Adaptive stabilization of switched affine systems with unknown equilibrium points: Application to power converters

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Abstract

The paper adresses the problem of designing a stabilizing control for switched affine systems with unknown parameters. We formulate the problem both in the case where the set of affine subsystems is finite and also in the case where the set of affine subsystems is not finite and given by a convex polytope, i.e., the convex hull of finitely many affine subsystems. The main contribution is a switched and adaptive control design methodology with a global asymptotic stability property. The difficulty is related to the fact that the equilibrium point is unknown a priori. We propose an observer-based control strategy that uses a parameter estimate to update the control law in real time. A DC/DC Flyback converter is considered to illustrate the effectiveness of the proposed method. We also show that the proposed strategy preserves the stability property when the Flyback converter works in the so-called discontinuous conduction mode (DCM).

Key words: Switched control, Switched affine systems, DC-DC power converter.

1 Introduction

A considerable interest has been devoted to switched systems by many researchers both for theoretical and practical reasons. These hybrid systems consists of continuous or discrete time dynamical subsystems and a switching rule that determines at each instant of time the active subsystem (Liberzon 2003.). They are encountered in many applications such as embedded systems, automotive, aerospace, and many other fields. From a theoretical point view, the analysis and design problems are very challenging and many contributions have been proposed during the last decades (see (Lin & Antsaklis 2009, Shorten et al. 2007) and references therein).

Here, we focus on a specific class of switched systems called switched affine systems. This class captures essential features of many applications including power systems and power electronics (Albea et al. 2015, Buisson et al. 2005, Naim et al. 1997, Sira-Ramirez & Ortega 1995). It is also characterized by the fact that many challenging problems have not been solved yet mainly because the affine nature of the linear subsystems introduces additional difficulties in the analysis and control design problems. Among these difficulties, the one related to equilibrium points is of particular interest. There are contributions dedicated to the design of stabilizing switching rules under the assumption that the equilibrium point is perfectly known in advance (Bolzern & Spinelli 2004, Deaecto et al. 2010, Hetel & Fridman 2013, Theunisse et al. 2015). However, these approaches cannot be applied in the realistic case where uncertainty or parameter variations are considered as it is the case of power converters with a lack of measurement on the input voltage and/or the output load (Shahin et al. 2010). The problem is open and one may find two types of contributions in the literature. The first type is switched systems oriented and there are very few papers dedicated to switched affine systems with uncertain or unknown equilibrium points. In a very recent paper, the problem of stabilization of switched affine systems is addressed in the case where the equilibrium is uncertain and characterized by a given convex combination (Senger & Trofino 2016). Under these assumptions, globally stabilizing switching rules are determined by solving an LMI problem. The second type of contributions is power converters oriented. More contributions can be found in this case (see (Sira-Ramirez et al. 2007) and references therein) but the results are specific to the considered application. As an example, for boost converters with unknown, piecewise constant, load resistance values an interesting contribution is proposed in (Sira-Ramirez et al. 2007) based on a nonlinear algebraic parameter estimation associated to a reduced order observer and a static output feedback.

Here, we propose a theoretical contribution dedicated to switched affine systems with unknown equilibrium points without restricting the set of equilibrium points to some given structure and we apply this approach to power converters with load and input voltage variations. We formulate the problem both in the case where the set of affine subsystems is finite and also in the case where the set of affine subsystems is not finite and given by a convex polytope, i.e., the convex hull of finitely many affine subsystems. The main contribution is an adaptive control design methodology with a global asymptotic stability property. A preliminary version of this work appeared in (Beneux et al. 2017*a*,*b*). The result in (Beneux et al. 2017*a*) concerns only the case where the set of affine subsystems is finite while (Beneux et al. 2017*b*) discusses this preliminary result in the case where a DC-DC power converter may operate in the so-called discontinuous conduction mode.

The paper is organized as follows. In the next section, we first recall the classical state feedback switching control design problem for switched affine systems and its solution in the case where the set of affine subsystems is finite. We also formulate the problem in the case where the set of affine subsystems is not finite and we give the solution in the case where the equilibrium point is perfectly known. Section 3 is dedicated to the main contribution of this paper which is an adaptive and switching control design methodology for switched affine systems with unknown parameters. Using a parametrization of the admissible set of equilibrium points and an estimation of the unknown parameters, we propose an adaptive and switched control law with global asymptotic stability guarantees. In section 4, the results are applied to a DC- DC power converter subject to load and input variations. We show how to take into account the fact that the load and the input voltage are not known a priori and we formulate the corresponding power converters stabilisation problem as a control problem for switched affine systems with unknown equilibrium points. We also show that the proposed method preserves the global asymptotic stability property when the DC-DC converter operates in the discontinuous conduction mode (DCM). We end the paper by a conclusion.

Notations: The set composed by the N first integers is denoted by $\mathbb{K} = \{1, \cdots, N\}$. The $(N-1)$-dimension simplex is denoted $\Lambda := \{\lambda \in \mathbb{R}^N \mid \forall i \in \mathbb{K},\ \lambda_i \geq 0,\ \sum_{i=1}^{N} \lambda_i = 1\}$. The convex combination of a set of matrices $\mathcal{A} = \{A_1, \cdots, A_N\}$ is denoted $A(\lambda) = \sum_{i=1}^{N} \lambda_i A_i$, with $\lambda \in \Lambda$. The transpose of a matrix M is denoted M^T. The identity matrix (or the null matrix) is denoted by I (or 0) for any dimension. For a square symmetric matrix, $M \succ 0$ ($M \prec 0$) indicates that it is positive (negative) definite. *co* stands for the convex hull. For a given vector v, the notation $v_{[k_1,k_2]}$ refers to a vector of dimension $k_2 - k_1 + 1$ formed by the components of v within the range $[k_1, k_2]$. Finally, $D = diag(d_i,\ i = 1, \cdots, n)$ is a diagonal matrix of dimension $n \times n$ whose entries are the $d_i's,\ i = 1, \cdots, n$.

2 State feedback stabilization

We consider the class of continuous−time switched affine systems given by:

$$\dot{x}(t) = A_\sigma x(t) + b_\sigma \tag{1}$$

where $x : \mathbb{R} \mapsto \mathbb{R}^n$ is the state and $\sigma : \mathbb{R}^n \mapsto \mathbb{K}$ refers to the state dependent switching law that selects at each time one of the N subsystems characterized by the pairs $(A_i,\ b_i),\ i \in \mathbb{K}$. Our main objective in this paper is the design of adaptive and global stabilizing control laws for this class of hybrid systems. By adaptive, we mean control laws that take into account some parameter variations to be defined in the next section.

We also consider the class of dynamical systems obtained by taking the convex combination of these N subsystems and which is given by:

$$\dot{x}(t) = A(\lambda)x(t) + b(\lambda). \tag{2}$$

The control is now λ and it takes its values in the whole simplex Λ. An important feature of this class of dynamical systems is the fact that it allows to characterize the set of equilibrium points of the switched affine systems (1) in already noticed in (Bolzern & Spinelli 2004, Deaecto et al. 2010, Hauroigne et al. 2011, Hetel & Fridman 2013). We recall this characterization.

Definition 1 *Let $\Lambda^{\mathcal{H}}$ be the subset of Λ such that $A(\lambda)$ is Hurwitz, that is:*

$$\Lambda^{\mathcal{H}} := \{\lambda \in \Lambda :\ \exists P \succ 0,\ A^T(\lambda)P + PA(\lambda) \prec 0\}.$$

The set $\Lambda^{\mathcal{H}}$ is open in Λ by construction. It is assumed to be nonempty

Definition 2 *Let X^e be the set of the equilibrium points related to system* (2) *and defined as:*

$$X^e := \{x^e \in \mathbb{R}^n,\ x^e = -A(\lambda^e)^{-1}b(\lambda^e),\ \lambda^e \in \Lambda^{\mathcal{H}}\}$$

As it has been shown in (Bolzern & Spinelli 2004, Deaecto et al. 2010, Hauroigne et al. 2011, Hetel & Fridman 2013), state x of system (1) can be stabilized on any equilibrium point $x_e \in X^e$ using the switched state feedback law, given in our framework by:

$$\sigma(x) \in \arg\min_{i \in \mathbb{K}} (x - x^e)^T P(A_i x + b_i), \tag{3}$$

where $P \succ 0$ satisfies the Lyapunov inequality

$$A(\lambda^e)^T P + PA(\lambda^e) \prec 0.$$

Another interest of the class of dynamical systems given by (2) is related to the property of density of the trajectories of system (1) into trajectories generated by (2) which allows to use (2) instead of (1) for stability analysis and/or control design (Bengea & Decarlo 2005, Hauroigne et al. 2011, Ingalls et al. 2003). To be more precise, this corresponds in (Bengea & Decarlo 2005), which is dedicated to optimal control design for switched systems, to the fact that the switched system is embedded into a larger family of systems and the optimization problem is formulated for the latter. The authors show that the set of trajectories of the switched system is dense in the set of trajectories of the embedded system and the relationship between the two sets of trajectories motivates the shift of focus from the original problem to the more general one. They also underly the engineering relevance of the study of the second problem. Indeed, this density property is useful to establish some bridges from control laws dedicated to (1) to the ones dedicated to (2) through the notion of averaging. Conversely, control methods designed for (2) can be applied to systems given by (1) using Pulse-Width Modulation techniques (PWM). Owing to these interests, we provide in the next proposition a way to design what we will call in the sequel the embedded control λ.

Proposition 1 *Consider the dynamical system (2). The equilibrium point corresponding to the pair (x^e, λ^e) is globally asymptotically stable under the state feedback law:*

$$\lambda_{[1,N-1]}(x, x^e) = \lambda^e_{[1,N-1]} - Ky(x, x^e),$$
$$\lambda_N(x, x^e) = \lambda^e_N + \mathbb{1}^T Ky(x, x^e),$$

with $y(x, x^e) = B^T(x)P(x - x^e)$, and $B(x)$ whose columns are given by $B_i(x) = (A_i - A_N)x + (b_i - b_N)$ and where $P \succ 0$ satisfies the Lyapunov inequality

$$A(\lambda^e)^T P + PA(\lambda^e) + Q \prec 0.$$

for some given $Q \succ 0$. The matrix K is such that $K = diag(k_i,\ i = 1, \cdots, N-1)$ with positive real numbers $k_i > 0$ and $\mathbb{1} = (1, \cdots, 1)$ ($N-1$ times).

PROOF. For a given (x^e, λ^e), system (2) can be rewriten:

$$\dot{x} = A(\lambda^e)(x - x^e) + A(\lambda - \lambda^e)x + b(\lambda - \lambda^e),$$

then, using relation : $\lambda_N = 1 - \sum_{i=1}^{N-1} \lambda_i$, we have:

$$\dot{x} = A(\lambda^e)(x - x^e) + \sum_{i=1}^{N-1} (\lambda - \lambda^e)_i B_i(x),$$

with $B_i(x) = (A_i - A_N)x + (b_i - b_N)$. By denoting $B(x)$ the matrix whose columns are the $B_i(x)$'s, $i = 1 \cdots, N-1$, we finally obtain:

$$\dot{x} = A(\lambda^e)(x - x^e) + B(x)(\lambda - \lambda^e)_{[1,N-1]}.$$

Consider the Lyapunov function

$$V(x, x^e) = (x - x^e)^T P(x - x^e),$$

where $P \succ 0$ satisfies the Lyapunov inequality

$$A(\lambda^e)^T P + PA(\lambda^e) + Q \prec 0,$$

for a given $Q \succ 0$. Denote by $y(x, x^e) = B^T(x)P(x - x^e)$ and let us define the weight matrix $K = diag(k_i,\ i = 1, \cdots, N-1)$ for some chosen positive real numbers $k_i > 0$. By a derivation along the trajectory of $V(x, x^e)$ in the direction defined by the control $\lambda_{[1,N-1]}(x, x^e) = \lambda^e_{[1,N-1]} - Ky(x, x^e)$, one can get the result:

$$\begin{aligned}\dot{V}(x, x^e) =& 2(x - x^e)^T P(A(\lambda^e)(x - x^e) - B(x)Ky(x, x^e)) \\ \leq& -(x - x^e)^T Q(x - x^e) - 2y(x, x^e)^T Ky(x, x^e) \\ <& 0 \text{ if } x - x^e \neq 0. \end{aligned} \quad (4)$$

As the control domain Λ is bounded, the embedded control of Proposition 1 must be limited by a saturation function as follows:

Definition 3 *For a given control of the form $\lambda = \lambda^e + \delta\lambda$ with $\lambda^e \in \Lambda$ and $\sum_{i \in \mathbb{K}} \delta\lambda_i = 0$, its saturation denoted by $sat(\lambda)$, is defined by its projection in the direction $\delta\lambda$ on Λ:*

$$sat(\lambda) = Proj(\lambda; \delta\lambda) = \lambda^e + \alpha\delta\lambda \quad (5)$$

where $\alpha = \max\{\alpha \in [0\ 1] : \lambda^e + \alpha\delta\lambda \in \Lambda\}$.

Lemma 1 *The values of α are determined by the following relations: $\alpha = \min_{j \in \mathbb{K}} \alpha_j$ and*

$$\alpha_j = \begin{cases} \min\left(1, \frac{1-\lambda^e_j}{\delta\lambda_j}\right) & \text{if } \delta\lambda_j > 0 \\ \min\left(1, \frac{-\lambda^e_j}{\delta\lambda_j}\right) & \text{if } \delta\lambda_j < 0 \\ 1 & \text{if } \delta\lambda_j = 0 \end{cases}$$

PROOF. As $\lambda^e \in \Lambda$ and as $\sum_{i \in \mathbb{K}} \delta\lambda_i = 0$, the sum $\lambda = \lambda^e + \delta\lambda$ is inside the set Λ if the components of $\delta\lambda$ satisfy for all $i \in \mathbb{K}$,

$$1 - \lambda^e_i \geq \delta\lambda_i \geq -\lambda^e_i.$$

Following the sign of $\delta\lambda_i$, we deduce the following conditions, for all $i \in \mathbb{K}$:

$$1 \leq \frac{-\lambda_i^e}{\delta\lambda_i} \text{ if } \delta\lambda_i < 0 \text{ and } 1 \leq \frac{1-\lambda_i^e}{\delta\lambda_i} \text{ if } \delta\lambda_i > 0.$$

If there exists at least one subscript i such that the above relations are not satisfied, then there exists a reduction ratio α, $0 < \alpha < 1$ such that these relations are satisfied when replacing $\delta\lambda$ by $\alpha\delta\lambda$. The greater reduction ratio is obviously given by $\alpha = \min_i \alpha_i$ where

$$\alpha_i = \begin{cases} \min\left(1, \frac{1-\lambda_i^e}{\delta\lambda_i}\right) & \text{if } \delta\lambda_i > 0 \\ \min\left(1, \frac{-\lambda_i^e}{\delta\lambda_i}\right) & \text{if } \delta\lambda_i < 0 \\ 1 & \text{if } \delta\lambda_i = 0 \end{cases}$$

We are now in position to state the main result of this section.

Theorem 2 *Consider the dynamical system (2). The equilibrium point corresponding to the pair (x^e, λ^e) is globally asymptotically stable under the state feedback control:*

$$sat(\lambda(x)) = \begin{cases} \lambda_{[1,N-1]}^e - \alpha K y(x, x^e), \\ \lambda_N^e + \mathbb{1}^T \alpha K y(x, x^e), \end{cases} \tag{6}$$

with $sat(\cdot)$ and α given by (5) and the control λ provided by Proposition 1.

PROOF. The proof can be easily obtained from the expression of the derivative of V and noticing that the function *sat* does not modify the sign of the derivative by substituting in (4) the term $-2y(x,x^e)^T K y(x,x^e)$ by $-2\alpha y(x,x^e)^T K y(x,x^e)$.

A key assumption behind the state feedback control laws (3) and (6) is the knowledge of the pair (x^e, λ^e) which allows to determine the feedback law in real time. A question of practical interest which one may ask is whether these control laws can be used in the case where the affine system under interest is subject to parameter variations or uncertainties. We answer this question in the next section.

3 Adaptive stabilization

Let a parameter dependent switched affine system described by:

$$\begin{aligned} \dot{x}(t) &= A_\sigma x(t) + b_\sigma + G_\sigma p \\ y(t) &= Cx \end{aligned} \tag{7}$$

where the matrices A_i, G_i and the vectors b_i, $i \in \mathbb{K}$, are constant and given, $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^r$ is a measured output, $\sigma : \mathbb{R}^n \mapsto \mathbb{K}$ is the switching control and $p \in \mathbb{R}^m$ is a constant ($\dot{p}(t) = 0$) but unknown parameter. In this setting, we assume that the dependency of the system with respect to the unknown parameter p is linear. We show in the section dedicated to the application the interest of this modeling in the case of power converters.

Now, we assume that only a part of the state has to be controlled to some prescribed values. Without loss of generality, we denote this part by $x_2 \in \mathbb{R}^{n_2}$. The remaining components of the state are denoted $x_1 \in \mathbb{R}^{n_1}$ and are such that $x = (x_1, x_2)$. This assumption is necessary in many applications and is related to the degrees of freedom provided by the control λ to fix the equilibrium by the relation $x^e = -A(\lambda)^{-1}[b(\lambda) + G(\lambda)p]$ for a given p. Thus, the dimension of x_2 is at most the number of free control variables, here $N-1$.

3.1 Parametrization of the equilibrium point

A non trivial difficulty in the case of parameter dependent switched affine systems is related to the definition of the equilibrium point which is a priori unknown. To define the set of admissible equilibrium points in this context, we introduce the following definition.

Definition 4 *For an arbitrary small $\xi \geq 0$ fixed, we define the set:*

$$\Lambda^{\mathcal{H}}(\xi) = \{\lambda \in \Lambda : A(\lambda + \nu) \textit{ is Hurwitz}, \forall \nu \in \mathcal{B}(0, \xi)\},$$

where $B(0,\xi)$ denoted a closed ball of radius ξ in $\mathbb{R}^N$ centered on 0.

As $\Lambda^{\mathcal{H}}$ is assumed to be non empty, there always exists $\xi > 0$ such that $\Lambda^{\mathcal{H}}(\xi) \neq \emptyset$. This set ensures, for any λ which belongs to the closure of $\Lambda^{\mathcal{H}}(\xi)$ in Λ, that the matrix $A(\lambda)$ is Hurwitz. In the sequel, it is useful in order to guarantee a bound on the decay rate of the Lyapunov function. One can notice that $\Lambda^{\mathcal{H}}(\xi) \subseteq \Lambda^{\mathcal{H}}(0) \subseteq \Lambda$.

The set of admissible equilibrium points is defined as follows:

Definition 5 *For $\xi \geq 0$ fixed, we define the set of the equilibrium points as:*

$$X^e(\xi) = \{x^e \in \mathbb{R}^n : \; x^e = -A(\lambda)^{-1}[b(\lambda) + G(\lambda)p], \\ (p, \lambda) \in \mathbb{R}^m \times \Lambda^{\mathcal{H}}(\xi)\}.$$

Definition 6 *The restriction of $X^e(\xi)$ to x_1 (respectively x_2) denoted by $X^e(\xi)|_{x_1}$ (respectively $X^e(\xi)|_{x_2}$) is defined by the set $X^e(\xi)|_{x_1} = \{x_1^e : x^e = (x_1^e, x_2^e) \in X^e(\xi)\}$. (respectively $X^e(\xi)|_{x_2} = \{x_2^e : x^e = (x_1^e, x_2^e) \in$*

$X^e(\xi)\}$).

Consider the C^∞-class function ψ defined by:

$$\psi : \Lambda^{\mathcal{H}}(\xi) \times X^e(\xi)|_{x_1} \times X^e(\xi)|_{x_2} \times \mathbb{R}^m \to \mathbb{R}^n$$
$$(\lambda, x_1, x_2, p) \mapsto A(\lambda)(x_1, x_2) + b(\lambda) + G(\lambda)p.$$

For any $\lambda \in \Lambda^{\mathcal{H}}(\xi)$, any fixed p, as $A(\lambda)$ is invertible, there exists a unique x such that:

$$\psi(\lambda, x_1, x_2, p) = 0.$$

For a given (x_2^e, p), we assume that there exists a parametrization of x_1^e and λ^e as follows.

Assumption 1 *For all $x_2^e \in X^e(\xi)|_{x_2}$, p, there exists a unique C^1-class function $\phi = (\phi_1, \phi_2)$ defined by:*

$$\begin{aligned} \phi_1 : X^e(\xi)|_{x_2} \times \mathbb{R}^m &\to X^e(\xi)|_{x_1} \\ (x_2^e, p) &\mapsto x_1^e = \phi_1(x_2^e, p) \end{aligned} \tag{8}$$

$$\begin{aligned} \phi_2 : X^e(\xi)|_{x_2} \times \mathbb{R}^m &\to \Lambda^{\mathcal{H}}(\xi) \\ (x_2^e, p) &\mapsto \lambda^e = \phi_2(x_2^e, p) \end{aligned} \tag{9}$$

fulfilling the relation

$$\psi(\lambda^e, x_1^e, x_2^e, p) = 0.$$

Moreover, we assume that there exists a constant $\eta(x_2^e, \xi) > 0$ such that:

$$\left\| \frac{\partial \phi_1(x_2^e, \cdot)}{\partial p} \right\|_\infty \leq \eta(x_2^e, \xi). \tag{10}$$

To overcome the fact that the equilibrium point depends on the parameter p which is unknown a priori, the control method we propose, uses an observer to estimate this equilibrium point in real time. This is explained in the next subsection.

3.2 Parameter estimation and reference point update

Here we present a switched observer which allows to estimate the state and the unknown parameter $z = (x, p)$. This switched observer will be used in the stabilizing control strategy proposed in this paper and it is given by:

$$\begin{cases} \dot{\hat{z}} = \tilde{A}_\sigma \hat{z} + \tilde{b}_\sigma + L_\sigma (y - \hat{y}) \\ \hat{y} = \tilde{C}\hat{z} \end{cases}, \tag{11}$$

where $\tilde{A}_i = \begin{bmatrix} A_i & G_i \\ 0 & 0 \end{bmatrix}$, $\tilde{b}_i = \begin{bmatrix} b_i \\ 0 \end{bmatrix}$, with $L_i = \begin{bmatrix} \overline{L}_i \\ \underline{L}_i \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$. Let the error between z and $\hat{z}$ be $e(t) = z(t) - \hat{z}(t)$. Its dynamics is given by:

$$\dot{e} = (\tilde{A}_\sigma - L_\sigma \tilde{C})e. \tag{12}$$

The gains L_i are designed to ensure that $\lim_{t\to\infty}(e(t) = z(t) - \hat{z}(t)) = 0$ for any switching law σ and this can be done using the following classical result (Boyd et al. 1994).

Proposition 3 *If there exist a matrix $P_{obs} \succ 0$, matrices R_i and scalars $\gamma_i > 0$ satisfying for all $i \in \mathbb{K}$:*

$$\begin{cases} \tilde{A}_i^T P_{obs} + P_{obs}\tilde{A}_i - \tilde{C}^T R_i^T - R_i \tilde{C} + \gamma_i P_{obs} < 0 \\ i = 1, 2, \ldots, N \end{cases}$$

then the observer gains $L_i = P_{obs}^{-1} R_i$ ensure that the error dynamics (12) is exponentially globally stable for any switching law σ and it is characterized by a decay rate at least equal to $\gamma = \min_{i\in\mathbb{K}} \gamma_i$.

PROOF. The proof is obvious from (Boyd et al. 1994) (chapter 5, section 5.1)

Remark 1 *The observer design is explained in the case of switched affine systems of the form (1). The same can be done for systems of the form (2) and one obtains a polytopic observer which is a relaxed version of the switched one.* One can now see the important role played by the parametrization (8)-(9) in the approach we propose. Indeed, it allows to estimate the reference point based on an estimation $\hat{p}$ of p and a given reference x_2^e for x_2 as follows:

$$\hat{x}_1^e = \phi_1(x_2^e, \hat{p}) \tag{13}$$
$$\hat{\lambda}^e = \phi_2(x_2^e, \hat{p}). \tag{14}$$

From now, we will refer to this estimation by $\hat{x}^e = (\hat{x}_1^e, x_2^e)$. The dynamics of $\hat{x}_1^e$ for x_2^e fixed is :

$$\dot{\hat{x}}_1^e = \frac{\partial \phi_1}{\partial p}\dot{\hat{p}} = \frac{\partial \phi_1}{\partial p}(\dot{\hat{p}} - \dot{p}).$$

Taking into account $\dot{p} = 0$ and replacing $\dot{\hat{p}}$ by its expression (11), we have:

$$\dot{\hat{x}}_1^e = \frac{\partial \phi_1}{\partial p} \underline{L}_\sigma \tilde{C} e \tag{15}$$
$$\dot{x}_2^e = 0. \tag{16}$$

3.3 Adaptive stabilizing control laws

We are now ready to present the main result of this paper. The following assumptions will be used.

Assumption 2 *For all $\lambda \in \Lambda$ such that $A(\lambda)$ is Hurwitz, there exists at least one subscript i such that $\lambda_i \neq 0$ and A_i is Hurwitz.*

Assumption 3 *Assume there exists a matrix $P \succ 0$ such that*

$$A_i^T P + PA_i + \alpha_i P \leq 0 \qquad i = 1, \ldots, N \tag{17}$$

with $\alpha_i = \begin{cases} \alpha_i > 0 & \text{if } A_i \text{ is Hurwitz} \\ \alpha_i = 0 & \text{if not} \end{cases}$

Assumptions 2 and 3 allow to impose a performance constraint and in particular to specify a decay rate index. The switched and adaptive stabilizing control law is given in the following Theorem.

Theorem 4 *Consider the switched affine system (7) with unknown parameters p and let $x^e = (x_1^e, x_2^e)$ with $x_1^e = \phi_1(x_2^e, p)$. Assume that Proposition 3 is satisfied. Under assumptions 1, 2 and 3, for all $x_2^e \in X^e(\xi)|_{x_2}$ with $\xi > 0$, the feedback law defined by*

$$\sigma^*(\hat{x}, \hat{x}^e, \hat{p}) \in \arg\min_{i \in \mathbb{K}} (\hat{x} - \hat{x}^e)^T P(A_i \hat{x} + b_i + G_i \hat{p}) \tag{18}$$

ensures global asymptotic stability of the equilibrium point $x^e = (\phi_1(x_2^e, p), x_2^e)$.

PROOF. For a given x_2^e and an estimated value $\hat{p}$ of p, denote by $w = (\hat{x} - \hat{x}^e)$ (see equation (13)). For $\hat{\lambda}^e = \phi_2(x_2^e, \hat{p}) \in \Lambda^{\mathcal{H}}(\xi)$ (see equation (14)), we have by (17) :

$$2w^T PA(\hat{\lambda}^e)w + \alpha(\hat{\lambda}^e) w^T P w \leq 0 \tag{19}$$

with $\alpha(\hat{\lambda}^e) = \sum_{i=1}^N \hat{\lambda}_i^e \alpha_i$.

By assumption 1, as $A(\hat{\lambda}^e)\hat{x}^e + b(\hat{\lambda}^e) + G(\hat{\lambda}^e)\hat{p} = 0$, it follows :

$$\begin{aligned} 2w^T P(A(\hat{\lambda}^e)\hat{x} + b(\hat{\lambda}^e) + G(\hat{\lambda}^e)\hat{p}) &\leq -\alpha(\hat{\lambda}^e) w^T P w \\ \sum_{i=1}^N \hat{\lambda}_i^e 2w^T P(A_i \hat{x} + b_i + G_i \hat{p}) &\leq -\alpha(\hat{\lambda}^e) w^T P w \end{aligned}$$

As $\alpha(\hat{\lambda}^e) = \sum_{i=1}^N \hat{\lambda}_i^e \alpha_i$ cannot be zero by Assumption 2, we can conclude that the control law (18) leads to:

$$\begin{aligned} 2w^T P(A_{\sigma^*}\hat{x} + b_{\sigma^*} + G_{\sigma^*}\hat{p}) &\leq -\alpha(\hat{\lambda}^e) w^T P w & (20) \\ &< 0, \text{ if } w \neq 0. & (21) \end{aligned}$$

Consider the Lyapunov function

$$V(x, x^e) = (x - x^e)^T P(x - x^e).$$

Using $(\hat{x}, \hat{x}^e)$ instead of (x, x^e) and taking into account the dynamics of $\hat{x}^e$ (see (15) and (16)), the directional derivative in a direction specified by λ is given by:

$$\begin{aligned} \dot{V}(\hat{x}, \hat{x}^e; \lambda) = \; & 2w^T P(A(\lambda)\hat{x} + b(\lambda) + G(\lambda)\hat{p} \\ & + (\overline{L}(\lambda) - \Phi \underline{L}(\lambda))\tilde{C}e) \end{aligned}$$

with $\Phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial p} \\ 0 \end{bmatrix}$.

Therefore, adding to both sides of the inequality (20) the term:

$$2w^T P(\overline{L}_{\sigma^*} - \Phi \underline{L}_{\sigma^*})\tilde{C}e,$$

we obtain:

$$\dot{V}(\hat{x}, \hat{x}^e; \sigma^*) \leq -\alpha(\hat{\lambda}^e) w^T P w + 2w^T P(\overline{L}_{\sigma^*} - \Phi \underline{L}_{\sigma^*})\tilde{C}e$$

As e converges exponentially to 0 for any switching rule, and thus, particularly for the switching rules (18), with $\bar{\gamma} = min_i \gamma_i$, there exists $K > 0$,

$$\|e(t)\| \leq K e^{-\bar{\gamma} t/2} \|e_0\|.$$

Hence:

$$\begin{aligned} \dot{V}(\hat{x}, \hat{x}^e, \sigma^*) \leq \; & -\alpha(\hat{\lambda}^e) w^T P w + 2\beta_{max} \|w\| \\ & \times (\|\overline{L}_{\sigma^*}\| + \left\| \tfrac{\partial \phi_1(x_2^e, \cdot)}{\partial p} \right\|_\infty \|\underline{L}_{\sigma^*}\|) \\ & \times \|\tilde{C}\| K e^{-\bar{\gamma} t/2} \|e_0\| \end{aligned}$$

with β_{max} the maximal eigenvalue of P. A sufficient condition to guarantee

$$\dot{V}(\hat{x}, \hat{x}^e; \sigma^*) \leq -\frac{\alpha(\hat{\lambda}^e)}{2} w^T P w < 0$$

is

$$\begin{aligned} \|w\| > & 4 \frac{\beta_{max}}{\alpha_{min} \beta_{min}} \max_i (\|\overline{L}_i\| + \left\| \frac{\partial \phi_1(x_2^e, \cdot)}{\partial p} \right\|_\infty \|\underline{L}_i\|) \\ & \times \|\tilde{C}\| \|e_0\| K e^{-\bar{\gamma} t/2} \end{aligned} \tag{22}$$

where β_{min} is the minimum eigenvalue of P and

$$\alpha_{min} = \min_{\lambda \in \Lambda^{\mathcal{H}}(\xi)} \alpha(\lambda).$$

Note that $\alpha_{min} > 0$ for $\xi > 0$ under Assumption 2 and by definition of $\Lambda^{\mathcal{H}}(\xi)$.

As there exists $\eta(x_2^e, \xi) > 0$ such that

$$\Big|\Big|\frac{\partial \phi_1(x_2^e, \cdot)}{\partial p}\Big|\Big|_\infty \leq \eta(x_2^e, \xi)$$

we can conclude that there exists a constant $M(x_2^e, \xi) > 0$ such that if

$$\|w\| > M(x_2^e, \xi) e^{-\bar{\gamma} t/2} \|e_0\|$$

then

$$\dot{V}(\hat{x}, \hat{x}^e; \sigma^*) \leq -\frac{\alpha(\hat{\lambda}^e)}{2} w^T P w < 0.$$

As we can claim that

$$\forall e_0,\ \forall \epsilon > 0, \exists t_1, \text{ such that } M e^{-\bar{\gamma} t/2} \|e_0\| < \epsilon, \forall t > t_1,$$

we necessarily have $\dot{V}(\hat{x}, \hat{x}^e; \sigma^*) \leq -\frac{\alpha_{min}}{2} w^T P w < 0$, for all w such that $\|w\| > \epsilon$ and $t > t_1$. This means that for any $w(0)$, there exists t_2 such that

$$\|w(t)\| \leq \epsilon$$

for all $t > t_2$. We can conclude that w converges asymptotically to 0. As e converges also to zero, we have

$$\hat{x} \to x \to \hat{x}^e = \begin{bmatrix} \hat{x}_1^e \\ x_2^e \end{bmatrix}$$

with $\hat{x}_1^e = \phi_1(x_2^e, p)$ because $\hat{p} \to p$.

Remark 2 *If Assumptions 2 and 3 are not satisfied (for example if the matrices A_i, $i = 1 \cdots, N$ are not Hurwitz then it is possible to weaken these assumptions by an alternative design of P. Assume that there exists a polytopic approximation $\Lambda_{poly} := \{\zeta \in \mathbb{R}^m \mid \forall i \in \{1, \cdots, m\},\ \zeta_i \geq 0,\ \sum_{i=1}^m \zeta_i = 1\}$ of the set $\Lambda^{\mathcal{H}}(\xi)$ such that :*

a) $\Lambda^{\mathcal{H}}(\xi) \subseteq \Lambda_{poly} \subset \Lambda$,
b) $A(\upsilon_i)$, $i = 1, \cdots, m$ are Hurwitz,

where the υ_i's, $i = \{1, \cdots, m\}$ denote the vertices of Λ_{poly} (as the vertices υ_i , $i = 1, \cdots, m$ of Λ_{poly} belong to Λ, there exists $\lambda(\upsilon_i) \in \Lambda$, $A(\upsilon_i) = \sum_{j=1,\ldots,N} \lambda_j A_j$, $i = 1, \cdots, m$ and the expression $A(\upsilon_i)$ is well defined).

Now, if there exists a matrix $P \succ 0$ such that

$$A(\upsilon_i)^T P + P A(\upsilon_i) + \alpha_{\upsilon_i} P \leq 0,\ i = 1, \cdots, m$$

for given $\alpha_{\upsilon_i} > 0$ then using this matrix P, the same steps of the proof of Theorem 4 can be applied to obtain the global stability property. The only difference concerns equation (19) where $\alpha(\hat{\lambda}^e)$ must be replaced by $\alpha_{min} = \min_i \alpha_{\upsilon_i}$ (since $\hat{\lambda}^e \in \Lambda_{poly}$).

We give in the next Theorem the expression of the adaptive stabilizing control for the polytopic version of (7) defined by:

$$\dot{x}(t) = A(\lambda) x(t) + b(\lambda) + G(\lambda) p \tag{23}$$

and where the control is now the adaptive and embedded control $\lambda(t) \in \Lambda$.

Theorem 5 *Consider the relaxed affine system (23) with unknown parameters p and let $x^e = (x_1^e, x_2^e)$ with $x_1^e = \phi_1(x_2^e, p)$. Assume that Proposition 3 is satisfied. Under assumptions 1, 2 and 3, for all $x_2^e \in X^e(\xi)|_{x_2}$ with $\xi > 0$, the feedback law $sat(\lambda^*)$ (see (5) for definition of sat) given by:*

$$sat(\lambda^*(\hat{x}, \hat{p})) = \begin{cases} \hat{\lambda}_{[1,N-1]}^e - \alpha K y(\hat{x}, \hat{p}), \\ \hat{\lambda}_N^e + \mathbb{1}^T \alpha K y(\hat{x}, \hat{p}), \end{cases}$$

with $y(\hat{x}, \hat{p}) = B^T(\hat{x}, \hat{p}) P (\hat{x} - \hat{x}^e)$, the columns of matrix $B(\hat{x}, \hat{p})$ given by $B_i(\hat{x}, \hat{p}) = (A_i - A_N)\hat{x} + (b_i - b_N) + (G_i - G_N)\hat{p}$, for $i = 1, \cdots, N-1$ and P satisfying (17), ensures global asymptotic stability of the equilibrium point x^e.

PROOF. We only provide a sketch of the proof since the steps are similar to the proof of Theorem 4. Applying the same type of calculus used in the proof of Proposition 1 allows to establish the dynamics of $w = \hat{x} - \hat{x}^e$ when a control λ is applied:

$$\dot{w} = A(\hat{\lambda}^e)(\hat{x} - \hat{x}^e) + B(\hat{x}, \hat{p})(\lambda - \hat{\lambda}^e)_{[1\ N-1]} + D(\lambda)\tilde{C}e$$

with columns of matrix $B(\hat{x}, \hat{p})$ given by $B_i(\hat{x}, \hat{p}) = (A_i - A_N)\hat{x} + (b_i - b_N) + (G_i - G_N)\hat{p}$ for $i = 1, \cdots, N-1$, $D(\lambda) = \sum_{i=1}^N \lambda_i (\overline{L}_i - \Phi \underline{L}_i)$ and $\Phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial p} \\ 0 \end{bmatrix}$.

Consider once again the Lyapunov function

$$V(x, x^e) = (x - x^e)^T P (x - x^e)$$

using $(\hat{x}, \hat{x}^e)$ instead of (x, x^e). The directional derivative of V in direction provided by the proposed feedback $sat(\lambda^*)$ leads to:

$$\begin{aligned} \dot{V}(\hat{x}, \hat{x}^e; sat(\lambda^*)) &= 2w^T P A(\hat{\lambda}^e) w \\ &\quad -2\alpha y^T(\hat{x}, \hat{p})\ K y(\hat{x}, \hat{p}) + 2w^T P D(sat(\lambda^*))\tilde{C}e \end{aligned}$$

with $y(\hat{x}, \hat{p}) = B^T(\hat{x}, \hat{p}) P (\hat{x} - \hat{x}^e)$ and where α is provided by the function $sat(\lambda^*)$ (see (5)).

As $\|D(sat(\lambda^*))\| \leq \max_i(\|\overline{L}_i\| + \left\|\frac{\partial \phi_1(x_2^e, \cdot)}{\partial p}\right\|_\infty \|\underline{L}_i\|)$, it appears that equation (22) is easily achieved and the proof follows the same steps as in Theorem 4.

The conditions on ϕ in Assumption 1 can be relaxed as follows.

Assumption 4 *We assume that the application $\phi = (\phi_1, \phi_2)$ in Assumption 1 is no more valid for all p but only for all p in a closed convex domain, denoted $\mathcal{P}(x_2, \xi)$ function of x_2 and ξ. We also assume that the nominal value of the parameter p belongs to $\mathring{\mathcal{P}}(x_2^e, \xi)$ (where $\mathring{\mathcal{P}}$ stands for the interior of $\mathcal{P}$) and $\lambda = \phi_2(x_2^e, p) \in \Lambda^{\mathcal{H}}(\xi)$.*

Proposition 6 *Under assumption 4, Theorem 4 and Theorem 5 hold true if the estimated values $(\hat{x}_1^e, \hat{\lambda}^e)$ are such that*

$$\begin{aligned} \hat{x}_1^e &= \phi_1(x_2^e, \Pi\hat{p}) \\ \hat{\lambda}^e &= \phi_2(x_2^e, \Pi\hat{p}) \end{aligned}$$

where $\Pi\hat{p}$ is the orthogonal projection of $\hat{p}$ on $\mathcal{P}(x_2^e, \xi)$.

PROOF. As $\mathcal{P}(x_2^e, \xi)$ is closed and convex, the orthogonal projection $\Pi\hat{p}$ of $\hat{p}$ on $\mathcal{P}(x_2^e, \xi)$ is continuous. We consider the function ϕ and we use the projection Π such that $\hat{p} \mapsto \phi_i(x_2^e, \Pi\hat{p})$, $i = 1, 2$ for x_2^e fixed. As $\hat{\lambda}^e = \phi_2(x_2^e, \Pi\hat{p}) \in \Lambda^{\mathcal{H}}(\xi)$, we see that

$$\begin{aligned} \hat{x}_1^e &= \phi_1(x_2^e, \Pi\hat{p}) \\ &= -C_{x_1} A(\hat{\lambda}^e)^{-1}[b(\hat{\lambda}^e) + G(\hat{\lambda}^e)\Pi\hat{p}] \end{aligned}$$

where the matrix C_{x_1} is such that $x_1 = C_{x_1}(x_1, x_2)$. The last right term is well defined since $A(\hat{\lambda}^e)$ is invertible. Thus, the projection Π allows the computation of an estimated point $(\hat{x}_1^e, \hat{\lambda}^e)$ for all t. Of course, the obtained estimated operating point $\hat{x}^e = (\hat{x}_1^e, x_2^e)$ with a projection of $\hat{p}$ is not an equilibrium point of the system when $\hat{p} \notin \mathcal{P}(x_2^e, \xi)$.

As $p \in \mathring{\mathcal{P}}(x_2^e, \xi)$ where $\mathring{\mathcal{P}}$ stands for the interior of $\mathcal{P}$, the convergence to zero of the error e for any switching rule and the continuity of ϕ ensure that there exists a time $t_1(e_0)$ such that $\hat{\lambda}^e = \phi_2(x_2^e, \hat{p}) \in \Lambda^{\mathcal{H}}(\xi)$ and $\hat{p} \in \mathcal{P}(x_2^e, \xi)$ for all $t > t_1$ as soon as $\hat{p}$ is close enough to p. For all $t > t_1$, $\Pi\hat{p} = \hat{p}$ and one concludes the proof following similar steps as in the proof of Theoreom 4 and Theorem 5.

4 Application to DC-DC power converters

The theoretical approach we propose is applied to a DC-DC power converter subject to load and input variations. The so-called Flyback converter is depicted in Fig. 1 and it is composed of passive components (a resistor R, an inductor L, a capacitor C), a transformer and two types of switches: a controlled switch (transistor S) and an uncontrolled switch (diode D). The behavior of the Flyback converter implies switching among three different modes summarized in Table 1 and the operating domain of the converter is the first orthant. The Mode $S = 1$

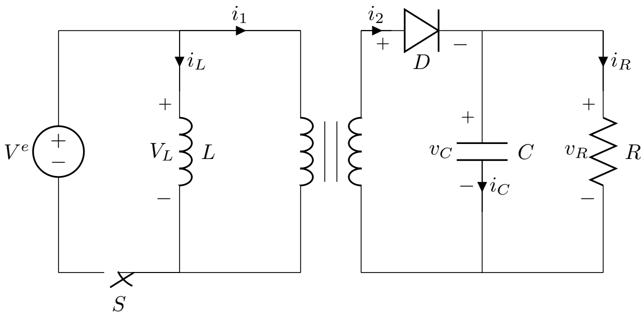


Fig. 1. A Flyback converter

Table 1
Possible mode of functioning

Mode	S	D
1	1	0
2	0	1
3	0	0

and $D = 1$ does not appear for physical reasons. Indeed, if $D = 1$, the current i_2 passing through the diode is positive and this means that the current i_1 is negative. As $S = 1$ means that the current i_1 is positive this means that it is impossible to have both $S = 1$ and $D = 1$. The switching rules between the three modes are summarized in Fig.2. If only Mode 1 and Mode 2 are active

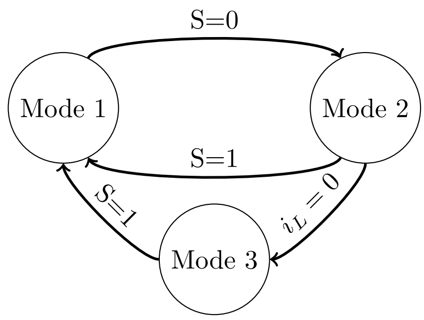


Fig. 2. Flyback modes and switching rules

during the cycle, the converter works in CCM (Continuous Conduction Mode). If Mode 3 is active, the system works in DCM (Discontinuous Conduction Mode) and this means that all the energy stored during Mode 1 in the inductor is transferred to the capacitor during Mode 2.

We recall that only Mode 1 and Mode 2 can be directly controlled. It is possible to have Mode 3 activated for physical reasons but its activation cannot be controlled.

The design of a stabilizing switching control law that takes into account these three modes is very challenging. We first explain how the approach we propose in this paper can be applied when the converter works in CCM. The DCM phase is discussed at the end of this section.

The Flyback converter in the CCM mode is a switched affine system given by:

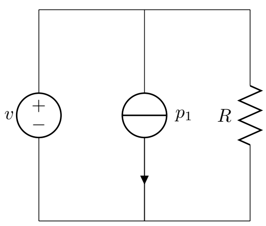
$$\dot{x} = A_\sigma x + B_\sigma u,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input considered as fixed ($u(t) = V^e$) and cannot be used as a control variable. Therefore, denoting $b_\sigma = B_\sigma u$ allows to use the framework developed in this paper. The control is the switching rule $\sigma : \mathbb{R}^n \to \mathbb{K}$ which indicates the active Mode at each time instant. The state vector $x^T = [i_L, v_C]$ is composed by i_L (the inductor current) and v_C (the capacitor voltage). Based on Fig.1 and using Kirchhoff's laws, the sets $\mathcal{A} = \{A_1, \cdots, A_N\}$ and $\mathcal{B} = \{B_1, \cdots, B_N\}$ with $N = 2$ are given by:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\frac{n_t}{L} \\ \frac{n_t}{C} & \frac{-1}{RC} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The interest of considering unknown parameters appears when robustness issues are formulated. For example, a lack of measurement on the input voltage and the output load can be represented by two parameters that can be linked to the resistor R and the voltage u as shown in Fig. 3 and 4. The voltage parameter p_2 allows to identify the input error directly as explained in (25), and the current parameter p_1 allows to identify the output load as explained in (24).

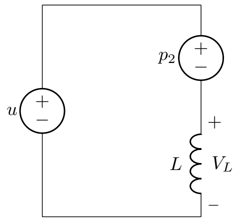


$$\begin{aligned} i = \tfrac{v}{R} &= p_1 + \tfrac{v}{R} \\ \hat{R} &= \frac{1}{\frac{1}{R} + \frac{p_1}{v}} \end{aligned} \tag{24}$$

Fig. 3. Parameter p_1 to estimate $\hat{R}$

To model the lack of measurement on the input voltage and the output load, the parameters given by (24) and (25) are virtually added to the Flyback converter as shown in Fig.5. This modified converter is a parameter dependent switched affine system given by:

$$\dot{x} = A_\sigma x + B_\sigma u + G_\sigma p,$$



$$\begin{aligned} \dot{x}_1 = \tfrac{v_L}{L} &= \tfrac{u - p_2}{L} \\ \hat{u} &= u - p_2 \end{aligned} \tag{25}$$

Fig. 4. Parameter p_2 to estimate $\hat{u}$

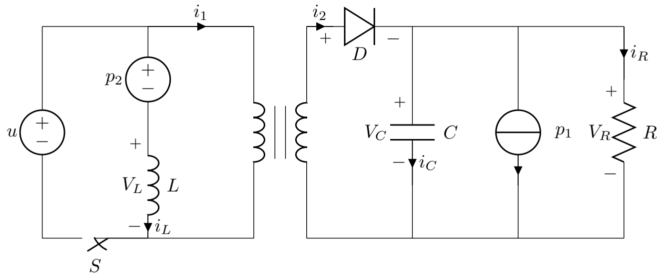


Fig. 5. Flyback converter with parameters estimation

with p the vector of unknown parameters and

$$G_1 = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{-1}{C} & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{-1}{C} & 0 \end{bmatrix}.$$

The application of the approach proposed in this paper and in particular the use of the observer allows to estimate the unknown parameters and gives a key information related to the steady state point which is not known a priori. This is clearly of practical interest and relevant for DC-DC power converters when the load or the input voltage are subject to variations.

4.1 Simulation results in CCM

The Flyback converter described by Fig. 5 depends on five parameters that, for simulation purpose, have been considered with the following nominal values : $V_e = 28V$, $R = 75\Omega$, $L = 200\mu H$, $C = 2.6\mu F$ and $n_t = 2$. Let the target point be $x_2^e = 15V$. We first present the simulation results in the case where the Flyback converter works with its nominal values to make a comparison between the switched based control σ of Theorem 4 and the embedded based control λ of Theorem 5. With $\alpha_1 = 0$, $\alpha_2 = 772$ the solution of the LMI conditions (17) is:
$P = \begin{pmatrix} 3.2170 & -0.0032 \\ -0.0032 & 0.0418 \end{pmatrix}$.
The additional parameter k_i in the expression $\lambda_i = \lambda_i^e - k_i y_i$ can be used to meet some performance requirements. Fig. 6 shows the transient for three values of this parameter $k_1 = 3.10^{-7}$, 1.10^{-6}, 7.10^{-6} and also the transient corresponding to the switched case. These simulations are obtained using a control λ and a Pulse-width modulation with a frequency $f_s = 10\,MHz$.

The switched control is simulated with a sampling time $T_e = \frac{1}{2f_s}$ in order to have for both σ and λ a similar number of switchings.

Compared to the switched based control, a better transient can be obtained using the control λ and an appropriate choice of k_1. The ability to tune the smoothness of the embedded control using the gain k_i is clearly of practical interest to meet performance requirements and not only stability properties. One can notice that large values for k_i lead to a saturation of the control and tends to recover the switched case. Conversely, very small values correspond to an open loop control $\lambda(t) \approx \lambda^e$ for all t. Obviously, the later is not suitable. To ease the presentation and comparison with the adaptive strategy, we decided to keep in the sequel only the simulation results of the embedded control λ with $k_1 = 1.10^{-6}$.

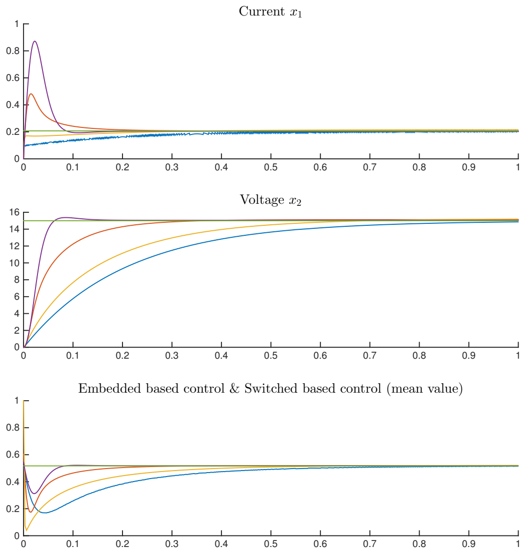


Fig. 6. Start-up transient: Plot 1 shows the current x_1, Plot 2 shows the voltage x_2. Plot 3 shows the control (in average for the switched based control case). Switched based control (Blue); Embedded based control (Purple, $k_1 = 3.10^{-7}$)(Red, $k_1 = 1.10^{-6}$)(Yellow, $k_1 = 7.10^{-6}$). Time is given in (ms).

Now in order to show the effectiveness of the proposed adaptive design, we assume that R and V_e may change and we do not measure these parameters. We use as a scenario a piecewise constant function with variations between 50Ω and 100Ω for the load, and variations between $20V$ and $40V$ for the input. First, let us show that all the required assumptions are satisfied. Assumption 2 is obvious. To show that Assumptions 4 is also satisfied, notice that for a given x_2^e and p, the functions ϕ_i, $i = 1, 2$ are provided by the relations:

$$x_1^e = \phi_1(x_2^e, p) = \frac{(x_2^e + Rp_1)(V_e + nx_2^e)}{nR(p_2 - V_e)}$$

and

$$\lambda^e = \phi_2(x_2^e, p) = \begin{cases} \lambda_1^e = \frac{p_2 + nx_2^e}{V_e + nx_2^e} \\ \lambda_2^e = 1 - \lambda_1^e \end{cases} \tag{26}$$

Note that the above relations for ϕ_i, $i = 1, 2$ are deduced from equation $0 = A(\lambda^e)x^e + b(\lambda^e) + G(\lambda^e)p$ assuming the pair (x_1^e, λ^e) as the unknowns and (p, x_2^e) are fixed. For a given $\xi > 0$, the set $\Lambda^{\mathcal{H}}(\xi)$ is determined by

$$\begin{cases} \lambda_1 \in [0 \; 1 - \xi] \\ \lambda_2 = 1 - \lambda_1 \end{cases}$$

since only A_1 is not Hurwitz. Then from (26), it is simple to check that if $\mathcal{P}(x_2^e, \xi)$ is defined by:

$$\begin{aligned} \mathcal{P}(x_2^e, \xi) = \{ & p \in \mathbb{R}^2 : |p_1| \leq p_{1_{max}}, \\ & - nx_2^e \leq p_2 \leq (1 - \xi)V_e - \xi n x_2^e \} \end{aligned}$$

where $p_{1_{max}}$ is chosen arbitrarily large, then $\lambda^e \in \Lambda^{\mathcal{H}}(\xi)$ and x_1^e is well defined for this set $\mathcal{P}(x_2^e, \xi)$. Moreover, as $\frac{\partial \phi_1}{\partial p}$ is given by:

$$\frac{\partial \phi_1}{\partial p} = \begin{bmatrix} \frac{V_e + nx_2^e}{n(p_2 - V_e)} \\ -\frac{(x_2^e + Rp_1)(V + nx_2^e)}{nR(p_2 - V_e)^2} \end{bmatrix},$$

it is well defined on the set $\mathcal{P}(x_2^e, \xi)$ and (10) is clearly satisfied by compacity of $\mathcal{P}(x_2^e, \xi)$. Hence, Assumption 4 is satisfied for our application.

We solve the LMI conditions of Proposition 3 and we obtain the following gains L_i with $\gamma_1 = \gamma_2 = 3.3 \times 10^4$:

$$L_i = 10^3 \begin{pmatrix} 362 & 20 \\ 20 & 1589 \\ 243 & -3152 \\ -3152 & -243 \end{pmatrix}, \; i \in \mathbb{K}.$$

Applying the adaptive control of Theorem 5, using Assumption 4 instead of Assumption 1 and thanks to Proposition 6, we obtain the results depicted in Fig.7.

We start the scenario of the simulation with the nominal values (ie $V_e = u = 28V$ and $R = 75\Omega$) to see the behavior of the closed loop system in the case where all the parameters are perfectly known. The variations of R and V_e are introduced after 0.5 ms. From 0 to 0.5ms, the reference x_2^e is reached for both the adaptive control

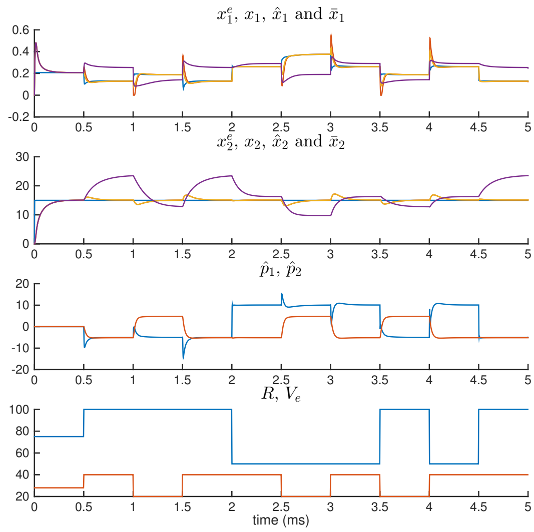


Fig. 7. Plot 1 shows respectively the current reference (Blue), the current x_1(Yellow) when the adaptive embedded control λ is used, its estimation $\hat{x}_1$ (Red) and the current $\bar{x}_1$(Purple) when only the embedded based control λ is used without load and input estimation. Time is given in (ms). Plot 2 shows the voltage x_2 using the same convention. Plot 3 shows the variation of the estimated value $\hat{p}_1$ (respectively $\hat{p}_2$) of parameter p_1 (p_2). Finally, Plot 4 shows the load R and input V_e variations.

(Theorem 5) and the non adaptive one (the embedded control λ with $k_1 = 1.10^{-6}$). After 0.5 ms, stability is preserved by these control laws in the presence of load and input voltage variations. However, a huge steady state error appears when using the non adaptive law. Using the control proposed in Theorem 5, the parameter $\hat{p}$ converges to a new value each time R or V_e changes and these changes have limited effect on the controlled output x_2. The adaptive embedded control λ rejects nicely the perturbations induced by the parameter variations and allows the controlled output x_2 to follow its reference while the current x_1 adapts its value to support these variations.

4.2 DCM mode case

The proof of global asymptotic stability of the proposed adaptive control laws (see Theorem 4 or Theorem 5) assumes that the DC-DC converter does not operate in the DCM mode (The uncontrolled Mode 3 is never activated). This means that the control law is established with $\mathbb{K}$ restricted to the set $\{1, 2\}$ instead of $\{1, 2, 3\}$. To prove stability when the converter enters in DCM mode, one has to take into account the fact that the uncontrolled mode (Mode 3) may appear when $x_1 = 0$, eventually many times, in the switching sequence. This mode is characterized by:

$$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 0 \\ \frac{-1}{C} & 0 \end{bmatrix}$$

Denote the derivative of $V(x)$ in the direction d by:

$$\dot{V}(x; d) = 2(x - x^e)^T P d.$$

and by $f_i(x) = A_i x + B_i V_e + G_i p$, $i = 1, 2, 3$.

For the case of switched based control as the one given by (3), we refer to two studies in the literature taking into account the DCM mode. The first one is discussed in (Beneux et al. 2017*b*) where we prove that Mode 3 may only occur along the line $\Delta := \{x : x_1 = 0\}$ when $\dot{V}(x; f_3(x)) < 0$ which preserves the stability property. The second study in presented in (Theunisse et al. 2015) where it is shown that a sufficient condition for the stability in DCM can be obtained by defining a Krasovskii regularization $\mathcal{F}_1(x) = \{f_1(x)\}$ and $\mathcal{F}_2(x) = co\{f_2(x), f_3(x)\}$ and fulfilling the following relation:

$$\min_{i=1,2}\Big(\max_{\xi \in \mathcal{F}_i(x)} \dot{V}(x; \xi)\Big) < 0 \qquad (27)$$

along the line Δ.

Here, we focus on the embedded based control of Theorem 5 and we show that stability property is also preserved when the DCM mode occurs. To this end, we first recall the following result.

Proposition 7 *When $x \in \Delta$, the vector field $f_3(x)$ satisfies:*

$$\begin{cases} f_3(x) = \gamma f_1(x) + (1 - \gamma) f_2(x) \\ \text{with } \gamma(x_2) = \frac{n x_2 + p_2}{n x_2 + V_e} \end{cases} \qquad (28)$$

where $f_i(x) = A_i x + B_i u + G_i p$, $i = 1, 2, 3$. Moreover, the function γ is a monotonically increasing function of x_2 and upper bounded by the value 1.

PROOF. The relation is easy to check, see (Beneux et al. 2017*b*).

Let $\Lambda^+(x) = \{\lambda \in \Lambda : [1\ 0] f(x, \lambda) \geq 0 \text{ with } x \in \Delta\}$, where $f(x, \lambda) = \lambda_1 f_1(x) + \lambda_2 f_2(x)$. This set defines the set of admissible controls along Δ (any $\lambda \notin \Lambda^+(x)$ is such that $\dot{x}_1 < 0$ and leads to $x_1 < 0$ which is not allowed by the diode). It is simple to show using Proposition 7 that the set $\Lambda^+(x)$ is characterized by:

$$\begin{cases} \lambda_1 \in [\max(0, \gamma(x_2))\ 1] \\ \lambda_2 = 1 - \lambda_1 \end{cases}$$

We are now in position to state the following result.

Proposition 8 *The embedded control preserves global asymptotic stability when Mode 3 is activated.*

PROOF. If $\gamma(x_2) \geq 0$ then, by linearity of the derivative and using (28), the relation (27) reduces to:

$$\min_{i=1,2}\big(\max_{\xi \in \mathcal{F}_i(x)} \dot{V}(x;\xi)\big) = \min_{i=1,3}\big(\dot{V}(x; f_i(x))\big) < 0 \quad (29)$$

and if $\gamma(x_2) < 0$,

$$\min_{i=1,2}\big(\max_{\xi \in \mathcal{F}_i(x)} \dot{V}(x;\xi)\big) = \min_{i=1,2}\big(\dot{V}(x; f_i(x))\big) < 0. \quad (30)$$

Note that the above relations are always satisfied: Eq. (29) as proved in (Beneux et al. 2017*b*) since if $\dot{V}(x; f_3(x)) \geq 0$ then $\dot{V}(x; f_1(x)) < 0$ and Eq. (30) by definition of V (it involves only modes 1 and 2).

Denote by η the embedded control ($\eta = sat(\lambda)$ obtained from (6)). Following the value of $\gamma(x_2)$, we have two cases.

- If $\gamma(x_2) < 0$, the control domain is not modified since $\Lambda^+(x) = \Lambda(x)$ and the diode cannot be blocking when applying the embedded control λ. Thus, Mode 3 will not occur and by Theorem 5, $\dot{V}(x; f_\eta(x)) < 0$ where $f_\eta(x) = \sum_{i=1,2} \eta_i f_i(x)$.
- If $\gamma(x_2) \geq 0$, Mode 3 occurs only if $\eta \notin \Lambda^+(x)$, or equivalently, only if the first component of η satisfies

$$0 \leq \eta_1 < \gamma(x_2) \ (< 1)$$

Therefore, if Mode 3 occurs, two cases must be distinguished from the relation (29):

a) If $S = 1$ where $S = \arg\min_{i=1,3}(\dot{V}(x; f_i(x)))$, we have clearly

$$\dot{V}(x; f_1(x)) \leq \dot{V}(x; f_3(x)) \leq \dot{V}(x; f_\eta(x)) < 0.$$

The inequalities are obtained by linearity of the derivative since $1 \geq \gamma(x_2) \geq \eta_1$ and using (28). The strict inequality is obtained from the global stability property of the control η.

b) If $S = 3$ then the relation (29) implies $\dot{V}(x; f_3(x)) < 0$.

As a conclusion, in both cases $\dot{V}(x; f_3(x)) < 0$ and the embedded control strategy of Theorem 5 preserves stability when the DCM occurs.

Finally, we illustrate this discussion in Figure 8 with a step variation leading to a transient in DCM mode. As expected, Mode 3 appears when the current x_1 vanishes, and stability property is preserved.

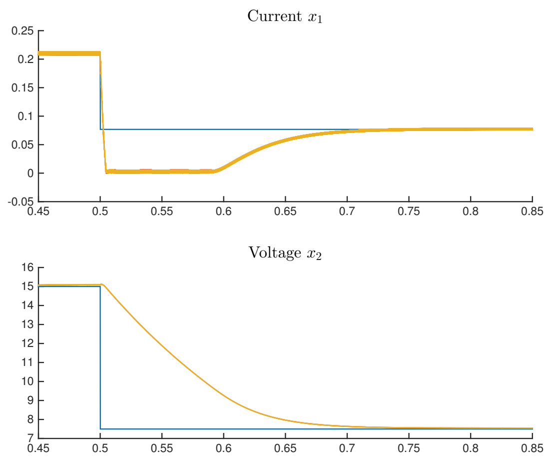


Fig. 8. DCM transient. Plot 1 shows the current x_1 (yellow), Plot 2 shows the voltage x_2 (yellow) when a step variation of the reference (blue) leads to DCM mode.

5 Conclusion

In this paper, it has been shown that an adaptive and switched-embedded based control can be designed to stabilize switched affine systems with unknown parameters. The results have been applied to a DC/DC Flyback converter where some of the parameters are not known (in practice, the load and/or the input voltage). The proposed simulations confirm the interest of our approach. Moreover, it has also been shown that stability of the adaptive and switched control is preserved when the discontinuous conduction mode occurs.

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