Constrained discounted Markov decision processes with Borel state spaces

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Abstract. We study discrete-time discounted constrained Markov decision processes (CMDPs) with Borel state and action spaces. These CMDPs satisfy either weak (W) continuity conditions, that is, the transition probability is weakly continuous and the reward function is upper semicontinuous in state-action pairs, or setwise (S) continuity conditions, that is, the transition probability is setwise continuous and the reward function is upper semicontinuous in actions. Our main goal is to study models with unbounded reward functions, which are often encountered in applications, e.g., in consumption/investment problems. We provide some general assumptions under which the optimization problems in CMDPs are solvable in the class of randomized stationary policies and in the class of chattering policies introduced in this paper. If the initial distribution and transition probabilities are atomless, then using a general "purification result" of Feinberg and Piunovskiy we show the existence of a deterministic (stationary) optimal policy. Our main results are illustrated by examples.

1. Introduction

In this paper, we study discrete-time constrained discounted Markov decision processes on Borel spaces with unbounded reward functions. A common feature of almost all previous studies of CMDPs with Borel state spaces is the assumption that the transition probabilities (reward functions) are jointly weakly continuous (upper semicontinuous) in state-action pairs. This condition played a significant role in the convex analytic approach in exploiting properties of the sets of occupancy measures, see Hernández-Lerma and González-Hernández (2000); Mao and Piunovskiy (2000); Piunovskiy (1997), Dufour and Prieto-Rumeau (2016) and Zhang (2013). The same continuity assumptions were also applied by Feinberg and Piunovskiy (2000, 2002), who examined Markov deterministic optimal policies in non-homogeneous Markov decision processes

(MDPs). Feinberg and Rothblum (2012) studied constrained absorbing (in particular, discounted) MDPs with both weakly and setwise continuous transition probabilities. MDPs with setwise continuous transition probabilities and multiple criteria were also considered in Feinberg and Piunovskiy (2019), because their results for such a model with finite action sets were used there to prove that, if an MDP is atomless (that is, initial and transition probabilities are atomless), for an arbitrary policy there exists a non-randomized stationary policy with the same vector of the expected total costs. This result is repoerted in Feinberg and Piunovskiy (2019) for uniformly absorbing (in particular, discounted) atomless MDPs without any continuity assumptions on costs and transition probabilities.

This paper studies discounted CMDPs with Borel state and action spaces and unbounded reward functions, where the transition probabilities are weakly or setwise continuous. A basic ingredient in our approach is the equivalence of the weak topology and the so-called ws^{∞} -topology (introduced by Schäl (1975)) on the space of probability measures induced by policies. This fact was first observed by Nowak (1988) and then generalized by Balder (1989), who dealt with the action sets depending on partial histories of the underlying process. The aforementioned results allow us to work in the space of strategic measures and to prove the existence of a solution, say π^* , to a discounted CMDP (with unbounded from above and below reward functions) in the class of history dependent policies. Furthermore, making use of certain lemmas on occupancy measures given under various assumptions by Borkar (1988), Piunovskiy (1997) and Feinberg and Rothblum (2012), we show that π^* can be replaced by a randomized stationary optimal policy φ^* . In the next step, we assume that the initial distribution and the transition probability are atomless. Applying a recent theorem on the performance sets in vector-valued MDPs due to Feinberg and Piunovskiy (2019) we "purify" φ^* and obtain a non-randomized stationary optimal policy.

In some sense this paper complements the results of Feinberg and Rothblum (2012), who studied absorbing CMDPs with rewards bounded from above. Absorbing MDPs are more general than discounted MDPs. In certain cases discounted MDPs with unbounded rewards can be transformed to discounted models with bounded and bounded from above rewards (see Chapter 5 in van der Wal (1981)). Then, the results of Feinberg and Rothblum (2012) can be also applied to those MDPs. In this paper, we use assumptions under which the standard transformations do not work. This paper is also relevant to the work of Dufour and Prieto-Rumeau (2016), who considered unbounded costs with weakly continuous transitions and some finer topology on the space of strategic measures. By imposing uniform integrability conditions for the positive parts of the reward functions in our model, we use the standard weak topology on the space of strategic measures induced by policies. We show that the assumptions imposed by Dufour and Prieto-Rumeau (2016) imply our assumptions and that the topology introduced there is the standard weak topology under assumptions of that paper.

The related literature on the models of CMDPs with finite or countable state spaces is quite large. The reader is referred to Altman (1998, 1999); Borkar (1994); Feinberg and Shwartz (1995, 1996, 1999); Kallenberg (1983); Piunovskiy (1997) and references cited therein.

The significance of CMDPs for various applications is very-well documented. The models with discrete state spaces are often used to study admission or flow control problems in queueing networks, see Altman (1999); Hordijk and Spieksma (1989); Lazar (1983); Piunovskiy (1997); Sennott (1991) and Vakil and Lazar (1987) among others. The models with uncountable Borel state spaces, on the other hand, are fundamental for inventory systems, consumption/investment problems, and some issues in mathematical finance or insurance, see Feinberg and Piunovskiy (2000, 2002); Mao and Piunovskiy (2000); Piunovskiy (1997); Zhang (2013). The methods which are used in the study of CMDPs are based on linear programming approach (see, e.g., Altman (1999); Kallenberg (1983); Hernández-Lerma and González-Hernández (2000); Dufour and Prieto-Rumeau (2016)), convex analysis (see, e.g., Borkar (1994); Feinberg and Rothblum (2012); Feinberg and Shwartz (1995, 1996); Mao and Piunovskiy (2000); Piunovskiy (1997); Zhang (2013), Lagrange multipliers (see Altman (1998, 1999); Piunovskiy (1997)), and dynamic programming techniques (see, e.g., Chen and Blankenship (2004); Chen and Feinberg (2007); Piunovskiy and Mao (2000)).

This paper is organized as follows. Section 2 describes the model. Section 3 contains results on the existence of a solution in the class of randomized stationary policies. In Section 4, we study some special classes of randomized stationary policies called "chattering" and present a result on the existence of deterministic stationary optimal policies in a class of CMDPs with atomless transitions. Section 5 contains three illustrative examples related to models developed in operations research or economics. Finally, in Sections 6 and 7 we give additional comments on our main assumptions.

2. The model

Let \mathbb{N} be the set of all positive integers and \mathbb{R} be the set of all real numbers. Moreover, put $\mathbb{R}_- := \mathbb{R} \cup \{-\infty\}$. By a Borel space, say Y, we mean a non-empty Borel subset of a complete separable metric space. Let $\mathcal{B}(Y)$ denote the σ -algebra of all Borel subsets of Y and $\Pr(Y)$ be the space of all probability measures on $\mathcal{B}(Y)$ endowed with the weak topology. This is the coarsest topology on $\Pr(Y)$ in which the functional $\nu \to \int_Y u d\nu$ is upper semicontinuous for every bounded above upper semicontinuous function $u: Y \to \mathbb{R}_-$. For any $B \in \mathcal{B}(Y)$, by $1_B(\cdot)$ we denote the indicator function of the set B.

By a Borel measurable transition probability from Y to a Borel space Z we mean a function $\gamma: \mathcal{B}(Z) \times Y \to [0,1]$ such that, for each $B \in \mathcal{B}(Z)$, $\gamma(B,\cdot)$ is a Borel measurable function on Y and $\gamma(\cdot,y) \in \Pr(Z)$ for each $y \in Y$. We shall write $\gamma(B|y)$ for $\gamma(B,y)$.

Let $I := \{1, \ldots, m\}$ and $I_0 = \{0\} \cup I$, where $m \in \mathbb{N}$.

A constrained Markov decision process (CMDP) is characterized by the objects: S, A, \mathbb{K} , p, μ , r_0 , r_1 , ..., r_m , β with the following meanings.

- (i) S is a Borel state space.
- (ii) A is the action space and is also assumed to be a Borel space.
- (iii) A(s) is a non-empty set of actions available in state $s \in S$. It is assumed that the set

$$\mathbb{K} := \{(s, a): \ a \in A(s), s \in S\}$$

is Borel in $S \times A$.

- (iv) p is a transition probability from \mathbb{K} to S.
- (v) $\mu \in \Pr(S)$ is an initial distribution.
- (vi) $r_0: \mathbb{K} \to \mathbb{R}_-$ is a Borel measurable stage reward function.
- (vii) $r_i: \mathbb{K} \to \mathbb{R}_-, i \in I$, are Borel measurable sources of constraints.
- (viii) $\beta \in (0,1)$ is a discount factor.

Let H_n be the space of all feasible histories up to the *n*-step, i.e.,

$$H_n = \mathbb{K}^{n-1} \times S$$
 for $n \ge 2$ and $H_1 = S$.

An element of H_n is called a partial history of the process and is of the form

$$h_1 = s_1$$
 and $h_n := (s_1, a_1, \dots, s_{n-1}, a_{n-1}, s_n), n \ge 2.$

A policy for the decision maker is a sequence $\pi = (\pi_n)$ of transition probabilities π_n from H_n into A such that $\pi_n(A(s_n)|h_n) = 1$ for all $h_n \in H_n$ and $n \in \mathbb{N}$. The set of all policies is denoted by Π . By Φ we denote the set of all Borel measurable mappings $\varphi : S \to \Pr(A)$ such that $\varphi(A(s))(s) = 1$ for each $s \in S$. Every $\varphi \in \Phi$ induces a transition probability $\varphi(\cdot|s) = \varphi(s)(\cdot)$ from S to A. Markov policy is a sequence $\pi = (\varphi_n)$, where each $\varphi_n \in \Phi$ for $n \in \mathbb{N}$. The set of all Markov policies is denoted by Π_M . A stationary policy is a constant sequence $\pi = (\varphi, \varphi, \ldots)$, where $\varphi \in \Phi$, and is identified with φ . Therefore, the set of all stationary policies will be also denoted by Φ . If the support of each measure $\varphi_n(s)(\cdot)$ is a single point for every $s \in S$, then $\pi = (\varphi_n)$ is called non-randomized or deterministic Markov (stationary) policy. If every set A(s) is compact, then by the Arsenin-Kunugui theorem (see Theorem 18.18 in Kechris (1995)) the correspondence $s \to A(s)$ admits a Borel measurable selector, that is, a mapping $f: S \to A$ such that $f(s) \in A(s)$ for every $s \in S$. We use F to denote both the set of all such selectors and the set of all deterministic stationary policies.

Let $((S \times A)^{\infty}, \mathcal{T})$ be the measurable space, where \mathcal{T} denotes the corresponding product σ -algebra. Due to the theorem of Ionescu-Tulcea (see Proposition V.1.1 in Neveu (1965)), for each policy $\pi \in \Pi$ there exists a unique probability measure P^{π}_{μ} on \mathcal{T} such that for all $D \in \mathcal{B}(A)$, $B \in \mathcal{B}(S)$ and $h_n = (s_1, a_1, \ldots, s_{n-1}, a_{n-1}, s_n)$ in H_n , $n \in \mathbb{N}$,

$$P_{\mu}^{\pi}(s_1 \in B) = \mu(B),$$

$$P_{\mu}^{\pi}(a_n \in D|h_n) = \pi_n(D|h_n),$$

$$P_{\mu}^{\pi}(s_{n+1} \in B|h_n, a_n) = p(B|s_n, a_n).$$

By E^{π}_{μ} we denote the expectation operator with respect to the probability measure P^{π}_{μ} . Define

$$J_i(\pi) := E_\mu^\pi \left(\sum_{n=1}^\infty \beta^{n-1} r_i(s_n, a_n) \right), \quad i \in I_0.$$
 (1)

Below, we formulate assumptions under which the functionals $J_i(\pi)$, $i \in I_0$ are well-defined for every $\beta \in (0,1)$.

Problem Statement. Let d_1, \ldots, d_m be fixed real numbers. Consider the following control problem:

(CP) maximize
$$J_0(\pi)$$

subject to $J_i(\pi) \ge d_i$, $i \in I$.

In the sequel, we shall tacitly assume that problem (CP) is non-trivial, i.e., the set of feasible policies is non-empty and there exists a feasible policy π such that $J_0(\pi) > -\infty$. We now make the following assumptions:

(A1) There exists a Borel measurable function $w: S \to [1, \infty)$ such that $r_i(s, a) \leq w(s)$ for each $(s, a) \in \mathbb{K}$ and for all $i \in I_0$.

Furthermore, we shall also consider the following stronger version of (A1).

(A1') $|r_i(s,a)| \leq w(s)$ for each $(s,a) \in \mathbb{K}$ and for all $i \in I_0$.

(A2)

$$\lim_{n \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=n}^{\infty} \beta^{k-1} w(s_k) \right) = 0, \tag{2}$$

and for each $n \in \mathbb{N}$

$$\lim_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(w(s_n) 1_{\{w(s_n) \ge l\}} \right) = 0 \tag{3}$$

Assumption (2) is used in Dufour and Prieto-Rumeau (2016) and is related to condition (C) in Schäl (1975); Feinberg and Rothblum (2012). Condition in (3) is a sort of the uniform integrability of the function w with respect to strategic measures. It is related with Assumption \mathbf{B} in Dufour and Genadot (2019). Clearly, here $1_{\{w(s_n) \geq l\}}$ is understood as $1_{\{w(s_n) \geq l\}}(s_n)$.

Remark 1 Observe that

$$0 \leq \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{\infty} \beta^{k-1} w(s_{k}) 1_{\{w(s_{k}) \geq l\}} \right)$$

$$\leq \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{n-1} \beta^{k-1} w(s_{k}) 1_{\{w(s_{k}) \geq l\}} \right) + \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=n}^{\infty} \beta^{k-1} w(s_{k}) \right),$$

for each $\pi \in \Pi$. Take any $\epsilon > 0$ and by (2) choose some n such that

$$0 \le \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{\infty} \beta^{k-1} w(s_k) 1_{\{w(s_k) \ge l\}} \right) \le \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{n-1} \beta^{k-1} w(s_k) 1_{\{w(s_k) \ge l\}} \right) + \epsilon.$$

This fact and (3) imply that

$$0 \le \limsup_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{\infty} \beta^{k-1} w(s_k) 1_{\{w(s_k) \ge l\}} \right) \le \epsilon.$$

Consequently, we have that

$$\lim_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=1}^{\infty} \beta^{k-1} w(s_k) 1_{\{w(s_k) \ge l\}} \right) = 0.$$
 (4)

From (3) with n=1 or (4), it follows that $\int_S w(s)\mu(ds) < \infty$.

If the functions r_i are bounded from above, then the function w in (A1) may be constant. In this case, (A2) is trivially satisfied. Some additional comments and examples, where (A1) and (A2) are satisfied, are given in Section 6 and 7.

Under assumptions (A1)-(A2) all expectations in (1) are well-defined, since for $r_i^+(s, a) := \max\{r_i(s, a), 0\}, (s, a) \in \mathbb{K}$, by (2) we have

$$0 \le J_i^+(\pi) := E_\mu^\pi \left(\sum_{n=1}^\infty \beta^{n-1} r_i^+(s_n, a_n) \right) \le E_\mu^\pi \left(\sum_{n=1}^\infty \beta^{n-1} w(s_n) \right) < \infty.$$
 (5)

Let $r_i^-(s, a) := \min\{r_i(s, a), 0\}, (s, a) \in \mathbb{K}, \text{ and }$

$$J_i^-(\pi) := E_\mu^\pi \left(\sum_{n=1}^\infty \beta^{n-1} r_i^-(s_n, a_n) \right).$$

Therefore, every $J_i(\pi)$, $i \in I_0$, can be written as

$$J_i(\pi) = J_i^{-}(\pi) + J_i^{+}(\pi). \tag{6}$$

3. Optimality of randomized stationary policies

In order to formulate our main results we need the following standard compactness and semicontinuity assumptions. For MDPs with weakly continuous transitions we assume:

- (W1) For each $s \in S$, the action set A(s) is compact and the set-valued mapping $s \to A(s)$ is upper semicontinuous, that is, the set $\{s \in S : A(s) \cap B \neq \emptyset\}$ is closed for every closed set $B \subset A$.
- (W2) For each bounded continuous function $v: S \to \mathbb{R}$, the function $(s, a) \to \int_S v(t) p(dt|s, a)$ is continuous on \mathbb{K} (weak continuity of p).
- (W3) The functions r_i , $i \in I_0$, are upper semicontinuous on \mathbb{K} .

For MDPs with setwise continuous transitions we assume:

- (S1) For each $s \in S$, the action set A(s) is compact and the set \mathbb{K} is Borel.
- (S2) For every $s \in S$ and for each $D \in \mathcal{B}(S)$, the function $p(D|s, \cdot)$ is continuous on A(s) (setwise continuity of p).
- (S3) The functions $r_i(s,\cdot)$, $i \in I_0$, are upper semicontinuous on A(s) for each $s \in S$.

We can now state our main result in this section.

Theorem 1 Assume that (A1)-(A2) and either (W1)-(W3) or (S1)-(S3) hold. Then, there exists an optimal stationary policy $\varphi^* \in \Phi$.

Consider the following additional assumption.

(A3) There exists $\delta > 0$ such that

$$\int_{S} w(t)p(dt|s,a) \le \delta w(s) \quad \text{for all} \quad (s,a) \in \mathbb{K}.$$

If $\beta \delta < 1$, then (A3) implies that (2) holds.

Corollary 1 Assume that (A1) and (3) and either (W1)-(W3) or (S1)-(S3) hold. Assume in addition that (A3) is satisfied and $\beta\delta$ < 1. Then, there exists an optimal stationary policy.

Remark 2 In models of MDPs (constrained or unconstrained) with weakly continuous transitions the following assumption is often made.

(W4) The function w in (A1) is continuous on S and $(s,a) \to \int_S w(t)p(dt|s,a)$ is continuous on \mathbb{K} .

In models with setwise continuous transitions the following additional assumption is sometimes imposed.

(S4) For every $s \in S$, the function $a \to \int_S w(t) p(dt|s,a)$ is continuous on A(s).

It should be noted that in general deterministic policies are not sufficient for solving the problem (CP), see for example Frid (1972).

Remark 3 Assumptions (A1) and (A3) were frequently used in the studies of discounted unconstrained MDPs with unbounded reward functions. The common approach is based on consideration of the weighted norm defined with the aid of the function w, see Wessels (1977); van der Wal (1981); Hernández-Lerma and Lasserre (1999). Under assumptions of Corollary 1, one can consider a standard transformation of the CMDP to a model with rewards bounded from above, see van der Wal (1981), Remark 2.5 in Dufour and Prieto-Rumeau (2016), or Section 10 in Feinberg and Piunovskiy (2019). Making use of this transformation, Corollary 1 can be deduced from Theorem 9.1 of Feinberg and Rothblum (2012), who studied CMDPs with reward functions r_i bounded

from above. However, one has to assume in addition that (W4) or (S4) holds. That is because the transition probability functions in the transformed model depend on w and should have the weak or setwise continuity property.

Remark 4 The two alternative sets of conditions (W1)-(W3) and (S1)-(S3) were exploited in the literature on stochastic dynamic programming, see for example Balder (1989); Bertsekas and Shreve (1978); Dynkin and Yushkevich (1979); Hernández-Lerma and Lasserre (1996); Schäl (1975, 1979). We would like to point out that there are several works on CMDPs with transition probabilities that are weakly continuous on \mathbb{K} (jointly continuous in (s, a)) and with reward/cost functions that are upper/lower semicontinuous on \mathbb{K} . Such assumptions are naturally satisfied in a number of models, where the transition law is determined by some difference equation with random shocks. For a survey of various results on CMDPs with weakly continuous transition probabilities the reader is referred to Piunovskiy (1997); Mao and Piunovskiy (2000); Hernández-Lerma and González-Hernández (2000); Dufour and Prieto-Rumeau (2016); Zhang (2013) and Feinberg and Piunovskiy (2000, 2002). CMDPs with both weakly and setwise continuous transitions were studied in Feinberg and Rothblum (2012), whereas the models with measurable transition and reward functions are examined in Feinberg and Piunovskiy (2002, 2019).

3.1. The set of strategic measures

Let $\Omega := (S \times A)^{\infty}$. It is known that Ω is a Borel space. Let $\pi \in \Pi$. We refer to P^{π}_{μ} as the *strategic probability measure* generated by the policy π and the initial distribution μ . Let \mathcal{P} be the *set of strategic probability measures*, i.e.,

$$\mathcal{P} := \{ P_{\mu}^{\pi} : \ \pi \in \Pi \} \subset \Pr(\Omega).$$

Under condition (S1), \mathbb{K}^{∞} is a Borel subset of Ω . Condition (W1) implies that \mathbb{K}^{∞} is a closed subset of Ω . For each $P^{\pi}_{\mu} \in \mathcal{P}$, $P^{\pi}_{\mu}(\mathbb{K}^{\infty}) = 1$. Let \mathcal{C}_n (\mathcal{U}_n) be the set of all bounded (bounded from above) Borel functions on $(S \times A)^n$ having the following property. A function u belongs to \mathcal{C}_n (\mathcal{U}_n), if $u(s_1, \cdot, ..., s_n, \cdot)$ is continuous (upper semicontinuous) on $A(s_1) \times \cdots \times A(s_n)$ for any sequence of states $(s_1, ..., s_n)$. Note that \mathcal{U}_n contains the class of all upper semicontinuous and bounded from above functions on $(S \times A)^n$. Schäl (1975) defined the ws^{∞} -topology on $\Pr(\Omega)$ as the coarsest topology in which the functionals $P \to \int_{\Omega} udP$ are continuous for each $u \in \mathcal{C}_n$ and $n \in \mathbb{N}$. Assuming (S2), A(s) = A for all $s \in S$ and A is compact, Schäl (1975) showed that \mathcal{P} is compact in the ws^{∞} -topology on $\Pr(\Omega)$. Schäl (1979) also discussed a more general case, in which A(s) may depend on s. However, no formal proofs were given. Later, Nowak (1988) observed that the relative ws^{∞} -topology on \mathcal{P} is equivalent to the weak topology on this space of measures. As a consequence, the functionals $P \to \int_{\Omega} udP$ are continuous on the compact space \mathcal{P} endowed with the weak topology for any $u \in \mathcal{C}_n$, $n \in \mathbb{N}$. It is worthy to mention that assumption

¹This result implies that the topology on \mathcal{P} is metrizable and one can think of convergence of sequences in \mathcal{P} instead of nets.

(S2), the initial distribution and an argument related to the Scorza-Dragoni theorem (see Kucia (1991)) play a fundamental role in the proof of Nowak (1988). Balder (1989) extended the result of Nowak (1988) by allowing the action spaces to depend on the partial histories of the process.

Let $\hat{\mathcal{U}}_n$ be the set of all functions from \mathcal{U}_n restricted to the set \mathbb{K}^n . Every $u \in \mathcal{U}_n$ can be viewed as a function on \mathbb{K}^{∞} that depends only on the first 2n coordinates.

From Theorem 2.1 and Proposition 3.2 in Balder (1989), we conclude the following statement.

Lemma 1 (a) Let assumptions (S1)-(S2) or (W1)-(W2) be satisfied. Then, the set \mathcal{P} is a compact subset of $Pr(\Omega)$ endowed with the weak topology.

- (b) If (S1)-(S2) hold, then the functional $P \to \int_{\mathbb{K}^{\infty}} u dP$ is upper semicontinuous on \mathcal{P} for each $u \in \hat{\mathcal{U}}_n$, $n \in \mathbb{N}$.
- (c) If (W1)-(W2) are satisfied, then the functional $P \to \int_{\mathbb{K}^{\infty}} udP$ is upper semicontinuous on \mathcal{P} for every bounded from above and upper semicontinuous function u on \mathbb{K}^n , $n \in \mathbb{N}$.

Note that equivalently the discounted reward functional may be written as follows

$$J_i(\pi) = \int_{\mathbb{K}^{\infty}} \sum_{n=1}^{\infty} \beta^{n-1} r_i^+(s_n, a_n) dP_{\mu}^{\pi} + \int_{\mathbb{K}^{\infty}} \sum_{n=1}^{\infty} \beta^{n-1} r_i^-(s_n, a_n) dP_{\mu}^{\pi}.$$
 (7)

Therefore, $J_i(\pi)$ can be viewed as a function of $P^{\pi}_{\mu} \in \mathcal{P}$. Sometimes, we shall write $J_i(P^{\pi}_{\mu})$ for $J_i(\pi)$.

We now state our basic lemma.

Lemma 2 Under assumptions of Theorem 1, the discounted reward functionals $J_i : \mathcal{P} \to \mathbb{R}_-$ are upper semicontinuous for all $i \in I_0$.

Proof First, we prove the lemma assuming (S1)-(S3) and that every function r_i is non-negative. Consider the truncated functions $r_i^l(s, a) = \min\{l, r_i(s, a)\}$ for $(s, a) \in \mathbb{K}$, $l \in \mathbb{N}$ and $i \in I_0$. Then, every function $r_i^l(s, \cdot)$ is upper semicontinuous on A(s) for every $s \in S$. Note that under condition (A1), for every $N \in \mathbb{N}$,

$$\sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{n=N+1}^{\infty} \beta^{n-1} r_i^l(s_n, a_n) \right) \le \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{n=N+1}^{\infty} \beta^{n-1} w(s_n) \right). \tag{8}$$

Let

$$J_i^l(\pi) := E_\mu^\pi \left(\sum_{n=1}^\infty \beta^{n-1} r_i^l(s_n, a_n) \right) \quad \text{and} \quad J_i^{l,N}(\pi) := E_\mu^\pi \left(\sum_{n=1}^N \beta^{n-1} r_i^l(s_n, a_n) \right).$$

From (8) and (2), it follows that $J_i^{l,N}(\pi)$ converges to $J_i^l(\pi)$ as $N \to \infty$, uniformly in $\pi \in \Pi$. Therefore, it is sufficient to prove that for each N, $J_i^{l,N}(\pi)$ (understood as a

function of P_{μ}^{π}) is upper semicontinuous on \mathcal{P} . Observe that

$$\sup_{\pi \in \Pi} \left| E_{\mu}^{\pi} \left(\sum_{n=1}^{N} \beta^{n-1} r_{i}(s_{n}, a_{n}) \right) - E_{\mu}^{\pi} \left(\sum_{n=1}^{N} \beta^{n-1} r_{i}^{l}(s_{n}, a_{n}) \right) \right| \leq \sum_{n=1}^{N} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(r_{i}(s_{n}, a_{n}) - r_{i}^{l}(s_{n}, a_{n}) \right) \leq \sum_{n=1}^{N} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(w(s_{n}) 1_{\{w(s_{n}) \geq l\}} \right). \tag{9}$$

By (3) in (A2), for each n = 1, ..., N,

$$0 \le \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(w^{l}(s_n) 1_{\{w(s_n) \ge l\}} \right) \to 0 \quad \text{as} \quad l \to \infty.$$

This fact and (9) imply that

$$\sup_{\pi \in \Pi} \left| E_{\mu}^{\pi} \left(\sum_{n=1}^{N} \beta^{n-1} r_i(s_n, a_n) \right) - E_{\mu}^{\pi} \left(\sum_{n=1}^{N} \beta^{n-1} r_i^l(s_n, a_n) \right) \right| \to 0 \quad \text{as} \quad l \to \infty.$$
 (10)

By Lemma 1(b), the functional $P \to \int_{\mathbb{K}^{\infty}} u_i^l dP$ with

$$u_i^l(s_1, a_1, \dots, s_N, a_N) = \sum_{n=1}^N \beta^{n-1} r_i^l(s_n, a_n)$$

is upper semicontinuous on \mathcal{P} . Since the uniform limit of a sequence of upper semicontinuous functionals is upper semicontinuous, using (10), one can easily conclude that $P \to \int_{\mathbb{K}^{\infty}} u_i dP$ with

$$u_i(s_1, a_1, \dots, s_N, a_N) = \sum_{n=1}^N \beta^{n-1} r_i(s_n, a_n)$$

is also upper semicontinuous on \mathcal{P} . Thus, we have proved the lemma under conditions (S1)-(S3) for non-negative functions r_i , $i \in I_0$. The proof for $r_i \geq 0$ under assumptions (W1)-(W3) makes use of Lemma 1(c) and proceeds along the same lines. Hence, we can conclude, in both cases under consideration, that the functionals $P_{\mu}^{\pi} \to J_i^+(\pi)$ are upper semicontinuous on \mathcal{P} . Since $r_i^- \leq 0$, every functional $P_{\mu}^{\pi} \to J_i^-(\pi)$ is also upper semicontinuous on \mathcal{P} . The assertion now follows because $J_i(\pi) = J_i^-(\pi) + J_i^+(\pi)$. \square

In Section 6 we show a relation of Lemma 2 under conditions (W1)-(W3) with some recent results on weak convergence of measures and unbounded mappings.

3.2. Occupancy measures and randomized stationary optimal policies

Let $Q^{\pi}_{\mu}(ds \times da)$ be the *occupancy measure* on $\mathcal{B}(S \times A)$ of a policy $\pi \in \Pi$, i.e., the measure defined as

$$Q_{\mu}^{\pi}(Z) := E_{\mu}^{\pi} \left(\sum_{n=1}^{\infty} \beta^{n-1} 1_{Z}(s_{n}, a_{n}) \right), \quad Z \in \mathcal{B}(S \times A).$$
 (11)

This measure is finite and concentrated on the set \mathbb{K} for $\beta \in (0,1)$. Since every $\pi \in \Pi$ determines uniquely the probability measure $P^{\pi}_{\mu} \in \mathcal{P}$, equality (11) shows how Q^{π}_{μ} is determined by P^{π}_{μ} .

Let \mathcal{Q}^{Π}_{μ} and \mathcal{Q}^{Φ}_{μ} be the sets of all occupancy measures of policies $\pi \in \Pi$ and $\varphi \in \Phi$, respectively. By Lemma 4.1 in Feinberg and Rothblum (2012) or Proposition D8 in Hernández-Lerma and Lasserre (1996), for any $\pi \in \Pi$ there exists some $\varphi \in \Phi$ such that

$$Q^{\pi}_{\mu}(B \times D) = \int_{B} \varphi(D|s) q^{\pi}_{\mu}(ds), \quad B \in \mathcal{B}(S), \ D \in \mathcal{B}(A),$$

where q_{μ}^{π} is the projection of Q_{μ}^{π} on S, i.e., $q_{\mu}^{\pi}(B) = Q_{\mu}^{\pi}(B \times A)$ for any $B \in \mathcal{B}(S)$. Moreover, $Q_{\mu}^{\pi} = Q_{\mu}^{\varphi}$. The proof of this fact is given in Borkar (1988) (see Lemma 3.1). Although Borkar (1988) considered models on a countable state space, his proof also applies to our framework, since it does not require any continuity assumptions of the transition probability. The same result for Borel state space models was reported in Lemma 24 in Piunovskiy (1997) and Theorem 1 in Zhang (2013) or Lemma 4.2 in Feinberg and Rothblum (2012). Therefore, we can formulate the following result (see Corollary 4.1 in Feinberg and Rothblum (2012)).

Lemma 3 $\mathcal{Q}_{\mu}^{\Pi}=\mathcal{Q}_{\mu}^{\Phi}.$

Lemma 3 directly implies the following statement, on which the convex analytic approach to MDPs is based; see Remark 6 and, e.g., Borkar (1988); Piunovskiy (1997); Mao and Piunovskiy (2000); Hernández-Lerma and González-Hernández (2000).

Lemma 4 For each $\pi \in \Pi$ there exists some $\varphi \in \Phi$ such that $J_i(\pi) = J_i(\varphi)$ for all $i \in I_0$.

Proof of Theorem 1 Lemmas 1 and 2 imply that

$$\mathcal{P}^* := \{ P_{\mu}^{\pi} \in \mathcal{P} : J_1(P_{\mu}^{\pi}) \ge d_1, \dots, J_m(P_{\mu}^{\pi}) \ge d_m \}$$

is a compact subset of \mathcal{P} . Therefore, there exists a strategic measure $P_{\mu}^{\pi^*} \in \mathcal{P}^*$ such that

$$\max_{P_{\mu}^{\pi} \in \mathcal{P}^*} J_0(P_{\mu}^{\pi}) = J_0(P_{\mu}^{\pi^*}).$$

By Lemma 4, there exists some $\varphi^* \in \Phi$ such that $J_i(\pi^*) = J_i(\varphi^*)$ for all $i \in I_0$. Clearly, φ^* is a solution to problem (CP). \square

Remark 5 Assume that S contains an absorbing state s^* with zero rewards. Then, $p(S \setminus \{s^*\} | s, a) \leq 1$ for all $(s, a) \in \mathbb{K}$. Assumptions (A1) and (A2) can be considered with $\beta = 1$ and w such that $w(s^*) = 0$ and $w(s) \geq 1$ for all $s \neq s^*$. Lemma 2 with this modification remains correct. If the other assumptions of Theorem 1 are satisfied, then Lemmas 1 and 2 imply the existence of an optimal policy. In general, Lemma 4 may not hold for $\beta = 1$; see examples in Feinberg and Sonin (1996). However, this lemma and

the suggested version of Theorem 1 with $\beta = 1$ hold for absorbing MDPs considered in Altman (1999); Feinberg and Rothblum (2012).

Remark 6 From (11), it follows that

$$\int_{\mathbb{K}} r(s, a) Q_{\mu}^{\pi}(ds \times da) = E_{\mu}^{\pi} \left(\sum_{n=1}^{\infty} \beta^{n-1} r(s_n, a_n) \right), \tag{12}$$

for all non-negative measurable function $r: S \times A \to \mathbb{R}$. Formulae (5) and (6) imply that (12) holds for $r = r_i$, $i \in I_0$. By Lemma 1(a) and (12), we conclude that \mathcal{Q}^{Π}_{μ} is compact in the weak topology on the space of measures ν on $\mathcal{B}(S \times A)$ such that $\nu(S \times A) = 1/(1-\beta)$. For a more detailed discussion the reader is referred to Section 4 in Feinberg and Rothblum (2012). Convexity of \mathcal{Q}^{Π}_{μ} follows from convexity of \mathcal{P} using standard arguments based on disintegration of measures on the product space, see, e.g., Piunovskiy (1997), Schäl (1979) or Corollary 4.1 in Feinberg and Rothblum (2012). We have shown that problem (CP) has a solution $\varphi^* \in \Phi$. It is now clear that $\mathcal{Q}^{\varphi^*}_{\mu}$ solves the following linear programming problem:

(CP0) maximize
$$\int_{\mathbb{K}} r_0(s,a) Q_\mu^\pi(ds \times da)$$
 subject to $Q_\mu^\pi \in \mathcal{Q}_\mu^\Pi$ and
$$\int_{\mathbb{K}} r_i(s,a) Q_\mu^\pi(ds \times da) \geq d_i, \quad i \in I.$$

Remark 7 Under additional assumptions one can give a characterization of the solution $Q_{\mu}^{\varphi^*}$ to problem (CP0) as in Piunovskiy (1997); Mao and Piunovskiy (2000) or Zhang (2013). The first assumption is (A1'). The second condition requires that $r_i(s,\cdot)$ is continuous on A(s) for all $s \in S$, $i \in I$. By Lemma 2 applied to r_i and $-r_i$, the functional $P_{\mu}^{\pi} \to J_i(\pi)$ is continuous on \mathcal{P} . The last assumption is Slater's condition demanding that there exists a policy $\pi' \in \Pi$ such that $J_i(\pi') > d_i$ for all $i \in I$. Using the Lagrange functional approach as in Piunovskiy (1997) and Zhang (2013), with minor modifications, one can prove that $Q_{\mu}^{\varphi^*} = \sum_{j=1}^{m+1} \xi_j Q_{\mu}^{f_j}$, where $f_j \in F$, $\xi_j \geq 0$ for all j and $\sum_{j=1}^{m+1} \xi_j = 1$. Three ingredients play a significant role in the proof: the relation between $P_{\mu}^{\pi} \in \mathcal{P}$ and Q_{μ}^{π} with $\pi = \varphi$ given in (11), the compactness and convexity of \mathcal{P} and finally, Lemma 3, which implies that $Q_{\mu}^{\varphi} \to \int_{\mathbb{K}} r_i(s,a)Q_{\mu}^{\varphi}(ds \times da)$ is continuous (upper semicontinuous) on Q_{μ}^{Φ} for each $i \in I$ (for i = 0). Linear programming problems for CMDPs with weakly continuous transitions and unbounded cost functions satisfying condition similar to (A1) and some additional assumptions were studied by Dufour and Prieto-Rumeau (2016). Feinberg and Rothblum (2012) and Feinberg and Piunovskiy (2019), on the other hand, extended many earlier results on CMDPs to a class of total reward models with absorbing states including discounted ones.

4. Chattering and deterministic optimal stationary policies

Let $f_1, ..., f_N \in F$ be any stationary deterministic policies and $\alpha_1, ..., \alpha_N$ be non-negative numbers such that $\sum_{j=1}^N \alpha_j = 1$. By $Q_{\mu}^{f_j}$ we denote the occupancy measure induced by $f_j \in F$. Its projection on S is denoted by $q_{\mu}^{f_j}$. Define

$$Q(\cdot) := \sum_{j=1}^{N} \alpha_j Q_{\mu}^{f_j}(\cdot).$$

Note that Q is also an occupancy measure since the set of occupancy measures is convex. By Proposition D8 in Hernández-Lerma and Lasserre (1996), there exists $\varphi \in \Phi$ such that

$$Q(B \times D) = \int_{B} \varphi(D|s)q(ds), \quad \text{for all} \quad B \in \mathcal{B}(S), \ D \in \mathcal{B}(A), \tag{13}$$

where q is the projection of Q on S. Since Q is an occupancy measure, we now write $Q = Q_{\nu}^{\varphi}$ and $q = q_{\mu}^{\varphi}$ in (13). Note that for any bounded Borel measurable function $g: \mathbb{K} \to \mathbb{R}$

$$\int_{\mathbb{K}} g(s,a) Q_{\mu}^{\varphi}(ds \times da) = \int_{S} \int_{A(s)} g(s,a) \varphi(da|s) q_{\mu}^{\varphi}(ds).$$

If $\delta_{f_i(s)}(\cdot)$ is the Dirac measure with support at the point $f_i(s)$, then

$$\int_{\mathbb{K}} g(s,a) Q_{\mu}^{f_j}(ds \times da) = \int_{S} \int_{A(s)} g(s,a) \delta_{f_j(s)}(da) q_{\mu}^{f_j}(ds) = \int_{S} g(s,f_j(s)) q_{\mu}^{f_j}(ds).$$

Lemma 5 There exist Borel measurable functions $\gamma_j: S \to [0,1]$ (j=1,...,N) such that $\sum_{j=1}^N \gamma_j(s) = 1$ and $\varphi(\cdot|s) = \sum_{j=1}^N \gamma_j(s) \delta_{f_j(s)}(\cdot)$ $(q_\mu^{\varphi}\text{-a.e.}).$

Proof Observe that $q_{\mu}^{\varphi} = \sum_{j=1}^{N} \alpha_{j} q_{\mu}^{f_{j}}$. Hence, $q_{\mu}^{f_{j}} \ll q_{\mu}^{\varphi}$ for every $j=1,\ldots,N$. Let $\rho_{j} := \frac{q_{\mu}^{f_{j}}}{q_{\mu}^{\varphi}}$ be a non-negative Borel measurable Radon-Nikodým derivative of $q_{\mu}^{f_{j}}$ with respect to q_{μ}^{φ} . Define $\gamma_{j}(s) = \alpha_{j}\rho_{j}(s)$, $s \in S$. Observe that for every $B \in \mathcal{B}(S)$

$$\int_{B} \left(\sum_{j=1}^{N} \gamma_{j}(s)\right) q_{\mu}^{\varphi}(ds) = \int_{B} \left(\sum_{j=1}^{N} \alpha_{j} \rho_{j}(s)\right) q_{\mu}^{\varphi}(ds) = \sum_{j=1}^{N} \alpha_{j} q_{\mu}^{f_{j}}(B) = q_{\mu}^{\varphi}(B).$$

This implies that $\sum_{j=1}^{N} \gamma_j(s) = 1$ for all $s \in B_1$ where $B_1 \in \mathcal{B}(S)$ and $q_{\mu}^{\varphi}(B_1) = 1$. For any $s \notin B_1$ we can modify our definition of $\gamma_j(s)$. Namely, we can put $\gamma_1(s) = 1$ and $\gamma_j(s) = 0$ for j = 2, ..., N.

Let

$$\overline{\varphi}(\cdot|s) := \sum_{j=1}^{N} \gamma_j(s) \delta_{f_j(s)}(\cdot). \tag{14}$$

For a bounded Borel measurable function g on \mathbb{K} , we have

$$\int_{S} \int_{A(s)} g(s,a) \varphi(da|s) q_{\mu}^{\varphi}(ds) = \int_{\mathbb{K}} g(s,a) Q_{\mu}^{\varphi}(ds \times da)$$

$$= \sum_{j=1}^{N} \alpha_{j} \int_{\mathbb{K}} g(s,a) Q_{\mu}^{f_{j}}(ds \times da) = \sum_{j=1}^{N} \int_{S} \int_{A(s)} \alpha_{j} g(s,a) \delta_{f_{j}(s)}(da) q_{\mu}^{f_{j}}(ds)$$

$$= \int_{S} \int_{A(s)} g(s,a) \overline{\varphi}(da|s) q_{\mu}^{\varphi}(ds). \tag{15}$$

Since (15) holds for every bounded Borel measurable function $g: \mathbb{K} \to \mathbb{R}$, we conclude that $\varphi(\cdot|s) = \overline{\varphi}(\cdot|s)$ (q_{μ}^{φ} -a.e.). \square

The following terminology is borrowed from the theory of variational calculus and control theory, see Roubiček (1997). A stationary policy $\phi \in \Phi$ is called *chattering*, if there exist a family of N Borel functions $\gamma_j : S \to [0,1]$ and a family of N deterministic stationary policies $f_j \in F$ such that

$$\phi(D|s) = \sum_{j=1}^{N} \gamma_j(s)\delta_{f_j(s)}(D)$$
 and $\sum_{j=1}^{N} \gamma_j(s) = 1$

for each $D \in \mathcal{B}(S)$ and for all $s \in S$.

Let $\pi \in \Pi$. Following Feinberg and Rothblum (2012), we define the *performance vector* $\mathcal{V}(\pi) := (J_0(\pi), J_1(\pi), ..., J_m(\pi))$ and the *performance set* $\mathcal{V} := {\mathcal{V}(\pi) : \pi \in \Pi}$.

We are now ready to state our main result in this section.

Theorem 2 Under assumptions of Theorem 1, there exists a chattering stationary policy $\phi \in \Phi$ with N = m + 1 that solves problem (CP).

Proof By Theorem 1, there exists an optimal stationary policy $f^* \in \Phi$. Using the same geometric arguments as in the proof of Theorem 9.2 in Feinberg and Rothblum (2012), one can conclude that $\mathcal{V}(f^*)$ lies on the boundary of the performance set \mathcal{V} and therefore the occupancy measure $Q_{\mu}^{f^*}$ can be represented as

$$Q_{\mu}^{f^*} = \sum_{j=1}^{m+1} \alpha_j Q_{\mu}^{f_j}$$

with some $f_1, ..., f_{m+1} \in F$ and non-negative numbers $\alpha_1, ..., \alpha_{m+1}$ such that $\sum_{j=1}^{m+1} \alpha_j = 1$. Let $\varphi \in \Phi$ be as in (13) with $Q = Q_{\mu}^{f^*}$. By Lemma 5, there exist Borel measurable functions $\gamma_j : S \to [0,1]$ (j=1,...,m+1) such that $\sum_{j=1}^{m+1} \gamma_j(s) = 1$ for all $s \in S$, and a family of m+1 deterministic stationary policies $f_j \in F$ such that for

$$\phi(\cdot|s) := \sum_{j=1}^{m+1} \gamma_j(s) \delta_{f_j(s)}(\cdot)$$

we have $\phi(\cdot|s) = \varphi(\cdot|s)$ $(q_{\mu}^{\varphi}\text{-a.e.})$. Moreover, it follows that $Q_{\mu}^{f^*} = Q_{\mu}^{\varphi} = Q_{\mu}^{\phi}$. This implies that $J_i(f^*) = J_i(\phi)$ for all $i \in I_0$. Since f^* is optimal, the chattering stationary policy ϕ is optimal as well. \square

Remark 8 Feinberg and Rothblum (2012) showed in Theorem 9.2(i) that for any feasible policy π there exist m+2 stationary deterministic policies $f_j \in F$ and non-negative numbers $\alpha_1, ..., \alpha_{m+2}$ such that

$$Q_{\mu}^{\pi} = \sum_{j=1}^{m+2} \alpha_j Q_{\mu}^{f_j}$$

and $\sum_{j=1}^{m+2} \alpha_j = 1$. Their proof can also be used under assumptions of Theorem 1. Applying Lemma 5 with N = m+2 one can easily conclude that $\overline{\varphi}$ defined in (14) has the property that $Q_{\mu}^{\pi} = Q_{\mu}^{\varphi} = Q_{\mu}^{\overline{\varphi}}$. Therefore, for the chattering policy $\overline{\varphi}$ in (14) with N = m+2 we have that $J_i(\pi) = J_i(\varphi) = J_i(\overline{\varphi})$ for all $i \in I_0$.

Combining Corollary 1 with a general "purification result" stated as Corollary 10.2 in Feinberg and Piunovskiy (2019), we can conclude the following fact.

Corollary 2 Assume that the initial distribution and transition probabilities are atomless and assumptions of Corollary 1 with (A1) replaced by (A1') are satisfied. Then, there exists a deterministic stationary policy $\tilde{f} \in F$ that solves problem (CP).

Proof Under assumptions of Corollary 1 with (A1') instead of (A1), the model can be transformed to an absorbing CMDP with bounded reward functions. This is mentioned in Section 10 of Feinberg and Piunovskiy (2019). By Corollary 1, there exists an optimal stationary policy f^* in the original model. (Since no continuity conditions are imposed on the function w, we cannot in this place conclude the existence of f^* in the transformed model established, e.g., in Feinberg and Rothblum (2012).) By Corollary 10.2 in Feinberg and Piunovskiy (2019), there exists some deterministic stationary policy $\tilde{f} \in F$ giving the same expected discounted rewards in both transformed and original model. Thus, we have $J_i(f^*) = J_i(\tilde{f})$ for all $i \in I_0$. Obviously, \tilde{f} is an optimal policy. \square

5. Examples

In this section, we provide three examples satisfying assumptions imposed in Corollaries 1-2. In addition, we indicate a class of examples in which for any $\beta < 1$ there exists some $\delta > 0$ such that (A3) is satisfied and $\delta \beta < 1$. This fact implies that (2) is satisfied. By Lemma 9 in Section 7 assumption (A3) and some continuity assumptions imply that (3) holds.

We start with two models of economic growth theory, see Stachurski (2009). In Example 1 the dynamics is deterministic, whereas Example 2 includes a stochastic component.

Example 1 Consider a dynamic growth model with $S = [0, +\infty)$ and A(s) = [0, s]. Assume that the level of the resource stock evolves according to the equation $s_{n+1} = \sqrt{s_n - a_n}$, where $n \in \mathbb{N}$. Assume that the utility or reward function for the economic agent is $r_0(s, a) = a - 1/a$ for $(s, a) \in \mathbb{K}$, with $a \in A(s) := [0, s]$, whilst the reward function of the authorities is $r_1(s, a) = \ln s$ for $(s, a) \in \mathbb{K}$. For s = a = 0, $r_1(s, a) := -\infty$. The agent's problem is to solve problem (CP) with the constant d_1 provided by the authorities.

Let w(s) = s + c for $s \in S$ with some constant $c \ge 1$ and let μ be an initial distribution on S such that $\int_S w(s)\mu(ds) < \infty$. Observe that (A1) holds. Moreover, it must hold

$$w(\sqrt{s-a}) = \sqrt{s-a} + c \le \delta(s+c)$$
 for all $a \in [0,s]$ and $s \in S$

with some $\delta > 0$. Note that, for all $(s, a) \in \mathbb{K}$, we have

$$\frac{\sqrt{s-a}+c}{s+c} \le \frac{\sqrt{s}+c}{s+c} = \le 1 + \frac{\sqrt{s}-s}{1+c} \le 1 + \frac{1}{4+4c}.$$

Put $\delta := 1 + 1/(4 + 4c)$. It is easily seen that for every $\beta < 1$ there exists $c \ge 1$ so that $\beta \delta < 1$. Obviously, (W1)-(W4) are satisfied and by Theorem 2 there exists an optimal stationary chattering policy.

Based on the aforementioned example, we may generalize the method for finding δ such that $\delta\beta < 1$ for the given discount factor $\beta < 1$. Suppose that we have some continuous function $w_0: S \to [1, +\infty)$ such that it holds

$$\int_{S} w_0(t)q(dt|s,a) \le w_1(s), \quad (s,a) \in \mathbb{K},$$

where w_1 is a non-negative continuous function such that $\theta := \sup_{s \in S} (w_1(s) - w_0(s)) < \infty$. Then, we define $w(s) := w_0(s) + c$ for $s \in S$ and some $c \ge 1$. Simple calculations give that

$$\frac{\int_{S} (c + w_0(t)) q(dt|s, a)}{w_0(s) + c} \le \frac{w_1(s) + c}{w_0(s) + c} \le 1 + \frac{w_1(s) - w_0(s)}{1 + c}$$

for any $(s, a) \in \mathbb{K}$. Define now

$$\delta := \max\{1 + \frac{\theta}{1+c}, 1\}.$$

Hence, if $\theta > 0$ for every $\beta < 1$ we may take c sufficiently large in order to have $\beta \delta < 1$. In the second case, when $\theta \leq 0$, the condition $\beta \delta < 1$ is always satisfied. Many examples, which we have encountered in the literature can be reduced to this case, when

- $S = [0, \infty),$
- the functions w_0 and w_1 are increasing,

- equation $w_0(s) = w_1(s)$ has a unique solution $s^* > 0$.
- $w_1(s) w_0(s) < 0$ for all $s > s^*$.

The next two examples are dedicated to the application of Corollary 2. The first example is given with weakly continuous transition probabilities, whereas in the second one the transition probabilities are setwise continuous.

Example 2 Consider the model from Example 1, but with different dynamics for the resource stock. Suppose that the level of the resource is described by the following equation

$$s_{n+1} = s_n - a_n + \sqrt{s_n - a_n} + \xi_n$$
, for $n \in \mathbb{N}$.

where $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables taking values in the interval $[0, +\infty)$. Moreover, every ξ_n has an atomless distribution ρ such that $\overline{m} := \int_0^\infty z \rho(dz) < \infty$. Additionally, assume the initial state is chosen at random according to an atomless measure μ . Suppose that $r_0(s,a) = \sqrt{a}$ and $r_1(s,a) = \ln(s+1)$ for every $(s,a) \in \mathbb{K}$. Let d_1 be a given number. The agent again faces problem (CP). Let us define w(s) = s + c with some $c \geq 1$. Then, (A1') is satisfied. Furthermore, we obtain

$$\int_{S} w(t)q(dt|s,a) = s - a + \sqrt{s - a} + c + \overline{m} \le s + \sqrt{s} + c + \overline{m}, \quad \text{for } (s,a) \in \mathbb{K}.$$

Consequently,

$$\frac{\int_{S} w(t)q(dt|s,a)}{s+c} \le 1 + \frac{\sqrt{s} + \overline{m}}{s+c} \le 1 + \frac{1}{2} \frac{1}{\sqrt{\overline{m}^2 + c} - \overline{m}}.$$

Thus, (A3) is satisfied with $\delta := 1 + \frac{1}{2} \frac{1}{\sqrt{\overline{m^2} + c - \overline{m}}}$. Note that for any value of the discount coefficient $\beta < 1$, we may choose $c \ge 1$ so that $\beta \delta < 1$. Hence, by Corollary 2, there exists an optimal deterministic stationary policy solving (CP).

Example 3 Assume that a system can be in a state $s \in [0, 1]$, where 0 denotes the perfect condition and 1 means that the system is completely broken. At the end of each month the system is checked, and its state is observed. The higher values of s, the worse condition of the system is. The controller each month decides about the repair. He chooses some $a \in A := [0,1]$, where a = 0 means no repair and a = 1 is the replacement of the old system by a new one. Hence, the larger a, the more serious repair is required. The transition probability p is absolutely continuous with respect to the Lebesgue measure on S. In other words, there exists a density $g(\cdot, s, a)$ for every $(s, a) \in \mathbb{K}$ such that

$$p(B|s,a) = \int_B g(t,s,a)dt$$
 for $(s,a) \in \mathbb{K}$ and $B \in \mathcal{B}(S)$.

Additionally, let $g(t, s, \cdot)$ be continuous on A for every $t, s \in S$. The cost associated with the repair is $c_0(s, a)$ for $(s, a) \in \mathbb{K}$. Moreover, $c_0(s, \cdot)$ is bounded and lower semicontinuous

on A for every $s \in S$. The management of the company requires that the sum of the discounted values describing the system's state cannot be greater that d_1 (i.e., $c_1(s, a) = s$ for $(s, a) \in \mathbb{K}$). The initial distribution μ is atomless. Hence, the constrained control problem is

(CP0) minimize
$$E_{\nu}^{\pi} \left(\sum_{n=1}^{\infty} \beta^{n-1} c_0(s_n, a_n) \right)$$

subject to $E_{\nu}^{\pi} \left(\sum_{n=1}^{\infty} \beta^{n-1} s_n \right) \leq d_1$.

Note that all assumptions in Corollary 2 are satisfied. Thus, there exists an optimal deterministic stationary policy for problem (CP0).

6. A comment on the approach of Dufour and Prieto-Rumeau

In this section, we show that assumptions imposed by Dufour and Prieto-Rumeau (2016) imply the uniform integrability condition (3) and that the w^{α} -topology introduced in their paper is equivalent with the standard weak topology on \mathcal{P} . We also demonstrate how Lemma 2 under assumptions (A1)-(A2) and (W1)-(W3) can be briefly deduced from recent results on weak convergence of finite measures.

Let \mathcal{M} be a family of finite measures on a Borel space Y. An extended real-valued Borel function $v:Y\to [-\infty,\infty]$ is called uniformly integrable with respect to the family \mathcal{M} if

$$\lim_{l \to \infty} \sup_{\eta \in \mathcal{M}} \int_{Y} |v(y)| 1_{\{|v(y)| \ge l\}} \eta(dy) = 0.$$

The following known lemma is useful for some considerations in this paper.

Lemma 6 For a sequence (η_n) of finite measures on a Borel space Y converging weakly to a measure η and for an upper semicontinuous function $v: Y \to [-\infty, \infty]$, the following statements hold:

(a) if $v^+ = \max\{v, 0\}$ is uniformly integrable with respect to $\mathcal{M} = \{\eta_n : n \in \mathbb{N}\}$, then

$$\int_{Y} v(y)\eta(dy) \ge \limsup_{n \to \infty} \int_{Y} v(y)\eta_n(dy),$$

(b) if v is continuous and uniformly integrable with respect to $\mathcal{M} = \{\eta_n\}$, then the inequality in (a) can be replaced by the equality.

Lemma 6(a) follows from Fatou's lemma for uniformly integrable functions and weakly converging measures (Theorem 2.4 in Feinberg et al. (2018)). Statement (b) is a particular case of Theorem 2 in Zapała (2008). A related result was given in Dufour and Genadot (2019), see Theorem 3.2. Statement (b) can be also concluded from point (a) applied to the functions v and -v.

Dufour and Prieto-Rumeau (2016) consider a CMDP that satisfies conditions (W1)-(W3), (A1), (2) (their Assumption A, (B.3), C) and some additional assumptions, called (B.1) and (B.2), involving a continuous function w. They endow the set of strategic measures \mathcal{P} with the so-called w^{α} -topology (α is a fixed discount factor). From their definition, it follows that the w^{α} -topology on \mathcal{P} is finer than the weak topology and by Theorem 3.10 in Dufour and Prieto-Rumeau (2016), \mathcal{P} is a metrizable compact space. From the definition of the w^{α} -topology on \mathcal{P} , it follows that the functional $P^{\pi}_{\mu} \to E^{\pi}_{\mu}(w(s_n))$ is continuous for any $n \in \mathbb{N}$. Since $w(s)1_{\{w(s)< l\}}(s)$ is bounded and lower semicontinuous on \mathcal{P} endowed with the weak topology. Therefore, it is lower semicontinuous on \mathcal{P} endowed with the w^{α} -topology. Thus,

$$P_{\mu}^{\pi} \to E_{\mu}^{\pi}(w(s_n)1_{\{w(s_n) \ge l\}}) = E_{\mu}^{\pi}(w(s_n)) - E_{\mu}^{\pi}(w(s_n)1_{\{w(s_n) < l\}})$$

is upper semicontinuous on the compact space \mathcal{P} in the w^{α} -topology. Note that

$$E_{\mu}^{\pi}(w(s_n)1_{\{w(s_n)>l\}})\downarrow 0$$
 as $l\to\infty$.

From Dini's theorem (see Chapter 9 in Royden (1988)), it follows that

$$\lim_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi}(w(s_n) 1_{\{w(s_n) \ge l\}}) = 0,$$

that is, our assumption (3) is satisfied. Theorem 1 shows that the standard weak topology on \mathcal{P} is sufficient to solve problem (CP) under conditions given by Dufour and Prieto-Rumeau (2016).

Let us assume that (3) holds and w is continuous. As follows from Lemma 6(b), the functional $P^{\pi}_{\mu} \to E^{\pi}_{\mu}(w(s_n))$ is continuous on \mathcal{P} endowed with the weak topology. Hence, it follows that the weak topology is finer than the w^{α} -topology on \mathcal{P} . Thus, the two topologies on \mathcal{P} are equivalent.

We now show how Lemma 2 under assumptions (A1)-(A2) and (W1)-(W3) can be deduced from Lemma 6. Let us define for each $i \in I_0$ the function $u_i(s_1, a, s_2, a_2, ...) := \sum_{n=1}^{\infty} \beta^{n-1} r_i(s_n, a_n)$ if $\sum_{n=1}^{\infty} \beta^{n-1} r^+(s_n, a_n) < \infty$, and $u_i(s_1, a, s_2, a_2, ...) := \infty$ otherwise. This function is upper semicontinuous on \mathbb{K}^{∞} . Formula (4) implies that u_i are uniformly integrable with respect to \mathcal{P} . Lemma 6(a) implies upper semicontinuity of the functional $P_{\mu}^{\pi} \to \int_{\mathbb{K}^{\infty}} u dP_{\mu}^{\pi}$.

7. Additional remarks on our basic assumptions

Conditions (W1)-(W2), (W4) and (A3) imposed in Lemma 9 below, are discussed in Remark 2.3 in Dufour and Prieto-Rumeau (2016). They imply their assumption (B.2) which plays an important role in proving compactness of \mathcal{P} in the w^{α} -topology. This result was used in Section 6 in our proof that the assumptions of Dufour and Prieto-Rumeau (2016) imply the uniform integrability condition (3). Below we show directly that (W1)-(W2), (W4) and (A3) imply (3) (see the proof of Lemma 9). We also discuss the alternative case with conditions (S1)-(S2), (S4) and (A3).

Let us define two classes of non-negative functions on S denoted by $\hat{B}_{+}(S)$ and $\hat{U}_{+}(S)$, respectively. A non-negative function v belongs to the class $\hat{B}_{+}(S)$ ($\hat{U}_{+}(S)$) if it is Borel measurable (upper semicontinuous) on S and there exists a constant c > 0 such that $v(s) \leq cw(s)$ for all $s \in S$. For any $v \in \hat{B}_{+}(S)$ define

$$(Mv)(s) := \sup_{a \in A(s)} \int_{S} v(t)p(dt|s,a), \quad s \in S.$$

$$(16)$$

Lemma 7 Assume that (A3) is satisfied.

- (a) If $v \in \hat{U}_+(S)$ and (W1)-(W2) and (W4) hold, then $Mv \in \hat{U}_+(S)$.
- (b) If $v \in \hat{B}_{+}(S)$ and (S1)-(S2) and (S4) hold, then $Mv \in \hat{B}_{+}(S)$.

Proof (a) There exists some c > 0 such that $cw - v \ge 0$. The function cw - v is lower semicontinuous and non-negative. Thus, it is a pointwise limit of a non-decreasing sequence of bounded continuous functions. Therefore, from (W2), it follows that the function $(s,a) \to \int_S (cw(t) - v(t))p(dt|s,a)$ is lower semicontinuous on \mathbb{K} . This fact and (W4) imply that $(s,a) \to \int_S v(t)p(dt|s,a)$ is upper semicontinuous on \mathbb{K} . From (A3), it follows that there exists some constant $c_1 > 0$ such that $(Mv)(s) \le c_1w(s)$ for all $s \in S$. Using (W1) and the maximum theorem of Berge (1963), we conclude that $Mv \in \hat{U}_+(S)$.

(b) The function cw-v is non-negative for some c>0 and is a pointwise limit of a non-decreasing sequence of bounded Borel measurable functions. By (S2), the function $a \to \int_S (cw(t)-v(t))p(dt|s,a)$ is lower semicontinuous on A(s) for each $s \in S$. This fact and (S4) imply that $s \to \int_S v(t)p(dt|s,a)$ is upper semicontinuous on A(s) for each $s \in S$. By (A3), $(Mv)(s) \le c_1w(s)$ for all $s \in S$ and for some $c_1 > 0$. Using Corollary 1 in Brown and Purves (1973) and (S1), we conclude that $Mv \in \hat{B}_+(S)$. \square

Lemma 8 Let (v^l) be a non-increasing sequence of functions on S such that $v^l(t) \downarrow 0$ for each $t \in S$ as $l \to \infty$. Let (A3) be satisfied. Assume that $v^l \in \hat{U}_+(S)$ for all $l \in \mathbb{N}$ and conditions (W1)-(W2) and (W4) hold, or $v^l \in \hat{B}_+(S)$ for all $l \in \mathbb{N}$ and assumptions (S1)-(S2) and (S4) are satisfied. Then, $(Mv^l)(s) \downarrow 0$ for each $s \in S$ as $l \to \infty$.

Proof From the proof of Lemma 7, we know that in both cases, the function $a \to \int_S v^l(t) p(dt|s,a)$ is upper semicontinuous on A(s) for each $s \in S$, $l \in \mathbb{N}$. By the monotone convergence theorem, $\int_S v^l(t) p(dt|s,a) \downarrow 0$ for all $(s,a) \in \mathbb{K}$ as $l \to \infty$. The assertion follows now from Dini's theorem. \square

Lemma 9 Let (A3) be satisfied. Assume that either conditions (W1)-(W2) and (W4) or (S1)-(S2) and (S4) are satisfied. Then, condition (3) in (A2) holds.

Proof Let $v^l(s_n) = w(s_n) 1_{\{w(s_n) \ge l\}}$. For n = 1 we have

$$0 \le E_{\mu}^{\pi}(v_l(s_1)) \le \int_S v^l(s_1)\mu(ds_1) \to 0 \text{ as } l \to \infty.$$

If $n \geq 2$, then

$$0 \le \sup_{\pi \in \Pi} E_{\mu}^{\pi}(v^{l}(s_{n})) \le \int_{S} (M^{n-1}v^{l})(s_{1})\mu(ds_{1}),$$

where M^{n-1} is the (n-1)st composition of the operator M defined in (16) with itself. By Lemmas 7 and 8 and the monotone convergence theorem, it follows by induction on n that

$$\lim_{l \to \infty} \int_{S} (M^{n-1}v^{l})(s_{1})\mu(ds_{1}) = 0.$$

Thus, (3) in (A2) follows. \square

We now consider MDPs similar to those studied in Jaśkiewicz and Nowak (2011). Let (X_k) be a sequence of non-empty Borel subsets of S such that $X_k \subset X_{k+1}$ for each $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} X_k = S$. Let $m_k = \sup_{s \in X_k} w(s), k \in \mathbb{N}$.

(A4) For each
$$x \in X_k$$
, $a \in A(s)$, $k \in \mathbb{N}$, $p(X_{k+1}|s,a) = 1$ and $\sum_{k=1}^{\infty} m_k \beta^{k-1} < \infty$.

Lemma 10 Under assumption (A4) conditions (A1) and (A2) hold with $w(s) := m_k$ for all $s \in X_k$, $k \in \mathbb{N}$.

Proof Clearly, (A1) and (2) are obviously satisfied. By the monotone convergence theorem, the condition in (3) holds for n=1. Assume that $n\geq 2$. For any nonnegative Borel measurable function \tilde{w} on S, we define by $E^{\pi}_{s_1}(\tilde{w}(s_n))$ the conditional expectation of \tilde{w} with respect to the nth state given the initial state s_1 . Then, we have $E^{\pi}_{\mu}(\tilde{w}(s_n)) = \int_S E^{\pi}_{s_1}(\tilde{w}(s_n))\mu(ds_1)$. Choose any $s_1 \in X_k$, $k \in \mathbb{N}$. Condition (A5) implies that $E^{\pi}_{s_1}(1_{X_{k+n-1}}(s_n)) = 1$ for all $\pi \in \Pi$. Therefore, for each $l > m_{k+n-1}$, we have $\sup_{\pi \in \Pi} E^{\pi}_{s_1}(w(s_n)1_{\{w(s_n)\geq l\}}) = 0$. By the monotone convergence theorem, we have $\lim_{l\to\infty} \int_S \sup_{\pi \in \Pi} E^{\pi}_{s_1}(w(s_n)1_{\{w(s_n)\geq l\}})\mu(ds_1) = 0$. Hence,

$$0 \le \lim_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi}(w(s_n) 1_{\{w(s_n) \ge l\}}) \le \lim_{l \to \infty} \int_{S} \sup_{\pi \in \Pi} E_{s_1}^{\pi}(w(s_n) 1_{\{w(s_n) \ge l\}}) \mu(ds_1) = 0,$$

which completes the proof of (3). \square

Lemma 9 implies that (A2) holds, if (A3) is satisfied and either $\beta\delta$ < 1 or (A4) holds. Lemma 10 provides an additional set of conditions under which (A1) and (A2) are satisfied.

Finally, we provide an example illustrating the importance of our assumptions. In particular, we show that if (3) does not hold, then an optimal policy may not exist even in a very simple model, where (2) is trivially satisfied.

Example 4 Consider the following unconstrained MDP, where

•
$$S = \{0, 0^*, 0^{**}\} \cup \mathbb{N};$$

- $A(0^*) = \{0\} \cup \bigcup_{n=1}^{\infty} \{1/n\} \text{ and } A(s) = \{a_0\} \text{ for all } s \in S \setminus \{0^*\};$
- the transition probabilities are as follows: $p(0|0^*, 1/n) = 1 q_n$, $p(n|0^*, 1/n) = q_n$, $p(0|0^*, 0) = 1$ and $p(0^{**}|s, a_0) = 1$ for all $s \in S \setminus \{0^*\}$; the value of $q_n \in (0, 1)$ for every $n \in \mathbb{N}$ will be specified later;
- the payoffs are as follows: $r(0^{**}, a_0) = r(0^*, a) = 0$ for every $a \in A(0^*)$ and $r(0, a_0) = 1$, $r(s, a_0) = s$ for $s \in \mathbb{N}$.

Then, the function

$$w(s) := s \text{ for } s \in \mathbb{N} \text{ and } w(s) := 1 \text{ for } s \in \{0, 0^*, 0^{**}\}\$$

is continuous on S and satisfies (A1). Let $\mu = \delta_{0^*}$ and let π_n denote the policy that chooses the action 1/n in state 0^* . We shall consider two cases.

(I) Let $q_n = 1/n$ for $n \in \mathbb{N}$. Note that p is both weakly and setwise continuous. We now check that (3) in (A2) does not hold. Fix $l \in \mathbb{N}$ and choose n > l. Clearly, we have that

$$E_{\mu}^{\pi_n}(w(s_2)1_{\{w(s_2)\geq l\}})=1.$$

Then,

$$\sup_{n\in\mathbb{N}} E_{\mu}^{\pi_n}(w(s_2)1_{\{w(s_2)\geq l\}}) = 1$$

and consequently,

$$\lim_{l \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi}(w(s_2)1_{\{w(s_2) \ge l\}}) = 1,$$

so (3) does not hold. On the other hand,

$$E_n^{\pi}w(s_k)=1$$
 for any policy $\pi\in\Pi$ and $k>2$.

Therefore,

$$\lim_{n \to \infty} \sup_{\pi \in \Pi} E_{\mu}^{\pi} \left(\sum_{k=n}^{\infty} \beta^{k} w(s_{k}) \right) = 0.$$

Thus, (2) in (A2) holds. Moreover, we observe that (W4) and (S4) are not met, since

$$\sum_{t \in S} w(t)p(t|0^*, 1/n) = 2 - 1/n \to \sum_{t \in S} w(t)p(t|0^*, 0) = 1 \text{ as } n \to \infty.$$

Further, it is not difficult to see that (A3) holds with $\delta = 2$. In this example, the optimal policy does not exist, since

$$\sup_{n \in \mathbb{N}} E_{\mu}^{\pi_n} \left(\sum_{n=1}^{\infty} \beta^k r(s_n, a_n) \right) = \sup_{n \in \mathbb{N}} \beta(2 - 1/n) = 2\beta$$

and for each policy π_n the expected discounted payoff is strictly less than 2β .

(II) Let $q_n = 1/2^n$ for $n \in \mathbb{N}$. Obviously, p is again weakly and setwise continuous. For these values of q_n condition (3) in (A2) holds, because for n > l we have that

$$\sup_{\pi \in \Pi} E^{\pi}_{\mu}(w(s_2)1_{\{w(s_2) \ge l\}}) = l/2^l \quad \text{and} \quad \lim_{l \to \infty} \sup_{\pi \in \Pi} E^{\pi}_{\mu}(w(s_2)1_{\{w(s_2) \ge l\}}) = 0.$$

Moreover, assumptions (W4) and (S4) are also satisfied, because

$$\sum_{t \in S} w(t)p(t|0^*, 1/n) = 1 - 1/2^n + n/2^n \to \sum_{t \in S} w(t)p(t|0^*, 0) = 1 \text{ as } n \to \infty.$$

Observe that condition (3) is satisfied and assumption (A3) holds with $\delta = 5/4$. An optimal policy exists, because

$$\sup_{n \in \mathbb{N}} E_{\mu}^{\pi_n} \left(\sum_{n=1}^{\infty} \beta^k r(s_n, a_n) \right) = \sup_{n \in \mathbb{N}} \beta (1 - 1/2^n + n/2^n) = 5\beta/4$$

and this supremum is realized by π_2 or π_3 . Obviously, we may replace the probabilities $1/2^n$, $n \in \mathbb{N}$, by arbitrary values of $q_n \in (0,1)$ such that $nq_n \to 0$ as $n \to \infty$.

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