Distributed Nash Equilibrium Seeking under Partial-Decision Information via the Alternating Direction Method of Multipliers

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Abstract

In this paper, we consider the problem of finding a Nash equilibrium in a multi-player game over generally connected networks. This model differs from a conventional setting in that players have partial information on the actions of their opponents and the communication graph is not necessarily the same as the players' cost dependency graph. We develop a relatively fast algorithm within the framework of inexact-ADMM, based on local information exchange between the players. We prove its convergence to Nash equilibrium for fixed step-sizes and analyze its convergence rate. Numerical simulations illustrate its benefits when compared to a consensus-based gradient type algorithm with diminishing step-sizes.

Key words: Game theory over networks, distributed algorithms, noncooperative games, communication graph.

1 Introduction

We consider distributed Nash equilibrium (NE) seeking in a setting where players have limited local information, over a communication network. In contrast to the classical setting where all players have access to all their opponents' decisions Scutari et al. (2014), Frihauf et al. (2012), Kannan and Shanbhag (2012), our interest lies in networked regimes where agents may only access or observe the decisions of their neighbours, and there is no central node that has bidirectional communication with all players. This is a topic of increasing recent interest given the proliferation of engineering networked applications requiring distributed protocols that operate under partial, local information (V2V, P2P, ad-hoc, smart-grid networks), and the deregulation of global markets. Applications that motivate us to consider NE seeking in such a setting range from spectrum access in cognitive radio networks, Wang et al. (2010), Cheng et al. (2014), Swenson et al. (2015), rate control and congestion games in wireless, vehicular or ad-hoc networks, Alpcan and Başar (2005), Yin et al. (2011), Tekin et al. (2012), to networked Nash-Cournot competition, Bimpikis et al. (2014), Koshal et al. (2016), and opinion dynamics in social networks, Ghaderi and Srikant (2014), Bimpikis et al. (2016). For example, in cognitive radio networks users adaptively adjust their operating parameters based on interactions with the environment and other users in the network, Wang et al. (2010). In a vehicular ad-hoc network setting, vehicles on the road often maintain a relatively stable topology and form clusters wherein vehicles can communicate one with another. A best-response algorithm for channel selection is proposed in Cheng et al. (2014), based on vehicles within a cluster communicating with each other, while selfishly trying to maximize their own utilities. A similar information exchange setting is considered in Swenson et al. (2015), where because players are unable to directly observe the actions of all others, they engage in local, non-strategic information exchange to eventually agree on a common Nash equilibrium. In an economic setting, Bramoullé et al. (2014) draws attention to the problem of "who interacts with whom" in a network and to the importance of communication with neighbouring players. An example is Nash-Cournot competition played over a network, between a set of firms that compete over a set of locations, Bimpikis et al. (2014). A communication network is formed between the firms, and prescribes how they communicate locally their production decision over those locations, Koshal et al. (2016). All these examples are non-cooperative in the way decisions are made (each agent is self-interested in minimizing only its own cost), but collaborative in information sharing (agents have the incentive to exchange information with their neighbours in order to mitigate the lack of global information on others' decisions). Unlike the classical setting, where all others' decisions are known and information exchange is not incentive compatible, in a partial-decision information setting (not usually considered in classical game theory), information exchange is motivated because of the limited data each player has. Such an information exchange setting has started to be considered very recently, Swenson et al. (2015), Koshal et al. (2016), Salehisadaghiani and Pavel (2016), but convergence is guaranteed only for diminishing step-sizes. *Motivated by the above, in this paper we develop a distributed NE seeking algorithm that operates under limited (partial) decision-information, and is guaranteed to converge to NE with constant step-sizes. Each player updates his decision and his estimate simultaneously in a single step, based on local information exchange with his neighbours.*

We use an ADMM approach to develop the algorithm. Originally developed in 1970s, the *Alternating Direction Method of Multipliers* (ADMM) method has become widely used in distributed optimization problems (DOP) after its re-introduction in Boyd et al. (2011), Wei and Ozdaglar (2012), Wei and Ozdaglar (2013), Shi et al. (2014), Chang et al. (2015), Hong et al. (2016). In a DOP, N agents cooperatively minimize a global objective, $f(y) := \sum_{i=1}^{N} f_i(y)$, over $y \in \Omega$, by communicating over a connected graph. The problem is reformulated by introducing a separate decision variable x_i for each agent i and imposing the equality constraint $x_i = x_j$ for all agents connected by an edge (neighbours), leading to

$$\begin{cases} \underset{x}{\text{minimize}} & f(x) := \sum_{i=1}^{N} f_i(x_i) \\ \text{subject to} & Ax = 0, \quad x_i \in \Omega \;\; \forall i = 1, \ldots, N \end{cases} \tag{1}$$

where $x = [x_i]_{i=1}^{N}$, A is the edge-node incidence matrix. ADMM for DOP relies on the additive structure and separability of the objective function (1) and on the linearity of the equality constraints. Each agent i will minimize its cost $f_i(x_i)$ over x_i and this f_i is independent (*decoupled*) of the other agents' x_j, $j \neq i$. Dual decomposition leads to N parallel decoupled dual ascent problems.

In this work, we exploit the benefits of ADMM in the context of finding a NE of a game, where each player (agent) i aims to minimize its own cost function J_i with respect to (w.r.t) its action x_i, given a profile of other players' actions, except himself, x_{-i},

$$\begin{cases} \underset{x_i}{\text{minimize}} & J_i(x_i, x_{-i}) \\ \text{subject to} & x_i \in \Omega_i \end{cases} \qquad \forall i = 1, \ldots, N. \tag{2}$$

We consider a networked information exchange setting as in the motivating examples, due to players' limited (partial) information on the others' decisions. There are several challenges when comparing a game to a DOP:

- A Nash game can be seen as a set of parallel *coupled* optimization problems, (2). Each player's cost is dependent on the other players' decisions, hence its decision is directly affected by these.
- Each player i updates only its own decision x_i, but he also requires an estimate of all others $[x_j]_{j=1,\ldots,N,\, j\neq i}$, or x_{-i}, in order to solve its optimization problem.

We introduce local estimates of players' actions and (virtual) constraints for their consensus. We reformulate the problem and relate it to a modified game (a set of optimization problems) with consensus constraints. As in classical ADMM we reduce the computational costs by using a linear approximation in players' action update rule (inexact-ADMM). However, direct application of ADMM is not possible because each minimization is not over the whole vector (estimate), but rather over part of it (action). Every player updates its *decision/action* as well as its *estimates* of the other players' actions by this synchronous, inexact ADMM-type algorithm, using its own estimates and those of its neighbours. We show that the algorithm converges to NE under fixed step-sizes, the first such algorithm to the best of our knowledge. A short version appears in Salehisadaghiani and Pavel (2017), Shi and Pavel (2017) under stronger assumptions.

Related Works. Distributed NE computation is relevant to many applications, such as sensor network coordination Stanković et al. (2012), flow control Alpcan and Başar (2005), optical networks, Pan and Pavel (2009). In the *classical setting* of distributed NE computation using best-response or gradient-based schemes, competitors' decisions are assumed to be observable by all players, Facchinei and Pang (2007), Frihauf et al. (2012). Recent work in the classical setting considered games with monotone pseudo-gradient Scutari et al. (2014); Kannan and Shanbhag (2012); Yousefian et al. (2016), Zhu and Frazzoli (2016), networked aggregative games with quadratic cost functions, Parise et al. (2015), or games with affine constraints, Yi and Pavel (2017, 2018). Players are assumed to have access to the others' decisions, so the issue of *partial information on the opponents' decisions* is not considered in the above works.

However, having access to all others' decisions can be impractical in distributed networks, Marden (2007), Li and Marden (2013). In recent years, there has been an increasing interest to consider this issue and how to deal with it via networked (partial)-information exchange. Our work is related to this literature. Consensus-based approaches have been proposed, by using fictitious play in congestion games, Swenson et al. (2015), or projected-gradient algorithms, either for the special class of *aggregative* games, Koshal et al. (2016), or for games where players' cost functions depend on others' actions in a general manner, in Salehisadaghiani and Pavel (2016), or are partially coupled, Salehisadaghiani and Pavel (2018). Convergence to the NE was shown only for diminishing step-sizes, under strict monotonicity of the pseudo-gradient. In contrast, our ADMM algorithm converges with fixed step-sizes and has faster convergence. We note that very recently continuous-time NE seeking dynamics are proposed based on consensus and gradient-type dynamics, for games with unconstrained action sets, Ye and Hu (2017), or based on passivity, Gadjov and Pavel (2018). Differently, we develop a discrete-time NE seeking algorithm for games with compact action sets, based on an ADMM approach.

Our work is related to the large literature on ADMM for DOP but there are several differences. In contrast to typical ADMM algorithms developed for DOP (1) which is separable, e.g. Wei and Ozdaglar (2012), Wei and Ozdaglar (2013), Shi et al. (2014), Chang et al. (2015), Hong et al. (2016), our ADMM algorithm for NE seeking of (2) has an extra step for estimate update, besides the update of each player's decision variable. This is because of the intrinsic coupling in J_i (2); the update of each player's action is coupled nonlinearly to this estimate and to the others' decisions. This is unlike ADMM for DOP where, due to decoupled costs, the update of each agent's decision is only linearly coupled to the others' decisions (via the constraints). This also leads to a technical difference on the convexity assumptions under which convergence is shown. Typical assumptions in DOP are individual (strict/strong) joint convexity of each agent's decoupled cost function f_i (1), Wei and Ozdaglar (2012), Wei and Ozdaglar (2013), Shi et al. (2014), Chang et al. (2015). In the augmented space, due to separability of DOP (1), monotonicity of the full gradient is automatically maintained. Different from that, in a game setup, individual joint convexity is too restrictive, unless the game is separable to start with, which is a trivial case. Rather, a typical assumption is individual *partial* convexity of each cost, and monotonicity (strict, strong) of the pseudo-gradient. Furthermore, in the augmented space of actions and estimates, due to the inherent coupling in (2), monotonicity is not necessarily maintained. We note that in ADMM for DOP (1), relaxation from strongly convex to merely convex f_i involves either an exact augmented minimization at each step, (hence f_i needs to have inexpensive proximal-operator), Wei and Ozdaglar (2013), or an extra quadratic augmentation, for a sufficiently large step-size, Hong et al. (2016). In ADMM for game (2), we use a similar idea as Hong et al. (2016) for the extended pseudo-gradient (which may be non-monotone in the augmented space). However, unlike Hong et al. (2016), since cost functions J_i are not separable, our analysis is novel, based on the decomposition of the augmented space into the consensus subspace and its orthogonal complement. Moreover, our proof techniques rely on the pseudo-gradient, since we cannot employ a common Lagrangian as in DOP. Another consequence is that, unlike Wei and Ozdaglar (2013), we are able to characterize the rate of convergence only in terms of the residuals relative to NE optimality.

The paper is organized as follows. The problem statement and assumptions are provided in Section 2. In Section 3, an inexact ADMM-type algorithm is developed. Its convergence to Nash equilibrium is analyzed in Section 4. Simulation results are given in Section 5 and conclusions in Section 6.

1.1 Notations and Background

A vector $x \in \mathbb{R}^n$ may be represented in equivalent ways as $x = [x_1, \ldots, x_n]^T = [x_1; \ldots; x_n]$, or $x = [x_i]_{i=1,\ldots,n}$, or $x = (x_i, x_{-i})$. All vectors are assumed to be column vectors. Given a vector $x \in \mathbb{R}^n$, x^T denotes its transpose and $\|x\| = \sqrt{x^T x}$ denotes its Euclidean norm. Given a symmetric $n \times n$ matrix A, $\|x\|_A$ denotes the weighted norm $\|x\|_A := \sqrt{x^T A x}$. Denote $\mathbf{1}_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$ and $\mathbf{0}_n = [0, 0, \ldots, 0]^T \in \mathbb{R}^n$. Let e_i denote the $N \times 1$, i-th unit vector in $\mathbb{R}^n$, whose i-th element is 1 and the rest are 0. Given a vector $x \in \mathbb{R}^n$, $diag([x_i]_{i=1,\ldots,n})$ denotes the $n \times n$ diagonal matrix with $x_1, \ldots, x_n$ on the diagonal. Similarly, given A_i, $i = 1, \ldots, n$ as $(p \times q)$ matrices, $diag([A_i]_{i=1,\ldots,n})$ denotes the $(np \times nq)$ block-diagonal matrix with A_i on the diagonal. I_n denotes the identity matrix in $\mathbb{R}^{n \times n}$. The Kronecker product of matrices A and B is denoted by $A \otimes B$. Given matrices $A, B \in \mathbb{R}^{n \times n}$, $A \succ 0$ ($A \succeq 0$) denotes that A is positive (semi-)definite. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and the maximum eigenvalue of matrix A, respectively. Given a $m \times n$ matrix A, $\|A\|_2$ denotes its induced 2-norm, $\|A\|_2 = \sigma_{\max}(A)$, $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)}$. For any $a, b \in \mathbb{R}^n$ and $\rho > 0$,

$$-\frac{1}{2\rho}\|a\|^2 - \frac{\rho}{2}\|b\|^2 \le a^T b \le \frac{1}{2\rho}\|a\|^2 + \frac{\rho}{2}\|b\|^2. \quad (3)$$

For every $a, b, c \in \mathbb{R}^n$ and $n \times n$ matrix $A \succeq 0$,

$$(a-b)^T A(a-c) = \frac{1}{2}\|a-c\|_A^2 + \frac{1}{2}\|a-b\|_A^2 - \frac{1}{2}\|b-c\|_A^2. \quad (4)$$

Given a set $\Omega \in \mathbb{R}^n$, $|\Omega|$ denotes the cardinality of Ω. The Euclidean projection of $x \in \mathbb{R}^n$ onto $\Omega \subset \mathbb{R}^n$ is denoted by $T_\Omega\{x\}$. Denote by prox_g^a the proximal operator for function g with a constant a, defined as follows:

$$\mathrm{prox}_g^a\{s\} := \arg\min_x \{g(x) + \frac{a}{2}\|x - s\|^2\}. \quad (5)$$

Let $\mathcal{I}_\Omega(x) := \begin{cases} 0 & \text{if } x \in \Omega \\ \infty & \text{otherwise} \end{cases}$, be the indicator function of a closed convex set $\Omega \in \mathbb{R}^n$. Then, $\mathrm{prox}_{\mathcal{I}_\Omega}^a\{\cdot\} = T_\Omega\{\cdot\}$. Let $\partial\mathcal{I}_\Omega(x)$ denote the subdifferential of $\mathcal{I}_\Omega$ at x, i.e., the set of all subgradients of $\mathcal{I}_\Omega$ at x . Then $\partial\mathcal{I}_\Omega(x)$ is a convex set and $\partial\mathcal{I}_\Omega(x) = N_\Omega(x)$, where $N_\Omega(x) = \{y \in \mathbb{R}^n | \, y^T(x' - x) \le 0, \forall x' \in \Omega\}$ is the normal cone to Ω at x. Moreover, $(y_1 - y_2)^T(x_1 - x_2) \ge 0$, $\forall x_1, x_2 \in \Omega$, $\forall y_1 \in \partial\mathcal{I}_\Omega(x_1)$, $\forall y_2 \in \partial\mathcal{I}_\Omega(x_2)$, Rockafellar (1970).

For an undirected graph $G(V, E)$, we denote by:

- V: Set of vertices in G,
- $E \subseteq V \times V$: Set of all edges in G. $(i, j) \in E$ if and only if i and j are connected by an edge,
- $N_i := \{j \in V | (i, j) \in E\}$: Set of neighbours of i in G,
- $A := [a_{ij}]_{i,j \in V}$: Adjacency matrix of G where $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise,
- $\mathbb{D} := \mathrm{diag}(|N_1|, \ldots, |N_N|)$: Degree matrix of G, $d = trace(\mathbb{D}) = \sum_{i=1}^N |N_i|$, $d^* = \max_i\{|N_i|\}$,
- $L : \mathbb{D} - A$: Laplacian matrix of G,

- $L_N := \mathbb{D}^{-\frac{1}{2}} L \mathbb{D}^{-\frac{1}{2}}$: Normalized Laplacian of G if G has no isolated vertex.

The following hold for a graph G with no isolated vertex: $\mathbb{D} \succ 0$, $\lambda_{\max}(L_N) \le 2$ and $2\mathbb{D} - L = \mathbb{D}^{\frac{1}{2}}(2I - L_N)\mathbb{D}^{\frac{1}{2}} \succeq 0$, Chung (1997). For a connected, undirected graph G with n vertices, $L \succeq 0$, with 0 a simple eigenvalue and $L\mathbf{1}_n = \mathbf{0}_n$, $\mathbf{1}_n^T L = \mathbf{0}_n^T$. The other eigenvalues of L are positive, with minimum one $\lambda_2(L)$ and maximum $\lambda_{\max}(L) \le 2d^*$. Also, $Ker(L) = span\{\mathbf{1}_n\}$, $Ker(L)^\perp = \{x | \mathbf{1}_n^T x = 0\}$, $x^T L x \ge \lambda_2(L)\|x\|^2 > 0, \forall x \ne 0, x \in Ker(L)^\perp$.

2 Problem Statement

Consider a networked game with N players, defined with the following parameters and denoted by $\mathcal{G}(V, \Omega_i, J_i)$:

- $V = \{1, \ldots, N\}$: Set of all players,
- $\Omega_i \subset \mathbb{R}$: Action set of player i, $\forall i \in V$,
- $\Omega = \prod_{i \in V} \Omega_i \subset \mathbb{R}^N$: Action set of all players, where $\prod$ denotes the Cartesian product,
- $J_i : \mathbb{R}^N \to \mathbb{R}$: Cost function of player i, $\forall i \in V$.

Players' actions are denoted as follows:

- $x_i \in \Omega_i$: Player i's action, $\forall i \in V$,
- $x_{-i} \in \Omega_{-i} := \prod_{j \in V \setminus \{i\}} \Omega_j$: All players' actions except player i's,
- $x = (x_i, x_{-i}) \in \Omega$: All players actions.

The game is played such that for a given $x_{-i} \in \Omega_{-i}$, every player $i \in V$ aims to minimize its own cost function (2) with respect to (w.r.t.) x_i. Note that each player's optimal action is dependent on the other players' decisions. A Nash equilibrium (NE) lies at the intersection of solutions to the set of problems (2) (fixed-point of best-response map), such that no player can reduce its cost by unilaterally deviating from its action.

Definition 1 *Consider an N-player game $\mathcal{G}(V, \Omega_i, J_i)$. An action profile (vector) $x^* = (x_i^*, x_{-i}^*) \in \Omega$ is called a Nash equilibrium (NE) of this game if*

$$J_i(x_i^*, x_{-i}^*) \le J_i(x_i, x_{-i}^*) \quad \forall x_i \in \Omega_i, \ \forall i \in V.$$

We state a few assumptions for the existence of a NE, Nash (1950); Debreu (1952); Glicksberg (1952).

Assumption 1 *For every $i \in V$,*

- *$\Omega_i \subset \mathbb{R}$ is non-empty, compact and convex,*
- *$J_i(x_i, x_{-i})$ is C^1 and convex in x_i, for every x_{-i}, and jointly continuous in x.*

Let $\nabla_i J_i(x) = \frac{\partial J_i}{\partial x_i}(x_i, x_{-i})$ be the partial gradient of J_i w.r.t. x_i and $F : \mathbb{R}^N \to \mathbb{R}^N$ be the pseudo-gradient of game (2) defined by

$$F(x) := [\nabla_i J_i(x)]_{i \in V} \tag{6}$$

Let $\partial \mathcal{I}_{\Omega_i}(x_i)$ be the subdifferential of $\mathcal{I}_{\Omega_i}$ at x_i and $G(x) := [\partial \mathcal{I}_{\Omega_i}(x_i)]_{i \in V}$, where $\partial \mathcal{I}_{\Omega_i}(x_i) = N_{\Omega_i}(x_i)$. Let $N_\Omega(x) = \prod_{i \in V} N_{\Omega_i}(x_i)$ be the normal cone to Ω at x.

A Nash equilibrium (NE) $x^* = (x_i^*, x_{-i}^*) \in \Omega$ satisfies the variational inequality, (cf. Proposition 1.4.2, Facchinei and Pang (2007))

$$(x - x^*)^T F(x^*) \ge 0, \forall x \in \Omega,$$

i.e., $-F(x^*) \in N_\Omega(x^*)$, or

$$-\nabla_i J_i(x^*) \in N_{\Omega_i}(x_i^*), \ \forall i \in V,$$

Thus, with $\partial \mathcal{I}_{\Omega_i}(x_i) = N_{\Omega_i}(x_i)$, a NE $x^* = (x_i^*, x_{-i}^*)$ of (2) can be characterized by

$$0 \in \nabla_i J_i(x^*) + \partial \mathcal{I}_{\Omega_i}(x_i^*), \ \forall i \in V, \tag{7}$$

Then, with (6), (7) can be written in compact form as,

$$\mathbf{0}_N \in F(x^*) + G(x^*). \tag{8}$$

Typically another assumption such as monotonicity (strict, strong) of the pseudo-gradient vector F, (6), is used to show that projected-gradient type algorithms converge to x^*, Facchinei and Pang (2007).

Assumption 2 *The pseudo-gradient F is strongly monotone and Lipschitz continuous: there exists $\mu > 0$, $\theta_0 > 0$ such that for any x and y, $\langle x - y, F(x) - F(y) \rangle \ge \mu \|x - y\|^2$ and $\|F(x) - F(y)\| \le \theta_0 \|x - y\|$.*

Under Assumptions 1 and 2, the game has a unique NE x^* (Theorem 2.3.3 in Facchinei and Pang (2007)). Strong monotonicity of F is a standard assumption under which convergence of projected-gradient type algorithms is guaranteed with fixed step-sizes, (Theorem 12.1.2 in Facchinei and Pang (2007)).

We assume that the cost function J_i and the action set Ω_i information are available to player i only, hence an incomplete-information game, Li and Başar (1987). The challenge is that each optimization problem in (2) is dependent on the solution of the other simultaneous problems. In the classical setting of a game with incomplete information but perfect monitoring, each agent can observe the actions of all other players, x_{-i}, Facchinei and Pang (2007), Scutari et al. (2014), Frihauf et al. (2012), Kannan and Shanbhag (2012). In this paper, we consider a game with *incomplete information and imperfect monitoring*, where no player is able to observe the actions of all others. We refer to this as a game with *partial-decision information.* To compensate for the lack of global decision information, players exchange some information in order to update their actions. We assume players can communicate only locally, with their neighbours. An undirected *communication graph* $G_c(V, E)$ is then defined with no isolated vertex. Let denote the set of neighbours of player i in G_c by N_i. Let also denote $\mathbb{D}$ and L be the degree and Laplacian matrices associated to G_c, respectively. The following assumption is used.

Assumption 3 *G_c is undirected and connected.*

We assume that players maintain estimates of the other players' actions and share them with their neighbours in order to update their estimates. In an engineering application, this step could be prescribed, e.g. as a network protocol in peer-to-peer networks or ad-hoc networks. Our goal is to develop a distributed algorithm for computing the NE of $\mathcal{G}(V, \Omega_i, J_i)$ with fixed step-sizes, while

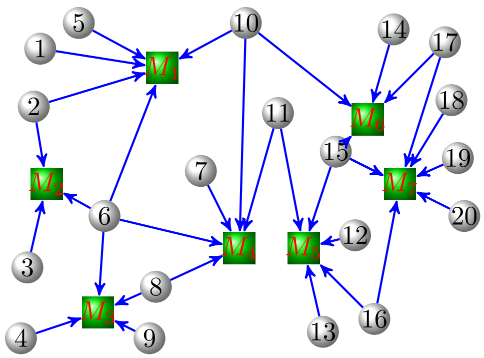

Figure 1. Network Cournot game: An edge from i to M_k on this graph implies that agent/firm i participates in Market M_k.

using only partial-decision information over the communication graph $G_c(V, E)$.

This is a topic of recent interest given the the proliferation of engineering networked applications requiring distributed protocols that operate under partial, local information (e.g. ad-hoc networks or smart-grid networks), and the deregulation of global markets. We provide next a motivating example, inspired by the one in Koshal et al. (2016), Kannan and Shanbhag (2012).

Example 1 *(Nash-Cournot Game Over a Network). Consider a networked Nash-Cournot game, as in Koshal et al. (2016), Bimpikis et al. (2014) between a set of N firms (players/agents) involved in the production of a homogeneous commodity that compete over m markets,* $M_1, \cdots, M_m$ *(Figure 1). Firm i,* $i \in V$ *participates in* $n_i \leq m$ *markets with* x_i *commodity amount that it supplies to each of its markets. Hence player i's action (strategy) is its commodity amount per market* $x_i \in \mathbb{R}$. *Its total amount it produces is* $x_{i,T} = n_i x_i$, *assumed to be limited to* Ω_T, *so that* $x_i \in \Omega_i \subset \mathbb{R}$, $\Omega_i = \Omega_T / n_i$. *Firm i has a local vector* $A_i \in \mathbb{R}^m$ *(with elements 1 or 0) that specifies which markets it participates in. The k-th element of* A_i *is 1 if and only if player i delivers* x_i *amount to market* M_k. *Therefore,* $A_1, \cdots, A_N$ *can be used to specify a bipartite graph that represents the connections between firms and markets (see Figure 1). Denote* $x = [x_i]_{i \in V} \in \mathbb{R}^N$, *and* $A = [A_1, \cdots, A_N] \in \mathbb{R}^{m \times N}$. *Then* $A\,x \in \mathbb{R}^m = \sum_{i=1}^N A_i x_i$ *is the total product supply to all markets, given the action profile x of all firms. Suppose that* $P : \mathbb{R}^m \to \mathbb{R}^m$ *is a price vector function that maps the total supply of each market to the corresponding market's price. Each firm i has a production cost* $c_i(x_{i,T}) : \Omega_T \to \mathbb{R}$, *function of its total production amount. Then the local objective function of firm (player) i is* $J_i(x_i, x_{-i}) = c_i(x_{i,T}) - P^T(Ax)A_i x_i$. *Overall, given the other firms' profile* x_{-i}, *each firm needs to solve the following optimization problem,*

$$\underset{x_i \in \Omega_i}{\text{minimize}} \quad c_i(x_{i,T}) - P^T(Ax)A_i x_i \tag{9}$$

In the classical, centralized-information setting each firm is assumed to have instantaneous access to the others' actions x_{-i}. *This may be impractical in a large network of geographically distributed firms, Bimpikis et al. (2014). For example, consider that Figure 1 depicts a group of* $N = 20$ *firms located on different continents that participate in* $m = 7$ *markets, with no centralized communication system between them. Since players are unable to directly observe the actions of all others, they engage in local, non-strategic information exchange, to mitigate their lack of global, centralized information. Firms/players may communicate with a local subset of neighbouring firms via some underlying communication infrastructure, hence a distributed partial-information setting. A communication network formed* G_c *between the firms prescribes how they communicate locally their production decision, Koshal et al. (2016). In this situation, the communication network is formed by the players who are viewed as the nodes in the network. In this example, we consider that most of the communication is between firms on the same continent, with one or at most two firms in each continent having a direct connection to another firm on another continent. One such instance of the communication network* G_c *is shown in Figure 2. Firms i and j are able to exchange their local* x^i *and* x^j *if there exists an edge between them on this graph. Various other topologies can be considered, with different connectivity.*

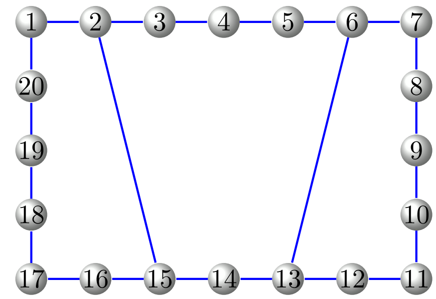

Figure 2. Communication graph G_c: Firms i and j are able to exchange their local x^i and x^j if there exists an edge between them on this graph.

A Nash-Cournot game (9) can describe other practical decision problems in engineering networks, for example rate control games in communication networks (Yin et al. (2011)) and demand-response games in smart-grid networks (Ye and Hu (2017), Yi and Pavel (2018)). Another example, of a rate control game over a wireless ad-hoc network, is presented in Section 5.

3 Distributed Inexact-ADMM Algorithm

In order to find the NE of $\mathcal{G}(V, \Omega_i, J_i)$ (2) over the communication graph $G_c(V, E)$, we propose a distributed ADMM-type algorithm with fixed step-sizes (Bertsekas and Tsitsiklis (1997), p. 255), based on introducing local estimates of players' actions and using (virtual) constraints for estimate consensus.

We define a few notations for players' estimates.

- $x^i_{-i} \in \mathbb{R}^{N-1}$: Player i's estimate of all other players' actions,
- $x^i_i \in \Omega_i \subset \mathbb{R}$: Player i's estimate of its action which is indeed its own action, i.e., $x^i_i = x_i$ for $i \in V$,
- $x^i = (x^i_i, x^i_{-i}) \in \mathbb{R}^N$: Player i's estimate of all players' actions (state variable),
- $\mathbf{x} = [x^i]_{i\in V}$ or $\mathbf{x} = [x^1; \dots; x^N] \in \mathbb{R}^{N^2}$: Augmented (stacked) vector of all estimates.

Note that all players' actions x can be represented as $x = [x^i_i]_{i\in V}$. In steady-state all local copies x^i should be equal. By employing the actions' estimates x^i, $\forall i \in V$, (local copies of x), we can reformulate game (2) as:

$$\begin{cases} \underset{x^i_i \in \Omega_i}{\text{minimize}} \quad J_i(x^i_i, x^i_{-i}), & \forall i \in V \\ \text{subject to} \quad x^i = x^j,\ \forall j \text{ s.t. } (i,j), (j,i) \in E \end{cases} \tag{10}$$

where E is the set of edges of G_c. For each agent i we consider the constraint that the estimate (local copy) x^i is equal to x^j of its neighbours for all cases where x^i is involved, i.e., player i is either a source or a destination, as specified via ordered pairs $(i,j) \in E$ and $(j,i) \in E$. Under Assumption 3 this ensures that $x^1 = x^2 \dots = x^N$, hence (10) recovers (2). The following distributed ADMM-type algorithm is proposed.

Algorithm 1 Inexact-ADMM Algorithm

1: **initialization**
$\quad x^i_i(0) \in \Omega_i$, $x^i_{-i}(0) \in \mathbb{R}^{N-1}$, $w^i(0) = \mathbf{0}_N$
2: **for** $k = 1, 2, \dots$ **do**
3: $\quad$ **for** each player $i \in V$ **do**
4: $\quad$ players i, j $\forall j \in N_i$ exchange $x^i(k-1)$, $x^j(k-1)$
5: $\quad w^i(k) = w^i(k-1) + c\sum_{j\in N_i}(x^i(k-1) - x^j(k-1))$
6: $\quad x^i_i(k) = T_{\Omega_i}\Big\{\alpha_i^{-1}\Big[-\nabla_i J_i(x^i_i(k-1), x^i_{-i}(k-1)) - w^i_i(k) + \beta_i x^i_i(k-1) + \bar{c}\sum_{j\in N_i}(x^i_i(k-1) + x^j_i(k-1))\Big]\Big\}$
7: $\quad x^i_{-i}(k) = \dfrac{\beta_i x^i_{-i}(k-1) + \bar{c}\sum_{j\in N_i}(x^i_{-i}(k-1) + x^j_{-i}(k-1)) - w^i_{-i}(k)}{\alpha_i}$
8: $\quad$ **end for**
9: **end for**

Here $\alpha_i = \beta_i + 2\bar{c}|N_i|$, $\bar{c} = c + c_0$, where $c > 0$, $c_0 > 0$, $\beta_i > 0$ are penalty parameters for the augmented Lagrangian and proximal first-order approximation. Besides its estimates, each player i needs to maintain only its dual variables w^i, and needs only the information on its Ω_i set. The derivation of Algorithm 1 following an ADMM approach adapted to a game setup is provided in Appendix A.

Algorithm 1 has a gradient-play structure for a game with estimate consensus constraints: Step 6 uses a projected-gradient (due to J_i) of an augmented Lagrangian with quadratic-penalty and proximal linearization (inexact ADMM) (see (A.10) in Appendix A). So in this sense, in the context of individual rationality, it can be interpreted as a better-response (as opposed to best-response) strategy, but in the modified game, due to partial-information setting. Step 5 updates the dual variables w^i via a dual ascent. Compared to a gossip-based algorithm, Koshal et al. (2016), Salehisadaghiani and Pavel (2016), the additional ADMM-induced modifications are instrumental to prove convergence under fixed step-sizes.

Remark 1 *We compare Algorithm 1 to typical ADMM DOP algorithms. Unlike Wei and Ozdaglar (2012), Wei and Ozdaglar (2013), where a minimization subproblem is solved exactly at each iteration, Algorithm 1 uses a gradient update based on inexact (proximal) approximation as in Hong et al. (2016). In contrast to either of these algorithms, the update of each player decision x^i_i (Step 6) is indirectly coupled nonlinearly to the previous decisions of other players, via its estimate x^i_{-i}, due to the intrinsic coupling in J_i. This is unlike ADMM for DOP where the update of each agent's decision variable x_i is only linearly coupled to the others agents' x_j. Moreover, in Algorithm 1 each player has an extra step for updating its estimate x^i_{-i} (Step 7), in addition to its decision variable x^i_i. Because of this coupling the convergence proof (based on the pseudo-gradient) is more involved.*

Next we analyze the fixed points of Algorithm 1.

Lemma 1 *Consider that Assumptions 1, 3 hold. Let $\overline{\mathbf{x}} := [\overline{x}^i]_{i\in V}$, $\overline{\mathbf{w}} := [\overline{w}^i]_{i\in V}$ be any fixed points of Algorithm 1. Then, $\overline{\mathbf{x}} = \mathbf{1}_N \otimes \overline{x}$, where $\overline{x} = x^*$ is NE of game (2), and $(\mathbf{1}_N^T \otimes I_N)\overline{\mathbf{w}} = \mathbf{0}_N$.*

Proof. From Step 5 of Algorithm 1, for all $i \in V$,

$$\overline{w}^i = \overline{w}^i + c\sum_{j\in N_i}(\overline{x}^i - \overline{x}^j) \ \Rightarrow (L \otimes I_N)\overline{\mathbf{x}} = \mathbf{0}_{N^2}$$

and by Assumption 3, this yields $\overline{\mathbf{x}} = \mathbf{1}_N \otimes \overline{x}$, for some $\overline{x} = [\overline{x}_i]_{i\in V} \in \mathbb{R}^N$, hence

$$\overline{x}^1 = \ldots = \overline{x}^N = \overline{x}, \tag{11}$$

Using (11) in Step 6 of Algorithm 1 yields, $(\beta_i + 2\bar{c}|N_i|)\overline{x}_i - \alpha_i \overline{x}_i \in \nabla_i J_i(\overline{x}) + \partial \mathcal{I}_{\Omega_i}(\overline{x}_i) + \overline{w}^i_i$. With $\alpha_i = \beta_i + 2\bar{c}|N_i|$ this leads to

$$0 \in \nabla_i J_i(\overline{x}) + \partial \mathcal{I}_{\Omega_i}(\overline{x}_i) + \overline{w}^i_i, \ \ \forall i \in V$$

Moreover, using (11) in Step 7 of Algorithm 1, we obtain $\alpha_i \overline{x}^i_{-i} = \alpha_i \overline{x}^i_{-i} - \overline{w}^i_{-i}$, hence

$$\overline{w}^i_{-i} = \mathbf{0}_{N-1} \ \forall i \in V$$

Combining the previous two relations into a single vector, yields

$$\mathbf{0}_N \in \nabla_i J_i(\overline{x}) e_i + \partial \mathcal{I}_{\Omega_i}(\overline{x}_i) e_i + \overline{w}^i, \ \ \forall i \in V \tag{12}$$

where e_i is the i-th unit vector in $\mathbb{R}^N$. Summing after $i \in V$, with $F(\overline{x}) = [\nabla_i J_i(\overline{x})]_{i\in V}$, yields

$$\mathbf{0}_N \in F(\overline{x}) + G(\overline{x}) + \sum_{i\in V} \overline{w}^i. \tag{13}$$

For $w^i(k)$, from Step 5 of Algorithm 1, it follows that

$$\sum_{i\in V} w^i(k) = \sum_{i\in V} w^i(k-1) + c\sum_{i\in V}\sum_{j\in N_i}(x^i(k-1)-x^j(k-1)) \tag{14}$$

Under Assumption 3 the second term in (14) is zero, hence $\sum_{i\in V} w^i(k) = \sum_{i\in V} w^i(k-1)$ for any $k \geq 1$. With initial conditions $w^i(0) = \mathbf{0}_N$ $\forall i \in V$ this implies that $\sum_{i\in V} w^i(k) = \mathbf{0}_N$ $\forall k \geq 1$, hence $\sum_{i\in V} \overline{w}^i = \mathbf{0}_N$. Thus (13) reduces to $\mathbf{0}_N \in F(\overline{x}) + G(\overline{x})$, hence by (8) $\overline{x} = x^*$ is NE and $\overline{\mathbf{x}} = \mathbf{1}_N \otimes x^*$. Moreover,

$$(\mathbf{1}_N^T \otimes I_N)\overline{\mathbf{w}} = \sum_{i\in V} \overline{w}^i = \mathbf{0}_N \tag{15}$$

■

Recall the pseudo-gradient (6), $F(x) = [\nabla_i J_i(x)]_{i\in V}$.

Definition 2 *Let $\mathbf{F} : \mathbb{R}^{N^2} \to \mathbb{R}^N$, be the extension of F to the augmented space, defined as*

$$\mathbf{F}(\mathbf{x}) := [\nabla_i J_i(x^i)]_{i\in V},\ \nabla_i J_i(x^i) = \frac{\partial J_i}{\partial x_i^i}(x_i^i, x_{-i}^i) \tag{16}$$

and called the extended pseudo-gradient. *Let also $\mathbf{G}(\mathbf{x}) := [\partial I_{\Omega_i}(x_i^i)]_{i\in V}$, so $\mathbf{G}(\mathbf{x}) = G(x)$.*

Note that $\mathbf{F}$ is the pseudo-gradient evaluated at estimates instead of actual actions, and $\mathbf{F}(\mathbf{1}_N \otimes x) = F(x)$. We write Algorithm 1 in a compact vector form.

Proposition 1 *Let $\mathbf{x} = [x^i]_{i\in V}$, $\mathbf{w} = [w^i]_{i\in V}$. Then the updates in Algorithm 1 have the following vector form:*

$$\mathbf{w}(k) = \mathbf{w}(k-1) + c\mathbf{L}\mathbf{x}(k-1) \tag{17}$$

$$\begin{aligned}\mathbf{0}_{N^2} \in \mathbf{R}\,(\mathbf{F}(\mathbf{x}(k-1)) + \mathbf{G}(\mathbf{x}(k))) + \mathbf{w}(k) \\ +((\mathbb{B}+2\bar{c}\mathbb{D})\otimes I_N)(\mathbf{x}(k)-\mathbf{x}(k-1)) + \bar{c}\,\mathbf{L}\mathbf{x}(k-1),\end{aligned} \tag{18}$$

where $\mathbf{R} = diag([e_i]_{i\in V})$ is $(N^2 \times N)$ block-diagonal matrix with unit vectors e_i, $\mathbf{L} = L \otimes I_N$, $\mathbf{F}$, $\mathbf{G}$ as in (16), $\mathbb{D}$ is the degree matrix of G_c, L is the Laplacian matrix of G_c, $\mathbb{B} := diag((\beta_i)_{i\in V})$ with $\beta_i > 0$, $\bar{c} = c + c_0$, $c, c_0 > 0$.

Proof. See Appendix B.

Properties of $\mathbf{F}$ and matrices $\mathbf{R}$ and $\mathbf{L}$ play a key role in the following. Matrix $\mathbf{R}^T$ allows action selection from the stacked $\mathbf{x}$: $\mathbf{R}^T\mathbf{x} = [e_i^T x^i]_{i\in V} = x$, based on $e_i^T x^i = x_i^i = x_i$. Also $(\mathbf{1}_N^T \otimes I_N)\mathbf{R} = I_N$, $\mathbf{R}^T\mathbf{R} = I_N$. For matrix $\mathbf{L}$, $Ker(\mathbf{L}) = Ker(L \otimes I_N) = span\{\mathbf{1}_N \otimes I_N\}$ (the consensus subspace), and $Ker(\mathbf{L})^\perp = span(\mathbf{L}) = Ker\{\mathbf{1}_N^T \otimes I_N\}$.

Note that by Lemma 1, $\overline{\mathbf{x}} = \mathbf{1}_N \otimes \overline{x} \in Ker(\mathbf{L})$ and $\overline{\mathbf{w}} \in span(\mathbf{L})$ (cf. (15)). Also, we can write (12) as,

$$\mathbf{0}_{N^2} \in [\nabla_i J_i(\overline{x})e_i]_{i\in V} + [\partial \mathcal{I}_{\Omega_i}(\overline{x}_i)e_i]_{i\in V} + \overline{\mathbf{w}}$$

or, with $\mathbf{F}$, $\mathbf{G}$ as in (16), compactly as,

$$\mathbf{0}_{N^2} \in \mathbf{R}\,(\mathbf{F}(\overline{\mathbf{x}}) + \mathbf{G}(\overline{\mathbf{x}})) + \overline{\mathbf{w}}. \tag{19}$$

4 Convergence Analysis

In this section we show global convergence of Algorithm 1 under the following assumption.

Assumption 4 *The extended pseudo-gradient $\mathbf{F}$, (16), is Lipschitz continuous: there exists $\theta > 0$ such that for any $\mathbf{x}$ and $\mathbf{y}$, $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq \theta\|\mathbf{x} - \mathbf{y}\|$.*

Remark 2 *In the general case when $\mathbf{F}$ is nonlinear, a sufficient condition for Assumption 4 is that $\mathbf{F}$ is C^1 with bounded Jacobian $\|D\mathbf{F}(\mathbf{x})\|_2$. For quadratic games, Assumption 4 is automatically satisfied ($\mathbf{F}$ is linear). Moreover, as shown below, $\theta = \theta_0$, where θ_0 is the Lipschitz constant of F (Assumption 2). Consider the Nash-Cournot game in Example 1, where firm i's production cost is a strongly convex quadratic function in its total production amount, $c_i(x_{i,T}) = q_i x_{i,T}^2 + b_i x_{i,T}$, i.e., $c_i(x_i) = n_i^2 q_i x_i^2 + n_i b_i x_i$, where $q_i > 0$, $b_i \in \mathbb{R}$. Consider that market M_k's price is a linear function of the total supplied commodity amount, $p_k(x) = \bar{P}_k - z_k[Ax]_k$ (known as a linear inverse demand function) with $\bar{P}_k, z_k > 0$. Denote $P = [p_k]_{k=1,m} : \mathbb{R}^N \to \mathbb{R}^m$, $\bar{P} = [\bar{P}]_{k=1,m} \in \mathbb{R}^m$, $Z = diag([z_k]_{k=1,m}) \in \mathbb{R}^{m\times m}$, so that $P = \bar{P} - ZAx$ is the vector price function, hence the objective function of player i is*

$$J_i(x_i, x_{-i}) = c_i(x_i) - (\bar{P} - ZAx)^T A_i x_i \text{ and} \tag{20}$$

$$\nabla_i J_i(x) = \nabla c_i(x_i) + A_i^T Z A_i x_i - A_i^T\,(\bar{P} - ZAx), \tag{21}$$

where $\nabla c_i(x_i) = 2n_i^2 q_i x_i + n_i b_i$. Then, with $x = [x_i]_{i\in V}$, $A = [A_1, \cdots A_N]$, $F(x) = [\nabla_i J_i(x)]_{i\in V}$ is given by

$$F(x) = \nabla c(x) + diag([A_i^T Z A_i]_{i\in V})\,x + A^T Z A\,x - A^T \bar{P}$$

where $\nabla c(x) = 2diag([n_i^2 q_i]_{i\in V})\,x + [n_i b_i]_{i\in V}$. Combining terms, $F(x)$ can be written compactly as,

$$F(x) = Q\,x + r, \quad \text{where } Q := \Sigma + A^T Z A, \tag{22}$$

$\Sigma := diag\big([2n_i^2 q_i + A_i^T Z A_i]_{i\in V}\big)$ and $r := [n_i b_i]_{i\in V} - A^T \bar{P}$. Since $q_i > 0$ and $z_k > 0$, $\Sigma \succ 0$ and $A^T Z A \succeq 0$, hence $Q \succ 0$. Thus $F(x)$ is Lipschitz continuous with $\theta_0 = \|Q\|_2$ where $\|Q\|_2 = \sigma_{\max}(Q)$.
Using (21) with x replaced by $x^i = (x_i, x_{-i}^i)$, yields

$$\nabla_i J_i(x^i) = \nabla c_i(x_i) + A_i^T Z A_i x_i - A_i^T\,(\bar{P} - ZAx^i).$$

Thus, $\mathbf{F}(\mathbf{x}) := [\nabla_i J_i(x^i)]_{i\in V}$, (16), is given by

$$\begin{aligned}\mathbf{F}(\mathbf{x}) = \nabla c(x) + diag\big([A_i^T Z A_i]_{i\in V}\big)x - A^T\bar{P} \\ + diag\big([A_i^T Z A]_{i\in V}\big)\mathbf{x}\end{aligned}$$

Using $\nabla c(x)$, Σ and r defined above, and $\mathbf{R}^T(I_N \otimes A^T Z A) = diag\big([e_i^T A^T Z A]_{i\in V}\big)$ for the last term, yields

$$\mathbf{F}(\mathbf{x}) = \Sigma\mathbf{R}^T\mathbf{x} + r + \mathbf{R}^T(I_N \otimes A^T Z A)\,\mathbf{x}.$$

where $x = \mathbb{R}^T\mathbf{x}$ was used. Since Σ is block-diagonal, $\Sigma\mathbf{R}^T = \mathbf{R}^T(I_N \otimes \Sigma)$, and with Q (22) we can write

$$\mathbf{F}(\mathbf{x}) = \overline{\mathbf{Q}}\,\mathbf{x} + r, \quad \text{where } \overline{\mathbf{Q}} := \mathbf{R}^T(I_N \otimes Q). \tag{23}$$

Hence, $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq \|\overline{\mathbf{Q}}\|_2\|\mathbf{x} - \mathbf{y}\|$, where $\|\overline{\mathbf{Q}}\|_2 = \|\mathbf{R}^T(I_N \otimes Q)\|_2 \leq \|\mathbf{R}^T\|_2\|I_N \otimes Q\|_2 = \|Q\|_2$ since $\|\mathbf{R}\|_2 = 1$ and $\|I_N \otimes Q\|_2 = \|Q\|_2$, and thus $\theta = \theta_0$.

Remark 3 *Note that Assumption 4 on $\mathbf{F}$ is weaker than* *strong monotonicity in Ye and Hu (2017)), or cocoercivty in Salehisadaghiani and Pavel (2017), Shi and Pavel (2017). In ADMM for DOP, cocoercivity of the full gradient is used, which is automatically satisfied by* joint convexity *of each decoupled f_i, (1), and Lipschitz continuity of its gradient (see Chang et al. (2015)). In contrast, in a game, because of coupling to the others' actions in J_i (2) and because of partial convexity (see Assumption 1), monotonicity of the pseudo-gradient is* not automatically satisfied *when extended to the augmented space. The next result shows how under Assumption 4, a monotonicity property can be achieved in the augmented space.*

Lemma 2 *Consider that Assumptions 1, 2, 3 and 4 hold and let*

$$\Psi = \begin{bmatrix} \frac{\mu}{N} & -\frac{\theta+\theta_0}{2\sqrt{N}} \\ -\frac{\theta+\theta_0}{2\sqrt{N}} & c_0\lambda_2(L)-\theta \end{bmatrix} \tag{24}$$

Then, for any $c_0 > c_{\min}$ where $c_{\min}\lambda_2(L) = \frac{(\theta+\theta_0)^2}{4\mu} + \theta$, $\Psi \succ 0$, and for any $\mathbf{x}$ and any $\mathbf{y} \in Ker(\mathbf{L})$,

$$(\mathbf{x}-\mathbf{y})^T\big(\mathbf{RF}(\mathbf{x})-\mathbf{RF}(\mathbf{y})+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big) \geq \bar{\mu}\|\mathbf{x}-\mathbf{y}\|^2, \tag{25}$$

where $\bar{\mu} := \lambda_{\min}(\Psi) > 0$.

Proof. See Appendix B.

Remark 4 *For quadratic games (see Remark 2), $\theta = \theta_0 = \|Q\|_2$, $F(x)$ is strongly monotone with $\mu = \lambda_{\min}(Q) > 0$ (by Assumption 2), hence the c_0 bound in Lemma 2 simplifies to $c_0\lambda_2(L) > \frac{\|Q\|_2^2}{\mu} + \|Q\|_2$. Furthermore, if Q is symmetric, $\|Q\|_2 = \sigma_{\max}(Q) = \lambda_{\max}(Q)$, and this reduces to $c_0 > \frac{\lambda_{\max}(Q)}{\lambda_2(L)}(\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)} + 1)$, showing the trade-off between game properties and communication graph connectivity.*

Lemma 2 shows that, for a sufficiently large c_0, $\mathbf{RF}(\mathbf{x})+c_0\mathbf{Lx}$ is strongly monotone in a *restricted* set of directions $\mathbf{x}-\mathbf{y}$, where $\mathbf{y} \in Ker(\mathbf{L})$. We can call $\mathbf{RF}$ a *pre*-$\mathbf{L}$-monotone function, related to the concept of *premonotone gradient* of a *prox-regular* function, Poliquin and Rockafellar (1996). In fact, being in a restricted set of directions, this is a weaker monotonicity property, similar to the notion of *restricted* strong convexity used in DOP and high-dimensional statistical estimation Zhang (2017), Agarwal et al. (2010), Negahban et al. (2009). To show this key property, an instrumental step is the decomposition of the augmented space $\mathbb{R}^{N^2}$ into the consensus subspace, $Ker(\mathbf{L})$ (where $\mathbf{F}$ is strongly monotone), and its orthogonal complement, $Ker(\mathbf{L})^\perp$ (where $\mathbf{L}$ is strongly monotone). Based on the Lipschitz continuity of $\mathbf{F}$, for a sufficiently large c_0, excess strong monotonicity of $c_0\mathbf{L}$ can balance the cross-terms (shortage of monotonicity of $\mathbf{RF}$) when $\mathbf{x}$ is off the consensus subspace $Ker(\mathbf{L})$, but not on $Ker(\mathbf{L})^\perp$, (see proof in Appendix B). We show next that this property is sufficient to prove convergence of Algorithm 1.

Theorem 1 *Consider that Assumptions 1, 2, 3 and 4 hold. Take an arbitrary $c > 0$, any $c_0 > c_{\min}$, with $c_{\min}$ as in Lemma 2, and let $\bar{c} = c + c_0$. If $\beta_i > 0$ are chosen such that*

$$\lambda_{\min}(\mathbb{B} + 2\bar{c}\,\mathbb{D} - c\,L) > \frac{\bar{\theta}^2}{2\bar{\mu}} \tag{26}$$

where $\mathbb{B} := diag((\beta_i)_{i\in V})$, $\mathbb{D}$ and L are the degree and Laplacian matrices of G_c, $\bar{\theta} = \theta + 2c_0 d^*$, d^* is the maximal degree of G_c and $\bar{\mu}$ is defined in Lemma 2, then the sequence $\{x^i(k)\}$ $\forall i \in V$, or $\{\mathbf{x}(k)\}$, generated by Algorithm 1 converges to NE of game (2) x^*, (or $\mathbf{1}_N \otimes x^*$).

Proof. Recall that $\mathbf{G}(\mathbf{x}(\mathbf{k})) = G(x(k)) = [\partial I_{\Omega_i}(x_i(k))]_{i\in V}$. From the $\mathbf{x}$-update (18) in the vector form of Algorithm 1 (see Proposition 1), it follows that there exists $y \in G(x(k)) \subset \mathbb{R}^N$ such that,

$$\begin{aligned}\mathbf{0}_{N^2} &= \mathbf{R}\,(\mathbf{F}(\mathbf{x}(k-1))+y)+\mathbf{w}(k)\\ &\quad+((\mathbb{B}+2\bar{c}\,\mathbb{D})\otimes I_N)(\mathbf{x}(k)-\mathbf{x}(k-1)) + \bar{c}\,\mathbf{Lx}(k-1)\end{aligned}$$

or, substituting $\mathbf{w}(k)$ by $\mathbf{w}(k+1) - c\,\mathbf{Lx}(k)$ (cf. (18)),

$$\begin{aligned}\mathbf{0}_{N^2} &= \mathbf{R}\,\mathbf{F}(\mathbf{x}(k-1)) + \mathbf{R}\,y + \mathbf{w}(k+1) - c\,\mathbf{Lx}(k)\\ &\quad+((\mathbb{B}+2\bar{c}\,\mathbb{D})\otimes I_N)(\mathbf{x}(k)-\mathbf{x}(k-1))+\bar{c}\,\mathbf{Lx}(k-1).\end{aligned}$$

Using $\bar{c} = c + c_0$ and $H := \mathbb{B} + 2\bar{c}\,\mathbb{D} - c\,L$ this yields

$$\begin{aligned}\mathbf{0}_{N^2} &= \mathbf{R}\,\mathbf{F}(\mathbf{x}(k-1)) + \mathbf{R}\,y + \mathbf{w}(k+1)\\ &\quad+(H\otimes I_N)(\mathbf{x}(k)-\mathbf{x}(k-1))+c_0\,\mathbf{Lx}(k-1).\end{aligned} \tag{27}$$

Note that $H := \mathbb{B} + c\,(2\mathbb{D} - L) + 2c_0\mathbb{D}$ hence $H \otimes I_N \succ 0$, since $\mathbb{B}\succ 0$ and $c\,(2\mathbb{D} - L) \succeq 0$.

Consider NE x^* and let $\mathbf{x}^* = \mathbf{1}_N \otimes x^*$. Then, from (8), with $\mathbf{F}(\mathbf{1}_N \otimes x^*) = F(x^*)$, and the definition of $\mathbf{R}$, it follows that $\mathbf{0}_{N^2} \in \mathbf{RF}(\mathbf{x}^*) + \mathbf{RG}(\mathbf{x}^*)$, i.e., there exists $y^* \in \mathbf{G}(\mathbf{x}^*)$ such that $\mathbf{R}\,\mathbf{F}(\mathbf{x}^*) + \mathbf{R}\,y^* = \mathbf{0}_{N^2}$. Moreover, there exists $\mathbf{q}^* \in \mathbb{R}^{N^2}$ (any $\mathbf{q}^* \in Ker(\mathbf{L})$ with structure $\mathbf{1}_N \otimes \widetilde{q}$, for $\widetilde{q} \in \mathbb{R}^N$) such that

$$\mathbf{0}_{N^2} = \mathbf{R}\,\mathbf{F}(\mathbf{x}^*) + \mathbf{R}\,y^* + c\mathbf{Lq}^*. \tag{28}$$

Subtracting (28) from (27), multiplying by $(\mathbf{x}(k)-\mathbf{x}^*)^T$ and using $\mathbf{R}^T\,\mathbf{x} = x$, yields

$$\begin{aligned}&\big(\mathbf{RF}(\mathbf{x}(k-1)) - \mathbf{RF}(\mathbf{x}^*)\big)^T(\mathbf{x}(k)-\mathbf{x}^*)\\ &+\big(y-y^*\big)^T(x(k)-x^*)\\ &+(\mathbf{w}(k+1)-c\mathbf{Lq}^*)^T(\mathbf{x}(k)-\mathbf{x}^*)\\ &+(\mathbf{x}(k)-\mathbf{x}(k-1))^T(H\otimes I_N)(\mathbf{x}(k)-\mathbf{x}^*)\\ &+\mathbf{x}(k-1)^T c_0\mathbf{L}(\mathbf{x}(k)-\mathbf{x}^*) = 0.\end{aligned} \tag{29}$$

Combine the first and fifth term in (29) and write,

$$\begin{aligned}&\big(\mathbf{RF}(\mathbf{x}(k-1))+c_0\mathbf{Lx}(k-1)-\mathbf{RF}(\mathbf{x}^*)\big)^T(\mathbf{x}(k)-\mathbf{x}^*)\\ &=\big(\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\big)^T(\mathbf{x}(k-1)-\mathbf{x}^*)\\ &+\big(\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\big)^T(\mathbf{x}(k)-\mathbf{x}(k-1)).\end{aligned} \tag{30}$$

For the first term on the right-hand side of (30) we use (25) in Lemma 2 (with $\mathbf{Lx}^* = \mathbf{0}_{N^2}$), while for the second term we use (3) for $\rho = \frac{\bar{\theta}^2}{2\bar{\mu}}$, $\bar{\theta} = \theta+2c_0 d^*$. This yields,

$$\big(\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\big)^T(\mathbf{x}(k)-\mathbf{x}^*)$$

$$\geq \bar{\mu}\|\mathbf{x}(k-1)-\mathbf{x}^*)\|^2$$
$$-\frac{\bar{\mu}}{\bar{\theta}^2}\|\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\|^2 \quad (31)$$
$$-\frac{\bar{\theta}^2}{4\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2.$$

Note that $\mathbf{L}$ is $\|\mathbf{L}\|_2$-Lipschitz with $\|\mathbf{L}\|_2 = \lambda_{max}(L) \leq 2d^*$. Based on this, with $\mathbf{Lx}^* = \mathbf{0}_{N^2}$, Assumption 4 for $\mathbf{F}$, $\|\mathbf{R}\|_2 = 1$ and the triangle inequality, we can write for the second term on the right-hand side of (31),

$$\|\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\| \leq \bar{\theta}\|\mathbf{x}(k-1)-\mathbf{x}^*)\|$$

where $\bar{\theta} = \theta + 2c_0 d^*$. Using this in (31) yields

$$\big(\mathbf{RF}(\mathbf{x}(k-1))-\mathbf{RF}(\mathbf{x}^*)+c_0\mathbf{Lx}(k-1)\big)^T(\mathbf{x}(k)-\mathbf{x}^*)$$
$$\geq -\frac{\bar{\theta}^2}{4\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2 \quad (32)$$

Back to (29), for the second term, since $y \in G(x(k))$, $y^* \in G(x^*)$, $\mathcal{I}_{\Omega_i}$ is convex (Assumption 1) it follows that

$$(y-y^*)^T(x(k)-x^*) \geq 0. \quad (33)$$

Finally for the third term in (29), we use the following. From (18) with zero initial conditions, it follows that $\mathbf{w}(k+1) = c\mathbf{Lq}(k)$, where $\mathbf{q}(k) = \sum_{t=0}^{k}\mathbf{x}(t)$. Thus,

$$\mathbf{q}(k)-\mathbf{q}(k-1) = \mathbf{x}(k) \quad (34)$$

Then, we can write

$$(\mathbf{w}(k+1)-c\mathbf{Lq}^*)^T(\mathbf{x}(k)-\mathbf{x}^*)$$
$$= (\mathbf{q}(k)-\mathbf{q}^*)^T c\mathbf{L}(\mathbf{q}(k)-\mathbf{q}(k-1)-\mathbf{x}^*)$$
$$= (\mathbf{q}(k)-\mathbf{q}^*)^T c\mathbf{L}(\mathbf{q}(k)-\mathbf{q}(k-1))$$

Using (32), (33) and this last relation in (29) yields

$$-\frac{\bar{\theta}^2}{4\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2+(\mathbf{q}(k)-\mathbf{q}^*)^T c\mathbf{L}(\mathbf{q}(k)-\mathbf{q}(k-1))$$
$$+(\mathbf{x}(k)-\mathbf{x}(k-1))^T(H\otimes I_N)(\mathbf{x}(k)-\mathbf{x}^*) \leq 0, \quad (35)$$

where $H\otimes I_N \succ 0$. Using (4) to deal with all cross-terms in the previous inequality, yields

$$\|\mathbf{x}(k)-\mathbf{x}^*\|^2_{H\otimes I_N}+\|\mathbf{q}(k)-\mathbf{q}^*\|^2_{c\mathbf{L}}-\|\mathbf{x}(k-1)-\mathbf{x}^*\|^2_{H\otimes I_N}$$
$$-\|\mathbf{q}(k-1)-\mathbf{q}^*\|^2_{c\mathbf{L}} \leq -\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2_{H\otimes I_N}$$
$$-\|\mathbf{q}(k)-\mathbf{q}(k-1)\|^2_{c\mathbf{L}}+\frac{\bar{\theta}^2}{2\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2.$$

Let us make the following notations:

$$\mathbf{z}(k) := \begin{bmatrix}\mathbf{x}(k)\\ \mathbf{q}(k)\end{bmatrix}, \mathbf{z}^* := \begin{bmatrix}\mathbf{x}^*\\ \mathbf{q}^*\end{bmatrix}, \Phi = \begin{bmatrix}H & \mathbf{0}_{N\times N}\\ \mathbf{0}_{N\times N} & cL\end{bmatrix} \quad (36)$$

Then we can write the last inequality as

$$\|\mathbf{z}(k)-\mathbf{z}^*\|^2_{\Phi\otimes I_N}-\|\mathbf{z}(k-1)-\mathbf{z}^*\|^2_{\Phi\otimes I_N}$$
$$\leq -\|\mathbf{z}(k)-\mathbf{z}(k-1)\|^2_{\Phi\otimes I_N}+\frac{\bar{\theta}^2}{2\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2.$$

Moreover, since $\frac{\bar{\theta}^2}{2\bar{\mu}}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2 \leq \frac{\bar{\theta}^2}{2\bar{\mu}\lambda_{\min}(H)}\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2_{H\otimes I_N} \leq \frac{\bar{\theta}^2}{2\bar{\mu}\lambda_{\min}(H)}\|\mathbf{z}(k)-\mathbf{z}(k-1)\|^2_{\Phi\otimes I_N}$, from the foregoing it follows that

$$\|\mathbf{z}(k)-\mathbf{z}^*\|^2_{\Phi\otimes I_N}-\|\mathbf{z}(k-1)-\mathbf{z}^*\|^2_{\Phi\otimes I_N} \leq \quad (37)$$
$$-\zeta\|\mathbf{z}(k)-\mathbf{z}(k-1)\|^2_{\Phi\otimes I_N} \leq 0,$$

where $\zeta = 1-\frac{\bar{\theta}^2}{2\bar{\mu}\lambda_{\min}(H)}$, and $0<\zeta<1$ by (26). Summing (37) over k from 1 to ∞ yields

$$\sum_{k=1}^{\infty}\|\mathbf{z}(k)-\mathbf{z}(k-1)\|^2_{\Phi\otimes I_N} \leq \frac{1}{\zeta}\|\mathbf{z}(0)-\mathbf{z}^*\|^2_{\Phi\otimes I_N} < \infty \quad (38)$$

From (38) it follows that $\|\mathbf{z}(k)-\mathbf{z}(k-1)\|^2_{\Phi\otimes I_N} \to 0$. From (37), since $\|\mathbf{z}(k)-\mathbf{z}^*\|^2_{\Phi\otimes I_N}$ or $\|\mathbf{x}(k)-\mathbf{x}^*\|^2_{H\otimes I_N}+\|\mathbf{q}(k)-\mathbf{q}^*\|^2_{c\mathbf{L}}$ is bounded and non-increasing, it follows that $\|\mathbf{x}(k)-\mathbf{x}^*\|^2_{H\otimes I_N}+\|\mathbf{q}(k)-\mathbf{q}^*\|^2_{c\mathbf{L}} \to v$, for some $v\geq 0$, where $(H\otimes I_N)\succ 0$, $\mathbf{L}\succeq 0$.

Any $\mathbf{q}\in\mathbb{R}^{N^2}$ can be decomposed as $\mathbf{q} = \mathbf{q}^{\|}+\mathbf{q}^{\perp}$, where $\mathbf{q}^{\|}\in Ker(\mathbf{L})$, $\mathbf{q}^{\perp}\in Ker(\mathbf{L})^{\perp}$, with $(\mathbf{q}^{\perp})^T\mathbf{L}\mathbf{q}^{\perp} > 0$, $\forall \mathbf{q}^{\perp}\neq 0$. Decomposing $\mathbf{q}(k)$ and $\mathbf{q}^*$ into $(\cdot)^{\|}$ and $(\cdot)^{\perp}$ components, $\|\mathbf{q}(k)-\mathbf{q}^*\|^2_{c\mathbf{L}} = \|\mathbf{q}(k)^{\perp}-\mathbf{q}^{*\perp}\|^2_{c\mathbf{L}}$, with $\mathbf{q}(k)^{\perp}, \mathbf{q}^{*\perp}\in Ker(\mathbf{L})^{\perp}$. Hence, from the above,

$$V_{x^*}(k) := \|\mathbf{x}(k)-\mathbf{x}^*\|^2_{H\otimes I_N}+\|\mathbf{q}(k)^{\perp}-\mathbf{q}^{*\perp}\|^2_{c\mathbf{L}}$$

is bounded and non-increasing and converges to v. Thus, since $H\succ 0$ and $\|\mathbf{q}(k)^{\perp}-\mathbf{q}^{*\perp}\|^2_{c\mathbf{L}} > 0$, $\forall \mathbf{q}^{\perp}(k)\neq 0$, the sequence $[\mathbf{x}(k)\,;\,\mathbf{q}(k)^{\perp}]$ is bounded, hence has at least a limit point $[\overline{\mathbf{x}}\,;\,\overline{\mathbf{q}}^{\perp}]$, where $\overline{\mathbf{q}}^{\perp}\in Ker(\mathbf{L})^{\perp}$. Also, $\mathbf{w}(k+1) = c\mathbf{Lq}(k) = c\mathbf{Lq}(k)^{\perp}$, has a limit point $\overline{\mathbf{w}} = c\mathbf{L}\overline{\mathbf{q}}^{\perp}$. By Lemma 1, any limit point of $\mathbf{x}(k)$ satisfies $\overline{\mathbf{x}} = \mathbf{1}_N\otimes x^*$, with x^* NE of the game and $\overline{\mathbf{w}}$ such that (19) holds, hence $\overline{\mathbf{x}} = \mathbf{x}^*$. Consider now

$$V_{\overline{x}}(k) := \|\mathbf{x}(k)-\mathbf{x}^*\|^2_{H\otimes I_N}+\|\mathbf{q}(k)^{\perp}-\overline{\mathbf{q}}^{\perp}\|^2_{c\mathbf{L}} \quad (39)$$

Similar to the first part of the proof for $V_{x^*}(k)$ with $\mathbf{x}^*$, $c\mathbf{Lq}^*$ and (28), it can be shown that $V_{\overline{x}}(k)$ is bounded and non-increasing, by using (19) for $\overline{\mathbf{x}} = \mathbf{x}^*$, $\overline{\mathbf{w}} = c\mathbf{L}\overline{\mathbf{q}}^{\perp}$ instead of (28). Hence, $V_{\overline{x}}(k)$ converges to some $\overline{v}\geq 0$ as $k\to\infty$. This $\overline{v}$ is the same along any subsequence. Since $[\mathbf{x}^*\,;\,\overline{\mathbf{q}}^{\perp}]$ is a limit point for $[\mathbf{x}(k)\,;\,\mathbf{q}(k)^{\perp}]$, there exists a subsequence $\{k_n\}$, $k_n\to\infty$ as $n\to\infty$, such that $\{[\mathbf{x}(k_n)\,;\,\mathbf{q}(k_n)^{\perp}]\}\to[\mathbf{x}^*\,;\,\overline{\mathbf{q}}^{\perp}]$ as $n\to\infty$. Taking limit of $V_{\overline{x}}(k_n)$, (39), along this $\{k_n\}$ subsequence, as $n\to\infty$, it follows that $\overline{v} = 0$. Since $(H\otimes I_N)\succ 0$ and $\overline{\mathbf{q}}^{\perp}, \mathbf{q}(k)^{\perp}\in Ker(\mathbf{L})^{\perp}$, taking limit in (39) as $k\to\infty$, it follows that $\mathbf{x}(k)\to\mathbf{x}^*$, with $\mathbf{x}^* = \mathbf{1}_N\otimes x^*$ and x^* NE. ■

Remark 5 *We note that, unlike ADMM for DOP, Wei and Ozdaglar (2013), Hong et al. (2016), to prove convergence we cannot use a common objective function* *and common Lagrangian. Instead our proof relies on properties of the extended pseudo-gradient* $\mathbf{F}$*. Compared to Salehisadaghiani and Pavel (2017), Shi and Pavel (2017), where* $c_0 = 0$ *and cocoercivity of* $\mathbf{RF}$ *was assumed, here we only assume Lipschitz continuity of* $\mathbf{F}$*. We achieved this relaxation by slightly modifying the ADMM algorithm so that the* $\mathbf{x}$*-update uses the extra* $c_0 > 0$ *parameter in the proximal-approximation (quadratic penalty). Convergence is proved for sufficiently large* c_0*. This is similar to the way in which non-convexity is overcome in ADMM for DOP in Hong et al. (2016), Wang et al. (2015)). Based on Gershgorin theorem, a sufficient diagonal-dominance condition can be derived for* (26) *to hold, and* β_i *parameters can be selected independently by players (see e.g. Yi and Pavel (2017)). We note that assumptions on* F *and* $\mathbf{F}$ *could be relaxed to hold only locally around* x^* *and* $\mathbf{x}^*$*, in which case all results become local. We also note that the class of quadratic games satisfies all asumptions globally.*

4.1 Convergence Rate Analysis

Next we investigate the convergence rate of Algorithm 1. We use the following result.

Proposition 2 (Shi et al. (2015)) *If a sequence* $\{a(k)\} \subset \mathbb{R}$ *is:* (i) *nonnegative,* $a(k) \geq 0$, (ii) *summable,* $\sum_{k=1}^{\infty} a(k) < \infty$, *and* (iii) *monotonically non-increasing,* $a(k+1) \leq a(k)$*, then we have:* $a(k) = o\left(\frac{1}{k}\right)$*, i.e.,* $\lim_{k\to\infty} k\, a(k) = 0$.

For $\|\mathbf{z}(k-1)-\mathbf{z}(k)\|^2_{\Phi\otimes I_N}$, with $\mathbf{z}(k)$, $\Phi \otimes I_N$ as in (36), (36), summability follows from (38) in Theorem 1. Next we provide a lemma showing that $\|\mathbf{z}(k-1)-\mathbf{z}(k)\|^2_{\Phi\otimes I_N}$ is monotonically non-increasing.

Lemma 3 *Under the assumptions of Theorem 1, the sequence* $\{\mathbf{z}(k)\}$, (36)*, generated by Algorithm 1 satisfies for all* $k \geq 1$,

$$\|\mathbf{z}(k) - \mathbf{z}(k+1)\|^2_{\Phi\otimes I_N} \leq \|\mathbf{z}(k-1) - \mathbf{z}(k)\|^2_{\Phi\otimes I_N},$$

where $\Phi \otimes I_N \succeq 0$ *is as in* (36).

Proof. Let $\Delta\mathbf{x}(k+1) := \mathbf{x}(k) - \mathbf{x}(k+1)$, $\Delta\mathbf{q}(k+1) := \mathbf{q}(k)-\mathbf{q}(k+1)$, $\Delta\mathbf{z}(k+1) := \mathbf{z}(k)-\mathbf{z}(k+1)$, $\Delta\mathbf{F}(\mathbf{x}(k+1)) := \mathbf{F}(\mathbf{x}(k))-\mathbf{F}(\mathbf{x}(k+1))$, $\Delta\mathbf{G}(x(k+1)) = \mathbf{G}(x(k))-\mathbf{G}(x(k+1))$. Following similar arguments as in the proof of Theorem 1, we obtain similar to (37), $\|\Delta\mathbf{z}(k)\|^2_{\Phi\otimes I_N} - \|\Delta\mathbf{z}(k+1)\|^2_{\Phi\otimes I_N} \geq \zeta\|\Delta\mathbf{z}(k)-\Delta\mathbf{z}(k+1)\|^2_{\Phi\otimes I_N} \geq 0$, where $\zeta = 1 - \frac{\bar{\theta}^2}{2\bar{\mu}\lambda_{\min}(H)} > 0$, by (26). ■

Theorem 2 *Under the same assumptions of Theorem 1, the following rates hold for Algorithm 1:*

$$\|\mathbf{x}^{\perp}(k)\|^2_{\mathbf{L}} = o\left(\tfrac{1}{k}\right),$$

where $\mathbf{x}^{\perp}(k) \in Ker(\mathbf{L})^{\perp}$, $\mathbf{L} = L \otimes I_N$, L *the Laplacian matrix of* G_c*, and for some* $y \in \mathbf{G}(x(k))$,

$$\|\mathbf{R}(\mathbf{F}(\mathbf{x}(k-1)) + y) + \mathbf{w}(k+1) + c_0\mathbf{L}\mathbf{x}^{\perp}(k-1)\|^2_{(H\otimes I_N)^{-1}} = o\left(\frac{1}{k}\right)$$

Proof. By Theorem 1 (equation (38)), $\sum_{k=0}^{\infty} \|\mathbf{z}(k-1) - \mathbf{z}(k)\|^2_{\Phi\otimes I_N} < \infty$. Moreover, Lemma 3 proves that $\|\mathbf{z}(k-1) - \mathbf{z}(k)\|^2_{\Phi\otimes I_N}$ is monotonically non-increasing. Then, it directly follows by Proposition 2 that,

$$\|\mathbf{z}(k-1) - \mathbf{z}(k)\|^2_{\Phi\otimes I_N} = o\left(\tfrac{1}{k}\right).$$

Expanding this by using (36) yields,

$$\|\mathbf{x}(k-1) - \mathbf{x}(k)\|^2_{H\otimes I_N} + c\|\mathbf{q}(k-1) - \mathbf{q}(k)\|^2_{\mathbf{L}} = o\left(\frac{1}{k}\right)$$

and, using (34) $\mathbf{q}(k-1) - \mathbf{q}(k) = \mathbf{x}(k)$, so that

$$\|\mathbf{x}(k-1) - \mathbf{x}(k)\|^2_{H\otimes I_N} + c\|\mathbf{x}(k)\|^2_{\mathbf{L}} = o\left(\frac{1}{k}\right)$$

As in the proof of Theorem 1, we can write this as

$$\|\mathbf{x}(k-1) - \mathbf{x}(k)\|^2_{H\otimes I_N} + c\|\mathbf{x}^{\perp}(k)\|^2_{\mathbf{L}} = o\left(\frac{1}{k}\right) \quad (40)$$

where $\mathbf{x}^{\perp}(k) \in Ker(\mathbf{L})^{\perp}$, $\mathbf{x}(k)^{\perp}\mathbf{L}\mathbf{x}^{\perp}(k) > 0$, for all $\mathbf{x}^{\perp}(k) \neq 0$. If the sum of two positive sequences is $o(\frac{1}{k})$, then each sequence is $o(\frac{1}{k})$, hence, by the foregoing,

$$\|\mathbf{x}^{\perp}(k)\|^2_{\mathbf{L}} = o\left(\frac{1}{k}\right).$$

From (40), (27), we can write $\|\mathbf{x}(k)-\mathbf{x}(k-1)\|^2_{H\otimes I_N} = \|\mathbf{RF}(\mathbf{x}(k-1)) + \mathbf{R}y + \mathbf{w}(k+1) + c_0\mathbf{L}\mathbf{x}^{\perp}(k-1)\|^2_{(H\otimes I_N)^{-1}} = o\left(\frac{1}{k}\right)$, for some $y \in \mathbf{G}(x(k))$. ■

Remark 6 *Since* $\mathbf{x}^{\perp}(k) \in Ker(\mathbf{L})^{\perp}$, $\|\mathbf{x}^{\perp}(k)\|^2_{\mathbf{L}}$ *can be interpreted as a weighted distance from the consensus subspace, while the second rate can be interpreted a weighted distance from the optimality conditions. Unlike DOP, e.g. Wei and Ozdaglar (2013), in a game we do not have a common merit function for all players and the convergence rate we are able to show is only in terms of optimality residuals.*

5 Simulation Results

In this section, we present two examples to illustrate the performance of Algorithm 1. The first one is a networked Nash-Cournot game (Example 1) as in Koshal et al. (2016), while the second one is a rate control game in Wireless Ad-Hoc Network (WANET), as in Salehisadaghiani and Pavel (2018); Alpcan and Başar (2005).

5.1 Example 1: Nash-Cournot game over networks

In this section, we examine the performance of the proposed algorithm on a networked Nash-Cournot game (Example 1). We consider $N = 20$ firms and $m = 7$ markets, (Fig. 1). Each firm i has a local normalized constraint as $0 \leq x_{T,i} \leq 1$ on its total production $x_{i,T} = n_i x_i$, where n_i is the number of markets it participates in, hence $\Omega_i = [0, 1/n_i]$. The local objective function is taken as (20), where the local production cost function of firm i is $c_i(x_i) = {n_i}^2 q_i x_i^2 + n_i b_i x_i$ where

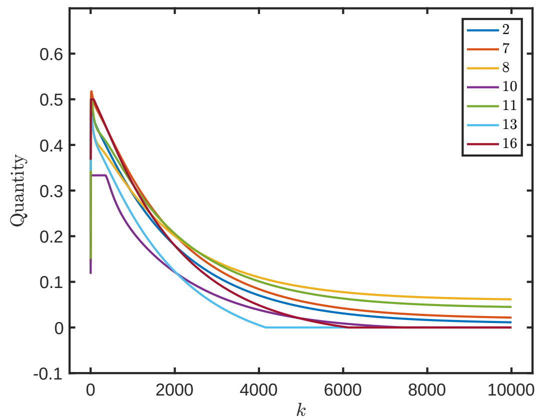

Figure 3. NE computation of a Nash-Cournot game by Algorithm 1 over G_c in Fig. 2. The figure shows the convergence paths of 7 randomly selected firms.

q_i, b_i are randomly drawn from the uniform distribution $U(1, 2)$. All assumptions hold globally. In the first case, each local cost function J_i is normalized by dividing by $n_i^2 q_i$, which yields a better condition number. The price function is taken as the linear function $P = \bar{P} - ZAx$, where $z_k = 0.01$ and $\bar{P}_k$ are randomly drawn from $ZA\Omega_i + U(1, 2)$. This yields $\lambda_{\min}(Q) = 1.001 = \mu$, $\lambda_{\max}(Q) = 1.099 = \theta$. For a communication graph G_c as in Fig. 2, $\lambda_2(L) = 0.102$, and the bound for c_0 in Lemma 2 is 22.5. We set $c_0 = 22.5$, an arbitrary step-size $c = 1$, and $\beta_i = 10$ for all firms, satisfying (26). The results for the implementation of Algorithm 1 in the partial decision setting, where the primal variables x^i are exchanged over the sparsely connected graph G_c (Fig. 2), are shown in Fig. 3. Initial conditions are randomly selected, for x_i from $U(0, 0.5)$, and for x_j^i $(i \neq j)$ from $U(0, 1)$. The results indicate fast convergence.

Next, we study an extreme case where the costs are widely different. This yields a high condition number, and we can study how the cost functions affect the outcome of market competition. The q_i and b_i for the first 10 firms are set at the level of 0.01 while those for the other firms are at the level of 100, and each firm i has a local constraint $0 \leq x_{i,T} \leq 100$. Intuitively, this will lead to high/low market occupation of the first/last 10 firms. The price function is taken as the linear function $P = \bar{P} - ZAx$, where z_k are randomly drawn from $U(1, 10)$ and $\bar{P}_k$ randomly generated such that $P_k > 0$ for all feasible x. This yields $\lambda_{\min}(Q) = 1.482 = \mu$, $\lambda_{\max}(Q) = 1005 = \theta$, and the bound for c_0 in Lemma 2 is $6e6$. However, we found this bound conservative in our experiments. We plot the sequences of relative error (vs. the 'true' NE obtained by (centralized) gradient method) under different parameter settings in the top sub-figure of Fig. 4. The trends reflected in the top sub-figure of Fig. 4 match the intuition of our algorithm design (see Appendix A) and the implication of (26) in Theorem 1. In general, smaller penalty parameters c, c_0, and β_i's lead to faster convergence but they have to be

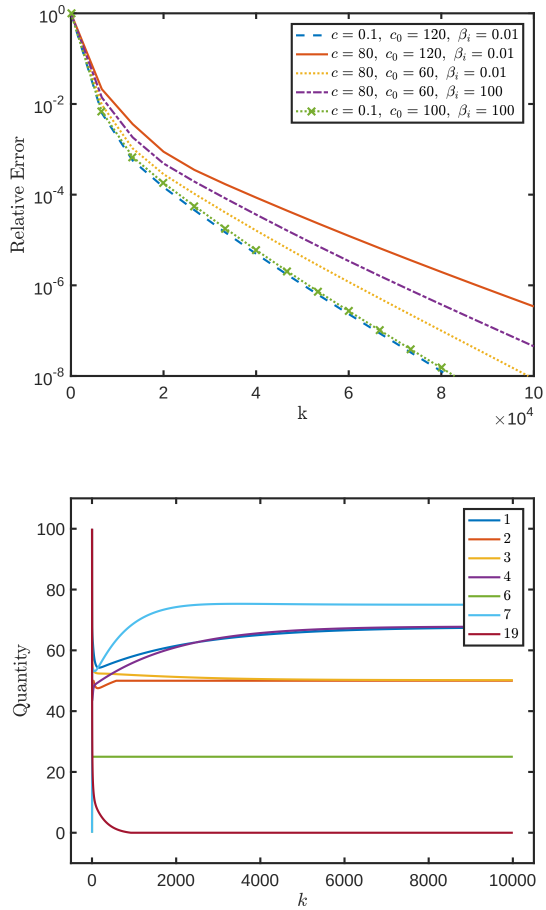

Figure 4. NE computation of a Nash-Cournot game by Algorithm 1 over G_c in Fig. 2. The top figure plots relative error $\frac{\|x(k)-x^*\|}{\|x(0)-x^*\|}$ vs. iteration number k. The bottom figure shows the convergence paths of 7 selected firms.

large enough to guarantee convergence. An instance of the first 10^4 iterations of trajectories of selected firms are plotted in the bottom sub-figure of Fig. 4. The initial estimate for x_{-i}^i's are randomly drawn from $(0, 200)$; to observe diverse firm trajectories, the first 5 and last 5 firms' local x_i^i's are initialized as the upper limits of their local constraints, respectively, while the other firms start from 0. This way, we can see four cases: (1-5) "aggressive start, rich end"; (6-10) "conservative start, rich end"; (11-15) "conservative start, poor end" (16-20) "aggressive start, poor end".

5.2 *Example 2: Rate control game over WANET*

In this section we consider an engineering network-protocol example for a rate control game over a *Wireless Ad-Hoc Network* (WANET), Alpcan and Basar (2002); Alpcan and Başar (2005). Consider 16 mobile nodes interconnected by multi-hop communication paths in a WANET Fig. 5 (a). $N = 15$ users/players want to transfer data from a source node to a destination node via this WANET. In Fig. 5 (a) solid lines represent physical links and dashed lines display unique paths that are

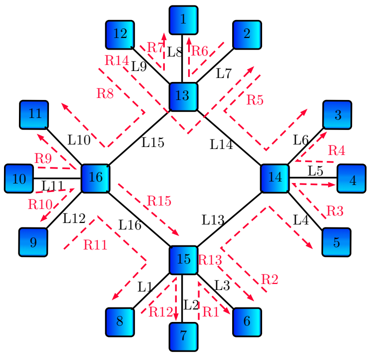

Figure 5. (a) Wireless Ad-Hoc Network (left). (b) Communication graph G_c (right) for Example 2.

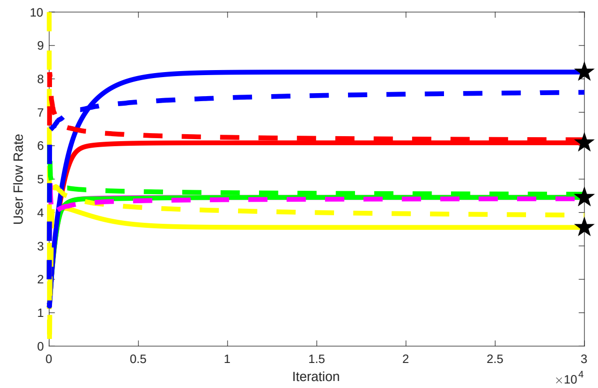

Figure 6. Flow rates of users 1, 3, 5, 8, 13 using Algorithm 1 (solid lines) vs. algorithm in Salehisadaghiani and Pavel (2018) (dashed lines). NE represented by black stars.

assigned to each user to transfer its data. Each link has a positive capacity that restricts the users' data flow. In this WANET scenario there is no central node. Each user/player is prescribed to communicate locally with its neighbours over a preassigned (sparse but connected) communication graph G_c, e.g. as depicted in Fig. 5 (b). In Fig. 5 (b) a node represents a player, and an edge in the graph signifies the ability for the two players to exchange information. Each user exchanges information locally (thus mitigating the lack of centralized information), so that he can estimate the other flows and, based on them, adjust its own decision which is his data rate. Here is the list of WANET notations:

(1) L_j: Link j, $j = 1, \ldots, 16$,
(2) R_i: The path assigned to user i, $i = 1, \ldots, 15$,
(3) $C_j > 0$: Link j's capacity, $j = 1, \ldots, 16$,
(4) $0 \leq x_i \leq 10$: The data flow of user i, $i = 1, \ldots, 15$.

Each path consists of a set of links, e.g., $R_1 = \{L_2, L_3\}$. Each user i decision (action) is its data rate flow $0 \leq x_i \leq 10$ to send over R_i, based on its cost function J_i,

$$J_i(x_i, x_{-i}) := \sum_{j: L_j \in R_i} \frac{\kappa}{C_j - \sum_{w: L_j \in R_w} x_w} - \chi_i \log(x_i + 1),$$

which depends on the flows of the other users, and where $\kappa > 0$ and $\chi_i > 0$ are network-wide known and user-specific parameters, respectively. In this case $\lambda_2(L) = 0.18$ and the other assumptions hold locally. For $\chi_i = 10$ $\forall i = 1, \ldots, 15$ and $C_j = 10$ $\forall j = 1, \ldots, 16$ we compute numerically $\|DF(x)\|_2$ and $\|D\mathbf{F}(x)\|_2$ over Ω, $\theta_0 = 4.7$, $\theta = 6.7$. The bound for c_0 in Lemma 2 is 30.7. We set $c_0 = 31$, hence $\bar{\mu} = 0.04$, an arbitrary step-size $c = 1$ and $\beta_i = 14$, $\forall i = 1, \ldots, 15$ (by (26) in Theorem 1). We compare our Algorithm 1 with the gossip-based one proposed in Salehisadaghiani and Pavel (2018) run over G_c, with diminishing-step sizes. The results are shown in Fig. 6, for uniform randomly selected initial conditions. The simulation results show that Algorithm 1 is about two orders of magnitude faster than the one in Salehisadaghiani and Pavel (2018).

6 Conclusions

We designed a distributed NE seeking algorithm in general games by using an inexact-ADMM approach. Each player maintains action estimates for the others, exchanged with its neighbours over a connected, undirected communication graph. The game was reformulated as a modified game with virtual consensus constraints for the action estimates, and solved within the framework of inexact-ADMM. An inexact-ADMM algorithm was designed which was shown to convergence to the NE of the game under strong monotonicity of the pseudo-gradient and Lipschitz continuity of the extended pseudo-gradient. The convergence rate of the algorithm was compared with that of an existing gossip-based NE seeking algorithm. To deal with the merely monotone case (thus with possibly multiple NEs) we leave for future work; this would require more involved computations even on the consensus subspace. As other directions for future work we can mention: (1) extension to directed communication graphs, (2) adaptation to games with networked structure (where computation and memory could be reduced if each player maintains only its own required estimates), (3) designing incentives to ensure that players truthfully share their states and auxiliary variables.

A Derivation of ADMM Algorithm 1

In the following we show how Algorithm 1 can be derived based on an ADMM approach adapted for a game setup. Using slack variables t^{ij}, t^{ji} associated with each edge to separate the constraints and $\mathcal{I}_{\Omega_i}(x_i^i)$ for the feasibility constraint $x_i^i \in \Omega_i$, we write (10) as

$$\begin{cases} \underset{x_i^i \in \mathbb{R}}{\text{minimize}} \quad J_i(x_i^i, x_{-i}^i) + \mathcal{I}_{\Omega_i}(x_i^i), & \forall i \in V \\ \text{subject to } \; x^i = t^{ij};\ \forall j \text{ s.t. } (i,j) \in E \\ \qquad\qquad\quad x^i = t^{ji};\ \forall j \text{ s.t. } (j,i) \in E \end{cases} \tag{A.1}$$

and similarly for player j. We cannot directly use a standard ADMM algorithm because each minimization in

(A.1) is not over the whole x^i as in DOPT, cf. (1), but rather only over part of it (action x_i^i), and inherent coupling to the others' decisions still exists indirectly, via the estimates x_{-i}^i. *Note that in* (A.1)*, since minimization of J_i is only over the action component x_i^i, for the other (estimate) components x_{-i}^i we can consider a zero objective function.* We will develop an ADMM-type algorithm to solve (A.1) in a distributed manner. Let $\{u^{ij}, v^{ji}\} \in \mathbb{R}^N$ be the Lagrange multipliers of player i, associated with the constraints in (A.1) (source, destination), and similarly, $\{u^{ji}, v^{ij}\} \in \mathbb{R}^N$, for player j, respectively. Let $N_i^{out} = \{j | (i,j) \in E\}$ and $N_i^{in} = \{j | (j,i) \in E\}$, and since G_c is undirected, $N_i^{in} = N_i^{out} = N_i$. We show next how the algorithm can be derived and simplified so that each player maintains only its dual variables $w^i := \sum_{j\in N_i} u^{ij} + v^{ji}$, not all its Lagrangian multipliers $\{u^{ij}, v^{ji}\}$. The auxiliary variables t^{ij}, t^{ji} are obtained based on minimizing the overall augmented Lagrangian,

$$\begin{aligned}
&\mathcal{L}^c\big(\{x^i\},\{t^{ij},t^{ji}\};\{u^{ij},v^{ji}\};\{u^{ji},v^{ij}\}\big) := \sum_{i\in V} J_i(x_i^i, x_{-i}^i) \\
&+\sum_{i\in V}\sum_{j\in N_i} {u^{ij}}^T(x^i - t^{ij}) + \sum_{i\in V}\sum_{j\in N_i} {v^{ji}}^T(x^i - t^{ji}) \quad \text{(A.2)}\\
&+\frac{c}{2}\sum_{i\in V}\sum_{j\in N_i}\|x^i - t^{ij}\|^2 + \frac{c}{2}\sum_{i\in V}\sum_{j\in N_i}\|x^i - t^{ji}\|^2
\end{aligned}$$

Based on (A.2), t^{ij}, t^{ji} are updated as:

$$\begin{aligned}
t^{ij}(k) &= \arg\min_{t^{ij}} \mathcal{L}^c\big(\{x^i(k-1)\},\{t^{ij}\};\{u^{ij}(k-1), v^{ij}(k-1)\}\big)\\
t^{ji}(k) &= \arg\min_{t^{ji}} \mathcal{L}^c\big(\{x^j(k-1)\},\{t^{ji}\};\{u^{ji}(k-1), v^{ji}(k-1)\}\big)
\end{aligned}$$

or, since $\mathcal{L}^c$ is quadratic in t^{ij}, t^{ji}, in closed-form as

$$\begin{aligned}
t^{ij}(k) &= \frac{u^{ij}(k-1)+v^{ij}(k-1)}{2c} + \frac{x^i(k-1)+x^j(k-1)}{2} \quad \text{(A.3)}\\
t^{ji}(k) &= \frac{u^{ji}(k-1)+v^{ji}(k-1)}{2c} + \frac{x^j(k-1)+x^i(k-1)}{2}
\end{aligned}$$

For each player $i \in V$ we consider an augmented Lagrangian associated to (A.1), where $c > 0$,

$$\begin{aligned}
&\mathcal{L}_i^c\big(x^i, t^{ij}, t^{ji}; \{u^{ij}, v^{ji}\}\big) := J_i(x_i^i, x_{-i}^i) + \mathcal{I}_{\Omega_i}(x_i^i)\\
&+\sum_{j\in N_i} {u^{ij}}^T(x^i - t^{ij}) + \sum_{j\in N_i} {v^{ji}}^T(x^i - t^{ji}) \quad \text{(A.4)}\\
&+\frac{c}{2}\sum_{j\in N_i}\|x^i - t^{ij}\|^2 + \frac{c}{2}\sum_{j\in N_i}\|x^i - t^{ji}\|^2
\end{aligned}$$

For player i its dual variables $\{u^{ij}, v^{ji}\}$ are updated by dual ascent with c as the step-size, based on (A.4),

$$\begin{aligned}
u^{ij}(k) &= u^{ij}(k-1) + c\big(x^i(k-1) - t^{ij}(k)\big), \quad \text{(A.5)}\\
v^{ji}(k) &= v^{ji}(k-1) + c\big(x^i(k-1) - t^{ji}(k)\big).
\end{aligned}$$

Similarly, for player j's, based on $\mathcal{L}_j^c$, its $\{u^{ji}, v^{ij}\}$ are,

$$\begin{aligned}
u^{ji}(k) &= u^{ji}(k-1) + c\big(x^j(k-1) - t^{ji}(k)\big), \quad \text{(A.6)}\\
v^{ij}(k) &= v^{ij}(k-1) + c\big(x^j(k-1) - t^{ij}(k)\big).
\end{aligned}$$

Considering $u^{ij}(0) = v^{ij}(0) = u^{ji}(0) = v^{ji}(0) = \mathbf{0}_N$ and using (A.3) in (A.5),(A.6) yields

$$\begin{aligned}
u^{ij}(k) + v^{ij}(k) &= \mathbf{0}_N \quad \forall k \geq 0\\
u^{ji}(k) + v^{ji}(k) &= \mathbf{0}_N \quad \forall k \geq 0.
\end{aligned}$$

so that (A.3) become,

$$t^{ij}(k) = \frac{x^i(k-1) + x^j(k-1)}{2} = t^{ji}(k). \quad \text{(A.7)}$$

Using (A.7) to eliminate $t^{ij}(k)$, $t^{ji}(k)$ from (A.5), yields

$$\begin{aligned}
u^{ij}(k) &= u^{ij}(k-1) + \frac{c}{2}\big(x^i(k-1) - x^j(k-1)\big)\\
v^{ji}(k) &= v^{ji}(k-1) + \frac{c}{2}\big(x^i(k-1) - x^j(k-1)\big).
\end{aligned}$$

Then for $w^i(k) := \sum_{j\in N_i} u^{ij}(k) + v^{ji}(k)$, this yields

$$w^i(k) = w^i(k-1) + c\sum_{j\in N_i}(x^i(k-1) - x^j(k-1)) \quad \text{(A.8)}$$

where $c > 0$, as in Step 5 of Algorithm 1. Thus, the auxiliary variables t^{ij}, t^{ji} can be eliminated using (A.7) and, based on (A.4), for each player $i \in V$ only the dual variables $w^i(k)$ (A.8) are needed to update his *action* $x_i^i(k)$ and *estimates* $x_{-i}^i(k)$, using his and his neighbours' previous estimates, $x^i(k-1)$, $x^j(k-1)$, $j \in N_i$. Specifically, minimizing (A.4) w.r.t. x_i^i and with (A.7), for each $i \in V$ the local update for $x_i^i(k) = x_i(k)$ (decision) is

$$\begin{aligned}
x_i^i(k) = \arg\min_{x_i^i\in\mathbb{R}} \Big\{ & J_i(x_i^i, x_{-i}^i(k-1)) + \mathcal{I}_{\Omega_i}(x_i^i) \quad \text{(A.9)}\\
&+ w_i^i(k)^T x_i^i + c\sum_{j\in N_i}\Big\| x_i^i - \frac{x_i^i(k-1) + x_i^j(k-1)}{2}\Big\|^2\Big\}.
\end{aligned}$$

Note that in DOP, where the cost $f_i(x_i)$ is separable in x_i (cf. (1)), in the update of x_i, coupling appears only linearly due to the quadratic terms, e.g., Wei and Ozdaglar (2013), Hong et al. (2016). In contrast, in (A.9) the x_i^i problem is coupled nonlinearly to $x_{-i}^i(k-1)$ (due to the inherent coupling in J_i) and linearly to $x_i^j(k-1)$. As in an *inexact ADMM*, instead of solving exactly an optimization sub-problem, each player uses a linear proximal approximation to reduce the complexity of each subproblem. We approximate $J_i(x_i^i, x_{-i}^i(k-1))$ in (A.9) by a proximal first-order approximation around $x_i^i(k-1)$, but inspired by Hong et al. (2016), with an extra quadratic penalty, which yields

$$\begin{aligned}
x_i^i(k) = \arg\min_{x_i^i\in\mathbb{R}} \Big\{ & \nabla_i J_i(x^i(k-1))^T(x_i^i - x_i^i(k-1))\\
&+\frac{\beta_i}{2}\|x_i^i - x_i^i(k-1)\|^2 + \mathcal{I}_{\Omega_i}(x_i^i) \quad \text{(A.10)}
\end{aligned}$$

$$+w_i^i(k)^T x_i^i+\bar{c}\sum_{j\in N_i}\Big\|x_i^i-\frac{x_i^i(k-1)+x_i^j(k-1)}{2}\Big\|^2\Bigg\},$$

where $\beta_i>0$, $\bar{c}=c+c_0$, $c_0\geq 0$. This is equivalent to

$$x_i^i(k)=\arg\min_{x_i^i\in\mathbb{R}}\Big\{\mathcal{I}_{\Omega_i}(x_i^i)+\frac{\alpha_i}{2}\Big\|x_i^i-\alpha_i^{-1}\Big[-\nabla_i J_i(x^i(k-1))$$
$$+\beta_i x_i^i(k-1)-w_i^i(k)+\bar{c}\sum_{j\in N_i}\big(x_i^i(k-1)+x_i^j(k-1)\big)\Big]\Big\|^2\Big\},$$

where $\alpha_i=\beta_i+2\bar{c}|N_i|$. Using (5) and $\text{prox}^{\alpha_i}_{\mathcal{I}_{\Omega_i}}\{\cdot\}=T_{\Omega_i}\{\cdot\}$, each player i updates its *action* $x_i^i(k)=x_i(k)$ as in Step 6 of Algorithm 1. Finally, each player i updates its *estimates* of the other players' actions $x_{-i}^i(k)$. This update is similar to (A.9) but ignoring the $J_i(x_i^i,x_{-i}^i)$ term (which is minimized only w.r.t. x_i^i, see (A.1)), i.e.,

$$x_{-i}^i(k)=\arg\min_{x_{-i}^i\in\mathbb{R}^{N-1}}\Big\{w_{-i}^i(k)^T x_{-i}^i$$
$$+\bar{c}\sum_{j\in N_i}\Big\|x_{-i}^i-\frac{x_{-i}^i(k-1)+x_{-i}^j(k-1)}{2}\Big\|^2\Big\}.$$

Following the same proximal approximation, the update for $x_{-i}^i(k)$ is obtained as in Step 7 of Algorithm 1.

B Proof of Proposition 1

The stacked vector update for $\mathbf{w}$, (18), follows from Step 5 of Algorithm 1. From Step 6 in Algorithm 1, $\alpha_i x_i^i(k)=T_{\Omega_i}\{y\}=\text{prox}_{\mathcal{I}_{\Omega_i}}\{y\}$, where y is the term in the square-bracket on the right-hand side of Step 6. From Proposition 16.34, Bauschke and Combettes (2011), $p=\text{prox}_{\mathcal{I}_{\Omega_i}}\{y\}\Leftrightarrow y-p\in\partial\mathcal{I}_{\Omega_i}(p)$. Thus, $\alpha_i x_i^i(k)=\text{prox}_{\mathcal{I}_{\Omega_i}}\{y\}\Leftrightarrow y-\alpha_i x_i^i(k)\in\partial\mathcal{I}_{\Omega_i}(\alpha_i x_i^i(k))$ $\Leftrightarrow y-\alpha_i x_i^i(k)\in\partial\mathcal{I}_{\Omega_i}(x_i^i(k))$ since $\alpha_i>0$. Expanding y as on the right-hand side of Step 6 and using $\alpha_i=\beta_i+2\bar{c}|N_i|>0$ this is equivalent to

$$\mathbf{0}\in\nabla_i J_i(x_i^i(k-1),x_{-i}^i(k-1))+\partial_i\mathcal{I}_{\Omega_i}(x_i^i(k))+w_i^i(k)+$$
$$(\beta_i+2\bar{c}|N_i|)x_i^i(k)-\beta_i x_i^i(k-1)-\bar{c}\sum_{j\in N_i}(x_i^i(k-1)+x_i^j(k-1))$$

i.e.,

$$\mathbf{0}\in\nabla_i J_i(x_i^i(k-1),x_{-i}^i(k-1))+\partial_i\mathcal{I}_{\Omega_i}(x_i^i(k))+w_i^i(k)+$$
$$(\beta_i+2\bar{c}|N_i|)(x_i^i(k)-x_i^i(k-1))+\bar{c}\sum_{j\in N_i}(x_i^i(k-1)-x_i^j(k-1)).$$

From Step 7 in Algorithm 1, one can obtain for x_{-i}^i,

$$\mathbf{0}_{N-1}=(\beta_i+2\bar{c}|N_i|)(x_{-i}^i(k)-x_{-i}^i(k-1))+w_{-i}^i(k)$$
$$+\bar{c}\sum_{j\in N_i}(x_{-i}^i(k-1)-x_{-i}^j(k-1)).$$

Combining these two relations into a single vector one for all components of $x^i=(x_i^i,x_{-i}^i)$, yields $\forall i\in V$

$$\mathbf{0}_N\in\nabla_i J_i(x^i(k-1))\,e_i+\partial_i\mathcal{I}_{\Omega_i}(x_i^i(k))\,e_i+w^i(k)+$$
$$(\beta_i+2\bar{c}|N_i|)(x_i^i(k)-x_i^i(k-1))+\bar{c}\sum_{j\in N_i}(x_i^i(k-1)-x_i^j(k-1)).$$

In stacked vector form for all $i\in V$ this yields

$$\mathbf{0}_{N^2}\in[\nabla_i J_i(x^i(k-1))e_i]_{i\in V}+[\partial_i\mathcal{I}_{\Omega_i}(x_i^i(k))e_i]_{i\in V}+\mathbf{w}(k)$$
$$+((\mathbb{B}+2\bar{c}\mathbb{D})\otimes I_N)(\mathbf{x}(k)-\mathbf{x}(k-1))$$
$$+\bar{c}(L\otimes I_N)\mathbf{x}(k-1),$$

where $\mathbf{x}=[x^i]_{i\in V}$, $\mathbf{w}=[w^i]_{i\in V}$, $\mathbb{B}=\text{diag}([\beta_i]_{i\in V})$, $\mathbb{D}=\text{diag}([|N_i|]_{i\in V})$, which can be written as (18). ■

Proof of Lemma 2

We decompose the augmented space $\mathbb{R}^{N^2}$ into the consensus subspace $Ker(\mathbf{L})=\{\mathbf{1}_N\otimes x\,|x\in\ \mathbb{R}^N\}$ and its orthogonal complement $Ker(\mathbf{L})^\perp$. Any $\mathbf{x}\in\mathbb{R}^{N^2}$ can be decomposed as $\mathbf{x}=\mathbf{x}^\parallel+\mathbf{x}^\perp$, with $\mathbf{x}^\parallel\in Ker(\mathbf{L})$, $\mathbf{x}^\perp\in Ker(\mathbf{L})^\perp$, $(\mathbf{x}^\parallel)^T\mathbf{x}^\perp=0$, Thus $\mathbf{x}^\parallel=\mathbf{1}_N\otimes x$, for some $x\in\mathbb{R}^N$, so that $\mathbf{L}\mathbf{x}^\parallel=0$, and $min_{\mathbf{x}^\perp\in Ker(\mathbf{L})^\perp}(\mathbf{x}^\perp)^T\mathbf{L}\mathbf{x}^\perp=\lambda_2(L)\|\mathbf{x}^\perp\|^2$, where $\lambda_2(L)>0$. Similarly, we can write any $\mathbf{y}\in Ker(\mathbf{L})$ as $\mathbf{y}=\mathbf{1}_N\otimes y$, for some $y\in\mathbb{R}^N$ and $\mathbf{L}\mathbf{y}=0$. Then, using $\mathbf{x}=\mathbf{x}^\parallel+\mathbf{x}^\perp$, $\mathbf{F}(\mathbf{x}^\parallel)=F(x)$, $\mathbf{F}(\mathbf{y})=F(y)$, $\mathbf{R}^T\mathbf{x}^\parallel=x$, $\mathbf{R}^T\mathbf{y}=y$ we can write

$$
\begin{aligned}
&(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big)\\
&\quad=(\mathbf{x}^\parallel-\mathbf{y})^T\mathbf{R}\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)+\mathbf{F}(\mathbf{x}^\parallel)-\mathbf{F}(\mathbf{y})\big)\\
&\qquad+(\mathbf{x}^\perp)^T\mathbf{R}\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)+\mathbf{F}(\mathbf{x}^\parallel)-\mathbf{F}(\mathbf{y})\big)\\
&\qquad+c_0(\mathbf{x}^\parallel+\mathbf{x}^\perp-\mathbf{y})^T\mathbf{L}(\mathbf{x}^\parallel+\mathbf{x}^\perp-\mathbf{y})\big)\\
&\quad=(x-y)^T\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\big)+(x-y)^T\big(F(x)-F(y)\big)\\
&\qquad+(\mathbf{x}^\perp)^T\mathbf{R}\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\big)+(\mathbf{x}^\perp)^T\mathbf{R}\big(F(x)-F(y)\big)\\
&\qquad+c_0(\mathbf{x}^\perp)^T\mathbf{L}\mathbf{x}^\perp \qquad\qquad\qquad\qquad\qquad\qquad\text{(B.1)}
\end{aligned}
$$

Using strong monotonicity of F (by Assumption 2) for the second term, and properties of $\mathbf{L}$ on $Ker(\mathbf{L})^\perp$ for the fifth term on the right-hand side we can write,

$$
\begin{aligned}
&(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big)\\
&\quad\geq(x-y)^T\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\big)+\mu\|x-y\|^2\\
&\qquad+(\mathbf{x}^\perp)^T\mathbf{R}\big(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\big)+(\mathbf{x}^\perp)^T\mathbf{R}\big(F(x)-F(y)\big)\\
&\qquad+c_0\lambda_2(L)\|\mathbf{x}^\perp\|^2 \qquad\qquad\qquad\qquad\qquad\qquad\text{(B.2)}
\end{aligned}
$$

We deal with the cross-terms by using $a^Tb\geq-\|a\|\|b\|$, for any a, b, and $\|\mathbf{R}b\|\leq\|\mathbf{R}\|_2\|b\|$, so that

$$
\begin{aligned}
&(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big)\geq\\
&\quad-\|x-y\|\,\|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\|+\mu\|x-y\|^2\\
&\quad-\|\mathbf{x}^\perp\|\|\mathbf{R}\|_2\|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{x}^\parallel)\|-\|\mathbf{x}^\perp\|\|\mathbf{R}\|_2\|F(x)-F(y)\|\\
&\quad+c_0\lambda_2(L)\|\mathbf{x}^\perp\|^2 \qquad\qquad\qquad\qquad\qquad\qquad\text{(B.3)}
\end{aligned}
$$

Using $\|\mathbf{R}\|_2=1$ and Lipschitz properties of F and $\mathbf{F}$,

$$
\begin{aligned}
&(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big)\geq\\
&\quad-\theta\|x-y\|\,\|\mathbf{x}^\perp\|+\mu\|x-y\|^2-\theta\|\mathbf{x}^\perp\|^2-\theta_0\|\mathbf{x}^\perp\|\|x-y\|
\end{aligned}
$$

$$+c_0\lambda_2(L)\|\mathbf{x}^\perp\|^2 \tag{B.4}$$

Using $\|x-y\| = \frac{1}{\sqrt{N}}\|\mathbf{x}^\parallel - \mathbf{y}\|$ and Ψ, (24), we can write

$$(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big) \geq \begin{bmatrix}\|\mathbf{x}^\parallel-\mathbf{y}\|\\ \|\mathbf{x}^\perp\|\end{bmatrix}^T \Psi \begin{bmatrix}\|\mathbf{x}^\parallel-\mathbf{y}\|\\ \|\mathbf{x}^\perp\|\end{bmatrix}$$

Thus,

$$(\mathbf{x}-\mathbf{y})^T\big(\mathbf{R}(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y}))+c_0\mathbf{L}(\mathbf{x}-\mathbf{y})\big) \geq \bar{\mu}\left(\|\mathbf{x}^\parallel-\mathbf{y}\|^2 + \|\mathbf{x}^\perp\|^2\right),$$

where $\bar{\mu} := \lambda_{\min}(\Psi)$. Using $\|\mathbf{x}^\parallel - \mathbf{y}\|^2 + \|\mathbf{x}^\perp\|^2 = \|\mathbf{x}-\mathbf{y}\|^2$, this can be written as (25). For c_0 as in the statement, $\Psi \succ 0$, hence $\bar{\mu} > 0$. ■

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