On Kalman filtering with linear state equality constraints [1]

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Abstract

This article is concerned with the state estimation problem for linear systems with linear state equality constraints. We re-examine constrained Kalman filter variations and propose an alternative derivation of the optimal constrained Kalman filter for time variant systems. This results in an oblique state projection that gives the smallest error covariance. A simple example illustrates the performance of the different Kalman filters.

1 Introduction

The inclusion of the constraint information should result in an improved estimate and a smaller error-covariance matrix (Marzetta, 1993). One way to include the additional information is to reduce the system model, and use the reduced state and the conventional KF (Simon, 2010). However, this approach may be difficult to implement, and may hinder insights into the original unconstrained problem (Stoica and Ng, 1998).
For a conventional time-invariant linear stochastic model with additive white process noise and linear equality constraints the process noise must have a singular covariance matrix in order to be consistent with the linear constraints on the state (Ko and Bitmead, 2007). This realization leads to a modification of the initial estimation error covariance and the process noise covariance. Thereafter, the conventional KF can be used. This approach is called the *system projection* approach (spKF) (Simon, 2010) [2].
Another way to include equality constraints into the state estimate was presented by Simon and Chia (2002). They use the unconstrained KF and project the results onto the constraints subspace. This approach is called *estimate projection* KF (epKF) (Simon, 2010).
A generalization of the epKF was presented by Teixeira et al. (2009), where they used, in contrast to Simon and Chia (2002), the projected state and error covariance estimates in the recursion. They called this *equality constrained* KF (ecKF).
It was proven that the state error covariance of these projection approaches is smaller than that of the unconstrained estimate (Simon and Chia, 2002; Ko and Bitmead, 2007). Teixeira et al. (2009) compared all three constrained KF numerically. All equality constrained methods produced similar results in their examples concerning their performance measures. However, the epKF produces less accurate and informative forecasts. The main reason that the epKF is outperformed by the two others is that the projected state and covariance matrix are not fed back into the recursion.
The contribution of this article is to show that the Cramér-Rao lower bound for the whole state history of a constrained systems can be calculated based on an oblique projection. In fact, the smallest error covariance can be calculated recursively in the same way as proposed by Teixeira et al. (2009) also for time-variant systems showing that the ecKF is optimal considering the whole state history. This results in an alternative derivation of the ecKF.

2 Four Kalman filter variants

Consider the discrete time-variant system given by

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k, \tag{1a}$$

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[2] In the original paper by Ko and Bitmead (2007) it is called *constrained Kalman filter*. We, however, will call it *system projection Kalman filter* as in Simon (2010) to avoid confusion with the *equality constrained Kalman filter* presented later.

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{v}_k, \tag{1b}$$

where k is the time index, and $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{y}_k \in \mathbb{R}^p$ represent the state and measurement vectors, respectively. The state $\mathbf{x}_k$ is known to be constrained such that

$$\mathbf{D}_k \mathbf{x}_{k+1} = \mathbf{0}. \tag{2}$$

The vectors $\mathbf{w}_k \in \mathbb{R}^n$ and $\mathbf{v}_k \in \mathbb{R}^p$ are mutually independent white processes with the covariances $\mathbf{Q}_k^e \in \mathrm{I\!R}^{\mathrm{n}\times\mathrm{n}}$ and $\mathbf{R}_k \in \mathrm{I\!R}^{\mathrm{p}\times\mathrm{p}}$. Furthermore, it is assumed that the initial state $\mathbf{x}_0$ has a known pdf $p(\mathbf{x}_0)$.
The matrix $\mathbf{D}_k \in \mathrm{I\!R}^{\mathrm{c}\times\mathrm{n}}$ (with $c < n$) is assumed to have full row rank. Moreover, it is assumed without loss of generality that the rows of $\mathbf{D}_k$ are unit vectors. The constraint (2) implies that $\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k \in \mathcal{N}(\mathbf{D}_k)$ (Ko and Bitmead, 2007). It is assumed that the noise $\mathbf{w}_k$ is uncorrelated with the input or state and

$$(\mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k,\ \mathbf{w}_k) \in \mathcal{N}(\mathbf{D}_k). \tag{3}$$

Hence, the model is suitable for modeling either physical or design constraints (Ko and Bitmead, 2005). In the latter the system cannot maintain the state constraints without corrective action of the input as briefly discussed in the numerical example in section 4.
In sections 2.1 – 2.3 we re-examine constrained Kalman filter variations from the literature, for time-invariant systems as in the sources. In Section 2.4 we provide a new derivation of the ecKF, also valid for time-variant systems.

2.1 The unconstrained Kalman filter

For completeness, the equations of the unconstrained Kalman predictor (KF), where $\mathbf{V}_k^{u,p,c,e} = (\mathbf{C}_k \mathbf{\Sigma}_k^{u,p,c,e} \mathbf{C}_k^T + \mathbf{R})^{-1}$, are [3]

$$\begin{aligned}
\mathbf{K}_k^u &= \mathbf{\Sigma}_k^u \mathbf{C}_k^T \mathbf{V}_k^u, \\
\hat{\mathbf{x}}_{k+1}^u &= \mathbf{A}_k (\mathbf{I} - \mathbf{K}_k^u \mathbf{C}_k) \hat{\mathbf{x}}_k^u + \mathbf{B}_k \mathbf{u}_k + \mathbf{A}_k \mathbf{K}_k^u \mathbf{y}_k \\
\mathbf{\Sigma}_{k+1}^u &= \mathbf{A}_k \mathbf{\Sigma}_k^u \mathbf{A}_k^T - \mathbf{A}_k \mathbf{\Sigma}_k^u \mathbf{C}_k^T \mathbf{V}_k^u \mathbf{C}_k \mathbf{\Sigma}_k^u \mathbf{A}_k^T + \mathbf{Q}_k^u,
\end{aligned} \tag{4}$$

where $\mathbf{Q}_k^u$ is the unconstrained process noise covariance matrix.

2.2 The estimate projection Kalman filter

The epKF approach to the constrained filtering problem of linear time-invariant systems is to project the unconstrained estimate $\hat{\mathbf{x}}_k^u$ of the KF onto the constraint subspace (Simon and Chia, 2002). The constrained estimate can be found by solving

$$\min_{\tilde{\mathbf{x}}_k^p} (\tilde{\mathbf{x}}_k^p - \hat{\mathbf{x}}_k^u) \mathbf{W} (\tilde{\mathbf{x}}_k^p - \hat{\mathbf{x}}_k^u)^T \quad \text{subject to} \quad \mathbf{D}\tilde{\mathbf{x}}_k^p = \mathbf{0},$$

where $\tilde{\mathbf{x}}_k^p$ and $\mathbf{W}$ are the constrained estimate and a positive-definite matrix, respectively. The solution to this problem is

$$\tilde{\mathbf{x}}_k^p = \mathbf{P}_{\mathcal{N}(\mathbf{D})}^{\mathbf{W}} \hat{\mathbf{x}}_k^u,$$

where $\mathbf{P}_{\mathcal{N}(\mathbf{D})}^{\mathbf{W}} \equiv \mathbf{I} - \mathbf{W}^{-1}\mathbf{D}^T(\mathbf{D}\mathbf{W}^{-1}\mathbf{D}^T)^{-1}\mathbf{D}$, which in general is an oblique projection. The smallest projected error covariance $\mathbf{\Sigma}_k^p = \mathbf{P}_k^{\mathbf{W}} \mathbf{\Sigma}_k^u (\mathbf{P}_k^{\mathbf{W}})^T = \mathbf{P}_k^{\mathbf{W}} \mathbf{\Sigma}_k^u$ is obtained if we set $\mathbf{W} = (\mathbf{\Sigma}_k^u)^{-1}$ where $\mathbf{\Sigma}_k^u$ is the error covariance matrix of the unconstrained KF (Simon and Chia, 2002; Simon, 2010).
The complete epKF is therefore (4) combined with the projections

$$\begin{aligned}
\tilde{\mathbf{x}}_{k+1}^p &= \mathbf{P}_{\mathcal{N}(\mathbf{D})}^{\mathbf{W}} \hat{\mathbf{x}}_k^u, \\
\mathbf{\Sigma}_{k+1}^p &= \mathbf{P}_{\mathcal{N}(\mathbf{D})}^{\mathbf{W}} \mathbf{\Sigma}_k^u.
\end{aligned} \tag{5}$$

Remark 1 *The epKF does neither use the projected error covariance nor the projected state in the recursion.*

2.3 The system projection Kalman filter

The spKF approach was also derived for linear time-invariant systems and is based on the observation that the system can be projected onto the null space of $\mathbf{D}$. Let us denote the orthogonal basis of $\mathbf{D}$ by $\mathbf{U}$, which satisfies

$$\mathbf{D}\mathbf{U} = 0, \quad \mathbf{U}^T\mathbf{U} = \mathbf{I}. \tag{6}$$

The projected system is (Ko and Bitmead, 2007):

$$\mathbf{x}_{k+1} = \mathbf{P}_{\mathcal{N}(\mathbf{D})}(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k),$$

where $\mathbf{P}_{\mathcal{N}(\mathbf{D})} \equiv \mathbf{U}\mathbf{U}^T$ is the orthogonal projector onto the null space of $\mathbf{D}$.
The authors considered the case of a system with physical constraints where each component of the state equation is constrained in $\mathcal{N}(\mathbf{D})$. An important consequence is that the system matrix $\mathbf{A}$ satisfies the following properties

$$\mathbf{A}\mathbf{x}_k = \mathbf{P}\mathbf{A}\mathbf{x}_k = \mathbf{A}\mathbf{P}\mathbf{x}_k,$$

where $\mathbf{P}$ is any projection matrix onto the null space of $\mathbf{D}$. Moreover, by taking a conditional expectation for any given measurement $\mathbf{Y}_k = [\mathbf{y}_0^T, \mathbf{y}_1^T, \ldots, \mathbf{y}_k^T]^T$ the following can be obtained (Ko and Bitmead, 2007)

$$\mathbf{P}\mathbf{A}\ E\{\mathbf{x}_k|\mathbf{Y}_k\} = \mathbf{A}\mathbf{P}\ E\{\mathbf{x}_k|\mathbf{Y}_k\}.$$

It follows that

$$\mathbf{A}\mathbf{P}\mathbf{\Sigma} = \mathbf{P}\mathbf{A}\mathbf{\Sigma} = \mathbf{A}\mathbf{\Sigma}, \tag{7a}$$

$$\mathbf{A}\mathbf{P}\mathbf{\Sigma}\mathbf{P}^T\mathbf{A}^T = \mathbf{P}\mathbf{A}\mathbf{\Sigma}\mathbf{A}^T\mathbf{P}^T = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T, \tag{7b}$$

[3] The superscripts denote the unconstrained (u), estimate projection (p), system projection (c) and equality constrained Kalman filter (e).

where $\mathbf{\Sigma}$ is the error covariance matrix.
The spKF is given by (Ko and Bitmead, 2007)

$$\begin{aligned}
\mathbf{K}_k^c =& \mathbf{\Sigma}_k^c \mathbf{C}^T \mathbf{V}_k^c, && \text{(8a)}\\
\tilde{\mathbf{x}}_{k+1}^c =& \mathbf{P}_{\mathcal{N}(\mathbf{D})} \mathbf{A}(\mathbf{I} - \mathbf{K}_k^c \mathbf{C})\tilde{\mathbf{x}}_k^c + \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{B}\mathbf{u}_k + \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{A}\mathbf{K}_k^c \mathbf{y}_k \\
=& \mathbf{A}(\mathbf{I} - \mathbf{K}_k^c \mathbf{C})\tilde{\mathbf{x}}_k^c + \mathbf{B}\mathbf{u}_k + \mathbf{A}\mathbf{K}_k^c \mathbf{y}_k && \text{(8b)}\\
\mathbf{\Sigma}_{k+1}^c =& \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{A}\mathbf{\Sigma}_k^c \mathbf{A}\mathbf{P}_{\mathcal{N}(\mathbf{D})} + \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{Q}^u \mathbf{P}_{\mathcal{N}(\mathbf{D})} \\
&- \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{A}\mathbf{\Sigma}_k^c \mathbf{C}^T \mathbf{V}_k^c \mathbf{C}\mathbf{\Sigma}_k^c \mathbf{A}^T \mathbf{P}_{\mathcal{N}(\mathbf{D})} \\
=& \mathbf{A}\mathbf{\Sigma}_k^c \mathbf{A}^T + \mathbf{Q}^c - \mathbf{A}\mathbf{\Sigma}_k^c \mathbf{C}^T \mathbf{V}_k^c \mathbf{C}\mathbf{\Sigma}_k^c \mathbf{A}^T, && \text{(8c)}
\end{aligned}$$

where $\mathbf{Q}^c = \mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{Q}^u \mathbf{P}_{\mathcal{N}(\mathbf{D})}$, which is singular. We see that it is only necessary to modify the initial estimation error covariance and the process noise covariance. If $\mathbf{Q}^c$ is the true process noise covariance it follows that this method gives the optimal state estimate (Simon, 2010).

Remark 2 *It was shown in Ko and Bitmead (2007) that the error covariance matrix of the spKF is less than or equal to that obtained by the epKF. The main reason for this is that in the epKF the projected covariance is not used in the recursion. Consequently, only information about the constraints in the most recent step of the epKF is used.*

Remark 3 *In Chen (2010) a missing necessary assumption of Theorem 2 of Ko and Bitmead (2007) was pointed out. It was shown that the orthogonal projection* $\mathbf{P}_{\mathcal{N}(\mathbf{D})} \equiv \mathbf{U}\mathbf{U}^T$ *has to be replaced by the oblique projection* $\mathbf{P}_{\mathcal{N}(\mathbf{D})}^{(\mathbf{Q}^u)^{-1}} \equiv \mathbf{I} - \mathbf{Q}^u \mathbf{D}^T(\mathbf{D}\mathbf{Q}^u\mathbf{D}^T)^{-1}\mathbf{D}$ *derived by Simon and Chia (2002).*

2.4 The equality constrained Kalman Filter

In this section we provide a derivation of the optimal constrained Kalman filter for linear time-systems based on the constrained Cramér-Rao bound of the whole state history (Andersson et al., 2017). The proofs of the following Lemmas can be found in Andersson et al. (2017); Stoica and Ng (1998) and Khatri (1966).

Lemma 4 *Considering the complete state history* $\mathbf{X}_k = [\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_k^T]^T$ *and the inverse of the error covariance matrix* $\tilde{\mathbf{\Sigma}}^{-1} = [diag(\mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_k)]^{-1}$ *of the state history with* $\tilde{\mathbf{U}} = [diag(\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_k)]$ *as defined in (6). If* $\tilde{\mathbf{U}}^T\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}}$ *is non-singular, the constrained Cramér-Rao Bound is*

$$E\left\{(\mathbf{X} - \hat{\mathbf{X}})(\mathbf{X} - \hat{\mathbf{X}})^T\right\} \geq \tilde{\mathbf{U}}(\tilde{\mathbf{U}}^T\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}})^{-1}\tilde{\mathbf{U}}^T. \tag{9}$$

Remark 5 *Lemma 4 was derived in Stoica and Ng (1998) for the estimation of non-random parameters. In Andersson et al. (2017) it was extended to estimation of random parameters. The right-hand side of (9) is the greatest lower bound, which was obtained by solving a maximization problem. Interestingly, Lemma 4 also holds for nonlinear systems subject to linear equality constraints.*

Lemma 6 *If* $\tilde{\mathbf{\Sigma}}$ *is positive definite (9) becomes*

$$\tilde{\mathbf{U}}(\tilde{\mathbf{U}}^T\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{U}})^{-1}\tilde{\mathbf{U}}^T = \mathbf{P}_{\mathcal{N}(\tilde{\mathbf{D}})}^{\tilde{\mathbf{\Sigma}}^{-1}}\tilde{\mathbf{\Sigma}}, \tag{10}$$

where $\mathbf{P}_{\mathcal{N}(\tilde{\mathbf{D}})}^{\tilde{\mathbf{\Sigma}}^{-1}} = \mathbf{I} - \tilde{\mathbf{\Sigma}}\tilde{\mathbf{D}}^T(\tilde{\mathbf{D}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{D}}^T)^{-1}\tilde{\mathbf{D}}$.

The projection of the unconstrained error covariance in (10) is exactly the same as in the estimate projection approach using $\mathbf{W} = (\mathbf{\Sigma}_k^u)^{-1}$, but considers the whole state history.
In a recursion the information matrix $\mathbf{\Sigma}^{-1}$ can be computed by (Simon, 2010)

$$\mathbf{\Sigma}_{k+1|k}^{-1} = \mathbf{Q}_k^{-1} - \mathbf{Q}_k^{-1}\mathbf{A}_k\left(\mathbf{F}_k + \mathbf{K}_k + \mathbf{\Sigma}_{k|k}^{-1}\right)^{-1}\mathbf{A}_k^T\mathbf{Q}_k^{-1}, \tag{11}$$

where $\mathbf{K}_k = \mathbf{C}_k^T\mathbf{R}_k^{-1}\mathbf{C}_k$ and $\mathbf{F}_k = \mathbf{A}_k^T\mathbf{Q}_k^{-1}\mathbf{A}_k$.

Theorem 7 *The error covariance of the ecKF can be computed by*

$$\bar{\mathbf{\Sigma}}_{k+1}^e = \mathbf{A}_k\mathbf{\Sigma}_k^e\mathbf{A}_k^T - \mathbf{A}_k\mathbf{\Sigma}_k^e\mathbf{C}_k^T\mathbf{V}_k^e\mathbf{C}_k\mathbf{\Sigma}_k^e\mathbf{A}^T + \mathbf{Q}_k^u, \tag{12a}$$

$$\mathbf{\Sigma}_{k+1}^e = \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{(\bar{\mathbf{\Sigma}}_{\mathbf{k+1}}^{\mathbf{e}})^{-1}}\bar{\mathbf{\Sigma}}_{k+1}^e. \tag{12b}$$

PROOF. Since the constraints $\tilde{\mathbf{D}} = diag(\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_k)$ are decoupled in time, the matrix $\tilde{\mathbf{U}} \in \mathbb{R}^{\mathrm{kn}\times\mathrm{k(n-c)}}$ is block-diagonal with $\mathbf{U}_k$ on the diagonal. The inverse of the unconstrained error covariance matrix of the whole state history (9) is given by (Andersson et al., 2017)

$$\tilde{\mathbf{\Sigma}}_{k+1|k}^{-1} = \left[\begin{array}{c|c} \mathbf{J}_{k+1|k}^{1,1} & \mathbf{J}_{k+1|k}^{1,2} \\ \hline \mathbf{J}_{k+1|k}^{2,1} & \mathbf{J}_{k+1|k}^{2,2} \end{array}\right]$$
$$= \left[\begin{array}{cccc|c} \mathbf{K}_0+\mathbf{F}_0+\mathbf{\Sigma}_{0|0}^{-1} & -\mathbf{A}_0^T(\mathbf{Q}_0^u)^{-1} & & & \\ -(\mathbf{Q}_0^u)^{-1}\mathbf{A}_0 & \ddots & \ddots & & \\ & \ddots & \mathbf{K}_k+\mathbf{F}_k+(\mathbf{Q}_{k-1}^u)^{-1} & & -\mathbf{A}_k^T(\mathbf{Q}_k^u)^{-1} \\ \hline & & -(\mathbf{Q}_k^u)^{-1}\mathbf{A}_k & & (\mathbf{Q}_k^u)^{-1} \end{array}\right]. \tag{13}$$

We proceed by showing equivalence between (12) and the right-hand side of (9) for $\mathbf{x}_1$, which by the structure of $\tilde{\mathbf{\Sigma}}_{k+1|k}^{-1}$ and $\tilde{\mathbf{U}}$ implies that it also hold for $\mathbf{x}_k$.
If we compute (9) for $\mathbf{x}_1$ using the matrix inversion lemma we obtain

$$\mathbf{\Sigma}_1^e = \mathbf{U}_1\left[\mathbf{L}_1 - \mathbf{G}_{0,1}\left\{\mathbf{U}_0\left[\mathbf{U}_0^T\mathbf{J}_{1|0}^{1,1}\mathbf{U}_0\right]^{-1}\mathbf{U}_0^T\right\}\mathbf{G}_{0,1}^T\right]^{-1}\mathbf{U}_1^T \tag{14}$$

where $\mathbf{L}_1 = \mathbf{U}_1^T(\mathbf{Q}_1^u)^{-1}\mathbf{U}_1$, $\mathbf{G}_{0,1} = \mathbf{U}_1^T(\mathbf{Q}_0^u)^{-1}\mathbf{A}_0$ and $\mathbf{J}_{1|0}^{1,1} = \mathbf{K}_0 + \mathbf{F}_0 + \mathbf{\Sigma}_{0|0}^{-1}$. Using the binomial inverse theorem this expression can be transformed to

$$\mathbf{\Sigma}_1^e = \mathbf{U}_1\left[\mathbf{L}_0 - \mathbf{G}_{0,1}\left\{\left(\left[\mathbf{\Sigma}_0^e\right]^{-1} + \mathbf{F}_0\right)^{-1}\right\}\mathbf{G}_{0,1}^T\right]^{-1}\mathbf{U}_1^T, \tag{15}$$

where $\mathbf{\Sigma}_0^e = \mathbf{U}_0\left(\mathbf{U}_0^T[\mathbf{\Sigma}_{0|0}^{-1} + \mathbf{K}_0]\mathbf{U}_0\right)^{-1}\mathbf{U}_0^T$. This is exactly the expression obtained using in (12) the left-hand side of (10) with the information matrix (11) as projection. □

The recursion (12) is the same as obtained for the ecKF by Teixeira et al. (2009). The proof establishes a direct connection between the error covariance matrix to the Fisher Information matrix and Cramér-Rao lower bound. It shows that, in fact, the ecKF is optimal also considering the whole state trajectory for time-variant systems. Moreover, the necessity to guarantee numerically positive definiteness of (12) is avoided in the derivation.

Remark 8 *For a time-invariant system only the process noise covariance matrix is projected. Consequently, the projection can be performed a priori.*

Remark 9 *In a similar fashion as in the proof of Theorem 7 it can be shown that (9) can be computed recursively for nonlinear systems subject to linear equality constraints.*

The ecKF is given by

$$\begin{aligned}
\mathbf{K}_k^e &= \mathbf{\Sigma}_k^e \mathbf{C}_k^T \mathbf{V}_k^e, \\
\bar{\mathbf{x}}_{k+1}^e &= \mathbf{A}_k(\mathbf{I} - \mathbf{K}_k^e \mathbf{C}_k)\tilde{\mathbf{x}}_k^e + \mathbf{B}_k \mathbf{u}_k + \mathbf{A}_k \mathbf{K}_k^e \mathbf{y}_k, \\
\bar{\mathbf{\Sigma}}_{k+1}^e &= \mathbf{A}_k \mathbf{\Sigma}_k^e \mathbf{A}_k^T + \mathbf{Q}_k^u - \mathbf{A}_k \mathbf{\Sigma}_k^e \mathbf{C}_k^T \mathbf{V}_k^e \mathbf{C}_k \mathbf{\Sigma}_k^e \mathbf{A}_k^T, \\
\tilde{\mathbf{x}}_{k+1}^e &= \mathbf{P}_{\mathcal{N}(\mathbf{D}_k)}^{(\bar{\mathbf{\Sigma}}_{k+1}^e)^{-1}} \bar{\mathbf{x}}_{k+1}^e, \\
\mathbf{\Sigma}_{k+1}^e &= \mathbf{P}_{\mathcal{N}(\mathbf{D}_k)}^{(\bar{\mathbf{\Sigma}}_{k+1}^e)^{-1}} \bar{\mathbf{\Sigma}}_{k+1}^e,
\end{aligned} \tag{16}$$

where the last two steps are the projection steps due to the time-variant constraints. In the next section, we show that this results in a smaller covariance.

3 Comparison of constrained Kalman filters

None of the constrained Kalman filters violate the constraints for a time-invariant system (Simon, 2010; Ko and Bitmead, 2007; Teixeira et al., 2009). In fact, for time-invariant systems the spKF with the correction by Chen (2010) and the ecKF are identical. The reduction of the constrained error covariance in comparison to the unconstrained one can be established easily as well as the error covariance sequence of the different constrained KF.

Lemma 10 *The constrained error covariance $\mathbf{\Sigma}^e$ is less than or equal to the unconstrained one.*

PROOF. $\mathbf{\Sigma}$ and $\mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}}\mathbf{\Sigma}$ are symmetric and $\mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}}$ and $\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}}$ are idempotent. Therefore, the following holds (Gorman and Hero, 1990):

$$\begin{aligned}
\mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}}\mathbf{\Sigma} &= \mathbf{\Sigma} - (\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}})\mathbf{\Sigma} \\
&= \mathbf{\Sigma} - (\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}})(\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}})\mathbf{\Sigma} \\
&= \mathbf{\Sigma} - (\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}})\mathbf{\Sigma}(\mathbf{I} - \mathbf{P}_{\mathcal{N}(\mathbf{D_k})}^{\mathbf{\Sigma}^{-1}}) \leq \mathbf{\Sigma}
\end{aligned}$$

□

Theorem 11 *For the constrained time-invariant system the error covariance sequence is*

$$\mathbf{\Sigma}_k^u \geq \mathbf{\Sigma}_k^p \geq \mathbf{\Sigma}_k^c \geq \mathbf{\Sigma}_k^e.$$

PROOF. The first inequality was shown in Simon and Chia (2002) and by Lemma 10. The second inequality was shown in Ko and Bitmead (2007) and is true with the correction proposed by Chen (2010). The third inequality was also shown by Chen (2010) since his correction results in the ecKF for time-invariant systems. Moreover, the third inequality can also be derived following the argumentation in Simon and Chia (2002).

4 Numerical example

A simple numerical example is presented to illustrate the performance differences of the four estimators. The following benchmark model (Simon and Chia, 2002; Ko and Bitmead, 2007; Simon, 2010) is used but slightly changed here to make it time-variant. It is a navigation problem with the following linear system and measurement equation

$$\begin{aligned}
\mathbf{x}_{k+1} &= \begin{pmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\mathbf{x}_k + \begin{pmatrix} 0 \\ 0 \\ T\sin\theta_k \\ T\cos\theta_k \end{pmatrix} u_k + \mathbf{w}_k, \\
\mathbf{y}_k &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\mathbf{x}_k + \mathbf{v}_k,
\end{aligned}$$

where T is the discretization step size and u_k is the acceleration input. The states are the positions and velocities in north and east direction, respectively. The unconstrained covariances of process and measurement noise are

$$\mathbf{Q}^u = \mathrm{diag}(4, 4, 2, 1), \quad \mathbf{R} = \mathrm{diag}(900, 900)$$

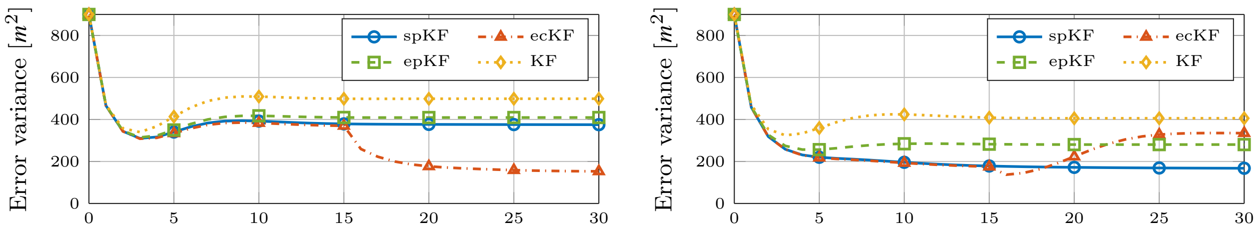
and the initial estimation error covariance is

$$\mathbf{P}_0^+ = \mathrm{diag}(900, 900, 4, 4).$$

It is known that the vehicle is on a road with a heading angle θ_k which can be described by the constraint equation

$$\mathbf{D}_k \mathbf{x}_{k+1} = \left[1 + \tan^2\theta_k\right]^{-1/2}\begin{pmatrix} 0 & 0 & 1 & -\tan\theta_k \end{pmatrix}\mathbf{x}_{k+1} = 0$$

At time point $k = n$ the road angle θ_k changes. The process has to fulfill (3) at time point $k = n - 1$ and $k = n$.



(a) Position - $\mathbf{x}_k(1)$

(b) Position - $\mathbf{x}_k(2)$

Fig. 1. Estimation error variance for the position of the system. At $k = 15$ the system constraint changes. The time-invariant spKF and epKF do not change their projections while the ecKF adapts to the new constraint.

This means that the process must be capable to reach the new null space $\mathcal{N}(\mathbf{D}_k)$ from the old null space $\mathcal{N}(\mathbf{D}_{k-1})$ in one time step.
Assuming that the system matrix A does not change, the input matrix B_k or the input u_k have to change at least twice. First, to transfer the process to the new null space and, secondly, to keep the process in the new null space. Considering that the process can only control the velocity directly, constraints on the position as, for example, used in Simon (2010) for a time-invariant system, will violate (3) at the moment of change for a time-variant system.
The ecKF does not violate the constraints at any time and adjusts the error covariance matrix according to the information change (Fig. 1). At first the change decreases the uncertainty in all state estimates.
The time-invariant constrained KF obviously violates the constraints at all times $k \geq n$ since the estimate is projected onto the wrong subspace. As important as the constraint violation and larger state estimation error is the wrong estimate of the error covariance which is not adjusted and indicates an incorrect confidence in the state estimate (Fig. 1).

5 Conclusion

In this paper linear state estimation with linear equality constraints is revisited. A simple derivation based on stochastic arguments of the covariance of the constrained Kalman filter for time-variant systems was presented. It was shown that the optimal oblique projection of the whole state history can be computed recursively resulting in the ecKF. A numerical example discussed briefly the implications that a change of constraints may have and compares the different Kalman filters.

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