_2 w(\varepsilon\tau)), y(\tau_0) \in \mathbb{R}^n.\end{aligned} \tag{11}$$

Remark 2. The coordinate transformation (10) plays an important role in stability analysis of linear vibrational control systems (7). Since the vibrational matrix $B_1(\frac{t}{\epsilon})$ has zero mean, it will disappear if we directly apply the standard first order averaging, leading to an unstable averaged system. This shows that the precision of first order averaging is not enough to conclude the stability for vibrational control systems. Intuitively, the transformation improves resolution: i.e., in the new coordinate, the averaging can provide a better approximation. ∘

Remark 3. In the cases of no disturbances ($w = 0$), Lemma 1 indicates that if the matrix A satisfies Assumption 1, there exists a vibrational matrix $B_1\left(\frac{t}{\varepsilon}\right)$ such that the average of nominal term $\Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)$ is Hurwitz. With the averaging technique, stability of the transformed system could be concluded. According to Bellman et al. (1986b), the transformation (10) is nonsingular and will preserve the stability of the original system (7), thus the original system (7) is v-stable as well. ∘

Transformed system (11) can be rewritten as

$$\frac{dy}{d\tau} = \varepsilon(\tilde{A}(\tau)y(\tau) + \tilde{w}(\tau)), y(\tau_0) = x(\tau_0) \in \mathbb{R}^n, \tag{12}$$

where $\tilde{A}(\tau) = \Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)$ and $\tilde{w}(\tau) = \Phi^{-1}(\tau, \tau_0)B_2 w(\varepsilon\tau)$. Since $\Phi(\tau, \tau_0)$ and $\Phi^{-1}(\tau, \tau_0)$ are periodic functions, $\tilde{A}(\tau)$ is T_τ periodic where $T_\tau = \frac{T}{\varepsilon}$. Moreover, as $\Phi^{-1}(\tau, \tau_0)$ is periodic and continuous, then $\|\Phi^{-1}(\tau, \tau_0)\|$ is bounded for any $\tau \ge \tau_0 \ge 0$. As $w \in \mathcal{L}_\infty$, the boundedness of $\Phi^{-1}(\cdot, \cdot)$ and the constant B_2 matrix lead to $\tilde{w} \in \mathcal{L}_\infty$.

The subsequent robustness analysis is based on the transformed system (12). First of all, perturbation technique is used to show the robustness of the system for bounded disturbances with a relatively conservative estimation of the trajectory bound (Theorem 1). When disturbances are also periodic, Fourier series expansion technique is further applied to show that the ultimate bound of trajectories can be made smaller for higher frequency disturbances. The estimation of the ultimate bound is less conservative under such a situation.

Theorem 1. *Suppose Assumption 1 holds and $w \in \mathcal{L}_\infty$. Let D be a compact set in $\mathbb{R}^n$, there exists $B_1(t)$ satisfying Assumption 2, $\varepsilon^* > 0$ such that for all $x_0 \in D$ and $\varepsilon \in (0, \varepsilon^*)$, the solutions of system (7) satisfy:*

$$|x(t)| \le MN|x_0|\mathrm{e}^{-\lambda(t-t_0)} + \frac{MN\|w\|_\infty}{\lambda}, \tag{13}$$

where N, M, λ are strictly positive constants.

Proof. See Proof of Theorem 1 in Appendix. □

Theorem 1 indicates that for any initial compact set $D \in \mathbb{R}^n$, the trajectories of the system will exponentially converge to a neighborhood of $\tilde{\gamma}(\|w\|_\infty) = MN\|w\|_\infty/\lambda$ around origin. This result shows the robustness of linear vibrational control systems with respect to any bounded additive state-independent disturbances. A direct outcome of Theorem 1 is the following proposition.

It is assumed that the averaged system of (12) without disturbance is:

$$\frac{dy_{av}}{d\tau} = \varepsilon\tilde{A}_{av}y_{av}, y_{av}(\tau_0) = y(\tau_0). \tag{14}$$

The closeness of solutions between (12) and (14) is stated in the following proposition:

Proposition 1 (*Closeness of Solutions*)**.** *Suppose Assumptions 1–2 hold. Let D be a compact set in $\mathbb{R}^n$. For any $y(\tau_0) \in D$, Ω_w and δ, there* *exist ε^* and T^* such that whenever $\varepsilon \in (0, \varepsilon^*)$, solutions of (12) $y(\tau, \varepsilon)$ and solutions of averaged system (14) $y_{av}(\tau, \varepsilon)$ satisfy:*

$$|y(\tau, \tau_0, \varepsilon) - y_{av}(\tau, \varepsilon)| < \delta + \tilde{\gamma}(\|w\|_\infty), \forall \tau \geq T^*,$$

where $\tilde{\gamma}(\|w\|_\infty) = MN\|w\|_\infty/\lambda$, and M and N are defined in (13).

Proof. See Proof of Proposition 1 in Appendix. □

Remark 4. Proposition 1 shows that due to the existence of disturbances, the solutions of transformed vibrational control system (12) converge to a $\tilde{\gamma}(\|w\|_\infty)$-neighborhood of solutions of averaged system without disturbances (14) in a finite time interval as ε is sufficiently small. ∘

Next we will show that if the bounded additive state-independent disturbances are periodic, it is possible to link the frequency of disturbances to the estimate of the trajectory bound. In particular, a class of periodic disturbances that have Fourier series expansion will be considered. It is well-known that the Fourier series expansion of a periodic signal exists if it satisfies Dirichlet conditions (see Kamen and Heck, 2007 for more details). Most periodic signals in engineering applications satisfy it.

Theorem 2. *Let Assumption 1 hold and $w \in \mathcal{L}_\infty$. Suppose disturbance $w(t)$ is a T_w-periodic function and its norm can be expressed in Fourier series form: $|w(t)| = a_0/2 + \sum_{k=1}^{\infty}[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$ where $\omega_0 = 2\pi/T_w$. Let D be a compact set in $\mathbb{R}^n$, there exists $B_1(t)$ satisfying Assumption 2 and there exist strictly positive real numbers M, N, λ and ε^* s.t. for all $x_0 \in D$ and $\varepsilon \in (0, \varepsilon^*)$, the solutions of system (7) satisfy:*

$$\begin{aligned}|x(t)| &\leq MN\mathrm{e}^{-\lambda t}|x_0| + \frac{a_0 MN}{2\lambda}(1 - \mathrm{e}^{-\lambda t}) \\ &+ \sum_{k=1}^{\infty} \frac{MNa_k}{\lambda^2 + k^2\omega_0^2}[\lambda \cos(k\omega_0 t) - \lambda \mathrm{e}^{-\lambda t} + k\omega_0 \sin(k\omega_0 t)] \qquad (15) \\ &+ \sum_{k=1}^{\infty} \frac{MNb_k}{\lambda^2 + k^2\omega_0^2}[\lambda \sin(k\omega_0 t) - k\omega_0 \cos(k\omega_0 t) + k\omega_0 \mathrm{e}^{-\lambda t}].\end{aligned}$$

Proof. See Proof of Theorem 2 in Appendix. □

Remark 5. Theorem 2 reveals a relationship between the trajectory bound and the frequency of periodic disturbances. As shown in (15), there are two major components in the estimate of the ultimate bound: one is related to a_0 and the other is related to the frequency of the periodic disturbances. It can be seen that disturbances with a higher frequency will have a smaller ultimate bound. ∘

4. Extension to state-dependent disturbances

This section aims at analyzing the robustness of the vibrational control systems with a more general form of additive disturbances, which are state-dependent as shown in (16):

$$\dot{x} = Ax + \frac{1}{\varepsilon}B_1\left(\frac{t}{\varepsilon}\right)x + \eta(x, w),\ x(t_0) = x_0 \in \mathbb{R}^n. \qquad (16)$$

In (16), matrices A and B_1 are same as in (7) and the term $\eta(x, w)$ represents the state-dependent disturbances terms. It is assumed that $\eta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and locally Lipschitz in x and w satisfying $\eta(0, 0) = 0$.

Due to existence of the nonlinear term $\eta(\cdot, \cdot)$, the standard averaging technique cannot be applied directly. The perturbation techniques cannot be applied either. Thus strong average and weak average techniques (Nešić & Teel, 2001) are used to show the robustness of (16). Next the definitions of the strong average and the weak average are provided.

4.1. Strong and weak average

Consider a parameterized time-varying system with disturbance

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x, w\right),\ x(t_0) = x_0 \in \mathbb{R}^n, \qquad (17)$$

where ε is a positive and sufficiently small real number. Definitions of strong and weak average are given below respectively:

Definition 4 (*Nešić & Teel, 2001*)**.** A locally Lipschitz function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the strong average of $f(t, x, w)$ if there exist $\beta_{sa} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq t_0 \geq 0$, $w \in \mathcal{L}_\infty$ and $T > T^*$ the following holds:

$$\begin{aligned}&\left|\frac{1}{T}\int_t^{t+T}[f_{sa}(x, w(s)) - f(s, x, w(s))]ds\right| \\ &\quad\leq \beta_{sa}(\max\{|x|, \|w\|_\infty, 1\}, T).\end{aligned}$$

Definition 5 (*Nešić & Teel, 2001*)**.** A locally Lipschitz function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the weak average of $f(t, x, w)$ if there exist $\beta_{wa} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq t_0 \geq 0$, $T > T^*$, the following holds:

$$\begin{aligned}&\left|f_{wa}(x, w) - \frac{1}{T}\int_t^{t+T} f(s, x, w)ds\right| \\ &\quad\leq \beta_{wa}(\max\{|x|, |w|, 1\}, T).\end{aligned}$$

Remark 6. Both strong and weak averaging techniques handle disturbances while doing averaging over a period. As discussed in Nešić and Teel (2001), strong average can handle any bounded disturbance while weak average can handle slowly time-varying bounded disturbances. For a general nonlinear time-varying system with disturbances, it is relatively easy to find its weak average while the strong average only exists for a special class of nonlinear time-varying systems (Nešić & Teel, 2001). ∘

Remark 7. Weak averaging techniques were used in Cheng, Tan, and Mareels (2016) to show the robustness of linear vibrational control systems with additive state-independent disturbances. The obtained bound of trajectory is relatively conservative compared with the results obtained in Theorems 1 and 2. Here we consider state-dependent disturbances where disturbances are coupled with system states. ∘

4.2. Bounded and slowly time-varying disturbances

Similar to Section 3, by using the coordinate transformation (10), the system (16) is transformed into the following form:

$$\begin{aligned}\frac{dy}{d\tau} &= \varepsilon\left(\Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)y\right. \\ &\quad \left.+ \Phi^{-1}(\tau, \tau_0)\eta(\Phi(\tau, \tau_0)y, w)\right). \qquad (18)\end{aligned}$$

Lemma 2. *The strong average of the system (18) does not exist but the weak average exists. The weak average system is:*

$$\dot{y}_{wa} = \varepsilon\left(\tilde{A}_{av}y_{wa} + \tilde{w}_{wa}\right), y_{wa}(\tau_0) = y(\tau_0), \qquad (19)$$

where $\tilde{A}_{av} = \frac{1}{T}\int_\tau^{\tau+T} \Phi^{-1}(s, \tau_0)A\Phi(s, \tau_0)ds$, $\tilde{w}_{wa} = \frac{1}{T}\int_\tau^{\tau+T} \Phi^{-1}(s, \tau_0)\eta(\Phi(s, \tau_0)y, w)ds$.

Proof. See Proof of Lemma 2 in Appendix. □

As the weak average of the system (18) exists, if it is Lyapunov-ISS, by applying (Nešić & Teel, 2001, Theorem 2), the following result is obtained.

Theorem 3. *Suppose Assumption 1 holds and $w \in \mathcal{L}_\infty$. Assume the weak average system (19) is Lyapunov-ISS with gain $\hat{\gamma}$. There exists $B_1(t)$ satisfying Assumption 2 and $\beta \in \mathcal{KL}$, for any given strictly positive real numbers Ω_x, Ω_w, $\Omega_{\dot{w}}$, δ, there exist positive constants ε^*, M_1, M_2 s.t. $\forall \varepsilon \in (0, \varepsilon^*)$ the solutions of (7) satisfy:*

$$|x(t)| \le M_1 \max\{\beta(|x_0|, t-t_0), M_2\hat{\gamma}(\|w\|_\infty)\} + M_1\delta, \tag{20}$$

whenever $t \ge t_0 \ge 0$, $|x_0| \le \Omega_x$, $\|w\|_\infty \le \Omega_w$ and $\|\dot{w}\|_\infty \le \Omega_{\dot{w}}$.

Proof. See Proof of Theorem 3 in Appendix. □

Remark 8. The assumption that weak average system (19) is Lyapunov-ISS in Theorem 3 is not very restrictive due to the fact that $\tilde{A}_{av}$ is Hurwitz from Assumption 1. Next, Corollary 1 provides a sufficient condition for the nonlinear mapping $\eta(\cdot,\cdot)$ to guarantee that the weak average system (19) is Lyapunov-ISS. ∘

Corollary 1. *Suppose Assumption 1 holds. If the nonlinear function $\eta(\cdot,\cdot)$ in (18) satisfies the following inequality*

$$|\eta(x, w)| \le |x|^c |w|, \tag{21}$$

for some $c \in [0, 1)$, then that weak average system (19) of is Lyapunov ISS.

Proof. See Proof of Corollary 1 in Appendix. □

5. Simulation results

The motivating example is used to illustrate the robustness of the linear vibrational control systems. A linearized inverted pendulum model with a vibrational control algorithm is considered in the presence of different types disturbances.

The state-space model of the inverted pendulum after linearization has the form of (7) with

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}, B_1\left(\frac{t}{\varepsilon}\right) = \begin{bmatrix} 0 & 0 \\ \frac{a}{l\varepsilon}\cos\frac{t}{\varepsilon} + \frac{ka}{ml}\sin\frac{t}{\varepsilon} & 0 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{22}$$

It can be easily verified that the matrix A satisfies Assumption 1 and $B_1(t)$ satisfies Assumption 2. Without disturbances, Lemma 1 indicates that vibrational control system is stable if ε is sufficiently small.

In the following simulations, parameters of linearized inverted pendulum are selected as $l = 0.185$, $m = 0.2$, $k = 1$, $g = 9.8$ and vibrational control parameters are chosen as $a = 20$, $\varepsilon = 0.0032$. Fig. 1 shows that the vibrational controller can stabilize the inverted pendulum locally while the original system is unstable.

Next it will be verified that this vibrational control algorithm is robust to different types of disturbances. Assumed that there are exogenous forces disturbing the stabilization of inverted pendulum, the system has additive, but state-independent disturbances in this situation. As Theorem 1 indicates, the vibrational control system is ISS if disturbances are bounded. The first disturbance is selected as

$$w_0(t) = \frac{t^2}{1+t^2}, \tag{23}$$

which satisfies $\|w_0\|_\infty = 1$. The simulated trajectory in l_2-norm in Fig. 2(a) shows that the states converge to a neighborhood of the origin.

Theorem 2 provides a less conservative estimation of the trajectory bound when the state-independent disturbances are periodic.

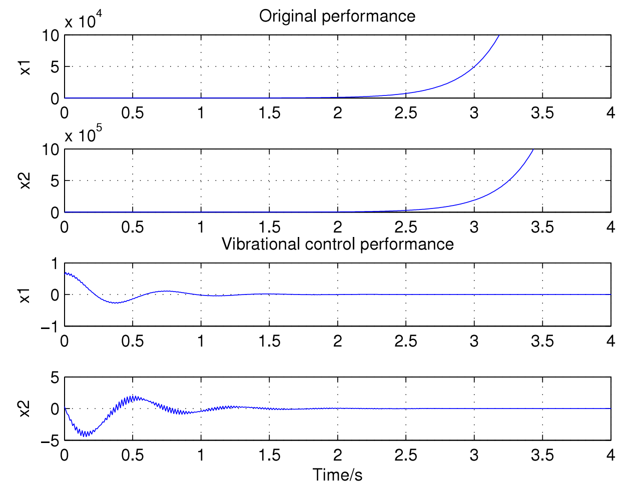

Fig. 1. Linearized inverted pendulum is stabilized by vibrational control while original system is unstable.

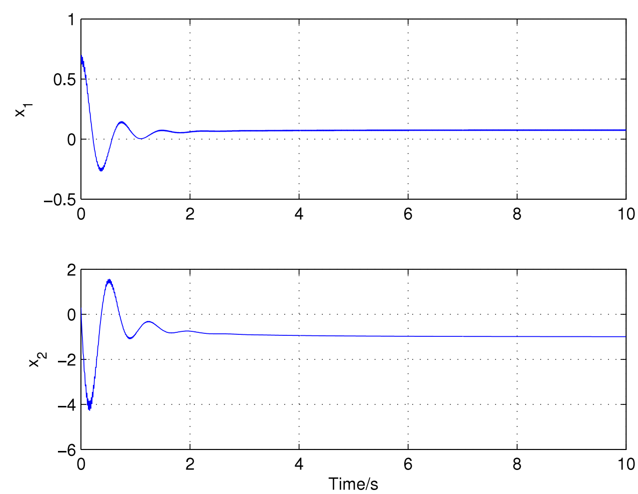

Fig. 2. Trajectories of the inverted pendulum in the existence of disturbances $w_0(t)$.

Under such a situation, both Theorems 1 and 2 are applicable. In order to compare the ultimate bounds for two results, $w_2(t) = \sin 50t$ is applied. By applying Theorem 1, the bound of the trajectories estimated from (13) is around 300. By applying Theorem 2, the bound from (15) is shown in Fig. 3, which is much less conservative (compared with 300 obtained from Theorem 1).

Other than providing a less conservative bound for trajectories, Theorem 2 also indicates that when disturbances are periodic, the ultimate bound is almost inversely proportional to the disturbance frequency when it is sufficiently large. Fig. 4 shows simulated trajectories with different-frequency sinusoidal disturbances (the angular frequency is 10, 50, 10^2, 10^3, 10^4 respectively). It indicates that the ultimate bound will reduce as the frequency increases.

It is possible to have the scenario when disturbances are coupled with states. For example, the friction coefficient may vary as humidity changes, leading to state-dependent disturbances.

Suppose state-dependent disturbance $\eta(x, w)$ takes the form of $\eta(x, w) = \left[\frac{|x|^{0.5} w_0(t)}{\sqrt{2}}, \frac{|x|^{0.5} w_0(t)}{\sqrt{2}}\right]^T$, where w_0 comes from (23). Obviously, the condition in Corollary 1 is satisfied. Thus the weak average (19) is Lyapunov-ISS. As the disturbance $w(t)$ is bounded and slowly time-varying, Theorem 3 shows that the trajectories of the system will be practically ISS. The trajectories of the vibrational

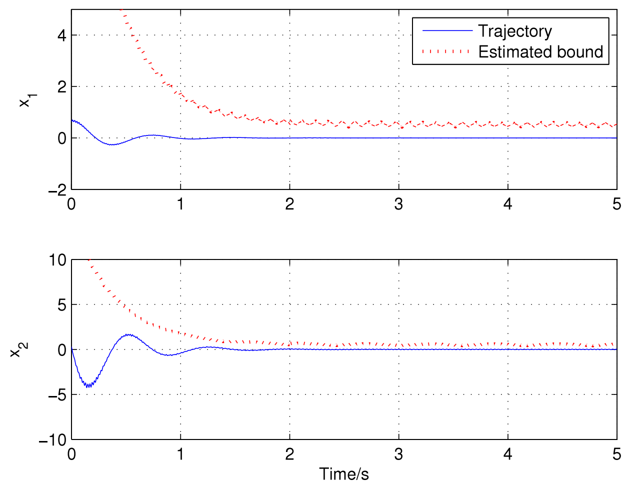

Fig. 3. Trajectories of x_1, x_2 and the bound of the trajectories from Theorem 2.

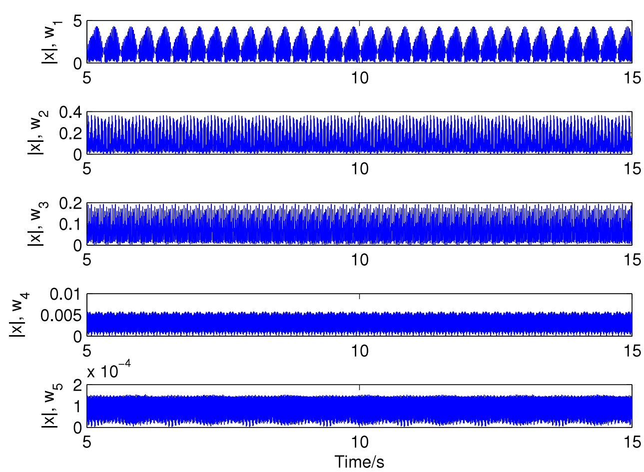

Fig. 4. The relationship between the ultimate bound and the frequency of disturbances.

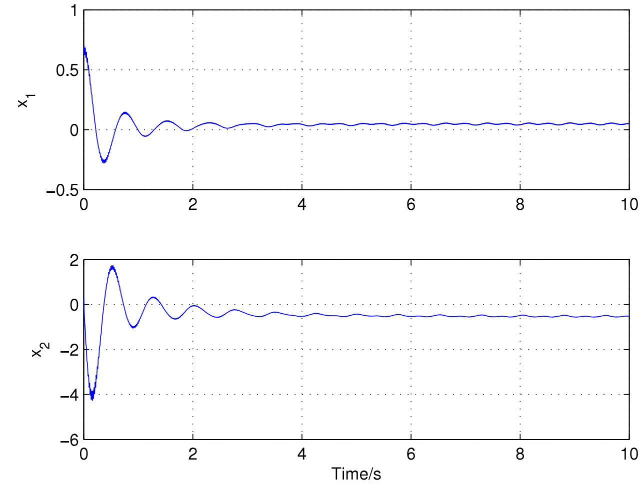

Fig. 5. System is practically input-to-state stable in the existence of state-dependent disturbance.

control system with such a disturbance are shown in Fig. 5. These simulation results are consistent with theoretic results.

6. Conclusion

In this paper, we present robustness analysis of linear vibrational control systems in the presence of additive disturbances. When the disturbances are state-independent, perturbation techniques can be used to show that the linear vibrational control system is input-to-state stable (ISS). In particular, when disturbances are periodic, a higher frequency leads to a smaller ultimate bound. When state-dependent disturbances are considered, weak averaging techniques are used to show the robustness of vibrational control systems when disturbances are slowly time-varying. Numerical results support theoretic findings. Future work will explore the robustness of vibrational control systems for general nonlinear systems.

Appendix

Proof of Theorem 1. Let us introduce the following system:

$$\frac{dy_1}{d\tau} = \varepsilon\tilde{A}(\tau)y_1(\tau), y_1(\tau_0) = y(\tau_0). \tag{24}$$

As the averaging result in Khalil (1996, Theorem 10.4) indicates, for all $y(\tau_0) \in D$ which is a compact set in $\mathbb{R}^n$, there exists a positive pair (ε^*, k) such that for all $\varepsilon \in (0, \varepsilon^*)$, the following inequality holds:

$$|y_1(\tau) - y_{av}(\tau)| \le k\varepsilon. \tag{25}$$

According to Lemma 1, if the matrix A satisfies Assumption 1, there exists $B_1(t)$ satisfying Assumption 2 such that the matrix $\tilde{A}_{av}$ is Hurwitz. Then we have that $|y_{av}(\tau)| \le e^{\lambda_{max}(\varepsilon\tilde{A}_{av})\tau}|y_{av}(\tau_0)|$, where $\lambda_{max}(\varepsilon\tilde{A}_{av})$ is the largest eigenvalue of $\varepsilon\tilde{A}_{av}$. Thus for any δ, there exists ε^* such that when $\varepsilon \in (0, \varepsilon^*)$ the solutions of system (24) satisfy:

$$|y_1(\tau)| \le e^{\lambda_{max}(\varepsilon\tilde{A}_{av})\tau}|y_{av}(\tau_0)| + \delta.$$

It means we can find positive real number N, λ such that $|y_1(\tau)| \le Ne^{-\varepsilon\lambda\tau}|y_1(\tau_0)|$. Consequently, the upper bound of the trajectories of (12) is obtained as

$$\begin{aligned}|y(\tau)| &\le Ne^{-\varepsilon\lambda\tau}|y(\tau_0)| + \varepsilon\int_0^\tau Ne^{-\varepsilon\lambda(\tau-s)}|\tilde{w}(s)|ds \\ &\le Ne^{-\varepsilon\lambda\tau}|y(\tau_0)| + \frac{1}{\lambda}N\|\tilde{w}\|_\infty(1 - e^{-\varepsilon\lambda\tau}) \\ &\le N|y(\tau_0)|e^{-\varepsilon\lambda\tau} + \frac{1}{\lambda}N\|\tilde{w}\|_\infty.\end{aligned}$$

From (10), $x(\tau) = \Phi(\tau, \tau_0)y(\tau)$, therefore, the solution of system (7) is ISS:

$$|x(t)| \le MN|x(\tau_0)|e^{-\lambda t} + \frac{MN}{\lambda}\|w\|_\infty, \forall t \ge t_0,$$

where $M = M_1M_2$ and M_1, M_2 are the $\mathcal{L}_\infty$ norms of $\Phi(\tau, \tau_0)$, $\Phi^{-1}(\tau, \tau_0)$ respectively. □

Proof of Proposition 1. From Proof of Theorem 1, it can be shown that for all $y(\tau_0) \in D$ which is a compact set in $\mathbb{R}^n$, there exists positive real number ε_1^* and k such that for all $\varepsilon \in (0, \varepsilon_1^*)$, it has (24),

$$|y_1(\tau) - y_{av}(\tau)| \le k\varepsilon.$$

where $y_1(\tau)$ comes from (12) and $y_{av}(\tau)$ comes from (24). Let $e(\tau) = y(\tau) - y_1(\tau)$, then take derivative on both sides:

$$\frac{de}{d\tau} = \varepsilon(\tilde{A}(\tau)e(\tau) + \tilde{w}(\tau)).$$

Using the similar procedure as the proof of Theorem 1, there exist positive numbers N and λ such that the upper bound of the solution $e(\tau)$ satisfies:

$$|e(\tau)| \le N|e(\tau_0)|\mathrm{e}^{-\varepsilon\lambda\tau} + \frac{1}{\lambda}N\|\tilde{w}\|_\infty.$$

Noting that $|y(\tau) - y_{av}(\tau)| \le |y(\tau) - y_1(\tau)| + |y_1(\tau) - y_{av}(\tau)|$, the closeness of solutions between (12) and (14) is thus established:

$$\begin{aligned}|y(\tau) - y_{av}(\tau)| &\le N|e(\tau_0)|\mathrm{e}^{-\varepsilon\lambda\tau} + \frac{N}{\lambda}\|\tilde{w}\|_\infty + k\varepsilon\\ &\le N|e(\tau_0)|\mathrm{e}^{-\varepsilon\lambda\tau} + \frac{N}{\lambda}\|\tilde{w}\|_\infty + k\varepsilon.\end{aligned}$$

Consequently, there exists T^* such that $N|e(\tau_0)|\mathrm{e}^{-\varepsilon\lambda T^*} < \frac{\delta}{2}$ and let $\varepsilon_2^* = \frac{\delta}{2k}$. If we choose $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$, for all $\varepsilon \in (0, \varepsilon^*)$, it follows that $|y(\tau, \varepsilon) - y_{av}(\tau, \varepsilon)| < \delta + \gamma(\|w\|_\infty)$, $\forall\tau \ge T^*$, which completes the proof. □

Proof of Theorem 2. From Proof of Theorem 1, for all $y(\tau_0) \in D$, there exist strictly positive real numbers N and λ s.t.

$$|y(\tau)| \le N\mathrm{e}^{-\varepsilon\lambda\tau}|y(\tau_0)| + \varepsilon\int_0^\tau NM_2\mathrm{e}^{-\varepsilon\lambda(\tau-s)}|w(\varepsilon s)|ds$$

where N, M_2 and λ are defined the same as in Theorem 1. Since $|w(t)|$ could be expressed in Fourier series: $|w(t)| = |w(\varepsilon\tau)| = a_0/2 + \sum_{k=1}^{\infty}[a_k\cos(k\omega_0\varepsilon\tau) + b_k\sin(k\omega_0\varepsilon\tau)]$. It leads to

$$\begin{aligned}|y(\tau)| \le N\mathrm{e}^{-\lambda\varepsilon\tau}|y(\tau_0)| &+ \frac{a_0NM_2\varepsilon}{2}\int_0^\tau \mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &+ NM_2a_k\sum_{k=1}^{+\infty}\varepsilon\int_0^\tau \cos(k\omega_0\varepsilon s)\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &+ NM_2b_k\sum_{k=1}^{+\infty}\varepsilon\int_0^\tau \sin(k\omega_0\varepsilon s)\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds.\end{aligned}$$

By calculations, we have:

$$\varepsilon\int_0^\tau \mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds = \frac{1}{\lambda}\left(1 - \mathrm{e}^{-\lambda\varepsilon\tau}\right),$$

$$\begin{aligned}&\varepsilon\int_0^\tau \cos(k\omega_0\varepsilon s)\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &= \varepsilon\int_0^\tau \frac{\mathrm{e}^{jk\omega_0\varepsilon s} + \mathrm{e}^{-jk\omega_0\varepsilon s}}{2}\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &= \frac{1}{\lambda^2 + k^2\omega_0^2}\left(\lambda\cos(k\omega_0\varepsilon\tau) + k\omega_0\sin(k\omega_0\varepsilon\tau) - \lambda\mathrm{e}^{-\lambda\varepsilon\tau}\right),\end{aligned}$$

$$\begin{aligned}&\varepsilon\int_0^\tau \sin(k\omega_0\varepsilon s)\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &= \varepsilon\int_0^\tau \frac{\mathrm{e}^{jk\omega_0\varepsilon s} - \mathrm{e}^{-jk\omega_0\varepsilon s}}{2}\mathrm{e}^{-\varepsilon\lambda(\tau-s)}ds\\ &= \frac{1}{\lambda^2 + k^2\omega_0^2}\left(\lambda\sin(k\omega_0\varepsilon\tau) - k\omega_0\cos(k\omega_0\varepsilon\tau) + k\omega_0\mathrm{e}^{-\lambda\varepsilon\tau}\right).\end{aligned}$$

Noting that $|x(\tau)| \le \|\Phi\|_\infty|y(\tau)|$, solutions of $x(t)$ satisfy

$$\begin{aligned}|x(t)| &\le MN\mathrm{e}^{-\lambda t}|x(t_0)| + \frac{a_0N}{2\lambda}(1 - \mathrm{e}^{-\lambda t})\\ &+ N\sum_{k=1}^{\infty}\frac{a_k}{\lambda^2 + k^2\omega_0^2}[\lambda\cos(k\omega_0t) - \lambda\mathrm{e}^{-\lambda t} + k\omega_0\sin(k\omega_0t)]\\ &+ N\sum_{k=1}^{\infty}\frac{b_k}{\lambda^2 + k^2\omega_0^2}[\lambda\sin(k\omega_0t) - k\omega_0\cos(k\omega_0t) + k\omega_0\mathrm{e}^{-\lambda t}]\end{aligned}$$

where $M = M_1M_2$. This completes the proof. □

Proof of Lemma 2. From Nešić and Teel (2001, Proposition 1), it can be seen that the strong average exists if and only if the system (17) has a specific structure $f(t, x, w) = \varphi(t, x) + \psi(x, w)$. However the transformed vibrational system (18) does not satisfy the needed form. Therefore the strong average does not exist. According to Definition 5, the proposed $f_{wa}(y, w)$ in (19) is the weak average of the system. □

Proof of Theorem 3. Since Assumption 1 holds, $\Phi(\tau, \tau_0)$ and $\Phi^{-1}(\tau, \tau_0)$ are periodic and bounded, let their bounds in $\mathcal{L}_\infty$ norm be M_1 and M_2 respectively, i.e. $\|\Phi\|_\infty = M_1$, $\|\Phi^{-1}\|_\infty = M_2$.

$$f(\tau, y, w) = \varepsilon\left(\Phi^{-1}(\tau)A\Phi(\tau)y + \Phi^{-1}(\tau)\eta(\Phi(\tau)y, w)\right)$$

for system (18), then

$$\begin{aligned}&|f(t, y_1, w) - f(t, y_2, w)| \le \|\Phi^{-1}A\Phi\||y_1 - y_2|\\ &+ \|\Phi^{-1}\||\eta(\Phi y_1, w) - \eta(\Phi y_2, w)| \le M_1M_2(\lambda(A) + L_y)|y_1 - y_2|,\\ &|f(t, y, w_1) - f(t, y, w_2)| \le M_2L_w|w_1 - w_2|.\end{aligned}$$

So the system (18) is Lipschitz in y and w uniformly in τ. Therefore, applying (Nešić & Teel, 2001, Theorem 2), there exists $\beta \in \mathcal{KL}$ such that the solutions of system (11) satisfy: $\forall\tau \ge \tau_0 \ge 0$

$$|y(\tau)| \le \max\{\beta(|y(\tau_0)|, \tau - \tau_0), M_2\tilde{\gamma}(\|w\|_\infty)\} + \delta.$$

Since $x(\tau) = \Phi(\tau, \tau_0)y(\tau)$ and $x(\tau_0) = y(\tau_0)$, $\forall\tau \ge \tau_0 \ge 0$, it leads to

$$|x(\tau)| \le M_1\max\{\beta(|x(\tau_0)|, \tau - \tau_0), M_2\tilde{\gamma}(\|w\|_\infty)\} + M_1\delta. \quad □$$

Proof of Corollary 1. As Assumption 1 holds, Lemma 1 indicates that there exists $B(\frac{t}{\varepsilon})$ such that $\tilde{A}_{av}$ is Hurwitz. According to (Khalil, 1996, Theorem 4.6), for any positive definite symmetric matrix Q, there exists positive definite symmetric P s.t. $PA_{av} + A_{av}^TP = -Q$. Choosing $Q = I$, a Lyapunov candidate is selected as $V(y) = y^TPy$. This Lyapunov candidate is used to show that the weak average system (19) is Lyapunov-ISS.

$$\begin{aligned}\dot{V}(t) &= -y^Ty + 2y^TP\frac{1}{T}\int_t^{t+T}\Phi^{-1}(\tau)\eta(\Phi(\tau)y, w)d\tau\\ &\le -y^Ty + 2|y|\|P\|\|\Phi\|_\infty\sup_{t\ge t_0}|\eta(\Phi(t)y, w)|\\ &\le -|y|^2 + 2|y|\|P\|\|\Phi\|_\infty\|\Phi^{-1}\|_\infty^c|y|^c|w|\\ &\le -(1 - \theta)|y|^2,\end{aligned} \tag{26}$$

whenever $|y| \ge \left(\frac{2\|P\|\|\Phi\|_\infty\|\Phi^{-1}\|_\infty^c|w|}{\theta}\right)^{1/(1-c)}$ for $0 \le c < 1$ and $0 < \theta < 1$. Applying (Khalil, 1996, Theorem 4.19) directly, it is concluded that the system (19) is Lyapunov-ISS. □

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