

Stochastic Stability in Max-Product and Max-Plus Systems with Markovian Jumps[☆]

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Abstract

We study Max-Product and Max-Plus Systems with Markovian Jumps and focus on stochastic stability problems. At first, a Lyapunov function is derived for the asymptotically stable deterministic Max-Product Systems. This Lyapunov function is then adjusted to derive sufficient conditions for the stochastic stability of Max-Product systems with Markovian Jumps. Many step Lyapunov functions are then used to derive necessary and sufficient conditions for stochastic stability. The results for the Max-Product systems are then applied to Max-Plus systems with Markovian Jumps, using an isomorphism and almost sure bounds for the asymptotic behavior of the state are obtained. A numerical example illustrating the application of the stability results on a production system is also given.

Keywords:

Stochastic Systems, Nonlinear Systems, Max-Plus Systems, Stochastic Stability

1. Introduction

Max-Plus systems are dynamical systems which satisfy the superposition principle in the Max-Plus algebra. The use of Max-Plus systems was

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proposed in various applications involving timing, such as communication and traffic management, queueing systems, production planning, multi-generation energy systems, et.c. (eg. Cuninghame-Green (1979), Baccelli et al. (1992), Heidergott et al. (2014), Goverde (2007), Baccelli and Hong (2000b)). Recently, the use of the closely related class of Max-Product systems (systems which satisfy the superposition principle in the Max-Product algebra) was proposed as a tool for the modelling of cognitive processes, such as detecting audio and visual salient events in multimodal video streams (Maragos and Koutras (2015)). Max-Plus and Max-Product algebras have also computational uses involving Optimal Control problems (McEneaney (2006)) and estimation problems in probabilistic models such as the max-sum algorithm in Probabilistic Graphical models and the Viterbi algorithm in Hidden Markov Models (eg. Bishop (2006)).

In this work, we study stochastic Max-Plus and Max-Product systems, where the system matrices depend on a finite state Markov chain. For the Max-Plus systems we focus on the asymptotic growth rate, whereas for the Max-Product systems on stochastic stability. A motivation to study Max-Plus systems with Markovian jumps is to model production systems, where the processing or holding times are random variables (not necessarily independent) or there are random failures and repairs, modeled as a Markov chain. The results on max-product stochastic systems will be used as an intermediate step. An independent motivation to study Max-Product systems is the modeling of cognitive processes interrupted by random events. Similar problems with Markovian delays for linear systems were studied in Beidas and Papavassilopoulos (1993), for random failures in Papavassilopoulos (1994) and for nonlinear time varying systems in Beidas and Papavassilopoulos (1995), in the context of distributed parallel optimization and routing applications. In the current work, we try to exploit the special (Max-Product or Max-Plus) structure of the system.

At first, deterministic Max-Product systems are considered and their asymptotic stability is characterized using Lyapunov functions. The Lyapunov function derived can be also used to study systems which are not linear in the Max-Product algebra. We then study Max-Product systems with Markovian Jumps and derive sufficient conditions for their stochastic stability. Further, necessary and sufficient conditions for the stochastic stability of Max-Product systems with Markovian Jumps are derived using many step Lyapunov functions. The results for the stochastic stability of

Max-Product systems are then used to derive bounds for the evolution of the state of Max-Plus systems with Markovian Jumps.

The results of this work relate to the literature for the approximation of the Lyapunov exponent of Max-Plus stochastic systems. The existence of the Lyapunov exponent was proved in Cohen (1988). Limit theorems for the scaled asymptotic evolution of stochastic Max-Plus systems were proved in Merlet (2007), Merlet (2008). Most of the works on the approximation of the Lyapunov exponent focus on the independent random matrix case. In Baccelli and Hong (2000a), Gaubert and Hong (2000) series expansions are used in order to approximate the Lyapunov exponent and Goverde et al. (2008), Goverde et al. (2011) use approximate stochastic simulation techniques to estimate the Lyapunov exponent. In Blondel et al. (2000) it is shown that the approximation of the Lyapunov exponent is an NP-hard problem. Bounds for the tail distributions of Max-Plus stochastic systems are proposed in Chang (1996). In Liu et al. (1995), a model of Max-Plus system with Markovian input is considered and bounds for the tail distributions are derived. A model where the Markov chain (branching process) evolves according to a Max-Plus stochastic system is analyzed in Altman and Fiems (2012). Bounds on the length of the transient phase of Max-Plus systems are proved in Nowak and Charron-Bost (2014).

Another related class of systems is Switching Max-Plus systems with deterministic or stochastic switching introduced in van den Boom and De Schutter (2006) and studied further in van den Boom and De Schutter (2012). The basic difference with the current work is that the current work focuses on stochastic stability properties whereas van den Boom and De Schutter (2006), van den Boom and De Schutter (2012) study stability under arbitrary switching. Several approximation methods in stochastic Max-plus systems control and identification were studied in Farahani (2012).

The techniques used in this work closely parallel the techniques used for the stability analysis of Markovian Jump Linear Systems (MJLS). The study of the stochastic stability of MJLS dates back at least to the 1960s (Bhaurucha (1961)) and today is a well-established field (eg. Costa et al. (2006), Fang and Loparo (2002), Beidas and Papavassilopoulos (1993), Papavassilopoulos (1994), Kordonis and Papavassilopoulos (2014)).

1.1. Background

The Max-Plus and Max-Product algebras are used. In the Max-Plus algebra the usual summation is substituted by maximum and the usual

multiplication is substituted by summation. In the Max-Product algebra the usual summation is substituted by maximum but the multiplication remains unchanged.

The Max-Plus algebra is defined on the set of extended reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the binary operations “ \vee ” and “ \boxplus ”. The operation “ \vee ” stands for the maximum i.e., for $x, y \in \bar{\mathbb{R}}$, it holds $x \vee y = \max\{x, y\}$. The operation “ \boxplus ” corresponds to the usual addition i.e., for $x, y \in \bar{\mathbb{R}}$ it holds $x \boxplus y = x + y$, where the convention $-\infty \boxplus \infty = -\infty$ is used. For a set $(x_i)_{i \in I}$ of extended reals “ \bigvee ” stands for the supremum i.e. $\bigvee_{i \in I} x_i = \sup_{i \in I} \{x_i\}$. For a pair of matrices $\mathbf{A} = [A_{ij}]$ and $\mathbf{B} = [B_{ij}]$, the operation “ \vee ” is their element-wise maximum, i.e.:

$$(\mathbf{A} \vee \mathbf{B})_{ij} = A_{ij} \vee B_{ij},$$

and similarly is the element-wise supremum for an arbitrary set of matrices.

For a pair of matrices $\mathbf{A} = [A_{ij}] \in \bar{\mathbb{R}}^{n \times m}$ and $\mathbf{B} = [B_{ij}] \in \bar{\mathbb{R}}^{m \times l}$ their Max-Plus product $\mathbf{A} \boxplus \mathbf{B}$ is an $n \times l$ matrix and its i, j -th element is given by:

$$(\mathbf{A} \boxplus \mathbf{B})_{ij} = \bigvee_{p=1}^m (A_{ip} + B_{pj}), \quad (1)$$

where “ \bigvee ” denotes the maximum of the m elements.

The Max-Product algebra is defined on $\bar{\mathbb{R}}_+ = [0, \infty]$, with the binary operations “ \vee ” and “ \boxtimes ”. The “ \boxtimes ” operation is the usual scalar multiplication with the convention $0 \boxtimes \infty = 0$. The “ \vee ” operation is defined exactly as in the Max-Plus algebra. The matrix multiplication in the Max-Product algebra is defined by:

$$[\mathbf{A} \boxtimes \mathbf{B}]_{ij} = \bigvee_{p=1}^m (A_{ip} B_{pj}),$$

The power of a square matrix is defined by $\mathbf{A}^k = \mathbf{A}^{k-1} \boxtimes \mathbf{A}$ and $\mathbf{A}^0 = \mathbf{I}$. For a given square matrix \mathbf{A} a new matrix \mathbf{A}^+ is defined as $\mathbf{A}^+ = \bigvee_{k=0}^{\infty} \mathbf{A}^k$. The subset $\mathbb{R}_+ = [0, \infty)$ of $\bar{\mathbb{R}}_+$ will be also used.

Max-Product multiplication distributes over “ \vee ”, i.e.:

$$\bigvee_{i \in I} \mathbf{A} \boxtimes \mathbf{B}_i = \mathbf{A} \boxtimes \left(\bigvee_{i \in I} \mathbf{B}_i \right). \quad (2)$$

The same property holds also for the Max-Plus multiplication.

Operation	Meaning
\vee	The maximum. Applies for scalars, vectors and matrices
\boxplus	Max-plus multiplication. Defined in (1)
\boxtimes	Max-plus multiplication. Defined in (2)

Table 1: The algebraic operations used.

In both algebras, the “ \vee ” operation has lower priority than “ $+$ ” or “ \boxplus ” in the Max-Plus algebra and “ \cdot ” or “ \boxtimes ” in the Max-Product algebra respectively. Let us note that there is an isomorphism $\exp(\cdot)$ between the Max-Plus algebra $(\bar{\mathbb{R}}, \vee, \boxplus)$ and the Max-Product algebra $(\bar{\mathbb{R}}_+, \cdot, \boxtimes)$.

A unifying algebraic framework to study Max-Plus and Max-Product systems (and also other systems) is the theory of Weighted Lattices (Maragos (2013), Maragos (2016)).

1.2. Notation

For a pair of vectors $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$, the inequality notation $\mathbf{x} \leq \mathbf{y}$ is used meaning that $x_i \leq y_i$, for all i . Similarly, the inequality notation $\mathbf{x} < \mathbf{y}$ stands for $x_i < y_i$, for all i . The infinity norm will be used i.e. $\|\mathbf{x}\| = \max_i |x_i|$. We denote by $\mathbf{1}$ the column vector of dimension n consisting of ones. The underlying probability space is denoted by (Ω, \mathcal{F}, P) .

A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ will be called class \mathcal{K} function if α is increasing and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ will be called class \mathcal{KL} function if, for each fixed t , the function $\beta(\cdot, t)$ is a class \mathcal{K} function and for any fixed s , the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \rightarrow 0$ as $t \rightarrow \infty$.

1.3. Problem Formulation

The first class of systems considered is Max-Product systems with Markovian jumps. The uncertainty of the system is described by a Markov chain y_k having a finite state space $\{1, \dots, M\}$ and transition probabilities c_{ij} . That is, the evolution of y_k is described by $c_{ij} = P(y_{k+1} = j | y_k = i)$. A Max-Product system with Markovian jumps is described by:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}(y_k) \boxtimes \mathbf{x}_k, \\ \mathbf{x}_0 &\in \mathbb{R}_+^n. \end{aligned} \tag{3}$$

That is, at each time step the system matrix A takes one of the M different values $A(1), \dots, A(M)$ according to the value of the Markov chain.

At first, the class of deterministic Max-Product systems will be considered. In these systems the matrix $\mathbf{A}(\cdot)$ does not depend on the Markov chain and takes a single value \mathbf{A} .

The other class of systems considered is Max-Plus systems with Markovian jumps in the form:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}(y_k) \boxplus \mathbf{x}_k, \\ \mathbf{x}_0 &\in \mathbb{R}^n.\end{aligned}\tag{4}$$

In the following definitions, some notions of stability and stochastic stability are recalled from the literature (eg. Khalil (2002), Maragos (2016) and Kozin (1969)).

Definition 1. *Consider the system:*

$$\mathbf{x}_{k+1} = (\mathbf{A} \boxtimes \mathbf{x}_k) \vee (\mathbf{B} \boxtimes \mathbf{u}_k),\tag{5}$$

$$\mathbf{z}_k = (\mathbf{C} \boxtimes \mathbf{x}_k) \vee (\mathbf{D} \boxtimes \mathbf{u}_k),\tag{6}$$

where \mathbf{x}_k , \mathbf{u}_k , \mathbf{z}_k , denote the system state, input and output and A, B, C, D are matrices of appropriate dimensions.

- (i) *The free system, i.e. (5) with $\mathbf{u}_k = 0$, is exponentially stable, if there exist constants $a > 1$ and $L > 0$ such that $\|\mathbf{x}_k\| \leq L\|\mathbf{x}_0\|/a^k$, for any initial conditions and any k .*
- (ii) *The system (5) is Input to State Stable (ISS) if there exist a class \mathcal{KL} function β and a class \mathcal{K} function α such that:*

$$\|\mathbf{x}_k\| \leq \beta(\|\mathbf{x}_0\|, k) + \alpha\left(\bigvee_{i=0}^k \|\mathbf{u}_i\|\right),$$

for any initial condition, any k and any input sequence \mathbf{u}_k .

- (iii) *The system (5), (6) is Bounded Input Bounded Output (BIBO) stable (Maragos (2016)) if $\bigvee_{k=0}^{\infty} \|\mathbf{u}_k\| < \infty$ implies $\bigvee_{k=0}^{\infty} \|\mathbf{z}_k\| < \infty$, for any initial conditions.*

Definition 2. *The system given by (3) is:*

- (i) *Almost surely stable if for any initial conditions, $\mathbf{x}_k \rightarrow 0$ almost surely.*

- (ii) Mean norm stable if $E[\|\mathbf{x}_k\|] \rightarrow 0$ as $k \rightarrow \infty$.
- (iii) Mean norm exponentially stable if there exist constants $a > 1$ and $L > 0$ such that $E[\|\mathbf{x}_k\|] \leq L\|\mathbf{x}_0\|/a^k$.

Conditions for the stochastic stability of systems in the form (3) will be derived. For Max-Plus systems bounds on the growth of \mathbf{x}_k will be derived.

2. Deterministic Max-Product Systems

In this section the asymptotic stability of deterministic Max-Product systems in the form:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A} \boxtimes \mathbf{x}_k, \\ \mathbf{x}_0 &\in \mathbb{R}_+^n, \end{aligned} \tag{7}$$

is studied.

The following Lemma presents a condition equivalent to the the exponential stability of (7) (for a definition of exponential stability see for example Khalil (2002)).

Lemma 1. *It holds:*

- (i) The function $f(\mathbf{x}) = \mathbf{A} \boxtimes \mathbf{x}$ is homogeneous of order 1, i.e. it holds $f(\rho\mathbf{x}) = \rho f(\mathbf{x})$ for any $\rho \in \mathbb{R}_+$.
- (ii) The system (7) is exponentially stable iff for some $a > 1$, the system $\mathbf{x}_{k+1} = a\mathbf{A} \boxtimes \mathbf{x}_k$ is stable.

Proof: The proof is immediate. □

A Lyapunov function will be constructed for the stable systems in the form (7). Consider the function:

$$V(\mathbf{x}) = \bigvee_{k=0}^{\infty} \boldsymbol{\lambda}^T \boxtimes \mathbf{A}^k \boxtimes \mathbf{x}, \tag{8}$$

where $\boldsymbol{\lambda}$ is a vector with positive entries. Equivalently, V can be written as

$$V(\mathbf{x}) = \bigvee_{k=0}^{\infty} \boldsymbol{\lambda}^T \boxtimes \mathbf{x}_k,$$

where \mathbf{x}_k is the state vector of (7) with initial condition $\mathbf{x}_0 = \mathbf{x}$. It is not difficult to see that if (7) is stable, then $V(\mathbf{x})$ is finite for any \mathbf{x} and $V(\mathbf{0}) = 0$. Furthermore, the sequence $V(\mathbf{x}_k)$ is non-increasing:

$$\bigvee_{k=k_0+1}^{\infty} \boldsymbol{\lambda}^T \boxtimes \mathbf{x}_k \leq \bigvee_{k=k_0}^{\infty} \boldsymbol{\lambda}^T \boxtimes \mathbf{x}_k.$$

Thus, V is a Lyapunov function.

The form of V can be computed using the following calculations:

$$\begin{aligned} V(\mathbf{x}) &= \bigvee_{k=0}^{\infty} \boldsymbol{\lambda}^T \boxtimes \mathbf{A}^k \boxtimes \mathbf{x} = \boldsymbol{\lambda}^T \boxtimes \left[\bigvee_{k=0}^{\infty} \mathbf{A}^k \right] \boxtimes \mathbf{x} \\ &= (\boldsymbol{\lambda}^T \boxtimes \mathbf{A}^+) \boxtimes \mathbf{x}. \end{aligned} \tag{9}$$

Thus, V has the form:

$$V(\mathbf{x}) = \mathbf{p}^T \boxtimes \mathbf{x}, \tag{10}$$

where \mathbf{p} is an n vector with positive entries.

Proposition 1. *The following are equivalent:*

- (i) *The system (7) is exponentially stable.*
- (ii) *There exists a vector \mathbf{p} , with positive entries, such that $\mathbf{A}^T \boxtimes \mathbf{p} < \mathbf{p}$*

Proof: In order to show the direct part, we use Lemma 1, to obtain a constant $a > 1$ such that $\mathbf{x}_{k+1} = a\mathbf{A} \boxtimes \mathbf{x}_k$ is stable. Using a Lyapunov function in the form (8) for that system, we obtain a positive vector \mathbf{p} such that $V(\mathbf{x}) = \mathbf{p}^T \boxtimes \mathbf{x}$. Then it holds:

$$a\mathbf{p}^T \boxtimes \mathbf{A} \boxtimes \mathbf{x} \leq \mathbf{p}^T \boxtimes \mathbf{x},$$

for any $\mathbf{x} \in \mathbb{R}_+^n$. Thus, $\mathbf{p}^T \boxtimes \mathbf{A} < \mathbf{p}^T$ or equivalently $\mathbf{A}^T \boxtimes \mathbf{p} < \mathbf{p}$.

The fact that (ii) implies (i) is shown with usual Lyapunov analysis. \square

Remark 1. *The condition $a\mathbf{p}^T \boxtimes \mathbf{A} \leq \mathbf{p}^T$ can be checked using Linear Programming.*

Remark 2. A Lyapunov function in the form (8) is the direct analogue of a Lyapunov function for a usual linear system $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$, $\mathbf{x}_0 = x$ in the form $V_L(\mathbf{x}) = \sum_{k=0}^{\infty} \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k$. Particularly, in the place of the summation, we have the supremum and in the place of the \mathbf{Q} -norm $\|\mathbf{x}\|_{\mathbf{Q}}^2 = \mathbf{x}^T \mathbf{Q} \mathbf{x}$ we have the λ -norm $\|\mathbf{x}\|_{\lambda} = \max\{\lambda_1 x_1, \dots, \lambda_n x_n\}$.

Remark 3. The asymptotic behaviour of Max-Plus deterministic systems, depends on the Max-Plus eigenvalue of the system matrix which under connectivity assumptions turns out to be unique (eg. Baccelli et al. (1992)). This eigenvalue can be computed in terms of the critical paths i.e. the paths with maximal average weight. This analysis can be transferred to Max-Product systems using the $\exp(\cdot)$ isomorphism of the Max-Plus and Max-Product algebras. The Lyapunov approach adopted here could, however, be extended to stochastic systems and systems which are not linear in the Max-Product algebra.

The following corollary studies the Input to State Stability (ISS) and the Bounded Input Bounded Output (BIBO) stability.

Corollary 1. Assume that the system given by (7) is exponentially stable. Then:

- (i) The system given by (5) is input to state stable.
- (ii) The system given by (5), (6) is BIBO stable.

Proof: (i) Consider a Lyapunov function V in the form (10). Then V satisfies Lemma 3.5 of Jiang and Wang (2001). Thus, the system is ISS.

(ii) Follows immediately from (i). □

The following example illustrates that the same Lyapunov functions can be used to analyze systems which are nonlinear in the Max-Product algebra.

Example 1. Consider the system:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 2/3 & 2 \\ 1/3 & 3/4 \end{bmatrix} \boxtimes \mathbf{x}_k. \quad (11)$$

We consider the Lyapunov function candidate $V(\mathbf{x}) = [2 \ 5] \boxtimes \mathbf{x}$. It holds:

$$[2 \ 5] \begin{bmatrix} 2/3 & 2 \\ 1/3 & 3/4 \end{bmatrix} = [5/3 \ 4] < [2 \ 5].$$

Thus, V is a Lyapunov function and the system (11) is exponentially stable.

Let us then consider the system:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 2/3 & 2 \\ 1/3 & 3/4 \end{bmatrix} \boxtimes \mathbf{x}_k \vee \begin{bmatrix} 2 \\ 3 \end{bmatrix} \boxtimes (\mathbf{x}_k^T \boxtimes \mathbf{x}_k), \quad (12)$$

which is not in the form of (7). The same Lyapunov function $V(\mathbf{x})$ can be used to show the local asymptotic stability of (12).

Furthermore, the same Lyapunov function $V(\mathbf{x})$ can be used to show the ISS of the system:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 2/3 & 2 \\ 1/3 & 3/4 \end{bmatrix} \boxtimes \mathbf{x}_k \vee \begin{bmatrix} 5 \\ 8 \end{bmatrix} \boxtimes u_k.$$

3. Max-Product Systems with Markovian Jumps

We then turn to Max-Product systems with Markovian Jumps in the form (3). Lyapunov functions in the form:

$$V(\mathbf{x}, y) = \mathbf{p}(y)^T \boxtimes \mathbf{x}, \quad (13)$$

generalizing (10) are considered.

Proposition 2. *Assume that there exist a constant $a > 1$ and vectors with positive entries $\mathbf{p}(1), \dots, \mathbf{p}(M)$ such that:*

$$a \sum_{j=1}^M c_{ij} \mathbf{p}(j)^T \boxtimes \mathbf{A}(i) \boxtimes \mathbf{v} \leq \mathbf{p}(i)^T \boxtimes \mathbf{v}, \quad (14)$$

for any vector \mathbf{v} with positive entries. Then, (3) is mean norm exponentially stable and almost surely stable.

Proof: Consider the function (13). It holds:

$$E[V(\mathbf{x}_{k+1}, y_{k+1}) | \mathbf{x}_k = \mathbf{x}, y_k = i] = \sum_{j=1}^M c_{ij} \mathbf{p}(j)^T \boxtimes \mathbf{A}(i) \boxtimes \mathbf{x}.$$

Condition (14) implies that V is a positive super-martingale. Furthermore, $V = 0$ implies $\mathbf{x} = 0$. Thus, the system is almost surely stable.

Condition (14) further implies that:

$$E[V(\mathbf{x}_{k+1}, y_{k+1})|V(\mathbf{x}_k, y_k)] \leq V(\mathbf{x}_k, y_k)/a.$$

Thus, using this inequality repeatedly and taking expectations in both sides we have:

$$E[V(\mathbf{x}_k, y_k)] \leq V(\mathbf{x}_0, y_0)/a^k.$$

Denoting by p_M and p_m the maximum and the minimum entry of $\mathbf{p}(1), \dots, \mathbf{p}(M)$, we obtain:

$$E[p_m \|\mathbf{x}_k\|] \leq p_M \|\mathbf{x}_0\|/a^k.$$

Thus, using $L = p_M/p_m$, the inequality in Definition 2 part (iii) holds and the system (3) is mean norm exponentially stable. \square

Condition (14) should hold for any $\mathbf{v} \in \mathbb{R}_+^n$ and thus, it could be difficult to check it in general. The following lemma may be used to simplify condition (14). The lemma will be used also in Section 4 which considers many step Lyapunov functions. Hence, the lemma will be stated using a possibly different timing (with t in the place of k), a possibly different set of system matrices, depending on an additional random variable w_t and a state vector \bar{x} in the place of x .

Lemma 2. *Consider a system in the form:*

$$\bar{\mathbf{x}}_{t+1} = \bar{\mathbf{A}}(y_t, w_t) \boxtimes \bar{\mathbf{x}}_t, \quad (15)$$

where y_t takes values in $\{1, \dots, M\}$ and w_t take values in $\{1, \dots, \bar{M}\}$. Assume also that (y_t, w_t) is a Markov chain and that w_t is independent of (w_{t-1}, y_{t-1}) given y_t . Denote by $\tilde{c}(i, j, i')$ the conditional probability $P(y_{t+1} = i', w_t = j | y_t = i)$. Consider also the function:

$$V(\bar{\mathbf{x}}, y) = \mathbf{p}(y)^T \boxtimes \bar{\mathbf{x}}, \quad (16)$$

with $\mathbf{p}(1), \dots, \mathbf{p}(M)$ vectors with positive entries. For some δ with $0 < \delta < 1$, the following are equivalent:

(i) *It holds*

$$E[V(\bar{\mathbf{x}}_{t+1}, y_{t+1})|\bar{\mathbf{x}}_t, y_t] \leq \delta V(\bar{\mathbf{x}}_t, y_t), \quad (17)$$

for all $\bar{\mathbf{x}}_t, y_t$.

(ii) It holds:

$$\sum_{j=1}^{\bar{M}} \sum_{i'=1}^M \tilde{c}(i, j, i') \mathbf{1}^T \boxtimes \tilde{\mathbf{A}}(i', i, j) \boxtimes \mathbf{1} \leq \delta, \quad (18)$$

for $i = 1, \dots, M$, where

$$\tilde{\mathbf{A}}(y_{t+1}, y_t, w_t) = \text{diag}(\mathbf{p}(y_{t+1})) \bar{\mathbf{A}}(y_t, w_t) \text{diag}(\mathbf{p}(y_t))^{-1}. \quad (19)$$

Proof: Consider the vector:

$$\mathbf{z}_t = \text{diag}(\mathbf{p}(y_t)) \boxtimes \bar{\mathbf{x}}_t.$$

Then, it holds:

$$V(\bar{\mathbf{x}}_t, y_t) = \mathbf{1}^T \boxtimes \mathbf{z}_t = \|\mathbf{z}_t\|. \quad (20)$$

Furthermore, \mathbf{z}_t evolves according to:

$$\mathbf{z}_{t+1} = \tilde{\mathbf{A}}(y_{t+1}, y_t, w_t) \boxtimes \mathbf{z}_t.$$

Let us first show that (i) can be expressed in terms of \mathbf{z}_t as:

$$E [\|\mathbf{z}_{t+1}\| \mid \mathbf{z}_t, y_t] \leq \delta \|\mathbf{z}_t\|. \quad (21)$$

Equation (20) shows that the both the right and the left hand side of (21) are equal to the corresponding terms of (17). Hence, it remains to prove that (ii) is equivalent to (21).

It holds:

$$\begin{aligned} E [\|\mathbf{z}_{t+1}\| \mid \mathbf{z}_t, y_t = i] &= F(\mathbf{z}_t, i) = \\ &= \sum_{j=1}^{\bar{M}} \sum_{i'=1}^M \tilde{c}(i, j, i') \mathbf{1}^T \boxtimes \tilde{\mathbf{A}}(i', i, j) \boxtimes \mathbf{z}_t. \end{aligned}$$

The function $F(\mathbf{z}, i)$ is 1-homogeneous in \mathbf{z} . Thus, (21) is equivalent to

$$\max_{\|\mathbf{z}\| \leq 1} F(\mathbf{z}, i) \leq \delta, \quad \text{for } i = 1, \dots, M.$$

Furthermore, $F(\mathbf{z}, i)$ is non-decreasing in \mathbf{z} . Thus, (i) is equivalent to $F(\mathbf{1}, i) \leq \delta$, which is equivalent to (ii). \square

Remark 4. Equation (18) is stated using the matrix \tilde{A} , which is computed in transformed coordinates (equation (19)). A similar transformation is used (in a different context) in van den Boom and De Schutter (2012), in order to define the ‘maximum autonomous growth rate’.

For the needs of the rest of the current section we shall use k in the place of $t, A(y)$ in the place of $\bar{A}(y, w)$ and x in the place of \bar{x} .

Corollary 2. *Assume that:*

$$\sum_{j=1}^M c_{ij} \mathbf{p}(j)^T \boxtimes \mathbf{A}(i) \boxtimes (\mathbf{p}^{-1}(i)) \leq \delta,$$

for $i = 1, \dots, M$, $\delta < 1$ and $\mathbf{p}^{-1}(i)$ is a vector having as entries the inverses of the entries of $\mathbf{p}(i)$. Then the system given by (3) is mean norm exponentially stable and almost surely stable.

Proof: Using k in the place of t , $A(y)$ in the place of $\bar{A}(y, w)$ and x in the place of \bar{x} in Lemma 2 we get that the conditions of Proposition 2 hold true. Thus, the system given by (3) is mean norm exponentially stable and almost surely stable. \square

We then consider Max-Product stochastic systems with inputs and outputs and the notion of BIBO stability in probability is introduced.

Definition 3. *Consider the system:*

$$\mathbf{x}_{k+1} = \mathbf{A}(y_k) \boxtimes \mathbf{x}_k \vee \mathbf{B}(y_k) \boxtimes \mathbf{u}_k, \quad (22)$$

$$\mathbf{z}_k = \mathbf{C}(y_k) \boxtimes \mathbf{x}_k. \quad (23)$$

The system is Bounded Input Bounded in probability Output (BIBipO) stable if for any $\varepsilon > 0$, $M_u > 0$ and any initial condition, there exist a bound $M_z > 0$ such that:

$$P(\|\mathbf{z}_k\| \leq M_z) > 1 - \varepsilon. \quad (24)$$

The following proposition shows that the exponential mean norm stability of the free system implies the BIBipO stability.

Proposition 3. *If the free system given by (3) is mean norm exponentially stable then the system (22)-(23) is BIBipO stable.*

Proof: Consider a pair of constants $\varepsilon > 0$, $M_u > 0$ and an initial condition $\mathbf{x}_0 \in \mathbb{R}_+^n$. Following Maragos (2016) the state vector can be written as:

$$\mathbf{x}_k = \Phi(k, 0) \boxtimes \mathbf{x}_0 \vee \left(\bigvee_{t=1}^k \Phi(k, t) \boxtimes \mathbf{B}(y_{t-1}) \boxtimes \mathbf{u}_{t-1} \right),$$

where Φ is the transition matrix given by:

$$\Phi(k_2, k_1) = \begin{cases} \mathbf{A}(y_{k_2-1}) \boxtimes \cdots \boxtimes \mathbf{A}(y_{k_1}) & \text{if } k_2 > k_1 \\ I & \text{if } k_2 = k_1. \end{cases}$$

For any given constant $M_x > 0$ it holds:

$$\begin{aligned} P[\|\mathbf{x}_k\| > M_x] &\leq P[\|\Phi(k, 0) \boxtimes \mathbf{x}_0\| > M_x] + \\ &+ \sum_{t=1}^k P[\|\Phi(k, t) \boxtimes \mathbf{B}(y_{t-1}) \boxtimes \mathbf{u}_{t-1}\| > M_x] \\ &\leq P[\|\Phi(k, 0) \boxtimes \mathbf{x}_0\| > M_x] + \\ &+ \sum_{t=1}^k P[\|\Phi(k, t) \boxtimes \bar{\mathbf{U}}\| > M_x], \end{aligned} \quad (25)$$

where $\bar{\mathbf{U}} = \max \{\|\mathbf{B}(i) \boxtimes \mathbf{u}\| : \|\mathbf{u}\| \leq M_u, i = 1, \dots, M\} \mathbf{1}$.

The following claim will be used:

Claim: There exists a value $M_x > 0$ such that the right hand side of the last inequality in (25) is less than ε for any positive integer k .

To prove the claim we first use the Markov inequality:

$$E[\|\mathbf{x}_k\| > M_x] \leq \frac{1}{M_x} \left[E[\|\Phi(k, 0) \boxtimes \mathbf{x}_0\|] + \sum_{t=1}^k E[\|\Phi(k, t) \boxtimes \bar{\mathbf{U}}\|] \right]. \quad (26)$$

The term $E[\|\Phi(k, 0) \boxtimes \mathbf{x}_0\|]$ is bounded, due to the mean norm exponential stability of the free system. Then, observe that it holds $E[\|\Phi(k, t) \boxtimes \bar{\mathbf{U}}\|] = E[\|\tilde{x}_{k-t}\|]$ where \tilde{x}_l satisfies:

$$\begin{aligned} \tilde{x}_{l+1} &= \mathbf{A}(y_{k-t+l})\tilde{x}_l, \\ \tilde{x}_0 &= \bar{\mathbf{U}}. \end{aligned} \quad (27)$$

The system (27) is mean norm exponentially stable. Thus:

$$\sum_{t=1}^k E[\|\Phi(k, t) \boxtimes \bar{U}\|] \leq \sum_{t=1}^{\infty} E[\|\Phi(k, t) \boxtimes \bar{U}\|] \leq \frac{aL}{a-1} \|\bar{U}\|, \quad (28)$$

where a and L the constants satisfying the mean norm exponential stability definition. Hence, the right hand side of (26) tends to zero as M_x increases, which completes the proof of the claim.

Hence, a constant M_z satisfying (24) is given by: $M_z = \max \{\|\mathbf{C}(i) \boxtimes \mathbf{x}\| : \|\mathbf{x}\| \leq M_x, i = 1, \dots, M\}$. \square

4. k -Step Lyapunov Functions

In the last section, Lyapunov functions were used for the stability analysis of Max-Product systems with Markovian jumps. In this section we consider k -step Lyapunov functions and derive necessary and sufficient conditions for the mean-norm exponential stability. It turns out that many step Lyapunov functions offer greater flexibility.

We shall consider Lyapunov functions $V : \mathbb{R}_+^n \times \{1, \dots, M\} \rightarrow \mathbb{R}_+$ with the following properties:

- P1. $V(\mathbf{x}, y)$ is 1-homogeneous in \mathbf{x} .
- P2. $V(\mathbf{x}, y)$ is continuous in \mathbf{x} .
- P3. It holds $V(\mathbf{x}, y) = 0$ iff $\mathbf{x} = 0$.

The following proposition gives necessary and sufficient conditions for the mean-norm exponential stability in terms of many step Lyapunov functions.

Proposition 4. *Consider a function $V(\mathbf{x}, y)$ satisfying (P1)-(P3). Then, the following are equivalent:*

- (i) *The system given by (3) is mean-norm exponentially stable.*
- (ii) *For each $\delta \in (0, 1)$, there exists a positive integer k_0 such that:*

$$E[V(\mathbf{x}_k, y_k)] \leq \delta V(\mathbf{x}_0, y_0), \quad (29)$$

for any $\mathbf{x}_0 \in \mathbb{R}_+^n$, $y_0 \in \{1, \dots, M\}$ and any $k \geq k_0$.

(iii) There exists a $\delta \in (0, 1)$ and a positive integer k_0 such that:

$$E[V(\mathbf{x}_{k_0}, y_{k_0})] \leq \delta V(\mathbf{x}_0, y_0), \quad (30)$$

for any $\mathbf{x}_0 \in \mathbb{R}_+^n$, $y_0 \in \{1, \dots, M\}$.

Proof: (i) \Rightarrow (ii). The following claim is first proved:

Claim: There exist positive constants b_{min} and b_{max} such that:

$$b_{min}\|\mathbf{x}\| \leq V(\mathbf{x}, y) \leq b_{max}\|\mathbf{x}\|. \quad (31)$$

From (P2) and (P3) the values of the constants b_{min} and b_{max} defined by:

$$\begin{aligned} b_{min} &= \min\{V(\mathbf{x}, y) : \mathbf{x} \in \mathbb{R}_+^n, \|\mathbf{x}\| = 1\}, \\ b_{max} &= \max\{V(\mathbf{x}, y) : \mathbf{x} \in \mathbb{R}_+^n, \|\mathbf{x}\| = 1\}, \end{aligned}$$

are finite and positive. Then, (P1) completes the proof of the claim.

Assume that the system given by (3) is mean-norm exponentially stable and a and L satisfy Definition 2 part (iii). Fix a $\delta \in (0, 1)$. It holds:

$$\begin{aligned} E[V(\mathbf{x}_k, y_k)] &\leq E[b_{max}\|\mathbf{x}_k\|] \leq \\ &\leq b_{max}L\|\mathbf{x}_0\|/a^k \leq \frac{b_{max}}{b_{min}}La^{-k}V(\mathbf{x}_0, y_0). \end{aligned}$$

Choosing k_0 such that $\frac{b_{max}}{b_{min}}La^{-k_0} < \delta$, inequality (29) is satisfied.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). Using the same arguments as in the first part of the proof it is easy to see that there exists a positive integer N_0 such that:

$$E[\|\mathbf{x}_{N_0k_0}\|] \leq \delta\|\mathbf{x}_0\|.$$

Consider the Euclidean division of k by N_0k_0 , i.e. $k = (N_0k_0)q + r$. Using repeatedly the following inequality:

$$\begin{aligned} E[\|\mathbf{x}_k\|] &= E\left[E\left[\|\mathbf{x}_k\| \middle| \mathbf{x}_{k-N_0k_0}, y_{k-N_0k_0}\right]\right] \\ &\leq \delta E[\|\mathbf{x}_{k-N_0k_0}\|], \end{aligned}$$

we obtain:

$$E[\|\mathbf{x}_k\|] \leq \delta^q E[\|\mathbf{x}_r\|]. \quad (32)$$

Furthermore, r as a remainder satisfies $0 \leq r < N_0 k_0$ and q as a quotient satisfies $q \geq \frac{k}{N_0 k_0} - 1$. A bound for $E[\|\mathbf{x}_r\|]$ is then derived using repeatedly the following inequality:

$$\|\mathbf{A}(y) \boxtimes \mathbf{x}\| \leq \left[\max_{i,j,y} A_{ij}(y) \right] \|\mathbf{x}\|.$$

Inequality 32 implies that:

$$E[\|\mathbf{x}_k\|] \leq \delta^{\frac{k}{N_0 k_0}} \left[\max_{i,j,y} A_{ij}(y) \right]^{N_0 k_0 - 1} / \delta.$$

Thus, using for a and L the values $a = \delta^{1/(N_0 k_0)}$ and $L = [\max_{i,j,y} A_{ij}(y)]^{N_0 k_0 - 1} / \delta$, the inequality in Definition 2 part (iii) holds true and the system is mean norm exponentially stable. \square

The following corollary uses Lyapunov functions in the form $V(\mathbf{x}, y) = p(\mathbf{y})^T \boxtimes \mathbf{x}$ and Lemma 2. Particularly, a system in the form (15) is considered with $\bar{\mathbf{x}}_t = \mathbf{x}_{k_0 t}$.

Corollary 3. *Fix a positive integer k_0 . Assume that there exists a set of vectors $\mathbf{p}(1), \dots, \mathbf{p}(M)$ such that:*

$$\sum_{(j_1, \dots, j_{k_0-1}), i'} \tilde{c}(i, (j_1, \dots, j_{k_0-1}), i') \mathbf{1}^T \boxtimes \boxtimes \tilde{\mathbf{A}}(i', i, (j_1, \dots, j_{k_0-1}), i') \boxtimes \mathbf{1} \leq \delta, \quad (33)$$

where $\delta < 1$, the matrix $\tilde{\mathbf{A}}$ is given by (19), the matrix $\bar{\mathbf{A}}$ by:

$$\bar{\mathbf{A}}(y_t, (j_1, \dots, j_{k_0-1}), i') = \mathbf{A}(j_{k_0-1}) \boxtimes \dots \boxtimes \mathbf{A}(j_1),$$

and the constants \tilde{c} by:

$$\tilde{c}(i, (j_1, \dots, j_{k_0-1}), i') = c_{ij_1} c_{j_1 j_2} \dots c_{j_{k_0-1} i'}.$$

Then (3) is mean norm exponentially stable. Furthermore, if (3) is mean norm exponentially stable then there exists a positive integer k_0 and a set of vectors $\mathbf{p}(1), \dots, \mathbf{p}(M)$ satisfying (33). \square

5. Max-Plus Systems with Markovian Jumps

5.1. Almost Sure Bounds for the Free System

We then turn to Max-Plus systems with Markovian Jumps in the form (4). An almost sure bound on the evolution of the state of (4) will be derived using the results of the previous sections.

For a given system in the form (4) and a positive constant γ , we construct an equivalent Max-Product system. Particularly, consider the vector $\mathbf{x}'_k = \exp(\mathbf{x}_k)/\gamma^k$, where the exponentiation is considered component-wise. Then, \mathbf{x}'_k evolves according to:

$$\begin{aligned} \mathbf{x}'_{k+1} &= \mathbf{A}'(y_k) \boxtimes \mathbf{x}'_k, \\ \mathbf{x}'_0 &\in \mathbb{R}_+^n, \end{aligned} \tag{34}$$

and $\mathbf{A}' = \exp(\mathbf{A})/\gamma$ where the exponentiation is again considered component-wise.

Remark 5. *A transformation of a Max-Plus system to a sub-linear system is used in Chang (1996). In contrast to the transformation to a sub-linear system the transformation to a max-product system is exact (invertible). Let us note that the proof of the following proposition uses similar techniques with the proof of Corollary 2.3 of Chang (1996).*

The mean norm exponential stability of (34) can be used to derive some almost sure bounds for the evolution of (4).

Proposition 5. *Assume that (34) is Mean-norm exponentially stable. Then almost all the sample paths of (4) satisfy:*

$$\mathbf{x}_k < (k \ln \gamma) \mathbf{1}, \tag{35}$$

for large k .

Proof: Consider the sets:

$$B_k = \{\omega \in \Omega : \mathbf{x}_k \not< (k \ln \gamma) \mathbf{1}\}. \tag{36}$$

It holds $E[\|\mathbf{x}'_k\|] \leq M/a^k$ for some $a > 1$. Thus, using Markov inequality $P[\|\mathbf{x}'_k\| > 1] \leq M/a^k$. Furthermore, $\|\mathbf{x}'_k\| > 1$ iff $\mathbf{x}_k \not< (k \ln \gamma) \mathbf{1}$. Hence, $P(B_k) \leq M/a^k$ and $\sum_{k=1}^{\infty} P(B_k) < \infty$.

Thus, 1st Borel-Cantelli lemma (eg. Billingsley (2008)) applies. Hence:

$$P(\limsup B_k) = 0, \quad (37)$$

which concludes the proof. \square

The results of Proposition 5 can be used to bound the Lyapunov Exponent of a Max-Plus systems with Markovian jumps. Conditions for the existence of the Lyapunov exponent are given in Cohen (1988).

Corollary 4. *Assume that the system described by (4) has a Lyapunov exponent ℓ . Furthermore, assume that (34) is mean norm exponentially stable. Then, $\ell < \ln \gamma$.*

Proof: It holds $\mathbf{x}_k/k < \gamma \mathbf{1}$ for large k , almost surely. \square

5.2. Systems with Inputs

In this section systems of the form:

$$\begin{aligned} \mathbf{x}_{k+1} &= (\mathbf{A}(y_k) \boxplus \mathbf{x}_k) \vee (\mathbf{B}(y_k) \boxplus u_k), \\ \mathbf{z}_k &= \mathbf{C}(y_k) \boxplus \mathbf{x}_k, \end{aligned} \quad (38)$$

are considered in the context of the multi machine production system example studied in the following section. Using the results of Theorem 1 of van den Boom and De Schutter (2012), we assume that the input signal u_k is scalar and that it grows in an approximately linear fashion:

$$u_k = kT + \delta_k, \quad (39)$$

with δ_k bounded and T a positive constant. In van den Boom and De Schutter (2012) it is proved that, under certain additional conditions, inputs in the form (39) stabilize the corresponding switching Max-Plus linear system.

The following proposition shows that the difference of the state vector entries from kT are bounded in probability. Let us note that the boundedness of these differences have been used in the literature to define a notion of stability for discrete-event systems van den Boom and De Schutter (2012), Passino and Burgess (1998).

Proposition 6. *If the system (34) with $\gamma = e^T$, where e is the basis of the natural logarithm, is mean norm exponentially stable then for any $\varepsilon > 0$ there exists a bound M_x such that:*

$$P[x_k^i - kT \leq M_x] > 1 - \varepsilon, \quad (40)$$

for any k , where x_k^i is the i -th component of the vector \mathbf{x}_k .

Proof: Consider the vector $\mathbf{x}'_k = \exp(\mathbf{x}_k)/\gamma^k = \exp(\mathbf{x}_k)/\exp kT$. This vector evolves according to:

$$\mathbf{x}'_{k+1} = (\mathbf{A}'(y_k) \boxtimes \mathbf{x}'_k) \vee (\mathbf{B}'(y_k) \boxtimes d_k), \quad (41)$$

where $d_k = e^{\delta_k}$, $\mathbf{A}' = \exp(\mathbf{A})/\gamma$ and $\mathbf{B}' = \exp(\mathbf{B})/\gamma$ where all the matrix exponentiations are considered component-wise. Then, the application of Proposition 3 to (41) completes the proof. \square

6. Numerical Examples

6.1. Deterministic Max-Product Systems

In this section we use the Lyapunov analysis of deterministic max-product systems to analyze slightly ‘nonlinear’ max-plus systems. Such systems may arise in the modeling of discrete event systems for which the transport, processing, holding or idle times depend on system operation. For example, the necessary cooling time for a machine in a production system may depend on the length of the previous cycle. Another example is the loading or boarding times in a rail transportation system which depend on the quantity of products or the number of passengers waiting to be served, which in turn may depend on the length of the last cycle. In this section we analyze a simple model of such a discrete event system.

Consider the two dimensional model:

$$\begin{aligned} x_{k+1}^1 &= \max(x_k^1 + \bar{a}_{11}, x_k^2 + \bar{a}_{12}) + f_1(x_{k+1}^1 - x_k^1), \\ x_{k+1}^2 &= \max(x_k^1 + \bar{a}_{21}, x_k^2 + \bar{a}_{22}) + f_2(x_{k+1}^2 - x_k^2), \end{aligned} \quad (42)$$

where x_k^i represents the instant of time at which an event takes place for the k -th time (eg. the train departs from station i) and f_1, f_2 terms represent the dependence on the length of the last cycle. For simplicity assume that f_1 and f_2 are linear: $f_1(z) = z_2(z) = \bar{\delta}z$, with $|\bar{\delta}| < 1/2$.

Then (1) can be written as:

$$\begin{aligned} x_{k+1}^1 &= \max(x_k^1 + a_{11}, x_k^2 + a_{12} - \delta(x_k^1 - x_k^2)), \\ x_{k+1}^2 &= \max(x_k^1 + a_{21} - \delta(x_k^2 - x_k^1), x_k^2 + a_{22}), \end{aligned} \quad (43)$$

where $a_{ij} = \bar{a}_{ij}/(1 - \delta)$ and $\delta = \bar{\delta}/(1 - \delta)$. This system is clearly **not** of the max-plus form. In order to analyze (43), consider the corresponding

exponentiated system:

$$\begin{aligned} x_{k+1}^1 &= \max(a'_{11}x_k^1, a'_{12}x_k^2(x_k^1/x_k^2)^{-\delta}), \\ x_{k+1}^2 &= \max(a'_{21}x_k^1(x_k^2/x_k^1)^{-\delta}, a'_{22}x_k^2), \end{aligned} \quad (44)$$

where $x_k^i = \exp(x_k^i)/\gamma^k$, $a'_{ij} = \exp(a_{ij})/\gamma$. The dynamics (44) can be written as:

$$\mathbf{x}'_{k+1} = \mathbf{A}'(x_k^1/x_k^2) \boxtimes \mathbf{x}'_k, \quad (45)$$

where

$$\mathbf{A}'(x_k^1/x_k^2) = \begin{bmatrix} a_{11} & a'_{12}(x_k^1/x_k^2)^{-\delta} \\ a'_{21}(x_k^2/x_k^1)^{-\delta} & a_{22} \end{bmatrix}.$$

Then, the stability of the dynamics (44) can be studied using the Lyapunov function of the max-product ‘linearized’ system:

$$\mathbf{x}'_{k+1} = \mathbf{A}'(1) \boxtimes \mathbf{x}'_k. \quad (46)$$

Example 2. Assume that $a_{11} = a_{22} = 1.5686$, $a_{12} = 1.7918$, $a_{21} = 1.3350$, $d = -0.15$ and $\gamma = 5$. Then, the exponentiated system is:

$$\begin{aligned} x_{k+1}^1 &= \max(0.96x_k^1, 1.2(x_k^1/x_k^2)^{0.15}x_k^2), \\ x_{k+1}^2 &= \max(0.76x_k^1(x_k^2/x_k^1)^{0.15}, 0.96x_k^2), \end{aligned} \quad (47)$$

and the ‘max-product linearized’ matrix is:

$$\mathbf{A}'(1) = \begin{bmatrix} 0.96 & 1.2 \\ 0.76 & 0.96 \end{bmatrix}.$$

Using (9), (10) we obtain a Lyapunov function for the max-plus linearized system:

$$V(\mathbf{x}) = [1 \quad 1.25] \boxtimes \mathbf{x}.$$

We then use V as a Lyapunov function candidate for (47). The function $f(\mathbf{x}) = A'(x^1/x^2) \boxtimes \mathbf{x}$ is 1-homogenous. Therefore, we need only to show that if $V(\mathbf{x}'_k) \leq 1$ implies $V(\mathbf{x}'_{k+1}) \leq 1$. Equivalently we need to show that $x_{k+1}^1 \leq 1$ and $x_{k+1}^2 \leq 0.8$, if $x_k^1 \leq 1$ and $x_k^2 \leq 0.8$. Indeed for such \mathbf{x}'_k it holds:

$$\begin{aligned} x_{k+1}^1 &= \max(0.96x_k^1, 1.2(x_k^1)^{0.15}(x_k^2)^{0.85}) \leq 1, \\ x_{k+1}^2 &= \max(0.76(x_k^1)^{0.85}(x_k^2)^{0.15}, 0.96x_k^2) \leq 0.8. \end{aligned}$$

Thus, the system (47) is stable. \square

Let us note it is not possible to analyze (43) using directly max-plus techniques and the transformation to a max-product system is essential.

6.2. Max-Product Systems with Markovian Jumps

In this section, we present a very simple numerical example of a Max-Product system with Markovian jumps. The Markov chain has two possible states $y_k \in \{1, 2\}$ and the values of matrix \mathbf{A} are given by:

$$\mathbf{A}(1) = \begin{bmatrix} 1.05 & 1.5 \\ 0.4 & 0.3 \end{bmatrix}, \mathbf{A}(2) = \begin{bmatrix} 0.5 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}, \quad (48)$$

and the Markov chain has transition probability matrix:

$$\mathbf{c} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}.$$

Using simple search techniques a Lyapunov function satisfying the conditions of Corollary 2 can be obtained. One of those Lyapunov functions is:

$$\mathbf{p}(y) = \begin{cases} [4 \ 6]^T & \text{if } y = 1 \\ [3 \ 2]^T & \text{if } y = 2. \end{cases}$$

Hence, the system is mean norm exponentially stable. Several sample paths of the system are given in Figure 1.

6.3. Application to Multi-Machine Production Systems

In this section we study a very simple example of a production system consisting of three machines analyzed in van den Boom and De Schutter (2012). The production system may produce two distinct outputs A and B. The order in which the machines process the raw material is different for the two products. Particularly, when the product A is produced, the machines are used with order $M_1 \rightarrow M_2 \rightarrow M_3$ while when the product B is produced the order is $M_2 \rightarrow M_1 \rightarrow M_3$. The production system is depicted in Figure 2. An important question is that of maximum throughput. Maximum throughput is the maximum rate at which the system can process the raw material and it is defined as the inverse of the minimum cycle time (Baccelli et al. (1992)).

A max-plus stochastic system describing the timing of the production system will be described. Each machine starts working as soon as possible, that is when the input material is available and also it has finished all the previous work. Let us denote by u_k the time instant at which the raw material

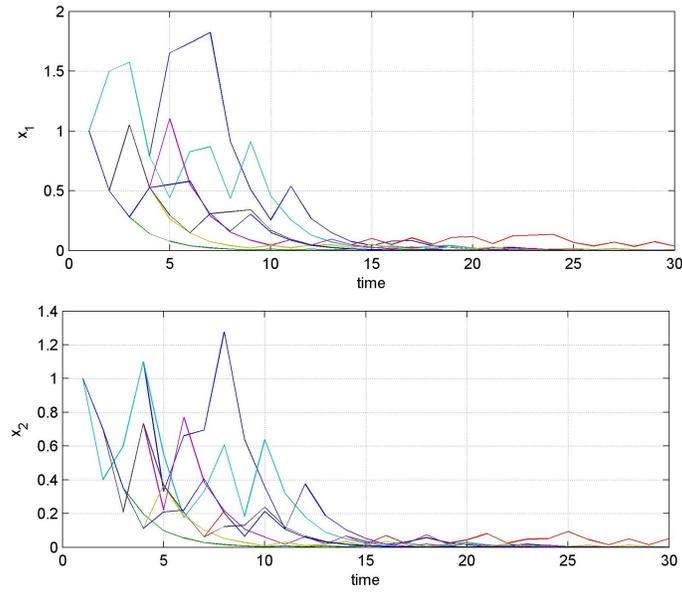


Figure 1: Several sample paths of the system described by (48). (Best viewed in color)

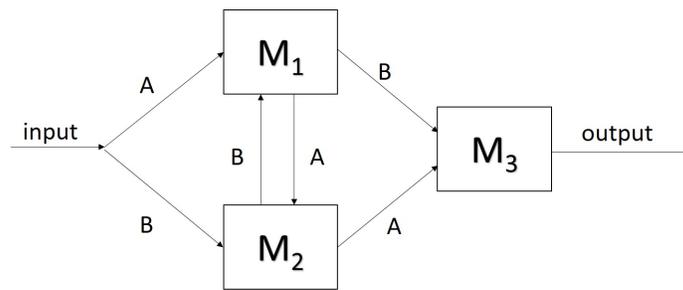


Figure 2: The production system

for the k -th product becomes available and by x_k^i the time instant at which the machine i starts working for the production of the k -th product. We assume that the processing time for the machines are $s_1 = 1$, $s_2 = 2$ and $s_3 = 1$. Furthermore, z_k denotes the time instant at which the product k becomes available.

The evolution of \mathbf{x}_k and \mathbf{z}_k is given by:

$$\begin{aligned}\mathbf{x}_{k+1} &= (\mathbf{A}(y_k) \boxplus \mathbf{x}_k) \vee (\mathbf{B}(y_k) \boxplus u_{k+1}), \\ \mathbf{z}_k &= \mathbf{C} \boxplus \mathbf{x}_k,\end{aligned}\tag{49}$$

where $y_k = 1$ when the product A is produced and $y_k = 2$ when product B is produced. The matrices $\mathbf{A}(1)$, $\mathbf{A}(2)$, $\mathbf{B}(1)$, $\mathbf{B}(2)$ and \mathbf{C} are given by:

$$\begin{aligned}\mathbf{A}(1) &= \begin{bmatrix} s_1 & -\infty & -\infty \\ 2s_1 & s_2 & -\infty \\ 2s_1 + s_2 & 2s_2 & s_3 \end{bmatrix}, \quad \mathbf{B}(1) = \begin{bmatrix} 0 \\ s_2 \\ s_1 + s_2 \end{bmatrix}, \\ \mathbf{A}(2) &= \begin{bmatrix} s_1 & 2s_2 & -\infty \\ -\infty & s_2 & -\infty \\ 2s_1 & s_1 + 2s_2 & s_3 \end{bmatrix}, \quad \mathbf{B}(2) = \begin{bmatrix} s_2 \\ 0 \\ s_1 + s_2 \end{bmatrix},\end{aligned}$$

and $\mathbf{C} = [-\infty \quad -\infty \quad s_3]$. The details can be found in van den Boom and De Schutter (2012).

We assume that which product is produced at each time step depends on exogenous orders which behave randomly. Particularly we assume that y_k is a Markov chain with transition probability matrix:

$$\mathbf{c} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}.$$

We further assume that the raw material arrives at a constant rate. Thus, the input signal has the form $u_k = kT$. Proposition 6 will be used to show that the raw material that has arrived to the production system but not yet fully processed remains bounded in probability. Thus the system is capable of processing the raw material at the given rate. The condition that all the buffer levels remain bounded has been used in the literature to define the stability of Discrete Event Systems (eg. van den Boom and De Schutter (2012)).

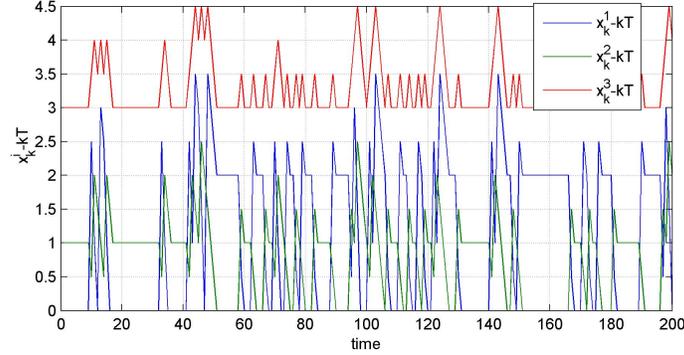


Figure 3: The differences $x_k^i - kT$ in a sample path of the system (49). (Should be viewed in color).

Example 3. Assume that the raw material arrival has a period $T = 2.5$. The matrices $\mathbf{A}'(y_k)$ are:

$$\mathbf{A}'(1) = \begin{bmatrix} 0.2231 & 0 & 0 \\ 0.6065 & 0.6065 & 0 \\ 4.4817 & 4.4817 & 0.2231 \end{bmatrix},$$

$$\mathbf{A}'(2) = \begin{bmatrix} 0.2231 & 4.4817 & 0 \\ 0 & 0.6065 & 0 \\ 0.6065 & 12.1825 & 0.2231 \end{bmatrix}.$$

The vectors $\mathbf{p}_1 = [12 \ 12 \ 1]$, $\mathbf{p}_2 = [3 \ 32 \ 1]$ satisfy the conditions of Corollary 2. Hence, Proposition 6 applies and $x_k^i - kT$ remains bounded in probability. Figure 3 illustrates the evolution of stock times.

Remark 6. A stability condition is also derived in van den Boom and De Schutter (2012). This stability condition resembles the stability under arbitrary switching property. It turns out that, in contrast to usual linear systems, the stability under arbitrary switching property is easier to check than the stochastic stability in the Max-Plus systems (Blondel et al. (2000)).

The minimum value for T satisfying the stability conditions of van den Boom and De Schutter (2012) can be computed using Linear Programming and for the current example has a value $T = 3$.

Thus, the stochastic stability conditions (40) are less restrictive and allow the system to operate at a higher rate, compared with the stability under arbitrary switching.

7. Conclusion

Max-Plus and Max-Product systems with Markovian jumps were considered. A Lyapunov function is constructed for asymptotically stable deterministic Max-Product systems. This Lyapunov function is found to have a simple form and the stability conditions derived can be checked using Linear Programming. Slightly modified Lyapunov functions are then used to derive sufficient conditions for the mean norm exponential stability of Max-Product systems with Markovian Jumps. A simpler form of these conditions can be derived based on the monotonicity of the Lyapunov functions. Necessary and sufficient conditions for the mean norm exponential stability are then derived using many step Lyapunov functions.

Bounds for the evolution of the state of Max-Plus systems with Markovian jumps are then derived, based on the results for the Max-Product systems. Finally a numerical example illustrates the application of the methods described on a production system.

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