State feedback control of Markov jump linear systems with hidden-Markov mode observation

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SUMMARY

In this paper, we study state-feedback control of Markov jump linear systems with partial information. In particular, we assume that the controller can only access the mode signals according to a hidden-Markov observation process. Our formulation generalizes various relevant cases previously studied in the literature on Markov jump linear systems, such as the cases with perfect information, no information, and cluster observations of the mode signals. In this context, we propose a Linear Matrix Inequalities (LMI) formulation to design feedback control laws for (stochastic) stabilization, H_2 , and H_{∞} control of discrete-time Markov jump linear systems under hidden-Markovian observations of the mode signals. We conclude by illustrating our results with some numerical examples. Copyright © 2010 John Wiley & Sons, Ltd.

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Markov jump linear systems; robust control; uncertain observation; linear matrix inequalities; hidden-Markov processes.

1. INTRODUCTION

Markov jump linear systems [1] are an important class of switched systems in which the *mode* signal, responsible for controlling the switch among dynamic modes, is modeled by a timehomogeneous Markov process. This type of systems has been widely used in multiple applications, such as robotics [2, 3], economics [4], networked control [5], and epidemiology [6]. Solutions to standard controller synthesis problems for Markov jump linear systems, such as state-feedback stabilization, quadratic optimal control, H_2 optimal control, and H_{∞} optimal control (see, e.g., the monograph [1]), can be found in the literature. These works, however, are based on the unrealistic assumption that the controller has full knowledge about the mode signal at any time instant.

To overcome this limitation, several papers investigate the effect of limited and/or uncertain knowledge about the mode signal. For example, the authors in [7] studied H_2 control of discretetime Markov jump linear systems when the state space of the mode signal is partitioned into subsets, called *clusters*, and the controller only knows in which cluster the mode signal is at a given time. Similar studies in the context of H_{∞} control can be found in [8,9]. In the extreme case of having a single mode cluster (in other words, when one cannot observe the mode), the authors in [10, 11] investigated quadratic optimal control problems. Most of the above works can be studied in a framework based on random and uncertain mode observations (see [12] for the description of this

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framework in the context of H_2 control). In a complementary line of work, we find some papers assuming that the mode signal can only be observed at particular sampling times, instead of at any time instant. In this direction, we find in the literature a variety of random sampling strategies of the mode signal. The authors in [13] designed almost-surely stabilizing state-feedback gains when the sampling times follow a renewal process. Similarly, the authors in [14, 15] derived stabilizing state-feedback gains using Lyapunov-like functions under periodic observations.

In this paper, we propose a framework to design state-feedback controllers for discrete-time Markov jump linear systems assuming that the mode signal can only be observed when a Markov chain (different than the one describing the mode signal) visits a particular subset of its state space. We call this observation process *hidden-Markov*, due to its similitude with hidden-Markov processes [16]. We show how hidden-Markov observation processes generalize many relevant cases previously studied in the literature, such as those in [7, 8, 12–15]. In this context, we propose a Linear Matrix Inequalities (LMI) formulation to design feedback control laws for (stochastic) stabilization, H_2 , and H_∞ control of discrete-time Markov jump linear systems under hidden-Markov observations of the mode signal. It is important to remark that, since the observation process is hidden-Markovian, existing control synthesis methods for Markov jump linear systems, such as those in [1,7,8], do not apply to our case.

The paper is organized as follows. In Section 2, we formulate the state-feedback control problem for Markov jump linear systems with hidden-Markovian observations of the mode signal. We show in Section 3 that the resulting closed-loop system can be reduced to a standard Markov jump linear system by embedding the (possibly non-Markovian) stochastic processes relevant to the controller into an extended Markov chain. In Section 4, we derive an LMI formulation to design state-feedback gains for stabilization, H_2 , and H_∞ control problems. Finally, in Section 5, we illustrate our results with some numerical examples.

Notation

The notation used in this paper is standard. Let $\mathbb N$ denote the set of nonnegative integers. Let $\mathbb R^n$ and $\mathbb R^{n\times m}$ denote the vector spaces of real n-vectors and $n\times m$ matrices, respectively. By $\|\cdot\|$, we denote the Euclidean norm on $\mathbb R^n$. $\Pr(\cdot)$ will be used to denote the probability of an event. The probability of an event conditional on another event $\mathcal A$ is denoted by $\Pr(\cdot\mid \mathcal A)$. Expectations are denoted by $E[\cdot]$. For a positive integer N, we define the set $[N]=\{1,\ldots,N\}$. For a positive integer T and an integer k, define $\lfloor k \rfloor_T$ as the unique integer in $\{0,\ldots,T-1\}$ such that $k-\lfloor k \rfloor_T$ is an integer multiple of T. When a real symmetric matrix A is positive (resp., negative) definite, we write A>0 (resp., A<0). The notations $A\geq 0$ and $A\leq 0$ are then understood in the obvious way. For sets of matrices $A=\{A_\lambda\}_{\lambda\in\Lambda}\subset\mathbb R^{n\times m}$ and $B=\{B_\lambda\}_{\lambda\in\Lambda}\subset\mathbb R^{m\times \ell}$ sharing the same index set Λ , we define another set of matrices $AB=\{A_\lambda B_\lambda\}_{\lambda\in\Lambda}\subset\mathbb R^{n\times \ell}$. The symbol \star will be used to denote the symmetric blocks of partitioned symmetric matrices. Finally, indicator functions are denoted by $\mathbb 1(\cdot)$.

2. PROBLEM FORMULATION

In this section, we formulate the problems under study. Let n, m, q, ℓ , and N be positive integers. For each $i \in [N]$, let $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{m \times n}$, $C_i \in \mathbb{R}^{\ell \times n}$, $D_i \in \mathbb{R}^{\ell \times m}$, and $E_i \in \mathbb{R}^{n \times q}$. Also, let $r = \{r(k)\}_{k=0}^{\infty}$ be the time-homogeneous Markov chain taking values in [N] and having the transition probability matrix $P \in \mathbb{R}^{N \times N}$. Consider the Markov jump linear system [1]:

$$\Sigma : \begin{cases} x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) + E_{r(k)}w(k), \\ z(k) = C_{r(k)}x(k) + D_{r(k)}u(k). \end{cases}$$

We call x and r the state and the mode of Σ , respectively. The signal w represents an exogenous disturbance, u is the control input, and z is the measured signal. The initial conditions are denoted by $x(0) = x_0$ and $r(0) = r_0$. We will assume that x_0 and r_0 are either deterministic constants or random variables, depending on the particular control problems considered.

2.1. State-feedback control with hidden-mode observation

In this paper, we consider the situation where the controller cannot measure the mode signal at every time instant. To study this case, we model the times at which the controller can observe the mode by the stochastic process $t = \{t_i\}_{i=0}^{\infty}$ taking values in $\mathbb{N} \cup \{\infty\}$. We call t the observation process and each t_i an observation time. For each i, we assume either $t_i < t_{i+1}$ or $t_i = t_{i+1} = \infty$. It is understood that, if $t_i < t_{i+1} = \infty$, then no observation will be performed after time t_i .

In this paper, we focus on the following class of observation processes:

Definition 2.1

We say that an observation process t is $hidden-Markov^{\dagger}$ if there exist an $M \in \mathbb{N}$, a Markov chain $s = \{s(k)\}_{k \geq 0}$ taking values in [M] (independent of the mode r), and a function $f : [M] \to \{0,1\}$ such that

$$t_0 = \min\{k \ge 0 : f(s(k)) = 1\}$$

and, for every i > 0,

$$t_{i+1} = \min\{k > t_i : f(s(k)) = 1\},\$$

where the minimum of the empty set is understood to be ∞ .

For example, if the image of f equals the set $\{1\}$, then the controller observes the mode at all time instants. On the other hand, if f maps into $\{0\}$, then the controller never observes the mode signal. In fact, the class of hidden-Markov observation processes contains many other interesting examples as will be seen below. Throughout the paper, we denote the transition probability matrix of s by $Q \in \mathbb{R}^{M \times M}$. In what follows, we provide three particular examples that can be formulated as hidden-Markovian observation processes:

Example 2.2 (Gilbert-Elliot channel)

Consider the case where the controller observes the mode through a Gilbert-Elliot channel [17]. This channel has two possible states: the good (G) and bad (B) states. When the channel is at the G state, it transmits the mode signal to the controller; in contrast, when it is at state B, it does not transmit. This channel switches its state according to a Markov chain, defined as follows. Let $p, q \in [0, 1]$ be the transition probabilities from G to B and B to G, respectively. We can formulate this channel as a hidden-Markovian observation process (Definition 2.1) using the following parameters:

$$M=2, \quad Q=\begin{bmatrix}1-p & p\\ q & 1-q\end{bmatrix}, \quad f(\lambda)=\begin{cases}1, & \text{if } \lambda=1,\\ 0, & \text{if } \lambda=2.\end{cases}$$

Our second example is closely related to the observation processes investigated in [12]:

Example 2.3 (Observations with independent and identically distributed failures)

Assume that, at each time instant, the controller attempts to observe the mode signal but it fails with probability $p_f \in [0,1]$, independently from the observations at other time instants. This observation process can be implemented as a hidden-Markovian observation process using the following parameters:

$$M=2, \quad Q=\begin{bmatrix} 1-p_f & p_f \\ 1-p_f & p_f \end{bmatrix}, \quad f(\lambda)=\begin{cases} 1, & \text{if } \lambda=1, \\ 0, & \text{if } \lambda=2. \end{cases}$$

We remark that, under a similar problem setting, the authors in [12] propose a framework for stochastic stabilization and H_2 control of Markov jump linear systems.

Our last example is concerned with periodic observation with failures:

[†]We adopt the terminology "hidden-Markov" because the process $\{f(s(k))\}_{k\geq 0}$ characterizing the observation process t is a hidden-Markov process [16].

Example 2.4 (Periodic observation with failures) Let ℓ be a positive integer and $p \in [0, 1]$. Define

$$M = \ell + 1, \quad Q = \begin{bmatrix} - - - - - \frac{1}{1} & - - - - \\ & \frac{1}{1} & - - - - \\ & & \ddots \\ - \frac{1}{p} & 1 - p & - - - - \frac{1}{1} \end{bmatrix}, \quad f(\lambda) = \begin{cases} 1, & \text{if } \lambda = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can see that $t_{i+1} - t_i$ is a positive integer multiple of ℓ with probability one and, also, $\Pr(t_{i+1} - t_i = k\ell) = (1-p)^{k-1}p$ for all $i \geq 0$ and $k \geq 1$. The corresponding observation process describes the situation where the controller tries to observe the mode signal every ℓ time units with a probability of success p for each observation. In particular, for p = 1, this observation process gives the periodic case considered in [14, 15].

In order to specify the behavior of the controller between two consecutive observation times, we introduce the following processes. Given an observation process t, we define the stochastic process $\tau = {\{\tau(k)\}_{k=0}^{\infty}}$ by

$$\tau(k) = \begin{cases} \max\{t_i : t_i \leq k, \ i \geq 0\}, & \text{if } k \geq t_0, \\ \tau_0, & \text{otherwise,} \end{cases}$$

where τ_0 is an integer satisfying

$$\begin{cases} \tau_0 = 0, & \text{if } t_0 = 0, \\ \tau_0 < 0, & \text{otherwise.} \end{cases}$$
 (1)

For each time k, the above defined $\tau(k)$ represents the most recent time the controller observed the mode. We, in particular, have $\tau(t_i) = t_i$ for every $i \geq 0$. Notice that, for $k < t_0$, we augment the process τ with a negative integer τ_0 . This is because, before time $k = t_0$, no observation is performed by the controller yet. This augmentation is not needed if $t_0 = 0$, in which case we set $\tau_0 = 0$ as in (1).

We also define the stochastic process $\sigma = {\{\sigma(k)\}_{k=0}^{\infty}}$ taking values in [N] by

$$\sigma(k) = \begin{cases} r(\tau(k)), & \text{if } k \ge t_0, \\ \sigma_0, & \text{otherwise,} \end{cases}$$

where σ_0 is an element in [N] satisfying

$$[t_0 = 0] \Rightarrow [\sigma_0 = r_0]. \tag{2}$$

For each k, the random variable $\sigma(k)$ represents the most-updated information about the mode signal kept by the controller at time k. We again notice that, by the same reason indicated above, σ is augmented by an arbitrary σ_0 before the time instant $k=t_0$ (i.e., before the first observation is performed). As is the case for τ_0 , if $t_0=0$, then this augmentation is not needed and thus we set $\sigma_0=r_0$ as in (2). See Figure 1 for an illustration of the stochastic processes described so far.

In what follows, we present the state-feedback control scheme studied in this paper. We assume that the controller has an access to the following pieces of information at each time $k \geq 0$: (i) the state variable x(k), (ii) the most recent observation $\sigma(k)$ of the mode r, and (iii) the quantity $k - \tau(k)$, which is the time elapsed since the last observation. Specifically, the state-feedback controller under consideration takes the form

$$u(k) = K_{\sigma(k), \lfloor k - \tau(k) \rfloor_T + 1} x(k), \tag{3}$$

where $K_{\gamma,\delta} \in \mathbb{R}^{m \times n}$ for each $\gamma \in [N]$ and $\delta \in [T]$. The first subindex of K in (3), i.e., $\sigma(k)$, allows the gain to be reset whenever the controller observes the mode. The second subindex[‡] of K in (3)

[‡]Notice that we add 1 to $\lfloor k - \tau(k) \rfloor_T$ in the second subindex of K in (3) to make the index δ of $K_{\gamma,\delta}$ start from 1, instead of 0.

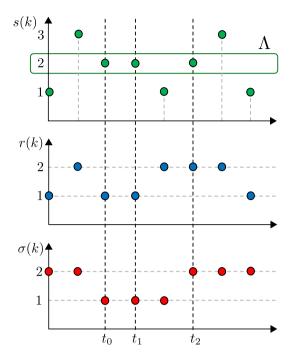


Figure 1. An observation of the mode r. The observation times t_0, t_1, t_2, \ldots are determined by the Markov chain s and the function $f \colon [3] \to \{0,1\}$ given by f(s) = 1 if $s \in \Lambda = \{2\}$ and f(s) = 0 otherwise. Until the first observation time $t_0 = 2$, the most recent observation σ is temporarily set to $\sigma_0 = 2$.

allows the controller to change its feedback gain between two consecutive observation times with period T as in [15], rather than keeping them to be constant. We will later see in Section 5 that, as the period T increases, the performance of the controller can in fact improve. Throughout the paper, we will use the notation

$$\rho(k) = |k - \tau(k)|_T + 1.$$

Notice that the initial condition of ρ is given by $\rho(0) = \rho_0 = |-\tau_0|_T + 1$.

2.2. Performance measures

We now introduce several performance measures used to evaluate the state-feedback control law (3). The feedback control law (3) applied to Σ yields the following closed-loop system

$$\Sigma_K : \begin{cases} x(k+1) = \left(A_{r(k)} + B_{r(k)} K_{\sigma(k), \rho(k)} \right) x(k) + E_{r(k)} w(k), \\ z(k) = \left(C_{r(k)} + D_{r(k)} K_{\sigma(k), \rho(k)} \right) x(k). \end{cases}$$

Let us introduce the compact notation

$$\bar{r}(k) = (r(k), s(k), \sigma(k), \rho(k)). \tag{4}$$

Also, define $\mathfrak X$ as the set of quadruples $(\alpha,\beta,\gamma,\delta)\in[N]\times[M]\times[N]\times[T]$ such that, if $f(\beta)=1$, then $\alpha=\gamma$ and $\delta=1$. The set $\mathfrak X$ contains all possible values that can be taken by the stochastic process $\bar r$. We denote the initial condition for $\bar r$ as $\bar r(0)=\bar r_0$. We sometimes denote the trajectories x and z of Σ_K by $x(\cdot;x_0,\bar r_0,w)$ and $z(\cdot;x_0,\bar r_0,w)$, respectively, whenever we need to clarify the initial conditions as well as the disturbance w. We finally remark that, by the conditions in (1) and (2), $\bar r_0$ is determined by r_0,s_0,σ_0 , and ρ_0 as

$$\bar{r}_0 = \begin{cases} (r_0, s_0, r_0, 1), & \text{if } f(s_0) = 1, \\ (r_0, s_0, \sigma_0, \rho_0), & \text{otherwise.} \end{cases}$$
 (5)

The first performance measure under consideration is mean square stability:

Definition 2.5 (Mean square stability)

We say that Σ_K is mean square stable if there exist C > 0 and $\lambda \in [0, 1)$ such that $E[\|x(k)\|^2] \le C\lambda^k \|x_0\|^2$ for all x_0, \bar{r}_0 , and $k \ge 0$, provided $w \equiv 0$.

In order to define the second performance under consideration, we first need to introduce the space of square summable stochastic processes, as follows. Let Θ_k be the σ -algebra generated by the random variables $\{r(k), s(k), \ldots, r(0), s(0)\}$. Define $\ell^2(\mathbb{R}^n)$ (ℓ^2 for short) as the space of \mathbb{R}^n -valued stochastic processes $f = \{f(k)\}_{k \geq 0}$ such that f(k) is an \mathbb{R}^n -valued and Θ_k -measurable random variable each $k \geq 0$ and, moreover, $\sum_{k=0}^{\infty} E[\|f(k)\|^2]$ is finite. For $f \in \ell^2$, define its ℓ^2 -norm $\|f\|_2$ by $\|f\|_2^2 = \sum_{k=0}^{\infty} E[\|f(k)\|^2]$. Then, we extend the definition of the H_2 norm of a Markov jump linear system introduced in [7], as follows:

Definition 2.6 (H_2 norm)

Assume that \bar{r}_0 follows the probability distribution $\bar{\mu}$. Define the H_2 norm of Σ_K by

$$\|\Sigma_K\|_2 = \sqrt{\sum_{i=1}^q \sum_{\chi \in \mathfrak{X}} \bar{\mu}(\chi) \|z(\cdot; 0, \chi, e_i \phi)\|_2^2},$$

where e_i denotes the *i*th standard unit vector in \mathbb{R}^q and ϕ is the function defined on \mathbb{N} by $\phi(0) = 1$ and $\phi(k) = 0$ for $k \ge 1$.

Our third and last performance measure is the H_{∞} norm. In our context, we use the following definition, which is an extension of the one for standard Markov jump linear systems [18]:

Definition 2.7 (H_{∞} norm)

Assume that Σ_K is mean square stable and $x_0 = 0$. Define the H_{∞} norm of Σ_K by

$$\|\Sigma_K\|_{\infty} = \sup_{\bar{r}_0 \in \mathfrak{X}} \sup_{w \in \ell^2(\mathbb{R}^q) \setminus \{0\}} \frac{\|z\|_2}{\|w\|_2}.$$

Having introduced these three performance measures, we can now formulate the problems under consideration. The first problem is concerned with stochastic stabilization, which is stated as follows:

Problem 2.8 (Stabilization)

Find a set of matrices $\{K_{\gamma,\delta}\}_{\gamma\in[N],\delta\in[T]}\subset\mathbb{R}^{m\times n}$ such that Σ_K is mean square stable.

The second problem is concerned with the stabilization of the closed-loop system with an upperbound on the H_2 norm. In this problem, we assume that the distributions of r_0 and s_0 are given. Thus, the parameters to be designed are feedback gains K and the distribution ν of the pair (σ_0, ρ_0) :

Problem 2.9 (H_2 control)

Assume that the distributions of r_0 and s_0 are known. For a given $\gamma > 0$, find a set of matrices $\{K_{\gamma,\delta}\}_{\gamma \in [N], \delta \in [T]} \subset \mathbb{R}^{m \times n}$ and a distribution ν such that Σ_K is mean square stable and $\|\Sigma_K\|_2 < \gamma$.

Remark 2.10

Given the distribution μ_r (resp., μ_s) of r_0 (resp., s_0), using (5) we can find $\bar{\mu}$ as

$$\bar{\mu}(\chi) = \begin{cases} \mu_r(\alpha)\mu_s(\beta) & \text{if } f(\beta) = 1, \\ \mu_r(\alpha)\mu_s(\beta)\nu(\gamma, \delta) & \text{otherwise,} \end{cases}$$
 (6)

for every $\chi = (\alpha, \beta, \gamma, \delta) \in \mathfrak{X}$.

The last problem is the stabilization of the closed-loop system with an upper-bound on the H_{∞} norm:

Problem 2.11 (H_{∞} control)

For a given $\gamma > 0$, find a set of matrices $\{K_{\gamma,\delta}\}_{\gamma \in [N], \delta \in [T]} \subset \mathbb{R}^{m \times n}$ such that Σ_K is mean square stable and $\|\Sigma_K\|_{\infty} < \gamma$.

We remark that Σ_K is no longer a standard Markov jump linear system, due to the nature of the processes σ and ρ . Therefore, we cannot use any of the techniques in the literature [1, 7, 8] to synthesize a control law. In fact, we cannot use existing techniques even to analyze the performance of Σ_K . Moreover, due to the generality of hidden-Markov observation processes, we cannot use any of the results in [12, 13, 15], recently proposed to design state-feedback control laws for Markov jump linear systems with partial mode observation.

3. ANALYSIS OF CLOSED-LOOP SYSTEM

In this section, we show how to analyze the closed-loop system Σ_K . In this direction, we reduce Σ_K to a standard Markov jump linear system by embedding stochastic processes appearing in the closed-loop system (which are not necessarily Markovian) into an extended Markov chain with a larger state space. Let us begin with the following observation:

Lemma 3.1

The stochastic process \bar{r} defined by (4) is a time-homogeneous Markov chain. Moreover, its transition probabilities are given by

$$\Pr(\bar{r}(k+1) = \chi' \mid \bar{r}(k) = \chi) = \begin{cases} \mathbb{1}(\alpha' = \gamma', \, \delta' = 1) p_{\alpha\alpha'} q_{\beta\beta'}, & \text{if } f(\beta') = 1, \\ \mathbb{1}(\gamma' = \gamma, \, \lfloor \delta' - \delta - 1 \rfloor_T = 0) p_{\alpha\alpha'} q_{\beta\beta'}, & \text{otherwise,} \end{cases}$$
(7)

for all $\chi = (\alpha, \beta, \gamma, \delta)$ and $\chi' = (\alpha', \beta', \gamma', \delta')$ in \mathfrak{X} .

Proof

Let $k_0 \in \mathbb{N}$, $k \ge k_0$, and $\chi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i) \in \overline{\mathfrak{X}}$ $(i = k_0, \dots, k+1)$ be arbitrary. For each i, define the events A_i and B_i as $A_i = \{\bar{r}(i) = \chi_i, \dots, \bar{r}(k_0) = \chi_{k_0}\}$ and $B_i = \{\bar{r}(i) = \chi_i\}$. Under the assumption that A_k is not the null set, we need to evaluate the conditional probability

$$\Pr(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{A}_k) = \Pr(\mathcal{A}_{k+1}) / \Pr(\mathcal{A}_k). \tag{8}$$

Remark that, since $A_i \neq \emptyset$, we have

$$\sigma(k) = \gamma_k, \quad \rho(k) = \delta_k. \tag{9}$$

First, assume that $f(\beta_{k+1}) = 1$. Then, by Definition 2.1, an observation occurs at time k+1, i.e., we have $\tau(k+1) = k+1$ and $\sigma(k+1) = r(k+1)$. This implies that

$$\mathcal{B}_{k+1} = \{ r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}, \alpha_{k+1} = \gamma_{k+1}, 1 = \delta_{k+1} \}.$$

Therefore, since $A_{k+1} = A_k \cap B_{k+1}$,

$$\mathcal{A}_{k+1} = \{\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1\} \cap \{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{A}_k \tag{10}$$

and hence

$$\Pr(\mathcal{A}_{k+1}) = \mathbb{1}(\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1) \Pr(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{A}_k).$$
 (11)

The probability appearing in the last term of this equation can be computed as

$$\Pr(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{A}_k)$$

$$= \Pr(\mathcal{A}_k) \Pr(r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1} \mid \mathcal{A}_k)$$

$$= \Pr(\mathcal{A}_k) \Pr(r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1} \mid r(k) = \alpha_k, s(k) = \beta_k)$$

$$= \Pr(\mathcal{A}_k) p_{\alpha_k \alpha_{k+1}} q_{\beta_k \beta_{k+1}},$$
(12)

where we have used the fact that both r and s are time-homogeneous Markov chains. Thus, from equations (8), (11), and (12), we conclude that for the case of $f(\beta_{k+1}) = 1$,

$$\Pr(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{A}_k) = \mathbb{1}(\alpha_{k+1} = \gamma_{k+1}, \delta_{k+1} = 1) p_{\alpha_k \alpha_{k+1}} q_{\beta_k \beta_{k+1}}. \tag{13}$$

Second, consider the case where $f(\beta_{k+1}) = 0$. In this case, the Markov mode r is not observed at time k+1, hence, we have $\tau(k+1) = \tau(k)$ and $\sigma(k+1) = \sigma(k)$. Therefore, using equations (9), in the same way as we derived (10), we can show that

$$\mathcal{A}_{k+1} = \{ \gamma_{k+1} = \gamma_k, \lfloor \delta_{k+1} - \delta_k - 1 \rfloor_T = 0 \} \cap \{ r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1} \} \cap \mathcal{A}_k \quad (14)$$
 and hence

$$\Pr(\mathcal{A}_{k+1}) = \mathbb{1}(\gamma_{k+1} = \gamma_k, |\delta_{k+1} - \delta_k - 1|_T = 0) \Pr(\{r(k+1) = \alpha_{k+1}, s(k+1) = \beta_{k+1}\} \cap \mathcal{A}_k).$$

Therefore, from equations (8), (12), and (14), we show that, if $f(\beta_{k+1}) = 0$, then

$$\Pr(\bar{r}(k+1) = \chi_{k+1} \mid \mathcal{A}_k) = \mathbb{1}(\gamma_k = \gamma_{k+1}, \lfloor \delta_{k+1} - \delta_k - 1 \rfloor_T = 0) p_{\alpha_k \alpha_{k+1}} q_{\beta_k \beta_{k+1}}. \tag{15}$$

Since the probabilities (13) and (15) do not depend on k_0 , letting $k_0 = k$ and $k_0 = 0$ in (13) and (15), we obtain

$$\Pr(\bar{r}(k+1) = \chi_{k+1} \mid \bar{r}(k) = \chi_k, \dots, \bar{r}(k_0) = \chi_{k_0}) = \Pr(\bar{r}(k+1) = \chi_{k+1} \mid \bar{r}(k) = \chi_k)$$

for every $k \ge 0$. This shows that \bar{r} is a Markov chain since $\chi_{k_0}, \ldots, \chi_{k+1} \in \mathfrak{X}$ are arbitrary. Moreover, since the probabilities (13) and (15) do not depend on k, we conclude that the Markov chain \bar{r} is time-homogeneous and its transition probabilities are given by (7).

Lemma 3.1 states that the closed-loop Σ_K can be represented as a Markov jump linear system with its mode being the extended Markov chain \bar{r} . This observation leads us to the following definitions. For $\chi, \chi' \in \mathfrak{X}$, we denote the transition probabilities of the Markov chain \bar{r} (eq. (7) in Lemma 3.1) by $\bar{p}_{\chi\chi'} = \Pr(\bar{r}(k+1) = \chi' \mid \bar{r}(k) = \chi)$. Then, we introduce the Markov jump linear system

$$\bar{\Sigma}_K: \begin{cases} \bar{x}(k+1) = A_{K,\theta(k)}\bar{x}(k) + E_{K,\theta(k)}\bar{w}(k), \\ \bar{z}(k) = C_{K,\theta(k)}\bar{x}(k), \end{cases}$$

where θ is the time-homogeneous Markov chain taking values in \mathfrak{X} whose transition probabilities are $\Pr(\theta(k+1) = \chi' \mid \theta(k) = \chi) = \bar{p}_{\chi\chi'}$, and the matrices $A_{K,\chi}$, $C_{K,\chi}$, and $E_{K,\chi}$ are defined by

$$A_{K,\chi} = A_{\alpha} + B_{\alpha} K_{\gamma,\delta}, \quad C_{K,\chi} = C_{\alpha} + D_{\alpha} K_{\gamma,\delta}, \quad E_{K,\chi} = E_{\alpha}, \tag{16}$$

for each $\chi = (\alpha, \beta, \gamma, \delta) \in \mathfrak{X}$. We sometimes denote \bar{x} and \bar{z} by $\bar{x}(\cdot; \bar{x}_0, \theta_0, \bar{w})$ and $\bar{z}(\cdot; \bar{x}_0, \theta_0, \bar{w})$ whenever we need to clarify initial conditions and disturbances \bar{w} .

The next corollary of Lemma 3.1 plays the key role in this paper.

Corollary 3.2

Assume that $x_0 = \bar{x}_0$, w and \bar{w} have the same probability distribution, and \bar{r}_0 and θ_0 have the same probability distribution. Then, the stochastic processes $x(\cdot; x_0, \bar{r}_0, w)$ and $\bar{x}(\cdot; \bar{x}_0, \theta_0, \bar{w})$ have the same probability distribution. Also, under the same assumption, the stochastic processes $z(\cdot; x_0, \bar{r}_0, w)$ and $\bar{z}(\cdot; \bar{x}_0, \theta_0, \bar{w})$ have the same probability distribution.

Proof

By the assumption, the Markov chains \bar{r} and θ have the same initial distribution. The chains also have the same transition probabilities from Lemma 3.1 and the definition of θ . Therefore, \bar{r} and θ have the same probability distribution. Also, we notice that, by the definition of the matrices in (16), the system Σ_K admits the following representation:

$$\begin{split} x(k+1) &= A_{K,\bar{r}(k)} x(k) + E_{\bar{r}(k)} w(k), \\ z(k) &= C_{K,\bar{r}(k)} x(k). \end{split}$$

Therefore, Σ_K has the same dynamics as $\bar{\Sigma}_K$. In conclusion, the claim holds true under the assumptions stated in the corollary.

Using Corollary 3.2, we can characterize the performance measures of the closed-loop system Σ_K . The next proposition provides a characterization for mean square stability:

Proposition 3.3

For $R = \{R_\chi\}_{\chi \in \mathfrak{X}} \subset \mathbb{R}^{n \times n}$ and $\chi \in \mathfrak{X}$, define $\bar{\mathcal{D}}_\chi(R) = \sum_{\chi' \in \mathfrak{X}} \bar{p}_{\chi'\chi} R_{\chi'} \in \mathbb{R}^{n \times n}$. Then, the following statements are equivalent:

- 1. Σ_K is mean square stable;
- 2. $\bar{\Sigma}_K$ is mean square stable;
- 3. There exist positive-definite matrices $Q_{\chi} \in \mathbb{R}^{n \times n}$ for every $\chi \in \mathfrak{X}$, such that Q_{χ} $\bar{\mathcal{D}}_{\chi}(A_K Q A_K^{\top}) > 0.$

Proof

The equivalence [2 \iff 3] immediately follows from the standard theory of Markov jump linear systems (see, e.g., [1]). Let us prove $[2 \Rightarrow 1]$. Assume that Σ_K is mean square stable. In order to show that Σ_K is mean square stable, let us take arbitrary $x_0 \in \mathbb{R}^n$ and $\bar{r}_0 \in \mathfrak{X}$. Then, by Corollary 3.2 and the mean square stability of $\bar{\Sigma}_K$, we can show $E[||x(k;x_0,\bar{r}_0,0)||^2] = E[||\bar{x}(k;x_0,\bar{r}_0,0)||^2] \le C\lambda^k ||x_0||^2$ for some C>0 and $\lambda \in [0,1)$, which implies mean square stability of Σ_K . We can prove $[1 \Rightarrow 2]$ in the same way.

The following proposition characterizes the H_2 norm of Σ_K :

Proposition 3.4

Let $\gamma > 0$ be arbitrary. Assume that \bar{r}_0 and θ_0 follow the same distribution $\bar{\mu}$. Then, the following statements are equivalent:

- 1. Σ_K is mean square stable and $\|\Sigma_K\|_2^2 < \gamma;$ 2. $\bar{\Sigma}_K$ is mean square stable and $\|\bar{\Sigma}_K\|_2^2 < \gamma;$
- 3. There exist a family of positive-definite matrices $Q_{\chi} \in \mathbb{R}^{n \times n}$ for every $\chi \in \mathfrak{X}$, such that $\sum_{\chi \in \mathfrak{X}} \operatorname{tr}(C_{K,\chi}Q_{\chi}C_{K,\chi}^{\top}) < \gamma$ and $\bar{\mathcal{D}}_{\chi}(A_{K}QA_{K}^{\top} + \bar{\mu}E_{K}E_{K}^{\top}) < Q_{\chi}$, where the set $\bar{\mu}E_{K}E_{K}^{\top} \subset \mathbb{R}$ $\mathbb{R}^{n \times n}$ indexed by \mathfrak{X} is defined as $(\bar{\mu} E_K E_K^{\top})_{\chi} = \bar{\mu}(\chi) E_{K,\chi} E_{K,\chi}^{\top}$.

Proof

The proof of the equivalence $[1 \Leftrightarrow 2]$ immediately follows from Corollary 3.2. Also, the equivalence $[2 \Leftrightarrow 3]$ is a direct consequence of a standard result [7, Proposition 4] in the theory of Markov jump linear systems. The details are omitted.

Finally, the following proposition characterizes the H_{∞} norm:

Proposition 3.5

For matrices $Z = \{Z_{\chi,\chi'}\}_{\chi,\chi' \in \mathfrak{X}} \subset \mathbb{R}^{n \times n}$, define $\bar{\mathcal{F}}_{\chi}(Z) = \sum_{\chi' \in \mathfrak{X}} \bar{p}_{\chi\chi'} Z_{\chi,\chi'} \in \mathbb{R}^{n \times n}$. Let $\gamma > 0$ be arbitrary. Consider the following statements:

- 1. Σ_K is mean square stable and $\|\Sigma_K\|_{\infty}^2 < \gamma$;
- Σ̄_K is mean square stable and ||Σ̄_K||²_∞ < γ;
 There exist matrices G_χ ∈ ℝ^{n×n}, H_χ ∈ ℝ^{n×n}, X_χ ∈ ℝ^{n×n}, and Z_{χ,χ'} ∈ ℝ^{n×n} (χ, χ' ∈ 𝒳) such that

$$\begin{bmatrix} G_{\chi} + G_{\chi}^{\top} - X_{\chi} & \star & \star & \star \\ O & \gamma I & \star & \star \\ A_{K,\chi}G_{\chi} & E_{K,\chi} & H_{\chi} + H_{\chi}^{\top} - \bar{\mathcal{F}}_{\chi}(Z) & \star \\ C_{K,\chi}G_{\chi} & O & O & I \end{bmatrix} > 0, \quad \begin{bmatrix} Z_{\chi,\chi'} & \star \\ H_{\chi} & X_{\chi'} \end{bmatrix} > 0,$$

for all $\chi, \chi' \in \mathfrak{X}$.

Then, 1 and 2 are equivalent. Moreover, 3 implies 1 and 2.

Proof

The implication [3 \Rightarrow 2] is a direct consequence of [8, Theorem 1]. Let us prove [1 \Leftrightarrow 2]. It is sufficient to show $\|\Sigma_K\|_{\infty} = \|\bar{\Sigma}_K\|_{\infty}$. Let $\bar{\Theta}_k$ denote the σ -algebra generated by the random variables $\{\theta(k),\ldots,\theta(0)\}$. Define $\bar{\ell}^2$ as the space of stochastic processes $\bar{f}=\{\bar{f}(k)\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} E[\|\bar{f}(k)\|^2]$ is finite and, for each $k \geq 0$, $\bar{f}(k)$ is an \mathbb{R}^n -valued and $\bar{\Theta}_k$ -measurable random variable. Define the norm of $\bar{f} \in \bar{\ell}^2$ by $\|\bar{f}\|_2^2 = \sum_{k=0}^{\infty} E[\|\bar{f}(k)\|^2]$. Then, the H_{∞} norm of $\bar{\Sigma}_K$ is given by

$$\|\bar{\Sigma}_K\|_{\infty} = \sup_{\theta_0 \in \mathfrak{X}} \sup_{\bar{w} \in \bar{\ell}^2 \setminus \{0\}} \frac{\|\bar{z}(\cdot; 0, \theta_0, \bar{w})\|_2}{\|\bar{w}\|_2}.$$

To show that this norm equals $\|\Sigma_K\|_{\infty}$, it is sufficient to show that, if $\bar{r}_0 = \theta_0$, then

$$\sup_{w \in \ell^2 \setminus \{0\}} \frac{\|z(\cdot; 0, \bar{r}_0, w)\|_2}{\|w\|_2} = \sup_{\bar{w} \in \bar{\ell}^2 \setminus \{0\}} \frac{\|\bar{z}(\cdot; 0, \theta_0, \bar{w})\|_2}{\|\bar{w}\|_2}.$$
 (17)

By Corollary 3.2, the only difference between both sides of the equality (17) is the spaces ℓ^2 and $\bar{\ell}^2$. In other words, to complete the proof, it is sufficient to show that Θ_k , the σ -algebra generated by the random variables $\{r(k), s(k), \ldots, r(0), s(0)\}$, coincides with the one generated by $\{\bar{r}(k), \ldots, \bar{r}(0)\}$. This is obvious because $\sigma(k)$ and $\rho(k)$ are images of $(r(k), s(k), r(k-1), s(k-1), \ldots, r(0), s(0))$ under measurable functions.

4. DESIGN OF FEEDBACK GAINS VIA LINEAR MATRIX INEQUALITIES

Based on the performance characterizations presented in the previous section, we now propose a formulation based on Linear Matrix Inequalities (LMI) to design feedback control laws for stabilization, H_2 , and H_∞ control of discrete-time Markov jump linear systems under hidden-Markovian observations of the mode signals.

The next theorem provides an LMI formulation to solve the stabilization problem stated in Problem 2.8:

Theorem 4.1

Assume that the matrices $R_{\chi} \in \mathbb{R}^{n \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$ $(\chi = (\alpha, \beta, \gamma, \delta) \in \mathfrak{X})$ satisfy the linear matrix inequality

$$\begin{bmatrix} R_{\chi} & A_{\alpha}G_{\gamma,\delta} + B_{\alpha}F_{\gamma,\delta} \\ \star & G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - \mathcal{D}_{\chi}(R) \end{bmatrix} > 0$$
 (18)

for every $\chi=(\alpha,\beta,\gamma,\delta)\in\mathfrak{X}.$ For each $\gamma\in[N]$ and $\delta\in[T],$ define

$$K_{\gamma,\delta} = F_{\gamma,\delta} G_{\gamma,\delta}^{-1}.$$
 (19)

Then, the resulting closed-loop system Σ_K is mean square stable.

Proof

Assume that $R_{\chi} \in \mathbb{R}^{n \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$ satisfy (18), and define K by (19). Our proof is based on an argument proposed in [7]. Since \mathfrak{X} is a finite set, there exists an $\epsilon > 0$ such that $Q_{\chi} = \mathcal{D}_{\chi}(R) + \epsilon I$ satisfies

$$\begin{bmatrix} R_{\chi} & A_{K,\chi} G_{\gamma,\delta} \\ \star & G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - Q_{\chi} \end{bmatrix} > 0, \tag{20}$$

where we have used $A_{\alpha}G_{\gamma,\delta} + B_{\alpha}F_{\gamma,\delta} = A_{K,\chi}G_{\gamma,\delta}$. Since this inequality implies $R_{\chi} > 0$, we have $Q_{\chi} \geq \epsilon I > 0$. We here recall that, for a positive definite matrix $A \in \mathbb{R}^{n \times n}$ and another matrix $B \in \mathbb{R}^{n \times n}$, it holds that (see, e.g., [7]) $BA^{-1}B^{\top} \geq B + B^{\top} - A$. Using this fact in (20), we obtain

$$\begin{bmatrix} R_{\chi} & A_{K,\chi} G_{\gamma,\delta} \\ \star & G_{\gamma,\delta}^{\top} Q_{\chi}^{-1} G_{\gamma,\delta} \end{bmatrix} > 0.$$
 (21)

Also, from (20), we see that $G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} > 0$ and therefore $G_{\gamma,\delta}$ is invertible. Hence, we can take the Schur complement of the positive-definite matrix in (21) with respect to R_{χ} to obtain $R_{\chi} - A_{K,\chi}Q_{\chi}A_{K,\chi}^{\top} > 0$. Applying to this inequality the operator $\bar{\mathcal{D}}_{\chi}$, we obtain $Q_{\chi} - \bar{\mathcal{D}}_{\chi}(A_K Q A_K^{\top}) > \epsilon I > 0$. Therefore, by Proposition 3.3, Σ_K is mean square stable.

Secondly, the next theorem provides an LMI formulation to solve the H_2 control problem stated in Problem 2.9:

Theorem 4.2

Let $\gamma > 0$ be arbitrary. Assume that $W_{\chi} \in \mathbb{R}^{\ell \times \ell}$, $R_{\chi} \in \mathbb{R}^{n \times n}$, $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $\nu(\gamma,\delta) \geq 0 \ (\chi=(\alpha,\beta,\gamma,\delta) \in \mathfrak{X})$ satisfy the following linear matrix inequalities

$$\begin{bmatrix}
R_{\chi} - \bar{\mu}(\chi)E_{\alpha}E_{\alpha}^{\top} & A_{\alpha}G_{\gamma,\delta} + B_{\alpha}F_{\gamma,\delta} \\
\star & G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - \bar{\mathcal{D}}_{\chi}(R)
\end{bmatrix} > 0,$$

$$\begin{bmatrix}
W_{\chi} & C_{\alpha}G_{\gamma,\delta} + D_{\alpha}F_{\gamma,\delta} \\
\star & G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - \bar{\mathcal{D}}_{\chi}(R)
\end{bmatrix} > 0,$$
(22)

$$\begin{bmatrix} W_{\chi} & C_{\alpha}G_{\gamma,\delta} + D_{\alpha}F_{\gamma,\delta} \\ \star & G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - \bar{\mathcal{D}}_{\chi}(R) \end{bmatrix} > 0,$$
 (23)

$$\sum_{\chi \in \mathfrak{X}} \operatorname{tr}(W_{\chi}) < \gamma, \tag{24}$$

$$\sum_{\gamma=1}^{N} \sum_{\delta=1}^{T} \nu(\gamma, \delta) = 1, \tag{25}$$

for every $\chi = (\alpha, \beta, \gamma, \delta) \in \mathfrak{X}$. Define the feedback matrix K by (19). Then, the closed-loop system Σ_K is mean square stable and satisfies $\|\Sigma_K\|_2^2 < \gamma$.

We remark that the (in)equalities in Theorem 4.2 are indeed linear with respect to the design variables. The linearity with respect to the matrix variables W_{χ} , R_{χ} , $F_{\gamma,\delta}$, and $G_{\gamma,\delta}$ is obvious. The linearity with respect to ν follows from (6). We also remark that the constraint (25) makes ν a probability measure.

Let us prove Theorem 4.2.

Proof of Theorem 4.2

Assume that $\gamma > 0$, $W_{\chi} \in \mathbb{R}^{\ell \times \ell}$, $R_{\chi} \in \mathbb{R}^{n \times n}$, $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $\nu(\gamma,\delta)$ satisfy (22)–(25), and let us define K by (19). In the same way as in the proof of Theorem 4.1, there exists an $\epsilon > 0$ such that $Q_{\chi} = \bar{\mathcal{D}}_{\chi}(R) + \epsilon I$ satisfies

$$\begin{bmatrix}
R_{\chi} - \bar{\mu}(\chi)E_{K,\chi}E_{K,\chi}^{\top} & A_{K,\chi}G_{\gamma,\delta} \\
\star & G_{\gamma,\delta}^{\top}Q_{\chi}^{-1}G_{\gamma,\delta}
\end{bmatrix} > 0,$$

$$\begin{bmatrix}
W_{\chi} & C_{K,\chi}G_{\gamma,\delta} \\
\star & G_{\gamma,\delta}^{\top}Q_{\chi}^{-1}G_{\gamma,\delta}
\end{bmatrix} > 0.$$
(26)

$$\begin{bmatrix} W_{\chi} & C_{K,\chi} G_{\gamma,\delta} \\ \star & G_{\gamma,\delta}^{\top} Q_{\chi}^{-1} G_{\gamma,\delta} \end{bmatrix} > 0.$$
 (27)

Applying Schur complement to the matrix in the left hand side of (26), we obtain $R_{\chi} - \bar{\mu}(\chi) E_{K,\chi} E_{K,\chi} - A_{K,\chi} Q_{\chi} A_{K,\chi}^{\top} > 0$. Applying the operator $\bar{\mathcal{D}}_{\chi}$ to this inequality yields $\bar{\mathcal{D}}_\chi(A_KQA_K^\top + \bar{\mu}E_KE_K^\top) < \bar{\mathcal{D}}_\chi(R) < Q_\chi$. In addition, from (27) it follows that $W_\chi > C_{K,\chi}Q_\chi C_{K,\chi}^\top$. Hence, we have $\sum_{\chi \in \mathfrak{X}} \operatorname{tr}(C_{K,\chi}Q_\chi C_{K,\chi}^\top) < \sum_{\chi \in \mathfrak{X}} \operatorname{tr}(W_\chi) < \gamma$ by (24). Therefore, by Proposition 3.4, Σ_K is mean square stable and satisfies $\|\Sigma_K\|_2^2 < \gamma$.

Finally, the next theorem provides an LMI formulation to solve the H_{∞} control problem stated in Problem 2.11:

Theorem 4.3

Assume that $\gamma > 0$, $H_{\chi} \in \mathbb{R}^{n \times n}$, $X_{\chi} \in \mathbb{R}^{n \times n}$, $Z_{\chi,\chi'} \in \mathbb{R}^{n \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$

 $(\chi = (\alpha, \beta, \gamma, \delta))$ and χ' in \mathfrak{X}) satisfy the linear matrix inequalities

$$\begin{bmatrix} G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - X_{\chi} & \star & \star & \star \\ O & \gamma I & \star & \star \\ A_{\alpha}G_{\gamma,\delta} + B_{\alpha}F_{\gamma,\delta} & E_{\alpha} & H_{\chi} + H_{\chi}^{\top} - \bar{\mathcal{F}}_{\chi}(Z) & \star \\ C_{\alpha}G_{\gamma,\delta} + D_{\alpha}F_{\gamma,\delta} & O & O & I \end{bmatrix} > 0,$$

$$(28)$$

$$\begin{bmatrix} Z_{\chi,\chi'} & \star \\ H_{\chi} & X_{\chi'} \end{bmatrix} > 0, \tag{29}$$

for all $\chi = (\alpha, \beta, \gamma, \delta)$ and χ' in \mathfrak{X} . Define the feedback matrix K by (19). Then, the closed-loop system Σ_K is mean square stable and satisfies $\|\Sigma_K\|_{\infty}^2 < \gamma$.

Proof

Assume that the inequalities (28) and (29) are satisfied by the matrices $H_{\chi} \in \mathbb{R}^{n \times n}$, $X_{\chi} \in \mathbb{R}^{n \times n}$, $Z_{\chi,\chi'} \in \mathbb{R}^{n \times n}$, $G_{\gamma,\delta} \in \mathbb{R}^{n \times n}$, and $F_{\gamma,\delta} \in \mathbb{R}^{m \times n}$. Define K by (19). Then, (28) implies

$$\begin{bmatrix} G_{\gamma,\delta} + G_{\gamma,\delta}^{\top} - X_{\chi} & \star & \star & \star \\ O & \gamma I & \star & \star \\ A_{K,\chi}G_{\gamma,\delta} & E_{K,\chi} & H_{\chi} + H_{\chi}^{\top} - \bar{\mathcal{F}}_{\chi}(Z) & \star \\ C_{K,\chi}G_{\gamma,\delta} & O & O & I \end{bmatrix} > 0.$$

By this inequality and (29), Proposition 3.5 immediately shows that Σ_K is mean square stable and satisfies $\|\Sigma_K\|_{\infty}^2 < \gamma$, as desired.

5. NUMERICAL EXAMPLES

The objective of this section is to illustrate Theorems 4.2 and 4.3 by numerical examples. We will also demonstrate how the periodicity of the feedback gain can be used to improve the performance of the closed-loop system Σ_K .

Example 5.1

In this example, we consider the Markov jump linear system studied in [12, Example 1]. The system has two modes and its parameters are given by

$$A_1 = \begin{bmatrix} 0.7017 & -1.227 & 0.3931 & -0.6368 \\ -0.4876 & -0.6699 & -1.7073 & -1.0026 \\ 1.8625 & 1.3409 & 0.2279 & -0.1856 \\ 1.1069 & 0.3881 & 0.6856 & -1.0540 \end{bmatrix}, A_2 = \begin{bmatrix} -0.0715 & -0.5420 & 0.6716 & 0.6250 \\ 0.2792 & 1.6342 & -0.5081 & -1.0473 \\ 1.3733 & 0.8252 & 0.8564 & 1.5357 \\ 0.1798 & 0.2308 & 0.2685 & 0.4344 \end{bmatrix}, \\ B_1 = B_2 = \begin{bmatrix} I_2 \\ O_{2\times 2} \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} I_4 \\ O_{2\times 4} \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} O_{4\times 2} \\ I_2 \end{bmatrix}, E_1 = E_2 = I_4, \\ P = \begin{bmatrix} 0.6942 & 0.3058 \\ 0.6942 & 0.3058 \end{bmatrix}, \mu_r = \begin{bmatrix} 0.6942 & 0.3058 \end{bmatrix},$$

where I_n and $O_{n \times m}$ denote the $n \times n$ identity matrix and the $n \times m$ zero matrix, respectively. Notice that the mode signal r is a sequence of independently and identically distributed random variables.

We assume that the controller observes the mode through a Gilbert-Elliot channel, described in Example 2.2. For simplicity in our presentation, we let p=q. Notice that, whatever value p takes, the limiting distribution of s is the uniform distribution on the set $\{1,2\}$; in other words, the asymptotic frequency of the controller observing the mode signal r is 1/2. In addition, the expected duration of the chain s staying at either Good or Bad state depends on p, and is equal to 1/p. We assume that the initial distribution μ_s of s is the uniform distribution.

We can use Theorem 4.2 to design stabilizing feedback gains and the initial distribution ν in order to achieve a small H_2 norm of the closed-loop system Σ_K by solving the following optimization

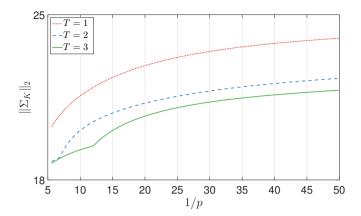


Figure 2. The H_2 norm of the closed-loop system Σ_K versus the expected duration 1/p

problem:

$$\begin{array}{ll} \underset{W_{\chi},\,R_{\chi},\,F_{\gamma,\delta},\,G_{\gamma,\delta},\nu(\gamma,\delta)}{\text{minimize}} & \gamma, \\ \text{subject to} & \text{(22)-(25)}. \end{array}$$

Figure 2 shows the H_2 norms of the optimized closed-loop systems. As expected, the larger the value of the period T, the smaller the attained H_2 norm. We can also see that the H_2 norm of the closed-loop system increases as the expected duration 1/p increases, although the stationary distribution of s does not depend on p. We remark that this feature arising from the Markov property of s cannot be captured by the framework in [12], where mode observations at different time instants are assumed to be independent events with identical probabilities.

Example 5.2

Consider the Markov jump linear system Σ with the following parameters:

$$A_{1} = \begin{bmatrix} -0.6 & -0.4 \\ -0.6 & -0.4 \end{bmatrix}, \ A_{2} = \begin{bmatrix} -0.8 & 0.4 \\ 0.8 & 0.2 \end{bmatrix}, B_{1} = \begin{bmatrix} -0.3 \\ -0.2 \end{bmatrix}, B_{2} = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix},$$

$$C_{1} = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}, C_{2} = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}, D_{1} = 0.1, D_{2} = -0.3,$$

$$E_{1} = \begin{bmatrix} -0.3 \\ -0.3 \end{bmatrix}, E_{2} = \begin{bmatrix} -0.2 \\ -0.1 \end{bmatrix}, P = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}.$$

From the standard theory of Markov jump linear systems [1], one can check that Σ is not mean square stable when $u \equiv 0$. We use the observation process with independent and identically distributed failures (described in Example 2.3). In order to design stabilizing feedback gains achieving a small H_{∞} norm of Σ_K , we solve the following optimization problem based on Theorem 4.3:

minimize
$$H_{\chi}, X_{\chi}, Z_{\chi\chi'}, G_{\gamma,\delta}, F_{\gamma,\delta}$$
 γ ,
subject to (28) and (29).

Figure 3 shows the H_{∞} norms of the resulting closed-loop systems for various values of period T and failure probability p_f . As we increase the period T, the H_{∞} norm tends to decrease. However, notice that when p_f is around 0.1, the H_{∞} norm attained by the controllers with T=3 are better than those with T=5. This phenomenon can happen because 5 is not an integer-multiple of 3. Remark that, for T=6 instead of T=5, such a phenomenon will not happen because the H_{∞} performance obtained by the feedback gains $K^{(3)}=\{K_{\gamma,\delta}^{(3)}\}_{\gamma\in[2],\delta\in[3]}$ with period T=3 is attained by the feedback gains $K^{(6)}$ with period T=6 given as $K_{\gamma,\ell}^{(6)}=K_{\gamma,\ell+3}^{(6)}=K_{\gamma,\ell}^{(3)}$ for every $\ell=1,2,3$. We also remark that, by the same reason, the performance of the resulting closed-loop system for any integer T is never worse than the performance for T=1 (since 1 divides T).

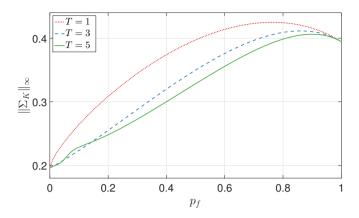


Figure 3. The H_{∞} norm of the closed-loop system Σ_K versus the failure probability p_f

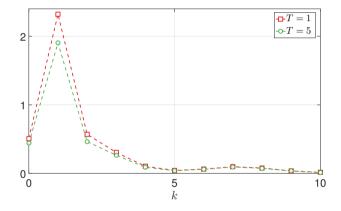


Figure 4. Sample averages of $||z(k)||^2$

Finally, Figure 4 shows the sample averages of $||z(k)||^2$ of the closed-loop systems for T=1 and T=5. For the computation of the sample averages, we fix $x_0=[1\quad 2]^{\top}$, $r_0=1$, $s_0=1$, and $p_f=0.5$. The disturbance signal is chosen as $w(k)=2\cos(k/2)$. We generate 300 sample paths of r and s. Using the sample paths, we then generate 300 sample paths of z for T=1 and T=5, respectively. We can see that the closed-loop system with T=5 attenuates the disturbance signal better than that with T=1.

6. CONCLUSION

In this paper, we have studied state-feedback control of Markov jump linear systems with hidden-Markovian observations of the mode signals. This observation model generalizes various relevant cases previously studied in the literature on Markov jump linear systems, such as the cases with perfect information, no information and cluster observations of the mode signal. We have then developed an optimization framework, based on Linear Matrix Inequalities, to design feedback gains for stabilization, H_2 and H_∞ control problems. Finally, we have illustrated the effectiveness of this optimization framework with several numerical examples.

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