Stabilization of a Linear Hyperbolic PDE with Actuator and Sensor Dynamics

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Abstract

We consider a scalar 1-D linear hyperbolic partial differential equation (PDE) for which infinite-dimensional backstepping controllers have previsously been designed based on boundary actuation and sensing, and incorporate first order actuator and sensor dynamics into the design. Two observer designs are proposed, and combined with a state-feedback into output-feedback control laws which render the origin of the closed-loop system exponentially stable with arbitrary convergence rate. The theory is verified in simulations.

Key words: Distributed parameter systems. Hyperbolic systems. Linear systems. Boundary control.

1 Introduction

Linear hyperbolic partial differential equations (PDEs) can be used to describe flow and transport phenomena. Typical examples are transmission lines [9], road traffic [1], heat exchangers [19], oil wells [15], multiphase flow [10], [12], time-delays [14] and predator-pray systems [18], to mention a few. These distributed parameter systems therefore give rise to important estimation and control problems.

The backstepping method for distributed systems is a relatively new method for controller and observer design for systems of partial differential equations. The method was originally developed for parabolic PDEs in [16], with the first result for hyperbolic PDEs published a few years later in [14]. In [14], a scalar 1–D linear PDE is stabilized using this technique, and convergence is achieved in a finite time that corresponds to the propagation time through the domain. Extensions to more complicated systems of hyperbolic PDEs were derived a few years later in [17], for two coupled linear hyperbolic PDEs, and more general systems in [11] and [13]. Backstepping has also found its use for adaptive systems containing linear hyperbolic PDEs in for instance [7], [8], [2], [3], [5]. When using backstepping for PDE controller synthesis, an invertible Volterra integral transformation and a control law are introduced, that map the system of interest into a "target" system designed with some desirable stability properties. The invertibility of the transform allows for stating equivalent stability properties of the two systems.

All aforementioned papers consider actuation signals that act directly at one or several of the PDEs boundaries, and hence neglect any actuator dynamics. Also, all previously derived observers assume sensor signals to be linear combinations of PDE boundary values, ignoring any dynamics in the sensors. Sometimes, however, actuator and sensors may be slow compared to the dominating dynamics of the plant, in which case their dynamics cannot be ignored in the control design. Common examples are slow control chokes in flow pipelines and noise filtering embedded in sensing devices.

In the present paper, we show how to incorporate actuator dynamics in the control design for a scalar 1–D linear hyperbolic PDE derived in [14]. In addition, we present two different observer designs that use a measurement with first-order dynamics. The observers are combined with the state-feedback into output-feedback stabilizing controllers. The finite-time convergence properties of the controller in [14] is, due to the actuator and sensor dynamics, lost. However, the derived controllers achieve exponential stability with arbitrary convergence rate.

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2 Problem statement

We consider a scalar system similar to the one investigated in [14], but with actuator and sensor dynamics

$$v_t(x,t) - \mu v_x(x,t) = \theta(x)v(0,t) \tag{1a}$$
$$v(1,t) = \eta(t) \tag{1b}$$
$$\dot\eta(t) + \rho_1\eta(t) = U(t) \tag{1c}$$
$$\dot y(t) + \rho_2 y(t) = v(0,t) \tag{1d}$$
$$v(x,0) = v_0(x) \tag{1e}$$
$$\eta(0) = \eta_0 \tag{1f}$$
$$y(0) = y_0 \tag{1g}$$

defined for $x \in [0,1]$, $t \geq 0$, where

$$\theta \in C^1([0,1]), \qquad \mu \in \mathbb{R}, \qquad \rho_1, \rho_2 \in \mathbb{R}. \tag{2}$$

We assume the initial conditions v_0, y_0, η_0 satisfy

$$v_0 \in L_2([0,1]) \qquad y_0, \eta_0 \in \mathbb{R} \qquad \eta_0 = v(1,t). \tag{3}$$

The goal is to design a stabilizing controller for system (1), using the measurement $y(t)$ only.

3 Controller design

3.1 Previous result: The backstepping controller

We will start by stating a state-feedback stabilizing controller for the case $\eta(t) = U(t)$. Such a controller was originally derived (for $\mu = 1$) in [14], but we include it here for later reference. Consider the control law

$$U_b(t) = \int_0^1 k(1-\xi)v(\xi,t)d\xi \tag{4}$$

where k is the solution to the Volterra integral equation

$$\mu k(x) = \int_0^x k(x-\xi)\theta(\xi)d\xi - \theta(x). \tag{5}$$

Lemma 1 *Consider system* (1)*, and suppose $U(t)$ is designed so that*

$$\eta(t) = U_b(t) \tag{6}$$

holds, where U_b is defined in (4)*. Then $v \equiv 0$ for $t \geq d_1$, where*

$$d_1 = \mu^{-1}. \tag{7}$$

PROOF. Consider the backstepping transformation

$$\begin{aligned}\alpha(x,t) &= T[v(t)](x) \\ &= v(x,t) - \int_0^x k(x-\xi)v(\xi,t)d\xi\end{aligned} \tag{8}$$

where k is the solution to (4). We will show that the transformation (8) and the control law (6) map the α-subsystem in (1) into

$$\alpha_t(x,t) - \mu\alpha_x(x,t) = 0 \tag{9a}$$
$$\alpha(1,t) = 0 \tag{9b}$$
$$\alpha(x,0) = \alpha_0(x) \tag{9c}$$

for some $\alpha_0 \in L_2([0,1])$.

From differentiating (8) with respect to time, inserting the dynamics (1a) and integration by parts, we find

$$\begin{aligned}v_t(x,t) &= \alpha_t(x,t) + \mu k(0)v(x,t) \\ &\quad - \left[\mu k(x) - \int_0^x k(x-\xi)\theta(\xi)d\xi\right] v(0,t) \\ &\quad + \mu \int_0^x k'(x-\xi)v(\xi,t)d\xi.\end{aligned} \tag{10}$$

Similarly, differentiating (8) with respect to space gives

$$\begin{aligned}v_x(x,t) &= \alpha_x(x,t) + k(0)v(x,t) \\ &\quad + \int_0^x k'(x-\xi)v(\xi,t)d\xi\end{aligned} \tag{11}$$

Inserting (10) and (11) into the dynamics (1a) yields

$$\begin{aligned}v_t(x,t) - \mu v_x(x,t) - \theta(x)v(0,t) &= \alpha_t(x,t) - \mu\alpha_x(x,t) \\ &\quad - \left[\mu k(x) - \int_0^x k(x-\xi)\theta(\xi)d\xi \right.\\ &\qquad \left. + \theta(x)\right] v(0,t) = 0.\end{aligned} \tag{12}$$

Using (5) gives the dynamics (9a). Inserting $x = 1$ into (8) and using (1b) gives

$$\alpha(1,t) = \eta(t) - \int_0^1 k(1-\xi)v(\xi,t)d\xi, \tag{13}$$

from which (6) gives (9b). From inserting $t = 0$ into (8), it is clear that the initial condition α_0 is given from v_0 as

$$\alpha_0(x) = T[v_0](x). \tag{14}$$

System (9) is identically zero for $t \geq d_1$ and from the invertibility of the backstepping transform (8), the result follows.

3.2 State-feedback controller

It is clear that we should seek to design $U(t)$ so that the relationship (6) is achieved. Towards that end, we define the state feedback

$$U_{sf}(t) = (k(0)\mu - \gamma_1)\eta(t) + \theta(1)v(0,t)$$

$$+\int_0^1 \left(k'(1-\xi)\mu + (\rho_1+\gamma_1)k(1-\xi)\right)v(\xi,t)d\xi \quad (15)$$

for some design gain $\gamma_1 > -\rho_1$.

Theorem 2 *Consider system* (1)*, and let the control law be taken as*

$$U(t) = U_{sf}(t) \quad (16)$$

where $U_{sf}(t)$ *is defined in* (15)*. Then, there exist constants* $c_1, c_2 > 0$ *so that*

$$|\eta(t)| \leq c_1(|\eta_0| + ||v_0||)e^{-(\rho_1+\gamma_1)t} \quad (17)$$

and for $t \geq d_1$*, where* d_1 *is defined in* (7)

$$|v(x,t)| \leq c_2\left(|\eta_0| + ||v_0||\right)e^{-(\rho_1+\gamma_1)t}, \ \forall x \in [0,1]. \quad (18)$$

PROOF. From differentiating U_b defined in (4) with respect to time, inserting the dynamics (1a) and integrating by parts, we find

$$\begin{aligned}\dot{U}_b(t) &= k(0)\mu v(1,t) - k(1)\mu v(0,t) \\ &\quad + \int_0^1 k'(1-\xi)\mu v(\xi,t)d\xi \\ &\quad + \int_0^1 k(1-\xi)\theta(\xi)d\xi v(0,t) \\ &= k(0)\mu\eta(t) + \theta(1)v(0,t) \\ &\quad + \int_0^1 k'(1-\xi)\mu v(\xi,t)d\xi \quad (19)\end{aligned}$$

where we in the last equality used the definition of k in (5). From (19) and (4), it is observed that the control law $U_{sf}(t)$ defined in (15) can be written as

$$U_{sf}(t) = -\gamma_1\eta(t) + \dot{U}_b(t) + (\rho_1+\gamma_1)U_b(t). \quad (20)$$

Inserting (20) into (1), we obtain

$$\begin{aligned}v_t(x,t) - \mu v_x(x,t) &= \theta(x)v(0,t) && (21a)\\ v(1,t) &= U_b(t) + p(t) && (21b)\\ \dot{p}(t) + (\rho_1+\gamma_1)p(t) &= 0 && (21c)\\ v(x,0) &= v_0(x) && (21d)\\ p(0) &= p_0 && (21e)\end{aligned}$$

where we have defined

$$p(t) = \eta(t) - U_b(t) \quad (22)$$

with initial condition

$$\begin{aligned}p_0 &= v(1,0) - U_b(0) \\ &= \eta(0) - \int_0^1 k(1-\xi)v_0(\xi)d\xi. \quad (23)\end{aligned}$$

The backstepping transformation (8) then straight forwardly produces the target system

$$\begin{aligned}\alpha_t(x,t) - \mu\alpha_x(x,t) &= 0 && (24a)\\ \alpha(1,t) &= p(t) && (24b)\\ \dot{p}(t) + (\rho_1+\gamma_1)p(t) &= 0 && (24c)\\ \alpha(x,0) &= \alpha_0(x) && (24d)\\ p(0) &= p_0. && (24e)\end{aligned}$$

Solving (24c) and (24e) explicitly for p, we find

$$p(t) = p_0 e^{-(\rho_1+\gamma_1)t}. \quad (25)$$

Now, we have using the triangle and Cauchy-Schwarz' inequalities

$$\begin{aligned}|p_0| &= \left|\eta(0) - \int_0^1 k(1-\xi)v_0(\xi)d\xi\right| \\ &\leq |\eta_0| + \left|\int_0^1 k(1-\xi)v_0(\xi)d\xi\right| \\ &\leq |\eta_0| + ||k||||v_0||, \quad (26)\end{aligned}$$

which gives the bound (17). Moreover, from (24a) and (24b), we have for $t \geq d_1(1-x)$

$$\begin{aligned}\alpha(x,t) &= \alpha(1, t - d_1(1-x)) = p(t - d_1(1-x)) \\ &= p_0 e^{(\rho_1+\gamma_1)d_1} e^{-(\rho_1+\gamma_1)d_1 x} e^{-(\rho_1+\gamma_1)t}. \quad (27)\end{aligned}$$

which gives

$$|\alpha(x,t)| \leq (|\eta_0| + ||k||||v_0||)e^{(\rho_1+\gamma_1)d_1}e^{-(\rho_1+\gamma_1)t} \quad (28)$$

valid for all $t \geq d_1$. The invertibility of the transform (8) gives the bound (18).

4 Observer designs

The control law of Theorem 2 requires $\eta(t)$ to be measured, as well as distributed measurements of $v(x,t)$. We will here present two different observer designs for estimating the states of system 1. The first one has the simplest dynamics, but estimates a state $u(x,t)$ which is a filtered transformation of the state $v(x,t)$. An actual estimate of $v(x,t)$ can be generated from inversely performing the filtered transformation. The second observer estimates the state $v(x,t)$ directly, but involves a slightly more complicated dynamics and analysis.

4.1 Design 1: Observer using a filtered transformation

4.1.1 Filtered transformation

Consider the system

$$\begin{aligned}u_t(x,t) - \mu u_x(x,t) &= \theta(x)u(0,t) && (29a)\\ u(1,t) &= c^T\sigma(t) && (29b)\end{aligned}$$

$$\dot{\sigma}(t) = A\sigma(t) + bU(t) \tag{29c}$$
$$y(t) = u(0,t) \tag{29d}$$
$$u(x,0) = u_0(x) \tag{29e}$$
$$\sigma(0) = \sigma_0 \tag{29f}$$

where $u(x,t)$ is defined for $x \in [0,1]$, $t \geq 0$, and is $\nu(t)$ defined for $t \geq 0$, and

$$\sigma(t) = \begin{bmatrix} \eta(t) \\ \nu(t) \end{bmatrix}, \qquad \sigma_0 = \begin{bmatrix} \eta_0 \\ \nu_0 \end{bmatrix} \tag{30a}$$
$$A = \begin{bmatrix} -\rho_1 & 0 \\ 1 & -\rho_2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{30b}$$

with

$$u_0(x) = y_0 e^{-d_1\rho_2 x} + d_1 \int_0^x e^{-d_1\rho_2(x-s)} \left(v_0(s) - \theta(s)y_0\right) ds \tag{31a}$$
$$\nu_0 = u_0(1). \tag{31b}$$

We note that the pair (A, c^T) is observable.

Lemma 3 *System u in* (29) *is related to* (1) *through the filtered transformation*

$$u_t(x,t) + \rho_2 u(x,t) = v(x,t). \tag{32}$$

PROOF. From differentiating (29a) with respect to time, we obtain

$$u_{tt}(x,t) - \mu u_{tx}(x,t) = \theta(x)u_t(0,t). \tag{33}$$

Moreover, from (32), we have

$$v_t(x,t) = u_{tt}(x,t) + \rho_2 u_t(x,t) \tag{34}$$

and

$$v_x(x,t) = u_{tx}(x,t) + \rho_2 u_x(x,t). \tag{35}$$

Using (34), (35) and (32) we form

$$\begin{aligned} v_t(x,t) &- \mu v_x(x,t) - \theta(x)v(0,t) \\ &= u_{tt}(x,t) - \mu u_{tx}(x,t) - \theta(x)u_t(0,t) \\ &\quad + \rho_2 \left[u_t(x,t) - \mu u_x(x,t) - \theta(x)u(0,t)\right], \end{aligned} \tag{36}$$

from which (29a) and (33) give (1a).

From inserting (29a) into (32), multiplying by $d_1 = \mu^{-1}$ and evaluating at $t = 0$, we find the following ODE for the initial condition u_0

$$u_0'(x) + d_1\rho_2 u_0(x) = d_1 v_0(x) - d_1\theta(x)u_0(0), \tag{37}$$

which, by requiring $u_0(0) = y_0$, can straightforwardly be solved to yield (31a). Similarly, inserting $x = 1$ into (32)

$$u_t(1,t) + \rho_2 u(1,t) = v(1,t) = \eta(t), \tag{38}$$

which can be written as (29b)–(29c) with initial condition $\nu(0) = \nu_0$ given as (31b).

Inserting $x = 0$ into (32) gives

$$u_t(0,t) + \rho_2 u(0,t) = v(0,t) \tag{39}$$

which is the same dynamics as for y in (1d), provided $u(0,0) = y_0$, which from (31a) is the case.

4.1.2 Observer equations

Consider now the observer for u

$$\hat{u}_t(x,t) - \mu\hat{u}_x(x,t) = \theta(x)y(t) + c^T e^{d_1 Ax}\kappa(y(t) - \hat{u}(0,t)) \tag{40a}$$
$$\hat{u}(1,t) = c^T\hat{\sigma}(t) \tag{40b}$$
$$\dot{\hat{\sigma}}(t) = A\hat{\sigma}(t) + bU(t) + e^{d_1 A}\kappa(y(t) - \hat{u}(0,t)) \tag{40c}$$
$$\hat{u}(x,0) = \hat{u}_0(x) \tag{40d}$$
$$\hat{\sigma}(0) = \hat{\sigma}_0 \tag{40e}$$

where

$$\hat{\sigma}(t) = \begin{bmatrix} \hat{\eta}(t) & \hat{\nu}(t) \end{bmatrix}^T \tag{41}$$

for some design vector $\kappa \in \mathbb{R}^2$ chosen so that the matrix $A - \kappa c^T$ is Hurwitz, and some initial conditions $\hat{u}_0 \in L_2([0,1])$, $\hat{\sigma} \in \mathbb{R}^2$. Consider also an estimate $\hat{v}$ of v generated from $\hat{u}$ as

$$\hat{v}(x,t) = \hat{u}_t(x,t) + \rho_2\hat{u}(x,t). \tag{42}$$

Theorem 4 *Consider system* (29) *and the estimate* (42) *generated using the observer* (40)*. Then, for $t \geq d_1$, the following inequalities hold*

$$|\tilde{\sigma}(t)| \leq |\tilde{\sigma}_0|e^{-\delta_1 t} \tag{43a}$$
$$|\tilde{v}(x,t)| \leq c_3|\tilde{\sigma}_0|e^{-\delta_1 t} \tag{43b}$$

for some constant $c_3 > 0$, where $\tilde{v} = v - \hat{v}$, $\tilde{\sigma}_0 = \begin{bmatrix} \tilde{\eta}_0 & \tilde{\nu}_0 \end{bmatrix}^T$ and $\delta_1 = -\max Re(eig(A - \kappa c^T))$.

PROOF. The observer errors $\tilde{u} = u - \hat{u}$ and $\tilde{\sigma} = \sigma - \hat{\sigma} = \begin{bmatrix} \tilde{\eta} & \tilde{\nu} \end{bmatrix}^T$ can straightforwardly, using (1) and (40), be shown to satisfy the dynamics

$$\tilde{u}_t(x,t) - \mu\tilde{u}_x(x,t) = -c^T e^{d_1 Ax}\kappa\tilde{u}(0,t) \tag{44a}$$

$$\tilde{u}(1,t) = c^T \tilde{\sigma}(t) \tag{44b}$$
$$\dot{\tilde{\sigma}}(t) = A\tilde{\sigma}(t) - e^{d_1 A}\kappa \tilde{u}(0,t) \tag{44c}$$
$$\tilde{u}(x,0) = \tilde{u}_0(x) \tag{44d}$$
$$\tilde{\sigma}(0) = \tilde{\sigma}_0. \tag{44e}$$

The error dynamics (44) is a special case of those considered in [14], for which the stability proof proceeds by showing that the transformation

$$w(x,t) = \tilde{u}(x,t) - c^T e^{-d_1 A(1-x)}\tilde{\sigma}(t) \tag{45}$$

maps (44) into the target system

$$w_t(x,t) - \mu w_x(x,t) = 0 \tag{46a}$$
$$w(1,t) = 0 \tag{46b}$$
$$\dot{\tilde{\sigma}}(t) = F\tilde{\sigma}(t) - e^{d_1 A}\kappa w(0,t) \tag{46c}$$
$$\tilde{w}(x,0) = \tilde{w}_0(x) \tag{46d}$$
$$\tilde{\sigma}(0) = \tilde{\sigma}_0 \tag{46e}$$

where

$$F = A - e^{d_1 A}\kappa c^T e^{-d_1 A} \tag{47}$$

has the same eigenvalues as $A - \kappa c^T$, and is thus Hurwitz. From (45), we find

$$\tilde{u}_t(x,t) = w_t(x,t) + c^T e^{-d_1 A(1-x)} A\tilde{\sigma}(t) - c^T e^{d_1 Ax}\kappa \tilde{u}(0,t) \tag{48}$$

and

$$\tilde{u}_x(x,t) = w_x(x,t) + c^T d_1 e^{-d_1 A(1-x)} A\tilde{\sigma}(t). \tag{49}$$

Inserting (48) and (49) into (44a), we find

$$\tilde{u}_t(x,t) - \mu\tilde{u}_x(x,t) + c^T e^{d_1 Ax}\kappa\tilde{u}(0,t) = w_t(x,t) - \mu w_x(x,t) = 0 \tag{50}$$

which gives the dynamics (46a). Evaluating (45) at $x = 1$ and inserting the boundary condition (44b) give

$$w(1,t) = \tilde{u}(1,t) - c^T\tilde{\sigma}(t) = c^T\tilde{\sigma}(t) - c^T\tilde{\sigma}(t) = 0 \tag{51}$$

verifying the boundary condition (46b). Inserting (45) into (44c), we obtain

$$\dot{\tilde{\sigma}}(t) = (A - e^{d_1 A}\kappa c^T e^{-d_1 A})\tilde{\sigma}(t) - e^{d_1 A}\kappa w(0,t) \tag{52}$$

which using the definition of F in (47), is the dynamics (46c). The fact that F is Hurwitz can be seen from using a similarity transformation $e^{d_1 A}$ on A.

From the target system (46), it is evident that $w \equiv 0$ for $t \geq d_1$, and hence (46c) reduces to

$$\dot{\tilde{\sigma}}(t) = F\tilde{\sigma}(t) \tag{53}$$

which is an exponentially stable system with bound (43a). Moreover, from (45) with $w \equiv 0$ for $t \geq d_1$, we get

$$\tilde{u}(x,t) = c^T e^{-d_1 A(1-x)}\tilde{\sigma}(t) \tag{54}$$

and

$$\tilde{u}_t(x,t) = c^T e^{-d_1 A(1-x)} F\tilde{\sigma}(t) \tag{55}$$

which together with the relationship

$$\tilde{v}(x,t) = \tilde{u}_t(x,t) + \rho_2\tilde{u}(x,t) \tag{56}$$

that immediately follows from (32) and (42), gives the bound (43b).

4.2 Design 2: Direct estimation of v

Consider the observer

$$\hat{v}_t(x,t) - \mu\hat{v}_x(x,t) = \theta(x)\hat{v}(0,t) + L(x)(y(t) - \hat{y}(t)) \tag{57a}$$
$$\hat{v}(1,t) = \hat{\eta}(t) - \theta(1)(y(t) - \hat{y}(t)) \tag{57b}$$
$$\dot{\hat{\eta}}(t) + \rho_1\hat{\eta}(t) = U(t) + \gamma_2(y(t) - \hat{y}(t)) \tag{57c}$$
$$\dot{\hat{y}}(t) + \rho_2\hat{y}(t) = \hat{v}(0,t) + \gamma_3(y(t) - \hat{y}(t)) \tag{57d}$$
$$\hat{v}(x,0) = \hat{v}_0(x) \tag{57e}$$
$$\hat{\eta}(0) = \hat{\eta}_0 \tag{57f}$$
$$\hat{y}(0) = \hat{y}_0 \tag{57g}$$

for some initial conditions $\hat{v}_0 \in L_2([0,1])$, $\hat{\eta}_0, \hat{y}_0 \in \mathbb{R}$, and where

$$L(x) = \mu\theta'(x) + (\rho_2 + \gamma_3)\theta(x) + \gamma_2 e^{d_1\rho_1(1-x)}. \tag{58}$$

Let γ_2, γ_3 be chosen so that the matrix

$$G = B - \begin{bmatrix}\gamma_2 \\ \gamma_3\end{bmatrix} c^T, \tag{59}$$

is Hurwitz,

$$B = \begin{bmatrix} -\rho_1 & 0 \\ e^{d_1\rho_1} & -\rho_2 + \theta(0)\end{bmatrix} \tag{60}$$

and c defined in (30b). This is always is possible since the pair (B, c^T) is observable.

Theorem 5 *Consider system* (1) *and the observer* (57)*. Then, for $t \geq d_1$, the following inequalities hold*

$$|\tilde{\eta}(t)| \leq (|\tilde{\eta}_0| + |\tilde{y}_0|)e^{-\delta_2 t} \tag{61a}$$
$$|\tilde{y}(t)| \leq (|\tilde{\eta}_0| + |\tilde{y}_0|)e^{-\delta_2 t} \tag{61b}$$
$$|\tilde{v}(x,t)| \leq c_4(|\tilde{\eta}_0| + |\tilde{y}_0|)e^{-\delta_2 t} \tag{61c}$$

where $\delta_2 = -\max Re(eig(G))$.

PROOF. The observer error $\tilde{v} = v - \hat{v}$, $\tilde{\eta} = \eta - \hat{\eta}$, $\tilde{y} = y - \hat{y}$ can straightforwardly, using (1) and (57) be shown to satisfy the dynamics

$$\tilde{v}_t(x,t) - \mu\tilde{v}_x(x,t) = \theta(x)\tilde{v}(0,t) - L(x)\tilde{y}(t) \tag{62a}$$
$$\tilde{v}(1,t) = \tilde{\eta}(t) + \theta(1)\tilde{y}(t) \tag{62b}$$
$$\dot{\tilde{\eta}}(t) + \rho_1\tilde{\eta}(t) = -\gamma_2\tilde{y}(t) \tag{62c}$$
$$\dot{\tilde{y}}(t) + \rho_2\tilde{y}(t) = \tilde{v}(0,t) - \gamma_3\tilde{y}(t) \tag{62d}$$
$$\tilde{v}(x,0) = \tilde{v}_0(x) \tag{62e}$$
$$\tilde{\eta}(0) = \tilde{\eta}_0 \tag{62f}$$
$$\tilde{y}(0) = \tilde{y}_0 \tag{62g}$$

where $\tilde{v}_0 = v_0 - \hat{v}_0$, $\tilde{\eta}_0 = \eta_0 - \hat{\eta}_0$, $\tilde{y}_0 = y_0 - \hat{y}_0$.

We will show that the transformation

$$z(x,t) = \tilde{v}(x,t) - e^{d_1\rho_1(1-x)}\tilde{\eta}(t) - \theta(x)\tilde{y}(t) \tag{63}$$

maps the error dynamics (62) into the target system

$$z_t(x,t) - \mu z_x(x,t) = 0 \tag{64a}$$
$$z(1,t) = 0 \tag{64b}$$
$$\dot{\tilde{\phi}}(t) = G\tilde{\phi}(t) + hz(0,t) \tag{64c}$$
$$z(x,0) = z_0(x) \tag{64d}$$
$$\tilde{\phi}(0) = \tilde{\phi}_0 \tag{64e}$$

where

$$\tilde{\phi}(t) = \begin{bmatrix}\tilde{\eta}(t)\\ \tilde{y}(t)\end{bmatrix}, \qquad \tilde{\phi}_0 = \begin{bmatrix}\tilde{\eta}_0\\ \tilde{y}_0\end{bmatrix}, \qquad h = \begin{bmatrix}0\\ 1\end{bmatrix}. \tag{65}$$

From differentiating (63) with respect to time and space, respectively, and inserting the dynamics (62b)–(62c), we find

$$\begin{aligned}\tilde{v}_t(x,t) &= z_t(x,t) - \theta(x)\tilde{v}(0,t) - e^{d_1\rho_1(1-x)}\rho_1\tilde{\eta}(t)\\ &\quad - (\rho_2 + \gamma_3 + e^{d_1\rho_1(1-x)}\gamma_2)\theta(x)\tilde{y}(t)\end{aligned} \tag{66}$$

and

$$\tilde{v}_x(x,t) = z_x(x,t) - d_1\rho_1 e^{d_1\rho_1(1-x)}\tilde{\eta}(t) + \theta'(x)\tilde{y}(t). \tag{67}$$

Substituting (66) and (67) into (62a) yields

$$\begin{aligned}&\tilde{v}_t(x,t) - \mu\tilde{v}_x(x,t) - \theta(x)\tilde{v}(0,t) + L(x)\tilde{y}(t)\\ &\quad = z_t(x,t) - \mu z_x(x,t)\\ &\qquad + \Big(L(x) - \mu\theta'(x) - (\rho_2+\gamma_3)\theta(x)\\ &\qquad\qquad - \gamma_2 e^{d_1\rho_1(1-x)}\Big)\tilde{y}(t) = 0.\end{aligned} \tag{68}$$

Inserting (58) gives the dynamics (64a). Evaluating (63) at $x = 1$ and inserting the boundary condition (62b) immediately gives (64b). Inserting (63) into (62c)–(62d), we find

$$\dot{\tilde{\eta}}(t) = -\rho_1\tilde{\eta}(t) - \gamma_2\tilde{y}(t) \tag{69a}$$
$$\begin{aligned}\dot{\tilde{y}}(t) &= e^{d_1\rho_1}\tilde{\eta}(t) + (-\rho_2 + \theta(0) - \gamma_3)\tilde{y}(t)\\ &\quad + z(0,t)\end{aligned} \tag{69b}$$

which can be written as (64c).

From the target system (64), we have $z \equiv 0$ for $t \geq d_1$, after which (64c) reduces to

$$\dot{\tilde{\phi}}(t) = G\tilde{\phi}(t) \tag{70}$$

which is an exponentially stable system with bounds (61a)–(61b). Moreover, from (63) with $z \equiv 0$ for $t \geq d_1$, we have

$$\tilde{v}(x,t) = e^{d_1\rho_1(1-x)}\tilde{\eta}(t) + \theta(x)\tilde{y}(t) \tag{71}$$

for $t \geq d_1$, from which the bound (61c) immediately follows.

Remark 6 *The transient performance of the observer of Theorem 5 can be slightly improved by choosing the initial condition $\hat{y}_0$ equal to y_0, which should be possible since y is measured.*

5 Output feedback controllers

Consider the controller $U(t) = U_{of}(t)$, where

$$\begin{aligned}U_{of}(t) &= (k(0)\mu - \gamma_1)\hat{\eta}(t) + \theta(1)\hat{v}(0,t)\\ &\quad + \int_0^1 \Big(k'(1-\xi)\mu\\ &\qquad + (\rho_1+\gamma_1)k(1-\xi)\Big)\hat{v}(\xi,t)d\xi\end{aligned} \tag{72}$$

with $\hat{v}$ and $\hat{\eta}$ being estimates of v and η.

5.1 Output feedback controller 1

Theorem 7 *Consider system* (1) *and let the controller be taken as*

$$U(t) = U_{of}(t) \tag{73}$$

where $U_{of}(t)$ is defined in (72)*, with $\hat{v}$ and $\hat{\eta}$ generated using the observer of Theorem 4. Then, there exists a constant $c_5 > 0$ so that*

$$|v(x,t)| \leq c_5(|\eta_0| + ||k||||v_0|| + |\tilde{\sigma}_0|)e^{-a_1 t} \tag{74}$$

for $t \geq 2d_1$, where

$$a_1 = \min\{\rho_1 + \gamma_1, \delta_1\}. \tag{75}$$

PROOF. Inserting the control law (73) into (1), and adding and subtracting terms, we obtain

$$
\begin{aligned}
v_t(x,t) - \mu v_x(x,t) &= \theta(x)v(0,t) && \text{(76a)}\\
v(1,t) &= p(t) + U_b(t) && \text{(76b)}\\
\dot{p}(t) + (\rho_1 + \gamma_1)p(t) &= \varepsilon(t) && \text{(76c)}\\
v(x,0) &= v_0(x) && \text{(76d)}\\
p(0) &= p_0 && \text{(76e)}
\end{aligned}
$$

where p is defined in (22) with p_0 given in (23), and ε is given as

$$
\begin{aligned}
\varepsilon(t) = &-(k(0)\mu - \gamma_1)\tilde{\eta}(t) - \theta(1)\tilde{v}(0,t)\\
&- \int_0^1 \Big(k'(1-\xi)\mu\\
&\qquad + (\rho_1+\gamma_1)k(1-\xi)\Big)\tilde{v}(\xi,t)d\xi \qquad (77)
\end{aligned}
$$

which from Theorem 4 converges exponentially to zero for $t \geq d_1$, and hence there exists a constant $c_7 > 0$ so that

$$|\varepsilon(t)| \leq c_7|\tilde{\sigma}_0|e^{-\delta_1 t} \qquad (78)$$

for $t \geq d_1$. The backstepping transformation (8) gives

$$
\begin{aligned}
\alpha_t(x,t) - \mu\alpha_x(x,t) &= 0 && \text{(79a)}\\
\alpha(1,t) &= p(t) && \text{(79b)}\\
\dot{p}(t) + (\rho_1+\gamma_1)p(t) &= \varepsilon(t) && \text{(79c)}\\
\alpha(x,0) &= \alpha_0(x) && \text{(79d)}\\
p(0) &= p_0. && \text{(79e)}
\end{aligned}
$$

The dynamics of p constitute an exponentially stable system, driven by an exponentially stable signal ε, and hence there exists some constant $c_8 > 0$ such that

$$|p(t)| \leq c_8(|\eta_0| + ||k||||v_0|| + |\tilde{\sigma}_0|)e^{-a_1 t} \qquad (80)$$

for $t \geq d_1$, where a_1 is defined in (75), and where we have used (26). Following the same steps as for the state-feedback case, we obtain the bound (74) for some constant $c_5 > 0$, valid for $t \geq 2d_2$.

5.2 Output feedback controller 2

Theorem 8 *Consider system* (1) *and let the controller be taken as*

$$U(t) = U_{of}(t) \qquad (81)$$

where $U_{of}(t)$ is defined in (72)*, with $\hat{v}$ and $\hat{\eta}$ generated using the observer of Theorem 5. Then, there exists a constant $c_6 > 0$ so that*

$$|v(x,t)| \leq c_6(|\eta_0| + ||k||||v_0|| + |\tilde{\eta}_0| + |\tilde{y}_0|)e^{-a_2 t} \qquad (82)$$

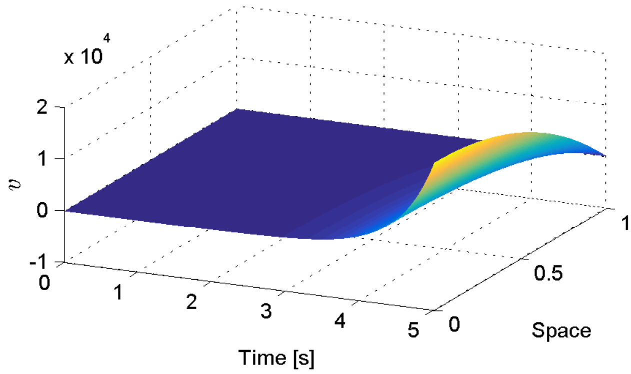

Fig. 1. System state v in the open-loop case.

for $t \geq 2d_1$, where

$$a_2 = \min\{\rho_1 + \gamma_1, \delta_2\}. \qquad (83)$$

PROOF. The proof is similar to the proof of Theorem 7 and therefore omitted.

6 Simulations

System (1), the state-feedback controller of Theorem 2, the observers of Theorems 4 and 5, and the output-feedback controllers of Theorems 7 and 8 were implemented in MATLAB, using the system parameters

$$\mu = 2, \quad \theta = 2 + 3\sin(x), \quad \rho_1 = 5, \quad \rho_2 = 1. \qquad (84)$$

The design parameters $\gamma_1, \kappa, \gamma_2$ and γ_3 were chosen so that

$$\gamma_1 = 1 \qquad \text{eig}(F) = \text{eig}(G) = \{-4, -2\}. \qquad (85)$$

The system's initial conditions were set to

$$v_0(x) = x, \qquad y_0 = 0 \qquad \eta_0 = 1 \qquad (86)$$

while all initial conditions for the observers were set to zero. System (1) with parameters (84) is unstable in the open-loop ($U \equiv 0$) case, as seen from the open-loop simulation in Figure 1.

In the closed loop cases, the system states are seen from Figures 2–4 to be bounded and converge to zero. The convergence time is faster for the state-feedback solution than for the output-feedback solutions, as seen from the state norms displayed in Figure 5. This is due to the convergence time of the observers, the estimation errors for which can be seen in Figure 6. It is observed that the convergence times of the observers are approximately the same, but that the initial transient of the observer of Theorem 4 is better than for the observer of Theorem 5. This difference in initial transient performance is reduced if the eigenvalues of F and G are moved further

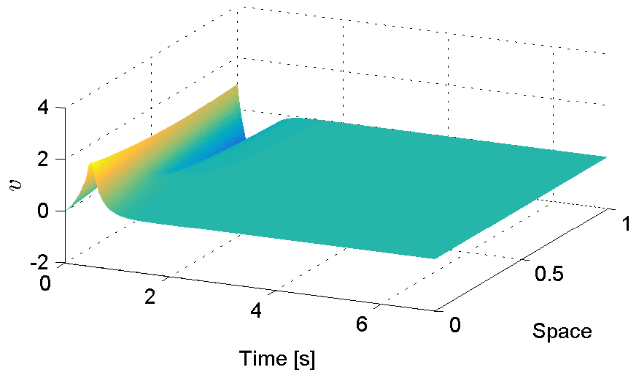

Fig. 2. System state v in the state-feedback case.

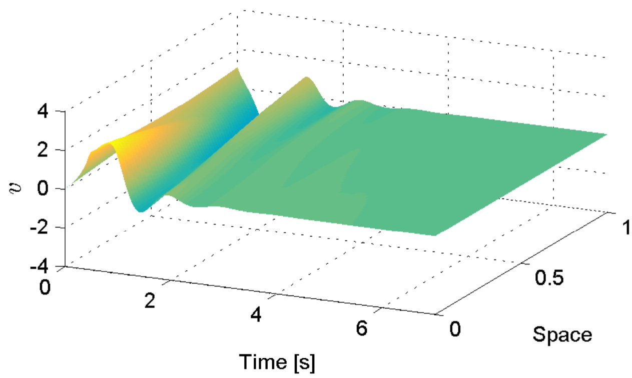

Fig. 3. System state v when using the output-feedback controller of Theorem 7.

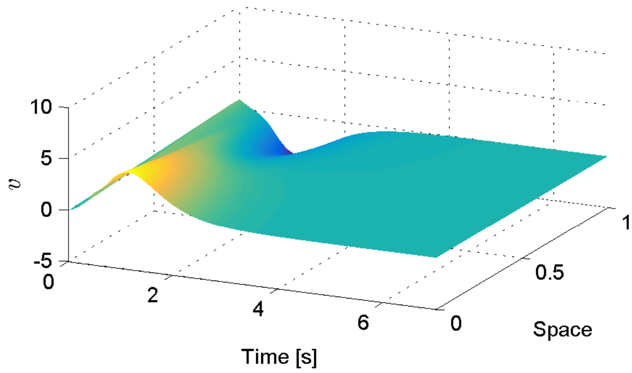

Fig. 4. System state v when using the output-feedback controller of Theorem 8.

to the left. This is illustrated in Figure 7, where the observers of Theorems 4 and 5 are implemented with gains chosen so that

$$\gamma_1 = 1 \qquad \mathrm{eig}(F) = \mathrm{eig}(G) = \{-20, -10\}. \tag{87}$$

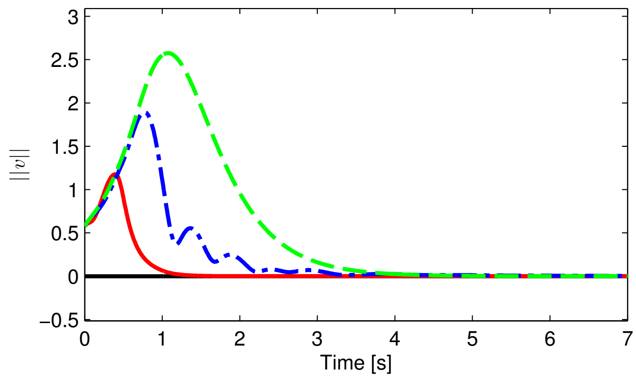

Fig. 5. System states' norm $||v||$ in the state-feedback (solid red) and output-feedback cases of Theorems 7 (dashed-dotted blue) and 8 (dashed green).

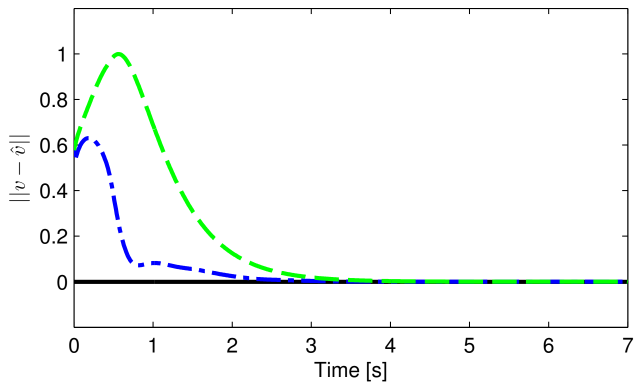

Fig. 6. Observer estimation error norm $||v - \hat{v}||$ when using the controllers of Theorems 7 (dashed-dotted blue) and 8 (dashed green).

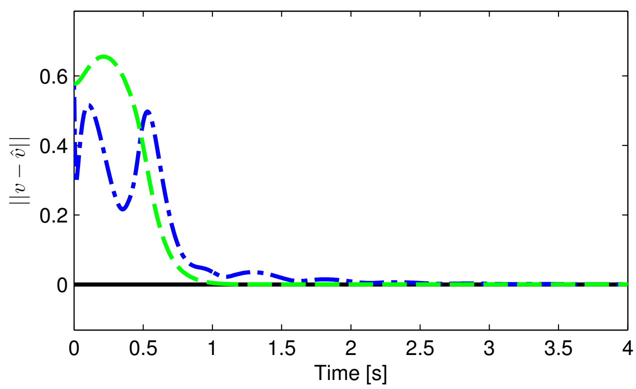

Fig. 7. Observer estimation error norm $||v - \hat{v}||$ when using the controllers of Theorems 7 (dashed-dotted blue) and 8 (dashed green) with faster convergence times.

7 Conclusions and further work

We have derived a controller for output-feedback stabilization of a 1–D linear hyperbolic PDE with first-order actuator and sensor dynamics. The convergence rate of the closed-loop is governed by the rate at which the

ODE-part of the observer converges, which can be arbitrarily chosed by design gains.

A possible extension is to solve the problem for more general actuator and sensor dynamics, that is, dynamics of any order. Also, systems of coupled PDEs can be considered. The extension to a more general 1-D linear hyperbolic PDE, with actuator and sensor dynamics, say for instance

$$u_t(x,t) - \mu(x)u_x(x,t) = f(x)u(x,t) + g(x)u(0,t) + \int_0^x h(x,\xi)u(\xi,t)d\xi \quad (88a)$$
$$u(1,t) = k_1\eta(t) \quad (88b)$$
$$\dot{\eta}(t) + \rho_1\eta(t) = U(t) \quad (88c)$$
$$\dot{y}(t) + \rho_2 y(t) = k_2 u(0,t) \quad (88d)$$
$$u(x,0) = u_0(x,t) \quad (88e)$$
$$y(0) = y_0 \quad (88f)$$

for some functions μ, f, g, h and nonzero constants k_1, k_2, with $u_0 \in L_2([0,1]), y_0 \in \mathbb{R}$, is covered by the above solution. This is due to the fact that system (88) is equivalent to system (1) by a series of invertible transformations and a scaling of the actuation signal, as was shown in [6].

The problem of incorporating pure actuator or sensor delays is solved for the non-adaptive case in [14], and the adaptive case in [4]. This problem is quite simple, since the delays can be modeled by simple linear hyperbolic PDEs, and incorporated by extending or remapping the PDE domain.

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