Controllability of Kronecker Product Networks

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Abstract

A necessary and sufficient condition is derived for the controllability of Kronecker product networks, where the factor networks are general directed graphs. The condition explicitly illustrates how the controllability of the factor networks affects the controllability of the composite network. For the special case where at least one factor network is diagonalizable, an easily-verifiable condition is explicitly expressed. Furthermore, the controllability of higher-dimensional multi-agent systems is revisited, revealing that some controllability criterion reported in the literature does not hold. Consequently, a modified necessary and sufficient condition is established. The effectiveness of the new conditions is demonstrated through several examples.

Key words: Network controllability, network observability, Kronecker product graph, multi-agent systems.

1 Introduction

Controllability is a fundamental issue to be addressed before considering how to control a dynamical system in applications [6]. This subject has been extensively studied over more than half a century with various criteria developed, such as the PBH test, Kalman and other kinds of rank conditions, substantial graphic properties, and so on [10,13,24].

For most large-scale networked systems, these criteria cannot be applied practically because of their complex structures and heavy computational burdens. Therefore, in recent years, the notion of network controllability has received compelling attention with some efficient criteria established [15,17,18,19,12,1,23,27,25,9,28,30,31,32,33,34]. In [31], an exact controllability framework was introduced to identify the minimum set of input nodes for a general network with an arbitrary link-weight distribution. The controllability of networks with specific topologies, such as path graphs, cycle graphs, circulant graphs, multi-chain graphs and grid graphs, was explored in [17,18,19,12]. Network controllability was investigated from a graph-theoretic perspective in [21,32,1,23]. It is noted that most of the above results are derived for the networks with one-dimensional nodes. Recently, the controllability of networks with higher-dimensional nodes has attracted a great deal of interest [9,25,27,28,30,32,33,34]. Some controllability conditions for networked LTI systems were presented in [34,33], which depend on the transmission zeros of every subsystem and the connection matrix. The controllability of diffusively coupled LTI systems was studied in [32] and [27]. In [25], the controllability condition for networked MIMO systems was established in terms of two algebraic matrix equations. Moreover, some easily-verifiable conditions were proposed in [9,28] for the controllability of networked LTI systems with a diagonalizable topology matrix.

Besides isolated networks, composite networks have come into play due to their broad applications in different areas of engineering [8,16]. There are many kinds

of composite networks, such as Cartesian product networks, Kronecker product networks, strong product networks, lexicographic product networks and so on. In addition to their importance of constructing 'larger' networks out of 'small' ones, they are useful in the sense that one can get insights about the properties of composite networks from the factor networks. Intuitively, the controllability of a composite network might be verified by checking some properties of the factor networks. The controllability and observability of Cartesian product networks were investigated in [5]. In [18], the controllability and observability of linear dynamical systems whose dynamics are induced by the Laplacian of a grid graph were studied. Note that many real-world networks are similar to stochastic Kronecker product graphs. For example, the Kronecker power of a simple generating matrix can yield a Kronecker product graph that fits the Internet (at the autonomous systems level) fairly well [14]. Large online social networks, web and blog graphs, peer-to-peer networks, etc., can also be modeled by Kronecker product networks. Moreover, every non-trivial graph has a prime factorization over the Kronecker product [8,16]. It has lower computational complexity to check the controllability of a large-scale network by examining some properties of the smaller factor networks. Therefore, the controllability analysis for Kronecker product networks has brought about renewed interest recently [2,4,29,11]. The eigenanalysis for the Kronecker product of two matrices was presented in [11], which shows that the Kronecker products of the factor matrices' eigenvectors are the eigenvectors of the composite matrix. However, not all the eigenvectors of the composite matrix are formulated therein. In [29], the Kronecker product of defective matrices was revisited, characterizing the number of the eigenvectors. Nevertheless, the explicit expressions of the eigenvectors, which are the cornerstone of controllability analysis, are not presented. Recently, a sufficient condition was established in [4] for the controllability of Kronecker product networks, where the topology matrices are required to be diagonalizable.

In this paper, the controllability of Kronecker product networks is revisited. The contribution of the paper is four-fold. First, the factor networks considered here are general, directed and weighted. Differing from the condition given in [4], which requires the topology matrix of the composite network to be diagonalizable, this paper removes the diagonalizability requirement. Second, a new necessary and sufficient condition on the controllability of Kronecker product networks is provided in terms of eigenvectors. Compared with the classical PBH test, the new condition typically has a much lower computational cost. Third, for the special case where at least one factor network is diagonalizable, a specified condition is explicitly expressed, which is easier to verify. Finally, this paper shows that the sufficiency of the controllability criterion given in [3] does not hold, thereby a modified necessary and sufficient condition is derived.

The remainder of this paper is organized as follows. Some notations and preliminaries are given in Section 2. The model is formulated in Section 3. Some conditions on the controllability of Kronecker product networks are developed in Section 4. The controllability of higher-dimensional multi-agent systems is reinvestigated in Section 5. Finally, conclusions are drawn in Section 6.

2 Notations and Preliminaries

In this section, notations and useful preliminaries are introduced.

2.1 Notations

The notations mostly follow [9]. Let e_i be the row vector with all zero entries except for $[e_i]_i = 1$. The linear span of row vectors $v_1, \cdots, v_k$ is a set of all the linear combinations of these vectors, i.e., $span\{v_1, v_2, \cdots, v_k\} = \left\{ \sum_{i=1}^{k} c_i v_i | c_i \in \mathbb{R} \right\}$. Let $U_1 \oplus U_2$ be the direct sum of two spaces U_1 and U_2. Matrices, if their dimensions are not explicitly indicated, are assumed to be compatible for algebraic operations.

A weighted digraph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m, where an edge exists from nodes i to j if $(i, j) \in E$ with edge weight $w_{ji} \in W$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called the adjacency matrix of $\mathcal{G}$ with $a_{ij} = w_{ij}$ if $(j, i) \in E$ and $a_{ij} = 0$ otherwise. The ith diagonal term of the adjacency matrix denotes the weight of the self-loop on node i.

2.2 Kronecker Product Network

The Kronecker product network is a kind of composite network that can be obtained by applying Kronecker product operation(s) to several smaller networks, called factor networks. Let $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ be two factor networks. The Kronecker product of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, has the node set $V_1 \times V_2$. There is an edge from node (i, p) to node (j, q) if and only if (i, j) is an edge of E_1 and (p, q) is an edge of E_2. If an edge exists, the corresponding weight is $w_{((i,p),(j,q))} = w_{ij} w_{pq}$. An example of the Kronecker product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is displayed in Fig. 1.

2.3 Useful Lemmas and Definitions

Lemma 1 [22] If u_1, u_2, $\cdots$, u_n are linearly independent vectors and v_1, v_2, $\cdots$, v_n are arbitrary vectors, then $\sum_{k=1}^{n} u_k \otimes v_k = 0$ implies that $v_k = 0$ for all $k =$

Fig. 1. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and their Kronecker product $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$

$1, 2, \cdots, n$. Moreover, the roles of u_k and v_k in the above statement can be exchanged.

Definition 1 *[22] A row vector x_m is called the mth-order generalized left eigenvector of matrix A corresponding to its eigenvalue λ if $x_m(A - \lambda I)^m = 0$, and $x_m(A - \lambda I)^{m-1} \neq 0$. Moreover, $x_1, \cdots, x_g$ form a left Jordan chain of A on top of x_1, where the maximum value of g is called the length of this Jordan chain.*

3 Model Formulation

Consider a network consisting of Nn nodes with a directed and weighted topology $\mathcal{G}$ in the following form [7,20,14]:

$$\dot{x}_i(t) = \sum_{j=1}^{Nn} c_{ij} x_j(t) + \delta_i u_i(t), \ i = 1, 2, \cdots, Nn, \qquad (1)$$

where $x_i \in \mathbb{R}$ is the state of node i, $c_{ij} \in \mathbb{R}$ represents the coupling strength between nodes i and j, $u_i \in \mathbb{R}$ is the control input to node i, and $\delta_i = 1$ if node i is under control, but otherwise $\delta_i = 0$, for all $i = 1, 2, \cdots, Nn$. Assume that $c_{ij} \neq 0$ if there is an edge from node j to node i, otherwise $c_{ij} = 0$, for all $i, j = 1, 2, \cdots, Nn$. Denote $A(\mathcal{G}) = \mathcal{A}(\mathcal{G}) = [c_{ij}] \in \mathbb{R}^{Nn \times Nn}$ and $B = diag\{\delta_1, \delta_2, \cdots, \delta_{Nn}\}$, which represent the topology and the external input channels of the network (1), respectively. Let $X = [x_1, x_2, \cdots, x_{Nn}]^T$ be the whole state of the network, and $U = [u_1, u_2, \cdots, u_{Nn}]^T$ be the total external control input. Then, the network (1) can be rewritten in a compact form as

$$\dot{X}(t) = A(\mathcal{G})X(t) + BU(t). \qquad (2)$$

In the following, consider the controllability of the network with topology graph $\mathcal{G}$ being the Kronecker product of two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. The dynamics of the factor networks are described by $\dot{Y}_1(t) = A(\mathcal{G}_1)Y_1(t) + B_1U_1(t)$ and $\dot{Y}_2(t) = A(\mathcal{G}_2)Y_2(t) + B_2U_2(t)$, where $Y_1 \in \mathbb{R}^N$ and $U_1 \in \mathbb{R}^D$ are the state vector and the control input of the first factor network, respectively; $Y_2 \in \mathbb{R}^n$ and $U_2 \in \mathbb{R}^d$ are the state vector and control input of the second factor network, respectively. According to the definition of Kronecker product graph, it is easy to

prove $A(\mathcal{G}) = A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1) \otimes A(\mathcal{G}_2)$. Therefore, for this Kronecker product network, its compact form can be formulated as (2) with

$$A(\mathcal{G}) = A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1) \otimes A(\mathcal{G}_2),$$
$$B = B_1 \otimes B_2. \qquad (3)$$

The analysis here is presented in terms of two factor networks, which can be extended to larger numbers of factor networks by sequential compositions. In the following, it will be shown that the controllability of the composite network can be revealed by examining some features of the smaller factor networks, which makes the computational complexity much lower.

Remark 1 *A graph $\mathcal{G}$ is prime if it is nontrivial and cannot be decomposed into the Kronecker product of two nontrivial graphs. An expression $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_k$, with each $\mathcal{G}_i$ being prime, is called a prime factorization of $\mathcal{G}$. Note that every nontrivial graph has a prime factorization over the Kronecker product [8]. It has lower computational complexity to check the controllability of a large-scale network by examining some properties of the smaller factor networks. Therefore, exploring low-dimensional controllability conditions for Kronecker product networks is of great significance in engineering applications, which gives insight into the controllability of large-scale networks.*

4 Main Results

In this section, controllability conditions for the composite network (2)-(3) are considered. Firstly, the general case that both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are non-diagonalizable is investigated. Then, a special case where at least one factor network is diagonalizable is further analyzed, with a direct and easily-verifiable condition derived.

4.1 Both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are non-diagonalizable

In this subsection, the general case that both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are non-diagonalizable is investigated. The connection between the eigenspaces of the factor networks and that of the composite network provides a mechanism to establish efficient controllability conditions. Firstly, all left eigenvectors of the Kronecker product of two defective matrices are characterized.

Theorem 1 *Let $P = \lambda I_p + N_p \in \mathbb{C}^{p \times p}$ and $Q = \mu I_q + N_q \in \mathbb{C}^{q \times q}$ be two defective matrices, where N_p and N_q are nilpotent matrices. The eigenvalues of $P \otimes Q$ are $\lambda\mu$. Moreover,*

- *if $\lambda\mu \neq 0$, the corresponding left eigenvectors are $\eta^1 = e_p \otimes \xi_1$, $\eta^2 = e_{p-1} \otimes \xi_1 + e_p \otimes \xi_2$, $\cdots$, $\eta^\theta = e_{p-\theta+1} \otimes \xi_1 + e_{p-\theta+2} \otimes \xi_2 + \cdots + e_p \otimes \xi_\theta$, where $\xi_1 = e_q$, $\xi_k = $*

$\frac{(-1)^{k+1}}{\lambda^{k-1}} \sum\limits_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+1}$, $k = 2, \cdots, \theta$, $\theta = \min\{p,q\}$;

- if $\lambda = 0$ and $\mu \neq 0$, the corresponding left eigenvectors are $\eta^k = e_p \otimes e_k$, $k = 1, \cdots, q$;
- if $\lambda \neq 0$ and $\mu = 0$, the corresponding left eigenvectors are $\eta^k = e_k \otimes e_q$, $k = 1, \cdots, p$;
- if $\lambda = \mu = 0$, the corresponding left eigenvectors are $\eta^1 = e_p \otimes e_1$, $\eta^2 = e_p \otimes e_2$, $\cdots$, $\eta^q = e_p \otimes e_q$, $\eta^{q+1} = e_{p-1} \otimes e_q$, $\cdots$, $\eta^{q+p-1} = e_1 \otimes e_q$.

Proof: According to the structure of matrix $P \otimes Q$, it is easy to verify that the eigenvalues of $P \otimes Q$ are $\lambda\mu$. In the following, the aforementioned four cases are proved respectively.

Case 1: $\lambda\mu \neq 0$. Since $\eta^1(P \otimes Q) = (e_p \otimes e_q)(P \otimes Q) = (\lambda e_p) \otimes (\mu e_q) = \lambda\mu\eta^1$, it follows that η^1 is the left eigenvector of $P \otimes Q$. Note that η^2 is the left eigenvector of $P \otimes Q$ if and only if

$$\eta^2(P \otimes Q) = \lambda\mu\eta^2. \tag{4}$$

This implies that $\lambda\xi_2(\mu I - Q) = \xi_1 Q$. Since $\lambda\xi_2(\mu I - Q) = \mu e_q$ and $\xi_1 Q = \mu e_q$, equation (4) holds, indicating that η^2 is the left eigenvector of $P \otimes Q$.

Assume that $\eta^1, \cdots, \eta^{k-1}$ are the left eigenvectors of $P \otimes Q$. Then, η^k is the left eigenvector of $P \otimes Q$ if and only if

$$\lambda\xi_k(\mu I - Q) = \xi_{k-1} Q. \tag{5}$$

Since $\lambda\xi_k(\mu I - Q) = \frac{(-1)^{k+2}}{\lambda^{k-2}} \sum\limits_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+2}$ and $\xi_{k-1} Q = \frac{(-1)^k}{\lambda^{k-2}} \sum\limits_{l=0}^{k-3} C_{k-3}^l \mu^{k-l-2}(\mu e_{q-k+l+2} + e_{q-k+l+3})$ $= \frac{(-1)^k}{\lambda^{k-2}} \sum\limits_{l=0}^{k-2} C_{k-2}^l \mu^{k-l-1} e_{q-k+l+2}$, one can verify that (5) holds. Therefore, η^k is the left eigenvector of $P \otimes Q$. It has been shown in [29] that the number of the left eigenvectors for this case is $\min\{p,q\}$, thus all the left eigenvectors of $P \otimes Q$ are $\eta^1, \cdots, \eta^\theta$.

Case 2: $\lambda = 0$ and $\mu \neq 0$. Since $\eta^k(P \otimes Q) = (e_p \otimes e_k)(P \otimes Q) = (e_p P) \otimes (e_k Q) = \mathbf{0}_p^T \otimes (e_k Q) = \mathbf{0}_{pq}^T = \lambda\mu\eta^k$, it follows that $\eta^k = e_p \otimes e_k$ is the left eigenvector of $P \otimes Q$, $k = 1, \cdots, q$. It has been shown in [29] that the number of the left eigenvectors for this case is q, thus all the left eigenvectors of $P \otimes Q$ are $\eta^1, \cdots, \eta^q$.

One can prove the results in Cases 3 and 4 similarly, thus the detail is omitted. This completes the proof. ■

Remark 2 *The number of the eigenvectors of $P \otimes Q$ associated with the sole eigenvalue $\lambda\mu$ was given in [29]. However, it does not present explicit expressions of the eigenvectors, which are characterized in the above theorem. Theorem 1 is the cornerstone of the following controllability analysis for Kronecker product networks.*

In what follows, the left eigenvectors of $A(\mathcal{G})$ are expressed through the generalized eigenspaces of the factor networks.

Theorem 2 *Let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be the eigenvalues of $A(\mathcal{G}_1)$, and $\mu_1, \cdots, \mu_t$ be the eigenvalues of $A(\mathcal{G}_2)$. Then, the eigenvalues of $A(\mathcal{G})$ are $\lambda_1\mu_1, \cdots, \lambda_1\mu_t, \cdots, \lambda_s\mu_1, \cdots, \lambda_s\mu_t$. Moreover, for the eigenvalue $\lambda_i\mu_j$ with geometric multiplicity θ_{ij},*

- *if $\lambda_i\mu_j \neq 0$, then $\theta_{ij} = \min\{p_i, q_j\}$, and the corresponding left eigenvectors are $\eta_{ij}^k = v_i^k \otimes w_j^1 - \frac{\mu_j}{\lambda_i} v_i^{k-1} \otimes w_j^2$ $+ v_i^{k-2} \otimes \left[\frac{\mu_j^2}{\lambda_i^2} w_j^3 + \frac{\mu_j}{\lambda_i^2} w_j^2\right] + \cdots + v_i^1 \otimes \left[\frac{(-1)^{k+1}}{\lambda_i^{k+1}} \sum\limits_{l=0}^{k-2} C_{k-2}^l \mu_j^{k-l-1} w_j^{k-l}\right]$, $k = 1, \cdots, \theta_{ij}$;*
- *if $\lambda_i = 0$ and $\mu_j \neq 0$, then $\theta_{ij} = q_j$, and the corresponding left eigenvectors are $\eta_{ij}^k = v_i^1 \otimes w_j^k$, $k = 1, \cdots, \theta_{ij}$;*
- *if $\lambda_i \neq 0$ and $\mu_j = 0$, then $\theta_{ij} = p_i$, and the corresponding left eigenvectors are $\eta_{ij}^k = v_i^k \otimes w_j^1$, $k = 1, \cdots, \theta_{ij}$;*
- *if $\lambda_i = 0$ and $\mu_j = 0$, then $\theta_{ij} = q_j + p_i - 1$, and the corresponding left eigenvectors are $\eta_{ij}^1 = v_i^1 \otimes w_j^1, \cdots, \eta_{ij}^{q_j} = v_i^1 \otimes w_j^{q_j}, \eta_{ij}^{q_j+1} = v_i^2 \otimes w_j^1, \cdots, \eta_{ij}^{\theta_{ij}} = v_i^{p_i} \otimes w_j^1$,*

where $v_i^1, \cdots, v_i^{p_i}$ is the left Jordan chain of $A(\mathcal{G}_1)$ associated with the eigenvalue λ_i, and $w_j^1, \cdots, w_j^{q_j}$ is the left Jordan chain of $A(\mathcal{G}_2)$ associated with the eigenvalue μ_j, $i = 1, \cdots, s$, $j = 1, \cdots, t$.

Proof: Let $V \in \mathbb{C}^{N \times N}$ be a nonsingular matrix such that $VA(\mathcal{G}_1)V^{-1} = P = blockdiag\{P_1, P_2, \cdots, P_s\}$, where P is the Jordan form of $A(\mathcal{G}_1)$ and the ith Jordan block $P_i = \lambda_i I_{p_i} + N_{p_i} \in \mathbb{C}^{p_i \times p_i}$ with N_{p_i} being a nilpotent matrix, $i = 1, \cdots, s$. The left Jordan chain of $A(\mathcal{G}_1)$ associated with the eigenvalue λ_i is denoted as $v_i^1, v_i^2, \cdots, v_i^{p_i}$, where v_i^1 is the top vector and p_i is the length of the Jordan chain.

Let $W \in \mathbb{C}^{n \times n}$ be a nonsingular matrix such that $WA(\mathcal{G}_2)W^{-1} = Q = blockdiag\{Q_1, Q_2, \cdots, Q_t\}$, where Q is the Jordan form of $A(\mathcal{G}_2)$ and the jth Jordan block $Q_j = \mu_j I_{q_j} + N_{q_j} \in \mathbb{C}^{q_j \times q_j}$, $j = 1, \cdots, t$. The left Jordan chain of $A(\mathcal{G}_2)$ associated with the eigenvalue μ_j is denoted as $w_j^1, w_j^2, \cdots, w_j^{q_j}$, where w_j^1 is the top vector and q_j is the length of the Jordan chain.

Since $(V \otimes W)[A(\mathcal{G}_1) \otimes A(\mathcal{G}_2)](V \otimes W)^{-1} = P \otimes Q = blockdiag\{P_1 \otimes Q_1, \cdots, P_1 \otimes Q_t, \cdots, P_s \otimes Q_1, \cdots, P_s \otimes Q_t\}$, the eigenvalues of $A(\mathcal{G})$ are $\lambda_1\mu_1, \cdots, \lambda_1\mu_t, \cdots, \lambda_s\mu_1, \cdots, \lambda_s\mu_t$. Let $V_i = [v_i^{p_i T} \cdots v_i^{1T}]^T$ and

$W_j = [w_j^{q_j T} \cdots w_j^{1T}]^T$. It is easy to derive that $(V_i \otimes W_j)A(\mathcal{G}) = (P_i \otimes Q_j)(V_i \otimes W_j)$. If $\zeta \in \mathbb{C}^{1 \times p_i q_j}$ is a left eigenvector of $P_i \otimes Q_j$, then $\zeta(V_i \otimes W_j)$ is the left eigenvector of $A(\mathcal{G})$, $i = 1, \cdots, s$, $j = 1, \cdots, t$. Thus, the results follow from Theorem 1 directly. ∎

The explicit expressions of the left eigenvectors given in Theorem 2 are the core of the following controllability criteria for Kronecker product networks. Let $U_{ij} = span\{\eta_{ij}^1, \cdots, \eta_{ij}^{\theta_{ij}}\}$ be the left eigenspace of $A(\mathcal{G})$ corresponding to the eigenvalue $\lambda_i \mu_j$. In what follows, a theorem is established for the controllability of the Kronecker product network (2)-(3).

Theorem 3 *Let* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_s\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_1)$, *and* $\Sigma = \{\mu_1, \cdots, \mu_t\}$ *be the set of the eigenvalues of* $A(\mathcal{G}_2)$. *The composite network (2)-(3) is controllable if and only if the following two conditions hold simultaneously:*

(1) $\forall \eta \in U_{ij}$ *and* $\eta \neq 0$, $\eta(B_1 \otimes B_2) \neq 0$, *for* $i = 1, \cdots, s$, $j = 1, \cdots, t$;
(2) if $\lambda_{i_1} \mu_{j_1} = \lambda_{i_2} \mu_{j_2} = \cdots = \lambda_{i_r} \mu_{j_r}$, *where* $\lambda_{i_k} \in \Lambda$, $\mu_{j_k} \in \Sigma$, $k = 1, \cdots, r$, $r > 1$, *then* $\forall \eta \in \bigoplus_{k=1}^{r} U_{i_k j_k}$ *and* $\eta \neq 0$, $\eta(B_1 \otimes B_2) \neq 0$.

Proof: Using Theorem 2 and the PBH test, one can prove this theorem easily. Thus, the detail is omitted. ∎

Remark 3 *Theorem 3 provides a precise and efficient criterion for determining the controllability of a general Kronecker product network, using the generalized eigenvectors of the factor networks with low dimensions. Compared with the classical PBH test, the new condition typically has a much lower computational cost. Due to linear systems duality, the results in Theorem 3 are equally applicable to the observability of Kronecker product networks.*

The effectiveness of the above condition can be illustrated by the following example.

Example 1 *Consider graph* $\mathcal{G}_1$ *depicted in Fig. 2. It is*

Fig. 2. Graph $\mathcal{G}_1$

easy to verify that $A(\mathcal{G}_1) = \begin{bmatrix} 8.5 & 4 & -0.5 \\ 3.5 & 8 & 0.5 \\ 3.5 & 3 & 5.5 \end{bmatrix}$. *The eigenvalues of* $A(\mathcal{G}_1)$ *are* $\lambda_1 = 12$, $\lambda_2 = 5$. *The left eigenvector associated with* λ_1 *is* $v_1 = -e_1 - e_2$ *and the left Jordan chain corresponding to* λ_2 *is* $v_2^1 = e_2 - e_3$, $v_2^2 = -e_1 + e_2$. *It is easy to obtain that* $A(\mathcal{G}_1)$ *is non-diagonalizable. Assume that the second node of* $\mathcal{G}_1$ *is under control. Thus, one has* $B_1 = e_2^T$. *Let* $A(\mathcal{G}) = A(\mathcal{G}_1) \otimes A(\mathcal{G}_1)$, $B = B_1 \otimes B_1$. *Then* $\eta_{22}^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$ *and* $\eta_{22}^2 = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$. *If* $c_1 = 0$, *one has* $(c_1 \eta_{22}^1 + c_2 \eta_{22}^2)(B_1 \otimes B_1) = c_1 = 0$. *It then follows from Theorem 3 that* $(A(\mathcal{G}), B)$ *is uncontrollable. This example fully demonstrates the effectiveness of the condition.*

Remark 4 *Theorem 6 in [4] has established a controllability condition for Kronecker product networks, which requires* $A(\mathcal{G})$ *to be diagonalizable. This assumption is conservative and the condition can only be applied to a very restricted type of networks. The new condition here removes this restriction, thus is more general and flexible. This nontrivial extension is of great significance in engineering applications.*

In the following, some more intuitive and easily-verifiable conditions are presented, which reveal how the controllability of the factor networks affects the controllability of the whole composite network.

Corollary 1 *If the composite network (2)-(3) is controllable, then the factor networks* $(A(\mathcal{G}_1), B_1)$ *and* $(A(\mathcal{G}_2), B_2)$ *are controllable.*

Corollary 2 *If* $A(\mathcal{G}_1)$ *has an eigenvalue 0, to ensure the controllability of the composite network (2)-(3), it is necessary that* $B_2 = I_n$. *Moreover, if* $A(\mathcal{G}_2)$ *has an eigenvalue 0, to ensure the controllability of the composite network, it is necessary that* $B_1 = I_N$.

Proof: If $A(\mathcal{G}_1)$ has an eigenvalue 0, without loss of generality, let $\lambda_i = 0$, then the left eigenvectors of $A(\mathcal{G})$ corresponding to 0 are $v_i^1 \otimes w_1^1$, $\cdots$, $v_i^1 \otimes w_1^{q_1}$, $\cdots$, $v_i^1 \otimes w_t^1$, $\cdots$, $v_i^1 \otimes w_t^{q_t}$, where v_i^1 is the left eigenvector of $A(\mathcal{G}_1)$ associated with the eigenvalue 0; w_1^1, $\cdots$, $w_1^{q_1}$, $\cdots$, w_t^1, $\cdots$, $w_t^{q_t}$ are all the left root vectors of $A(\mathcal{G}_2)$. If the composite network (2)-(3) is controllable, then $\left[\sum_{j=1}^{t} \sum_{k=1}^{q_k} (c_j^k v_i^1 \otimes w_j^k)\right](B_1 \otimes B_2) \neq 0$, for any scalars c_1^1, $\cdots$, $c_1^{q_1}$, $\cdots$, c_t^1, $\cdots$, $c_t^{q_t}$, which are not all zero. That is, $(v_i^1 B_1) \otimes \left[\sum_{j=1}^{t} \sum_{k=1}^{q_k} (c_j^k w_j^k B_2)\right] \neq 0$, which implies $\left[\sum_{j=1}^{t} \sum_{k=1}^{q_k} c_j^k w_j^k\right] B_2 \neq 0$, for any nonzero

$[c_1^1, \cdots, c_1^{q_1}, \cdots, c_t^1, \cdots, c_t^{q_t}]$. Since w_1^1, $\cdots$, $w_1^{q_1}$, $\cdots$, w_t^1, $\cdots$, $w_t^{q_t}$ are all the left root vectors of $A(\mathcal{G}_2)$, which span $\mathbb{C}^n$, one has $B_2 = I_n$.

The second part can be proved similarly. ∎

Hereinafter, the special case where $A(\mathcal{G}_1)$ or $A(\mathcal{G}_2)$ is diagonalizable is analyzed in detail, for which a simple and effective condition is obtained. Compared with the conditions for the general case, this easy-to-verify condition allows to check the network controllability more efficiently.

4.2 $A(\mathcal{G}_1)$ or $A(\mathcal{G}_2)$ is diagonalizable

In this subsection, a controllability criterion is established for the Kronecker product network (2)-(3) with one diagonalizable factor network.

Corollary 3 *Assume that $A(\mathcal{G}_1)$ or $A(\mathcal{G}_2)$ is diagonalizable. Let $\Lambda=\{\lambda_1, \cdots, \lambda_s\}$ be the set of the eigenvalues of $A(\mathcal{G}_1)$, and $\Sigma = \{\mu_1, \cdots, \mu_t\}$ be the set of the eigenvalues of $A(\mathcal{G}_2)$. The composite network (2)-(3) is controllable if and only if the following four conditions hold simultaneously:*

(1) $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable;
(2) if $A(\mathcal{G}_1)$ has an eigenvalue 0, then $B_2 = I_n$;
(3) if $A(\mathcal{G}_2)$ has an eigenvalue 0, then $B_1 = I_N$;
(4) if $\lambda_{i_1}\mu_{j_1} = \lambda_{i_2}\mu_{j_2} = \cdots = \lambda_{i_r}\mu_{j_r} \neq 0$, where $\lambda_{i_k} \in \Lambda$, $\mu_{j_k} \in \Sigma$, for $k = 1, \cdots, r$, $r > 1$, then $(v_{i_1}B_1) \otimes (w_{j_1}B_2), \cdots, (v_{i_r}B_1) \otimes (w_{j_r}B_2)$ are linearly independent, where v_{i_k} is the left eigenvector of $A(\mathcal{G}_1)$ corresponding to the eigenvalue λ_{i_k}; w_{j_k} is the left eigenvector of $A(\mathcal{G}_2)$ corresponding to the eigenvalue μ_{i_k}, $k = 1, \cdots, r$.

Proof: The case that $A(\mathcal{G}_1)$ is diagonalizable is firstly proved. In this case, $s = N$. Then with a similar method, one can prove the other case easily.

Necessity: First of all, from Corollaries 1 and 2, it follows that conditions (1)-(3) are necessary for the controllability of the composite network (2)-(3).

Assume that $\lambda_{i_1}\mu_{j_1} = \lambda_{i_2}\mu_{j_2} = \cdots = \lambda_{i_r}\mu_{j_r} = \sigma \neq 0$, where $\lambda_{i_k} \in \Lambda$, $\mu_{j_k} \in \Sigma$, $k = 1, \cdots, r$, $r > 1$. Then, any left eigenvector of $A(\mathcal{G})$ corresponding to σ can be expressed in the form of $\sum_{k=1}^{r} \alpha_k(v_{i_k} \otimes w_{j_k})$, where v_{i_k} is the left eigenvector of $A(\mathcal{G}_1)$ corresponding to the eigenvalue λ_{i_k}; w_{j_k} is the left eigenvector of $A(\mathcal{G}_2)$ corresponding to the eigenvalue μ_{i_k}; $\alpha_k \in \mathbb{R}$ ($k = 1, \cdots, r$) are some scalars which are not all zero. If the composite network (2)-(3) is controllable, then $\left[\sum_{k=1}^{r} \alpha_k(v_{i_k} \otimes w_{j_k})\right] (B_1 \otimes B_2) \neq 0$. Consequently, $\sum_{k=1}^{r} \alpha_k(v_{i_k}B_1) \otimes (w_{j_k}B_2) \neq 0$,

for any scalars $\alpha_1, \cdots, \alpha_r$, which are not all zero. Therefore, $(v_{i_1}B_1) \otimes (w_{j_1}B_2), \cdots, (v_{i_r}B_1) \otimes (w_{j_r}B_2)$ are linearly independent.

Sufficiency: One needs to prove that, if the composite network (2)-(3) is uncontrollable, then at least one condition in Corollary 3 does not hold. Based on the PBH test, if the composite network (2)-(3) is uncontrollable, then there exists a left-eigenpair of $A(\mathcal{G})$, denoted as (σ, ξ), such that

$$\xi B = 0. \tag{6}$$

This will be discussed in the following 3 cases.

- If $\sigma \neq 0$ is an eigenvalue with the geometric multiplicity being 1, assume that $\sigma = \lambda_i\mu_j$, it then follows from Theorem 2 that $\xi = v_i \otimes w_j^1$, where $\lambda_i \in \Lambda \neq 0$ with the corresponding left eigenvector v_i; $\mu_j \in \Sigma \neq 0$ with the corresponding left eigenvector w_j^1. From equality (6), it can be easily deduced that $(v_i \otimes w_j^1)(B_1 \otimes B_2) = (v_iB_1) \otimes (w_j^1B_2) = 0$, which yields $v_iB_1 = 0$ or $w_j^1B_2 = 0$. It can be then derived that $(A(\mathcal{G}_1), B_1)$ is uncontrollable or $(A(\mathcal{G}_2), B_2)$ is uncontrollable.
- If $\sigma \neq 0$ is an eigenvalue with the geometric multiplicity more than 1, assume that $\lambda_{i_1}\mu_{j_1} = \cdots = \lambda_{i_r}\mu_{j_r} = \sigma$, where $\lambda_{i_k} \in \Lambda \neq 0$, $\mu_{j_k} \in \Sigma \neq 0$, $k = 1, \cdots, r$, $r > 1$. It follows from Theorem 2 that ξ can be expressed in the form of $\sum_{k=1}^{r} \hat{\alpha}_k(v_{i_k} \otimes w_{j_k})$, where v_{i_k} is the left eigenvector of $A(\mathcal{G}_1)$ associated with the eigenvalue λ_{i_k}; w_{j_k} is the left eigenvector of $A(\mathcal{G}_2)$ associated with the eigenvalue μ_{j_k}; $\hat{\alpha}_k \in \mathbb{R}$ ($k = 1, \cdots, r$) are some scalars, which are not all zero. From equality (6), it follows that there exists a nonzero vector $[\hat{\alpha}_1, \cdots, \hat{\alpha}_r]$, such that $\left[\sum_{k=1}^{r} \hat{\alpha}_k(v_{i_k} \otimes w_{j_k})\right] (B_1 \otimes B_2) = \sum_{k=1}^{r} \hat{\alpha}_k(v_{i_k}B_1) \otimes (w_{j_k}B_2) = 0$. It indicates that $(v_{i_1}B_1) \otimes (w_{j_1}B_2), \cdots, (v_{i_r}B_1) \otimes (w_{j_r}B_2)$ are linearly dependent.
- If $\sigma = 0$, assume that $\lambda_1 = \cdots = \lambda_p = 0$ and $\mu_1 = \cdots = \mu_l = 0$. It follows from Theorem 2 that ξ can be expressed in the form of $\sum_{i=1}^{p}\sum_{j=1}^{t}\sum_{k=1}^{q_j}(\hat{\alpha}_{ij}^k v_i \otimes w_j^k) + \sum_{j=1}^{l}\sum_{i=p+1}^{N}(\hat{\beta}_{ij}v_i \otimes w_j^1)$, where v_i is the left eigenvector of $A(\mathcal{G}_1)$ associated with the eigenvalue λ_i, $i = 1, \cdots, N$; q_j, w_j^k ($k = 1, \cdots, q_j$) are defined as in Theorem 2; $\hat{\alpha}_{ij}^k$ ($k = 1, \cdots, q_j$, $j = 1, \cdots, t$, $i = 1, \cdots, p$) and $\hat{\beta}_{ij}$ ($j = 1, \cdots, l$, $i = p + 1, \cdots, N$) are some scalars, which are not all zero. From equality (6), it follows that there exist scalars $\hat{\alpha}_{ij}^k$ ($k = 1, \cdots, q_j$, $j = 1, \cdots, t$, $i = 1, \cdots, p$) and $\hat{\beta}_{ij}$ ($j = 1, \cdots, l$, $i = p + 1, \cdots, N$), which are not all zero, such that

$$\left[\sum_{i=1}^{p}\sum_{j=1}^{t}\sum_{k=1}^{q_j}(\hat{\alpha}_{ij}^k v_i \otimes w_j^k) + \sum_{j=1}^{l}\sum_{i=p+1}^{N}(\hat{\beta}_{ij}v_i \otimes w_j^1)\right](B_1 \otimes B_2) = 0,$$

which implies that

$$\sum_{j=1}^{l}\left[(\sum_{i=1}^{p}\hat{\alpha}_{ij}^1 v_i B_1 + \sum_{i=p+1}^{N}\hat{\beta}_{ij}v_i B_1) \otimes (w_j^1 B_2)\right] +$$
$$\sum_{j=1}^{l}\sum_{k=2}^{q_j}\left[(\sum_{i=1}^{p}\hat{\alpha}_{ij}^k v_i B_1) \otimes (w_j^k B_2)\right] + \quad (7)$$
$$\sum_{j=l+1}^{t}\sum_{k=1}^{q_j}\left[(\sum_{i=1}^{p}\hat{\alpha}_{ij}^k v_i B_1) \otimes (w_j^k B_2)\right] = 0.$$

From Lemma 1, it can be derived that equality (7) holds if and only if at least one of the following conditions holds:

(1) $w_1^1 B_2, \cdots, w_1^{q_1} B_2, \cdots, w_t^1 B_2, \cdots, w_t^{q_t} B_2$ are linearly dependent.

(2) $v_1 B_1, \cdots, v_N B_1$ are linearly dependent.

Since $w_1^1, \cdots, w_1^{q_1}, \cdots, w_t^1, \cdots, w_t^{q_t}$ are all the left root vectors of $A(\mathcal{G}_2)$, which span $\mathbb{C}^n$, the first condition implies that $B_2 \neq I_n$. Moreover, since $v_1, \cdots, v_N$ are all the left eigenvectors of $A(\mathcal{G}_1)$, which span $\mathbb{C}^N$, the second condition indicates that $B_1 \neq I_N$. Thus, in this case, if the composite network (2)-(3) is uncontrollable, then $B_1 \neq I_N$ or $B_2 \neq I_n$.

Therefore, if the composite network (2)-(3) is uncontrollable, then at least one condition in Corollary 3 does not hold. This completes the proof. ■

The effectiveness of this criterion can be illustrated by the following example.

Example 2 *Consider two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, which are depicted in Fig.3. It is easy to verify that $A(\mathcal{G}_1) =$*

(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

Fig. 3. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$

$$\begin{bmatrix} 8.5 & 6 & -2.5 \\ 3.5 & 6 & 2.5 \\ 3.5 & 6 & 2.5 \end{bmatrix} \text{ and } A(\mathcal{G}_2) = \begin{bmatrix} 8.5 & 4 & -0.5 \\ 3.5 & 8 & 0.5 \\ 3.5 & 3 & 5.5 \end{bmatrix}.$$ *Note that*

$A(\mathcal{G}_1)$ *is diagonalizable. The eigenvalues of $A(\mathcal{G}_1)$ are $\lambda_1 = 12$, $\lambda_2 = 5$ and $\lambda_3 = 0$ with the corresponding left eigenvectors $v_1 = e_1 + e_2$, $v_2 = -e_1 + e_3$ and $v_3 = -e_2 + e_3$, respectively. Moreover, the eigenvalues of $A(\mathcal{G}_2)$ are $\mu_1 = 12$, $\mu_2 = 5$. The left eigenvector associated with μ_1 is $w_1 = -e_1 - e_2$ and the left Jordan chain*

corresponding to μ_2 is $w_2^1 = e_2 - e_3$, $w_2^2 = -e_1 + e_2$. Assume that the first two nodes of $\mathcal{G}_1$ have control inputs and all the nodes of $\mathcal{G}_2$ are under control. Thus, one has $B_1 = [e_1^T, \ e_2^T]$ and $B_2 = [e_1^T, \ e_2^T, \ e_3^T]$. Firstly, it is easy to verify that $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable. Since $A(\mathcal{G}_1)$ has an eigenvalue 0 and $B_2 = I_3$, the second condition holds. Further, noting that $\lambda_1\mu_2 = \lambda_2\mu_1 = 60$, one needs to check whether $(v_1 B_1) \otimes (w_2^1 B_2)$ and $(v_2 B_1) \otimes (w_1 B_2)$ are linearly dependent. Since $B_2 = I_3$, $w_2^1 B_2$ and $w_1 B_2$ are linearly independent. Moreover, $v_1 B_1 \neq 0$ and $v_2 B_1 \neq 0$. Thus, one can verify that $(v_1 B_1) \otimes (w_2^1 B_2)$ and $(v_2 B_1) \otimes (w_1 B_2)$ are linearly independent. It then follows from Corollary 3 that $(A(\mathcal{G}), B)$ is controllable. This result coincides with the conclusion derived by the Kalman rank condition.

If $A(\mathcal{G})$ is diagonalizable, then both $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are diagonalizable. The conditions in Corollary 3 are also effective for diagonalizable Kronecker product networks, as demonstrated by the following example.

Example 3 *Consider two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, which are depicted in Fig. 4. It is easy to verify that $A(\mathcal{G}_1) =$*

(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

Fig. 4. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \text{ and } A(\mathcal{G}_2) = \begin{bmatrix} 6 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}.$$ *The eigenvalues of*

$A(\mathcal{G}_1)$ *are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 3$ with the corresponding left eigenvectors $v_1 = e_2$, $v_2 = e_1 - e_3$ and $v_3 = e_2 + e_3$, respectively. The eigenvalues of $A(\mathcal{G}_2)$ are $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 6$ with the corresponding left eigenvectors $w_1 = e_2$, $w_2 = 5e_2 + e_3$, $w_3 = 3/2 e_1 + 13/8 e_2 + e_3$, respectively. It is easy to verify that $A(\mathcal{G})$ is diagonalizable. Assume that the first two nodes of $\mathcal{G}_1$ have control inputs. Node 2 and node 3 of $\mathcal{G}_2$ are under control. Thus, one has $B_1 = [e_1^T, \ e_2^T]$ and $B_2 = [e_2^T, \ e_3^T]$. Both pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable. Since $\lambda_1\mu_2 = \lambda_2\mu_3 = \lambda_3\mu_1 = 6$, the value of $[c_1(v_1 \otimes w_2) + c_2(v_2 \otimes w_3) + c_3(v_3 \otimes w_1)](B_1 \otimes B_2)$ is required to be checked, which gives $[c_1(v_1 \otimes w_2) + c_2(v_2 \otimes w_3) + c_3(v_3 \otimes w_1)](B_1 \otimes B_2) = [\frac{13}{8}c_2, \ c_2, \ 5c_1 + c_3, \ c_1] \neq 0$, for any nonzero $[c_1, \ c_2, \ c_3]$. It then follows from Corollary 3 that $(A(\mathcal{G}), B)$ is controllable, which coincides with the conclusion derived by the Kalman rank condition.*

This example fully demonstrates the effectiveness of the condition.

Remark 5 *A controllability criterion for diagonalizable Kronecker product networks was established in [4], which was claimed to be necessary and sufficient. However, Example 3 shows that the necessity of that criterion does not hold. Corollary 3 provides a new controllability condition, which is not only sufficient but also necessary.*

5 Controllability of Higher-dimensional Multi-agent Systems Revisited

5.1 Problem Statement

Consider a multi-agent system consisting of N agents labeled by the set $V = \{1, 2, \cdots, N\}$. Assign the roles of leaders and followers to the agents by denoting $V_L = \{v_1, v_2, \cdots, v_m\}$ and $V_F = V \backslash V_L$ as the sets of indices of the leaders and followers, respectively, where $m \leq N$.

To each follower $i \in V_F$, associate a dynamical system $\dot{x}_i = z_i$, and to each leader $i \in V_L$, we associate a dynamical system $\dot{x}_i = z_i + Bu_i$, where $x_i \in \mathbb{R}^n$ is the state of agent i; $u_i \in \mathbb{R}^p$ is the external input to agent $i \in V_L$; $z_i \in \mathbb{R}^n$ is the coupling input from other agents.

Agent i is said to be a neighbor of agent j if its state is known by agent j. Here, assume that the neighboring relationships are fixed, which can be described by a directed and weighted graph $\mathcal{G} = (V, E, W)$. The coupling input z_i to each agent $i \in V$ is determined by the diffusive coupling rule based on the neighboring relations as follows: $z_i = H \sum_{(j,i) \in E} w_{ij}(x_j - x_i)$, where $H \in \mathbb{R}^{n \times n}$ is the matrix describing inner-coupling between different components, and $w_{ij} \in \mathbb{R}$ is the strength of the information link.

The Laplacian matrix of $\mathcal{G}$ and the external input channels of the multi-agent system are denoted by $L \in \mathbb{R}^{N \times N}$ and $\Delta = diag\{d_1, d_2, \cdots, d_N\}$, respectively, where $d_i = 1$ for $i \in V_L$, but otherwise $d_i = 0$, for all $i = 1, 2, \cdots, N$. Let $X = [x_1^T, x_2^T, \cdots, x_N^T]^T$ be the whole state of the multi-agent system, and $U = [u_1^T, u_2^T, \cdots, u_N^T]^T$ be the total external control input. Then, the above multi-agent system can be rewritten in a compact form as

$$\dot{X} = FX + GU, \tag{8}$$

with

$$F = -L \otimes H, \quad G = \Delta \otimes B. \tag{9}$$

Note that matrix F has the form of the Kronecker product of two matrices and so does G. This higher-dimensional multi-agent system can be seen as a special

case with $A(\mathcal{G}_1)$ being a Laplacian matrix. In the following, conditions for ensuring the controllability of the multi-agent system (8)-(9) are specified.

Remark 6 *The controllability of networked LTI systems or multi-agent systems with linear dynamics has been investigated in [32,9,27,28,30]. The state matrices for those systems have the form of $I \otimes A + L \otimes H$ rather than a pure Kronecker product. The higher-dimensional multi-agent systems investigated here can be seen as a special case with nodes having no internal dynamics.*

5.2 A Counterexample

Recall the controllability condition for the multi-agent system (8)-(9) given in [3], where it was assumed that the first N_l agents are leaders. Then, $\Delta = [e_1^T, \cdots, e_{N_l}^T, \mathbf{0}_N, \cdots, \mathbf{0}_N] \in \mathbb{R}^{N \times N}$ and L can be partitioned as $L = \begin{bmatrix} L_{ll} & L_{lf} \\ L_{fl} & L_{ff} \end{bmatrix}$, where $L_{ll} \in \mathbb{R}^{N_l \times N_l}$ and $L_{ff} \in \mathbb{R}^{(N-N_l) \times (N-N_l)}$ with subscripts 'l' and 'f' denoting 'leader' and 'follower', respectively.

Theorem 4 *(Theorem 1 of [3]) For an LTI swarm system described by (8)-(9), suppose the first N_l agents are leaders. Then, the system is completely controllable if and only if*

(1) (H, B) is a controllable matrix pair;
(2) L represents a controllable graph, i,e, $[L_{ff}\ L_{fl}]$ is a controllable matrix pair.

However, while this condition is necessary for the controllability of the network, it may not be sufficient, as shown in the following example.

Fig. 5. A path network

Consider the undirected path network with three nodes depicted in Fig. 5. Node 1 is selected to be the leader. One has $L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $H = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. It is easy to verify that (H, B) and (L_{ff}, L_{fl}) are controllable. From Theorem 1 in [3], it follows that this undirected path network is controllable.

However, if one checks the controllability of this system by using the classical Kalman rank condition, one can

find that $(-L \otimes H, \Delta \otimes B)$ is actually uncontrollable. Therefore, its sufficiency does not hold.

This example is a modification of the example presented in [27], which was used to demonstrate that the controllability condition for diffusive networks proposed in [32] is not always sufficient. The incomplete eigenanalysis of networks presented in [32] and [3] leads to errors in controllability analysis. In the following, a modified controllability condition is proposed, which is both necessary and sufficient.

5.3 New Controllability Criteria

Note that the conditions proposed in Section 4 are not restricted to checking the controllability of Kronecker product networks. They can tackle the controllability problem for any system represented by the Kronecker product of two matrices. Based on the results in Section 4, a modified controllability condition for the multi-agent system (8)-(9) is established as follows.

Corollary 4 *The multi-agent system (8)-(9) is controllable if and only if the following three conditions hold simultaneously:*

(1) (L, Δ) is controllable;
(2) $rank(B) = n$;
(3) if H has an eigenvalue 0, then $\Delta = I_N$.

Proof: Note that L has an eigenvalue 0. If the multi-agent system (8)-(9) is controllable, then $rank(B) = n$. From Corollaries 1 and 2, it follows that the conditions (1) and (3) are necessary for the controllability of the multi-agent system (8)-(9).

For sufficiency, one needs to prove that, if the conditions (1)-(3) hold, then no left eigenvectors of F are orthogonal to G, thus the multi-agent system (8)-(9) is controllable. It is easy to verify that if the conditions (1)-(3) hold, then no left eigenvectors of F associated with eigenvalue 0 are orthogonal to G. For a nonzero eigenvalue $\sigma = -\lambda\mu$, if it is not a common eigenvalue, it follows from Theorem 2 that any corresponding left eigenvector can be expressed as

$$\xi = \left(\sum_{i=1}^{\theta} c_i v^i\right) \otimes w^1 + \sum_{k=2}^{\theta}\left[\left(\sum_{i=k}^{\theta}\sum_{j=1}^{i-k+1} c_i l_{ij} v^j\right) \otimes w^k\right],$$

where $v^1, \cdots, v^p$ is the left Jordan chain of L associated with the eigenvalue λ, and $w^1, \cdots, w^q$ is the left Jordan chain of H associated with the eigenvalue μ, $\theta = \min\{p, q\}$, l_{ij} is a nonzero scalar about λ and μ, $c_1, \cdots, c_\theta$ are scalars, which are not all zero. Next, the value of $\xi(\Delta \otimes B)$ will be checked.

It is easy to verify that $\xi(\Delta \otimes B) = \left(\sum_{i=1}^{\theta} c_i v^i \Delta\right) \otimes$

$(w^1 B) + \sum_{k=2}^{\theta}\left[\left(\sum_{i=k}^{\theta}\sum_{j=1}^{i-k+1} c_i l_{ij} v^j \Delta\right) \otimes (w^k B)\right]$. Since

$rank(B) = n$, $w^1 B, \cdots, w^\theta B$ are linearly independent. Since (L, Δ) is controllable, it follows that $\sum_{i=1}^{\theta} c_i v^i \Delta$ and

$\sum_{i=k}^{\theta}\sum_{j=1}^{i-k+1} c_i l_{ij} v^j \Delta$ $(k = 2, \cdots, \theta)$ are not all zero. From Lemma 1, it can be easily deduced that $\xi(\Delta \otimes B) \neq 0$. Thus, for any nonzero eigenvalue of F, if it is not a common eigenvalue, no corresponding left eigenvectors are orthogonal to G. Similarly, one can prove that, for each nonzero common eigenvalue of F, no corresponding left eigenvectors are orthogonal to G. Consequently, no left eigenvectors of F are orthogonal to G. Therefore, the multi-agent system (8)-(9) is controllable. This completes the proof. ∎

According to Corollary 4, one can easily verify that the undirected path network in the above counterexample is uncontrollable. This example fully demonstrates the effectiveness of the condition.

6 Conclusions

The controllability of Kronecker product networks has been investigated, in which the factor networks have general directed topologies. A necessary and sufficient condition for the controllability of the composite network has been derived, which is effective and has a much lower computational cost as compared to existing criteria. For the special case where at least one factor network is diagonalizable, a specified condition has also been established, which is simple and easier to verify. Moreover, the controllability of higher-dimensional multi-agent systems has been reinvestigated. It is found that the sufficiency of the controllability criterion given in [3] does not hold. Consequently, a modified condition is derived, which is necessary and sufficient. In future studies, the controllability and observability of other types of network-of-networks will be further considered.

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