# Stabilization of cycles for difference equations with a noisy PF control 

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#### Abstract

Difference equations, such as a Ricker map, for an increased value of the parameter, experience instability of the positive equilibrium and transition to deterministic chaos. To achieve stabilization, various methods can be applied. Proportional Feedback control suggests a proportional reduction of the state variable at every $k$ th step. First, if $k \neq 1$, a cycle is stabilized rather than an equilibrium. Second, the equation can incorporate an additive noise term, describing the variability of the environment, as well as multiplicative noise corresponding to possible deviations in the control intensity. The present paper deals with both issues, it justifies a possibility of getting a stable blurred $k$-cycle. Presented examples include the Ricker model, as well as equations with unbounded $f$, such as the bobwhite quail population models. Though the theoretical results justify stabilization for either multiplicative or additive noise only, numerical simulations illustrate that a blurred cycle can be stabilized when both multiplicative and additive noises are involved.


Key words: Stochastic difference equations; proportional feedback control; multiplicative noise; additive noise; Ricker map; stable cycles

## 1 Introduction

A difference equation

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right), \quad x_{0}>0, \quad n \in \mathbb{N}_{0}=\{0,1,2, \ldots\} \tag{1}
\end{equation*}
$$

for a variety of maps $f$, for example, logistic or Ricker, can exhibit unstable and even chaotic behavior. For unstable (1), several control methods were developed in the literature, e.g. [6,7,8,10,11,17]. These methods include Proportional Feedback (PF) control in the deterministic

[^0][8] and stochastic [4] versions, Prediction-based control $[1,10,11,17]$ and Target Oriented control [6,7]. Some of these methods were used to stabilize cycles rather than an equilibrium in $[2,3,11]$. Stochastic versions of these control methods, applied to stabilize a blurred equilibrium, were considered in $[1,4]$. In addition, there are control methods where stabilization is achieved by noise only, see the recent papers $[5,9]$ and references therein. In the present paper, we concentrate on a stochastic version of PF control, applied to stabilize blurred cycles.

We consider the control by the proportional feedback (PF) method. This method, first introduced in [8], involves reduction of the state variable at each $k$-th step, $k \in \mathbb{N}$, when $n$ is divisible by $k(n \mid k)$, proportional to the size of the state variable $x_{n}$

$$
\begin{equation*}
x_{n+1}=f\left(\nu x_{n}\right), n \mid k, x_{n+1}=f\left(x_{n}\right), n \nmid k \tag{2}
\end{equation*}
$$

where $x_{0}>0, n \in \mathbb{N}_{0}, \nu \in(0,1], k \in \mathbb{N}$.

However, the reduction coefficient may involve a stochastic component, describing uncertainties in the control
process, resulting in a multiplicative noise

$$
x_{n+1}= \begin{cases}f\left(\left(\nu+\ell_{1} \chi_{n+1}\right) x_{n}\right), & n \mid k,  \tag{3}\\ f\left(x_{n}\right), & n \nmid k,\end{cases}
$$

$x_{0}>0, n \in \mathbb{N}_{0}, \nu \in(0,1], k \in \mathbb{N}$. We can also consider the case when the reduction coefficient is deterministic but there are random fluctuations of $x_{n}$ at the control step, describing variability of the environment

$$
x_{n+1}= \begin{cases}\max \left\{f\left(\nu x_{n}\right)+\ell_{2} \chi_{n+1}, 0\right\}, & n \mid k,  \tag{4}\\ f\left(x_{n}\right), & n \nmid k,\end{cases}
$$

$x_{0}>0, n \in \mathbb{N}_{0}, \nu \in(0,1], k \in \mathbb{N}$. Here $\chi_{n+1} \in[-1,1]$ is the bounded random variable, while $\ell_{j}, j=1,2$ describes the bound of the noise.

The deterministic version of cycle stabilization by PF control was justified in [2]. Stabilization of a positive equilibrium with PF method shifts an equilibrium closer to zero and is achieved in an interval $\nu \in(\alpha, \beta) \subset(0,1)$. For smaller values of $\nu$, zero becomes the only stable equilibrium, for higher values, a positive equilibrium still can be unstable. When we applied PF control on each $k$ th step [2], it led to an asymptotically stable $k$-cycle, with all the values between zero and a positive equilibrium. Here we construct a stochastic analogue of this process. Stabilization of stochastic equations with proportional feedback was recently explored in the continuous case [13], as well as the idea of periodic controls [18].

The paper is organized as follows. In Section 2, we introduce main assumptions and discuss properties of a $k$ iteration of function $f$. Section 3 contains results on the existence of a blurred $k$-cycle in the presence of stochastic multiplicative perturbations of the control parameter $\nu$ when the level of noise $\ell$ is small, while Section 4 deals with the controlled equation for additive stochastic perturbations. Section 5 contains examples with computer simulations illustrating the results of the paper, along with some generalizations. In particular, a modification of PF method "centered" at an unstable equilibrium $K$ instead of zero, is developed and applied to construct a blurred $k$-cycle in the neighborhood of $K$, when both stochastic, multiplicative and additive perturbations, are present.

## 2 Definitions and Auxiliary Results

In this paper, we impose an assumption on the map $f$ in a right neighbourhood of zero.

Assumption 1 The function $f:[0, \infty) \rightarrow[0, \infty)$ is continuous, $f(0)=0$, and there is a real number $b>0$ such that $f(x)$ is strictly monotone increasing, while the
function $f(x) / x$ is strictly monotone decreasing on $(0, b]$, $f(b)>b$, while $f(b) / b>f(x) / x$ for any $x \in(b, \infty)$.

Remark 1 Note that, once Assumption 1 holds for a certain $b>0$, it is also satisfied for any $b_{1} \in(0, b]$.

Many functions in (1) used in applications satisfy Assumption 1 , see [16] and examples below. We truncate values at zero, when necessary, to satisfy $f:[0, \infty) \rightarrow$ $[0, \infty)$, which is a common practice [14]. Examples include the Ricker model

$$
\begin{equation*}
x_{n+1}=f_{1}\left(x_{n}\right)=x_{n} e^{r\left(1-x_{n}\right)} \tag{5}
\end{equation*}
$$

for $r>1$, with any $b \leq 1 / r$, the logistic model (truncated at zero) $x_{n+1}=f_{2}\left(x_{n}\right)=\max \left\{r x_{n}\left(1-x_{n}\right), 0\right\}$ for $r>2$, with $b \leq 1 / 2$. In these maps, $f_{i}$ are unimodal, increasing on $\left[0, x_{\max }\right]$ and decreasing on $\left[x_{\max }, \infty\right)$, with the only critical point on $[0, \infty)$, which is a global maximum. However, Assumption 1 can hold for functions which have more than one critical point, for example, for the map developed in [12] to describe the growth of the bobwhite quail population

$$
\begin{equation*}
f_{3}(x)=x\left(A+\frac{B}{1+x^{\gamma}}\right), \quad A, B>0, \quad \gamma>1 \tag{6}
\end{equation*}
$$

which, generally, has two critical points, first a local maximum, then a global minimum, then increases, and $f_{3}(x) \rightarrow \infty$ as $x \rightarrow \infty$.

We denote by $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{m}\right)_{m \in \mathbb{N}}, \mathbb{P}\right)$ a complete filtered probability space, $\chi:=\left(\chi_{m}\right)_{m \in \mathbb{N}}$ is a sequence of independent random variables with the zero mean. The filtration $\left(\mathcal{F}_{m}\right)_{m \in \mathbb{N}}$ is naturally generated by the sequence $\left(\chi_{m}\right)_{m \in \mathbb{N}}$, i.e. $\mathcal{F}_{m}=\sigma\left\{\chi_{1}, \ldots, \chi_{m}\right\}$. The standard abbreviation "a.s." is used for both "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. A detailed discussion of stochastic concepts and notation can be found in [15]. We consider (3) and (4), where the sequence $\left(\chi_{m}\right)_{m \in \mathbb{N}}$ satisfies the following condition.

Assumption $2\left(\chi_{m}\right)_{m \in \mathbb{N}}$ is a sequence of independent and identically distributed continuous random variables, with the density function $\phi(x)$ such that $\phi(x)>0$ for $x \in[-1,1]$ and $\phi(x)=0$ for $x \notin[-1,1]$.

Remark 2 In fact, Assumption 2 can be relaxed to the condition $\mathbb{P}\{\chi \in[1-\varepsilon, 1]\}>0$ for any $\varepsilon>0$, which would allow to include discrete distributions, where $\mathbb{P}\{\chi=1\}>0$.

In numerical simulations, we also consider the combination of (3) and (4)

$$
x_{n+1}= \begin{cases}f\left(\left(\nu+\ell_{1} \chi_{n+1}\right) x_{n}\right)+\ell_{2} \chi_{n+1}, & n \mid k  \tag{7}\\ f\left(x_{n}\right), & n \nmid k\end{cases}
$$

$x_{0}>0, n \in \mathbb{N}_{0}, k \in \mathbb{N}, \nu \in(0,1]$.
Let us start with some auxiliary results on $f^{k}(x)=$ $f\left(f^{k-1}(x)\right)$ and $g(x):=f^{k}(\nu x)$ for any $\nu \in(0,1]$, where Assumption 1 holds. Obviously $f:[0, b] \rightarrow[0, f(b)]$ is increasing and continuous, and there is an increasing and continuous inverse function $f^{-1}:[0, f(b)] \rightarrow[0, b]$. As $f(b)>b$ and $f(x) / x$ is decreasing on $(0, b]$ by Assumption $1, f(x)>x$ for $x \in[0, b]$, and also $f$ is increasing. Thus $f^{-1}(b):[0, f(b)] \rightarrow[0, b]$ is well defined, and $f^{-1}(b) \in(0, b)$. Evidently $f^{2}:\left[0, f^{-1}(b)\right] \rightarrow[0, f(b)]$ is continuous and increasing, since $f$ is increasing on $[0, b]$, and $f^{2}(x) \in[0, f(b)]$ for $x \in\left[0, f^{-1}(b)\right]$. Therefore $f^{-2}:[0, f(b)] \rightarrow\left[0, f^{-1}(b)\right]$ is also well defined and increasing. Similarly, $f^{-k}:[0, f(b)] \rightarrow\left[0, f^{1-k}(b)\right]$ exists and is increasing for any $k \in \mathbb{N}$. Denote

$$
\begin{equation*}
b_{j}:=f^{1-j}(b), \quad j \in \mathbb{N}, \quad j \neq 1, \quad b_{1}=b, \tag{8}
\end{equation*}
$$

then $f\left(b_{j+1}\right)=b_{j}, j=1,2, \ldots, k$, and

$$
\begin{equation*}
b=b_{1}>b_{2}>\cdots>b_{k}>0 \tag{9}
\end{equation*}
$$

Lemma 2.1 If $f$ satisfies Assumption 1, this assumption also holds for $f^{k}$ with $b_{k}$ instead of $b$, where $b_{k}$ is defined in (8).

Proof. The function $f:[0, b] \rightarrow[0, f(b)]$ is continuous and monotone increasing, so is $f^{k}:\left[0, b_{k}\right]=$ $\left[0, f^{1-k}(b)\right] \rightarrow[0, f(b)]$. Next, let us prove that $f^{k}(x) / x$ is monotone decreasing on $\left[0, b_{k}\right]$. Let $0<$ $x_{1}<x_{2} \leq b_{k}$. Then $f\left(x_{1}\right) \leq b_{k-1}, \ldots, f^{j}\left(x_{1}\right) \leq$ $b_{k-j}, j=1, \ldots, k-1$. Since $f(x) / x$ is decreasing on $[0, b]$, while $f$ is increasing, $f\left(x_{1}\right) / x_{1}>$ $f\left(x_{2}\right) / x_{2}, \quad f\left(f\left(x_{1}\right)\right) / f\left(x_{1}\right)>f\left(f\left(x_{2}\right)\right) / f\left(x_{2}\right), \ldots$, $f^{k}\left(x_{1}\right) / f^{k-1}\left(x_{1}\right)>f^{k}\left(x_{2}\right) / f^{k-1}\left(x_{2}\right)$ and

$$
\begin{aligned}
& \frac{f^{k}\left(x_{1}\right)}{x_{1}}=\frac{f^{k}\left(x_{1}\right)}{f^{k-1}\left(x_{1}\right)} \ldots \frac{f^{2}\left(x_{1}\right)}{f\left(x_{1}\right)} \frac{f\left(x_{1}\right)}{x_{1}} \\
& >\frac{f^{k}\left(x_{2}\right)}{f^{k-1}\left(x_{2}\right)} \ldots \frac{f^{2}\left(x_{2}\right)}{f\left(x_{2}\right)} \frac{f\left(x_{2}\right)}{x_{2}}=\frac{f^{k}\left(x_{2}\right)}{x_{2}} .
\end{aligned}
$$

Also, $f(0)=0$ implies $f^{k}(0)=0$, and $f(b)>b$, by $(9)$, yields that $f^{k}\left(b_{k}\right)=f^{k}\left(f^{1-k}(b)\right)=f(b)>b>b_{k}$.

Finally, let us justify that $f^{k}(x) / x<f^{k}\left(b_{k}\right) / b_{k}$ for any $x>b_{k}$ by induction. For $k=1, f(x) / x<f(b) / b$ follows from Assumption 1.

For $k=2$ and $x>f^{-1}(b)=b_{2}$, we consider two possible cases: $f(x)<b$ and $f(x) \geq b$. In the former case, $f(f(x))<f(b)$, as $f$ increases on $[0, b]$, and

$$
\frac{f^{2}(x)}{x}<\frac{f(b)}{x}<\frac{f(b)}{b_{2}}=\frac{f^{2}\left(b_{2}\right)}{b_{2}}
$$

For $f(x) \geq b$, by Assumption $1, f(x) / x<f\left(b_{2}\right) / b_{2}$ for any $x>b_{2}$, as $b_{2}<b$, and $f(f(x)) / f(x) \leq f(b) / b$. Thus

$$
\begin{gathered}
\frac{f(f(x))}{x}=\frac{f(f(x))}{f(x)} \frac{f(x)}{x}<\frac{f(b)}{b} \frac{f\left(b_{2}\right)}{b_{2}} \\
=\frac{f(b)}{b} \frac{b}{b_{2}}=\frac{f(b)}{b_{2}}=\frac{f^{2}\left(b_{2}\right)}{b_{2}} .
\end{gathered}
$$

Next, let us proceed to the induction step. Assume $\frac{f^{n}(x)}{x}<\frac{f^{n}\left(b_{n}\right)}{b_{n}}=\frac{f(b)}{b_{n}}$ for any $x>b_{n}$. Consider $x>b_{n+1}$. Then either $f^{n}(x)<b$ or $f^{n}(x) \geq b$. In the former case $f^{n}(x)<b$, we have $f\left(f^{n}(x)\right)<f(b)$ due to monotonicity of $f$ on $[0, b]$ and

$$
\frac{f^{n+1}(x)}{x}<\frac{f(b)}{x}<\frac{f(b)}{b_{n+1}}=\frac{f^{n+1}\left(b_{n+1}\right)}{b_{n+1}}
$$

In the latter case $f^{n}(x) \geq b$ we get

$$
\begin{gathered}
\frac{f^{n+1}(x)}{x}=\frac{f\left(f^{n}(x)\right)}{f^{n}(x)} \frac{f^{n}(x)}{x}<\frac{f(b)}{b} \frac{f^{n}\left(b_{n+1}\right)}{b_{n+1}} \\
\quad=\frac{f^{n+1}\left(b_{n+1}\right)}{b} \frac{b}{b_{n+1}}=\frac{f^{n+1}\left(b_{n+1}\right)}{b_{n+1}}
\end{gathered}
$$

where in the inequality we used $\frac{f^{n}(x)}{x}<\frac{f^{n}\left(b_{n+1}\right)}{b_{n+1}}$ for any $x>b_{n+1}$ by the induction assumption. Also, $f(u) / u \leq f(b) / b$ for any $u=f^{n}(x) \geq b$ by Assumption 1, while equalities applied notation (8).

Define the function $\Psi_{k}$ as

$$
\begin{equation*}
\Psi_{k}(x):=\frac{x}{f^{k}(x)}, \quad x \in\left(0, b_{k}\right), \quad k \in \mathbb{N} \tag{10}
\end{equation*}
$$

and formally introduce the limit

$$
\begin{equation*}
\Psi_{k}(0):=\lim _{x \rightarrow 0^{+}} \frac{x}{f^{k}(x)} \tag{11}
\end{equation*}
$$

Lemma 2.2 Let Assumption 1 hold, $k \in \mathbb{N}$ and $\Psi_{k}$ be defined as in (10), (11). Then
(1) $\Psi_{k}:\left(0, b_{k}\right) \rightarrow\left(\Psi_{k}(0), \Psi_{k}\left(b_{k}\right)\right)$,
$\Psi_{k}^{-1}:\left(\Psi_{k}(0), \Psi_{k}\left(b_{k}\right)\right) \rightarrow\left(0, b_{k}\right) ;$
(2) $0 \leq \Psi_{k}(0)<\Psi_{k}\left(b_{k}\right)<1$;
(3) both $\Psi_{k}$ and its inverse $\Psi_{k}^{-1}$ are increasing and continuous on their domains.

Proof. By Lemma 2.1, the function $\Psi_{k}$ defined in (10) is increasing, continuous and hence has a unique inverse function on $\left(0, b_{k}\right)$. Following Assumption 1, we notice
that the limit $\lim _{x \rightarrow 0^{+}} \frac{f(x)}{x}$ exists (finite or infinite), is positive and greater than 1 , since $f(x) / x$ is decreasing on $\left(0, b_{k}\right)$ and $f^{k}\left(b_{k}\right)>b_{k}$. Note that $\frac{f^{k}(x)}{x}=\frac{1}{\Psi_{k}(x)}$ and $\lim _{x \rightarrow 0^{+}} \frac{f^{k}(x)}{x}=\lim _{x \rightarrow 0^{+}} \frac{1}{\Psi_{k}(x)}$, where $\lim _{x \rightarrow 0^{+}} \frac{1}{\Psi_{k}(x)}=0$ if $\lim _{x \rightarrow 0^{+}} \frac{f^{k}(x)}{x}=+\infty$. Thus (11) is well defined, and Part (1) is valid. Also, $\Psi_{k}$ and its inverse are continuous monotone increasing in their domains and by Lemma 2.1, $\Psi_{k}\left(b_{k}\right)<1$, which implies Parts (2) and (3).

To apply known results from [4], for each point $x^{*} \in$ $(0, f(b))$, we are looking for the control parameter $\nu=$ $\nu\left(x^{*}\right) \in(0,1)$ such that $x^{*}$ is a fixed point of the function $g(x):=f^{k}(\nu x)$. We recall from (8) that $b_{1}=b$ and introduce

$$
\begin{align*}
& \hat{x}=f^{-k}\left(x^{*}\right), \nu=\nu\left(x^{*}\right):=\Psi_{k}\left(f^{-k}\left(x^{*}\right)\right) \\
& \nu\left(x^{*}\right)=\Psi_{k}(\hat{x}), \hat{x}=\nu\left(x^{*}\right) x^{*}=\Psi_{k}^{-1}(\nu) \tag{12}
\end{align*}
$$

Lemma 2.3 Let Assumption 1 hold, $k \in \mathbb{N}$ and $x^{*} \in$ $(0, f(b))$. The function $\nu\left(x^{*}\right)$ defined in (12) satisfies the following conditions:
(1) $x^{*}$ is a fixed point of $g(x):=f^{k}(\nu x)$, i.e. $f^{k}\left(\nu\left(x^{*}\right) x^{*}\right)=x^{*}, \nu\left(x^{*}\right)=\Psi_{k}(\hat{x}), \hat{x} \in\left(0, b_{k}\right) ;$
(2) $\nu\left(x^{*}\right) \in\left(\Psi_{k}(0), \Psi_{k}\left(b_{k}\right)\right) \subset(0,1)$;
(3) $\nu\left(x^{*}\right)$ is an increasing function of $x^{*}$ on $(0, f(b))$.

Proof. (1). Let $x^{*} \in(0, f(b))$, then $f^{-k}\left(x^{*}\right) \in$ $\left(0, b_{k}\right)$, thus $\Psi_{k}\left(f^{-k}\left(x^{*}\right)\right)$ is well defined and $\nu\left(x^{*}\right)=$ $\Psi_{k}\left(f^{-k}\left(x^{*}\right)\right)$
$=\frac{f^{-k}\left(x^{*}\right)}{f^{k}\left(f^{-k}\left(x^{*}\right)\right)}=\frac{f^{-k}\left(x^{*}\right)}{x^{*}}=\frac{\hat{x}}{f^{k}(\hat{x})}=\Psi_{k}(\hat{x})$, hence $\hat{x}=\nu\left(x^{*}\right) x^{*} \in\left(0, b_{k}\right)$ and
$f^{k}\left(\nu\left(x^{*}\right) x^{*}\right)=f^{k}\left(\frac{f^{-k}\left(x^{*}\right)}{x^{*}} x^{*}\right)=f^{k}\left(f^{-k}\left(x^{*}\right)\right)=x^{*}$.
(2). We have $x^{*} \in(0, f(b))$ and $\hat{x}=f^{-k}\left(x^{*}\right) \in$ $\left(0, b_{k}\right)$. Thus Lemma 2.2, Part 1 implies $\nu\left(x^{*}\right) \in$ $\left(\Psi_{k}(0), \Psi_{k}\left(b_{k}\right)\right) \subset(0,1)$.
(3). By Lemma 2.2 and Assumption 1, for any $k \in \mathbb{N}$, both $\Psi_{k}$ and $f^{-k}$ are increasing functions on $\left(0, b_{k}\right)$ and $(0, f(b))$, respectively. Therefore $\nu\left(x^{*}\right)=\Psi_{k}\left(f^{-k}\left(x^{*}\right)\right)$ is increasing as a function of $x^{*}$ on $(0, f(b))$.

## 3 Multiplicative perturbations

Consider the deterministic PF with variable intensity $\nu_{m} \in(0,1]$, applied at each $k$-th step, for a fixed $k \in \mathbb{N}$,

$$
x_{n+1}=\left\{\begin{array}{ll}
f\left(\nu_{n} x_{n}\right), & n \mid k,  \tag{13}\\
f\left(x_{n}\right), & n \nmid k,
\end{array}, x_{0}>0, n \in \mathbb{N}_{0} .\right.
$$

Investigation of (13) will allow to analyze corresponding stochastic equation (3) with a multiplicative noise. For each $x^{*} \in(0, f(b))$, we establish the control $\nu=\nu\left(x^{*}\right)$ and define an interval such that a solution of (3) remains in this interval, once the level of noise $\ell$ is small enough. This method was applied, for instance, in [1].

Further, we apply the result obtained in [4] for

$$
\begin{equation*}
z_{m+1}=g\left(\nu_{m} z_{m}\right)=f^{k}\left(\nu_{m} z_{m}\right), z_{0}>0, m \in \mathbb{N} \tag{14}
\end{equation*}
$$

to explore stochastic equation (3) with a multiplicative noise.

For any $\mu_{1}, \mu_{2}$ such that

$$
\begin{equation*}
\Psi_{k}(0)<\mu_{1}<\mu_{2}<\Psi_{k}\left(b_{k}\right) \tag{15}
\end{equation*}
$$

we define

$$
\begin{equation*}
y_{1}:=\Psi_{k}^{-1}\left(\mu_{1}\right), \quad y_{2}:=\Psi_{k}^{-1}\left(\mu_{2}\right) \tag{16}
\end{equation*}
$$

Lemma 3.1 [4, Lemma 3.1] Let Assumption 1 hold for $f^{k}, k \in \mathbb{N}, \mu_{1}$ and $\mu_{2}$ satisfy (15) and, for each $m \in \mathbb{N}$,

$$
\begin{equation*}
\nu_{m} \in\left[\mu_{1}, \mu_{2}\right] \tag{17}
\end{equation*}
$$

Then, for any $z_{0}>0$ and $\varepsilon, 0<\varepsilon<\min \left\{y_{1}, b_{k}-y_{2}\right\}$, where $y_{1}, y_{2}$ are defined in (16), there is $m_{0}=m_{0}\left(x_{0}, \varepsilon\right)$, $m_{0} \in \mathbb{N}$, such that the solution $z_{n}$ of equation (14) for any $m \geq m_{0}$ satisfies

$$
\begin{equation*}
\nu_{m} z_{m} \in\left(y_{1}-\varepsilon, y_{2}+\varepsilon\right) \tag{18}
\end{equation*}
$$

Remark 3 Lemma 3.1 actually states (see its proof in [4]) that, for a prescribed $k \in \mathbb{N}$, for a small $\varepsilon>0$, once a solution of (13) satisfies $\nu_{k m} x_{k m} \in\left(y_{1}-\varepsilon, y_{2}+\varepsilon\right)$, $m \in \mathbb{N}$, all the subsequent $k$-iterates $\nu_{m+j} x_{(m+j) k}, j \in \mathbb{N}$, are also in this interval. This is also true for the results based on Lemma 3.1, in particular, for Lemma 3.3 and Theorem 3.4.

Lemma 3.2 Let Assumption 1 hold, $\mu_{1}, \mu_{2}$ satisfy (15) and, for each $m \in \mathbb{N}$, (17) be fulfilled. For any $x_{0}>0$ and $\varepsilon>0$, there is $m_{0}=m_{0}\left(x_{0}, \varepsilon\right) \in \mathbb{N}$ such that for $m \geq m_{0}$, the solution of (13) satisfies

$$
\begin{equation*}
x_{m k+j} \in\left(f^{j}\left(y_{1}\right)-\varepsilon, f^{j}\left(y_{2}\right)+\varepsilon\right), \quad j=1, \ldots, k \tag{19}
\end{equation*}
$$

where $y_{1}$ and $y_{2}$ are defined in (16).

Proof. Note that $y_{1}, y_{2} \in\left(0, b_{k}\right)$ and $f^{j}$ are continuous and monotone increasing on $\left(0, b_{k}\right)$ for $j=0, \ldots, k-1$. Therefore for any $\varepsilon>0$ there is an $\varepsilon_{1}>0$ such that for $j=1, \ldots, k, u \in\left(y_{1}-\varepsilon_{1}, y_{2}+\varepsilon_{1}\right) \Rightarrow$
$f^{j}(u) \in\left(f^{j}\left(y_{1}\right)-\varepsilon, f^{j}\left(y_{2}\right)+\varepsilon\right)$. Choose $z_{0}=x_{0}$ and $\varepsilon_{2}<\min \left\{y_{1}, b_{k}-y_{2}, \varepsilon_{1}\right\}$ instead of $\varepsilon$ in Lemma 3.1. Then for $m>m_{0}$, by (18), $\nu_{m k} x_{m k} \in\left(y_{1}-\varepsilon_{2}, y_{2}+\varepsilon_{2}\right)$. Since, by the above implication,
$x_{m k+1}=f\left(\nu_{m k} x_{m k}\right) \in\left(f\left(y_{1}\right)-\varepsilon, f\left(y_{2}\right)+\varepsilon\right), \ldots$, $x_{m k+k}=f^{k}\left(\nu_{m k} x_{m k}\right) \in\left(f^{k}\left(y_{1}\right)-\varepsilon, f^{k}\left(y_{2}\right)+\varepsilon\right)$, this implies (19).

Let us proceed to stochastic equation (3).
We start with an auxiliary result which follows from Lemma 3.2.

Lemma 3.3 Let $k \in \mathbb{N}$ be fixed, Assumptions 1 and 2 hold, $\Psi_{k}$ be defined in (10), $x^{*} \in(0, f(b)), \nu=\nu\left(x^{*}\right)$ be as in (12), and

$$
\begin{gather*}
\ell \in\left(0, \min \left\{\Psi_{k}\left(b_{k}\right)-\nu, \nu-\Psi_{k}(0)\right\}\right),  \tag{20}\\
\underline{y}:=\Psi_{k}^{-1}(\nu-\ell), \bar{y}:=\Psi_{k}^{-1}(\nu+\ell), 0<\underline{y}<\bar{y}<b_{k} . \tag{21}
\end{gather*}
$$

Let $x_{n}$ be a solution to equation (3) with $\nu, \ell$ satisfying (12) and (20), respectively.

Then, for any $\varepsilon>0$ there is a $m_{0}=m_{0}\left(\varepsilon, x^{*}, x_{0}\right) \in \mathbb{N}$ such that, for all $m \geq m_{0}, m \in \mathbb{N}$,

$$
x_{m k+j} \in\left(f^{j}(\underline{y})-\varepsilon, f^{j}(\bar{y})+\varepsilon\right), \quad j=1, \ldots, k, \quad \text { a.s. }
$$

Proof. Since $x^{*}<f\left(b_{k}\right)<f(b)$, Lemma 2.3 implies $\nu\left(x^{*}\right)=\Psi_{k}\left(f^{-k}\left(x^{*}\right)\right) \in\left(\Psi_{k}(0), \Psi_{k}\left(b_{k}\right)\right)$. Thus the right segment bound $\nu-\Psi_{k}(0)$ in (20) is positive. By Assumption 2 we have, a.s.,

$$
\nu_{m}=\nu+\ell \chi_{m+1} \leq \nu+\ell, \quad \nu_{m}=\nu+\ell \chi_{m+1} \geq \nu-\ell
$$

and $\nu_{m}=\nu+\ell \chi_{m k+1} \geq \nu-\ell$, thus $\nu_{m} \in[\nu-\ell, \nu+\ell]$, a.s. Let $\mu_{1}:=\nu-\ell, \mu_{2}:=\nu+\ell$. With $\nu$ as in (12) and $\ell$ satisfying (20), we have

$$
\Psi_{k}(0)-\ell<\mu_{1}<\mu_{2}<\Psi_{k}\left(b_{k}\right)+\ell
$$

then Lemma 3.2 implies the statement of the lemma.
Lemma 3.3 leads to the main result of this section, which states that for any $k \in \mathbb{N}$ and $x^{*} \in(0, f(b))$, we can find a control $\nu$ and a noise level $\ell$, such that the solution eventually reaches some neighbourhood of a $k$-cycle, a.s., and stays there.

Theorem 3.4 Let Assumptions 1 and 2 hold, $\Psi_{k}$ be defined in (10), (11), $x^{*} \in(0, f(b))$ be an arbitrary point,
$\nu=\nu\left(x^{*}\right)$ be denoted in (12), $\underline{y}$ and $\bar{y}$ be defined in (21), $x_{0}>0$ and $\ell \in \mathbb{R}$ satisfy inequality (20). Then for the solution $x_{n}$ of equation (3), the following statements hold. (i) For each $\varepsilon>0$ there exists a nonrandom $m_{0}=$ $m_{0}\left(\varepsilon, x^{*}, x_{0}\right) \in \mathbb{N}$ such that, for all $m \geq m_{0}$,
$x_{m k+j} \in\left(f^{j}(\underline{y})-\varepsilon, f^{j}(\bar{y})+\varepsilon\right), \quad j=1, \ldots, k$, a.s. (ii) $\liminf _{m \rightarrow \infty} x_{m k+j} \geq f^{j}(\underline{y}), \limsup _{m \rightarrow \infty} x_{m k+j} \leq f^{j}(\bar{y}), j=$ $1, \ldots, k$, a.s.

Proof. Note that from condition (20) we have $\nu>\ell$. By Lemma 3.3, for any $x_{0}>0$ and $\varepsilon>0$, there is $m_{0}=$ $m_{0}(\varepsilon) \in \mathbb{N}$ such that, a.s., $x_{m k+j}>f^{j}(\underline{y})-\varepsilon, x_{m k+j}<$ $f^{j}(\bar{y})+\varepsilon, m \geq m_{0}, j=1, \ldots, k$, which immediately implies (i).

Choosing a sequence of $\varepsilon_{m}=\frac{1}{m}, m \in \mathbb{N}$ in (i), we deduce (ii).

Next, let us assume that the level of noise can be chosen arbitrarily small. Theorem 3.5 below confirms the intuitive feeling that, as the noise level $\ell$ is getting smaller, the solution of stochastic equation (3) behaves similarly to the solution of corresponding deterministic equation (2) in terms of approaching its stable cycle $\left\{f^{j}(\hat{x})\right\}$, $j=1, \ldots, k$, where $\hat{x}$ is defined in (12).

Theorem 3.5 Let Assumptions 1 and 2 hold, $k \in \mathbb{N}$ be fixed, $\hat{x} \in\left(0, b_{k}\right)$ be an arbitrary point, $x^{*}=f^{k}(\hat{x})$, $\nu=\nu\left(x^{*}\right)$ be defined as in (12), and $x_{0}>0$. Then, for any $\varepsilon>0$, there exists the level of noise $\ell(\varepsilon)>0$ such that for each $\ell<\ell(\varepsilon)$, there is a nonrandom $m_{1}=m_{1}\left(\varepsilon, \ell, \hat{x}, x_{0}\right)$ such that the solution $x$ of equation (3) satisfies $x_{m k+j} \in$ $\left(f^{j}(\hat{x})-\varepsilon, f^{j}(\hat{x})+\varepsilon\right), j=1, \ldots, k$ for $m \geq m_{1}$, a.s.

Proof. First of all, from monotonicity of $f^{k}$ notice that the map $x^{*}=f^{k}(\hat{x})$ is one-to-one, and an arbitrary $x^{*} \in(0, f(b))$ corresponds to a certain $\hat{x} \in\left(0, b_{k}\right)$. Next, by continuity of all $f^{j}$, for any $\nu=\nu\left(x^{*}\right)$ defined as in (12), there is a $\delta>0$ such that

$$
\begin{equation*}
|x-\hat{x}|<\delta \Rightarrow\left|f^{j}(x)-f^{j}\left(x^{*}\right)\right|<\frac{\varepsilon}{2}, j=1, \ldots, k \tag{22}
\end{equation*}
$$

Also, from the choice of $\nu$ in (12) and continuity of $\Psi_{k}$, there is $\ell(\varepsilon)>0$ such that for $\ell<\ell(\delta)$,

$$
|\underline{y}-\hat{x}|<\delta, \quad|\bar{y}-\hat{x}|<\delta
$$

since $\underline{y}$ and $\bar{y}$ defined in (21) continuously depend on $\ell$. Thus, by (22),

$$
\begin{equation*}
\left|f^{j}(\hat{x})-f^{j}(\underline{y})\right|<\frac{\varepsilon}{2},\left|f^{j}(\bar{y})-f^{j}(\hat{x})\right|<\frac{\varepsilon}{2} \tag{23}
\end{equation*}
$$

$j=1, \ldots, k$. Next, let us apply Theorem 3.4, Part (i), with $\frac{\varepsilon}{2}$ instead of $\varepsilon$. Then, $\forall m \geq m_{0}, j=1, \ldots, k$, a.s.,

$$
\begin{equation*}
x_{m k+j} \in\left(f^{j}(\underline{y})-\frac{\varepsilon}{2}, f^{j}(\bar{y})+\frac{\varepsilon}{2}\right) . \tag{24}
\end{equation*}
$$

In view of (23) and (24), $x_{m k+j}>f^{j}(\hat{x})-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=$ $f^{j}(\hat{x})-\varepsilon$ and $x_{m k+j}<f^{j}(\hat{x})+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=f^{j}(\hat{x})+\varepsilon$, therefore $x_{m k+j} \in\left(f^{j}(\hat{x})-\varepsilon, f^{j}(\hat{x})+\varepsilon\right), j=1, \ldots, k$, a.s.

## 4 Additive perturbations

In this section we investigate similar problems for stochastic equation with additive perturbations (4), where $f$ satisfies Assumption 1. Our purpose remains the same: to achieve pseudo-stabilization of a blurred cycle $\left\{f^{j}\left(x^{*}\right)\right\}, j=1, \ldots, k$. Here $x^{*}$ is an arbitrary point $x^{*} \in(0, f(b))$.

Denoting again $g(x):=f^{k}(\nu x)$, we can connect (4) to the equation with $x_{m k}=z_{m}, x_{0}=z_{0}>0$,

$$
\begin{equation*}
z_{m+1}=\max \left\{g\left(z_{m}\right)+\ell \chi_{m+1}, 0\right\}, m \in \mathbb{N} \tag{25}
\end{equation*}
$$

Let $x^{*} \in(0, f(b)), \nu=\Psi_{k}^{-1}\left(x^{*}\right), \hat{x}=\nu x^{*}=f^{-k}\left(x^{*}\right) \in$ $\left(0, b_{k}\right)$. Note that $b_{k} / \nu>x^{*}$ and, for a fixed $\nu$, by Lemma 2.1, $g(x)=f^{k}(\nu x)$ satisfies Assumption 1 for $\nu x \in\left(0, b_{k}\right]$, so $g\left(b_{k} / \nu\right) /\left(b_{k} / \nu\right)<g\left(x^{*}\right) / x^{*}$. Here the equality $g\left(x^{*}\right)=x^{*}$ is due to Lemma 2.3, Part 1. Thus $\frac{b_{k}}{\nu}-g\left(\frac{b_{k}}{\nu}\right)>0$. In addition, $g(x)>x$ for $x \in\left(0, x^{*}\right)$ and $g(x)>x, x \in\left(x^{*}, b_{k} / \nu\right)$. For $\ell=0$, from monotonicity of $g$ on $\left(0, b_{k} / \nu\right), x^{*}$ is an attractor of $g$ on $\left(0, b_{k} / \nu\right)$. Moreover, $g(x)<x$ for any $x>x^{*}$ implies $x^{*}$ is an attractor for any $z_{0}>0$. Our purpose is to choose $\ell>0$ small enough, to have $z_{m+1} \in\left(0, b_{k} / \nu\right)$, once $z_{m}$ is in this interval.

However, attractivity of a positive equilibrium in a deterministic case, in the presence of the zero equilibrium, does not imply that zero is a repeller in the stochastic case, see $[5,9]$ and references therein. Generally, with a positive probability, a solution can still stay in the right neighbourhood of zero. Assumption 2 and its generalized version in Remark 2 allow to make a conclusion on attractivity of $x^{*}$, a.s.

We choose $\delta_{0}>0$ satisfying

$$
\begin{equation*}
\delta_{0}<\min \left\{\frac{b_{k}}{\nu}-g\left(\frac{b_{k}}{\nu}\right), \max _{x \in\left[0, x^{*}\right]}[g(x)-x]\right\} \tag{26}
\end{equation*}
$$

Define the numbers $y_{1}, y_{2}, \hat{x}_{1}, \hat{x}_{2}$ as

$$
\begin{aligned}
& y_{1}:=\sup \left\{x \in\left[0, x^{*}\right] \mid g(x)-x \geq \delta_{0}\right\}, \\
& \hat{x}_{1}:=\nu y_{1} \in\left(0, b_{k}\right), y_{1} \in\left(0, x^{*}\right), \\
& y_{2}:=\inf \left\{x \in\left[x^{*}, b_{k} / \nu\right] \mid g(x)-x \leq-\delta_{0}\right\}, \\
& \hat{x}_{2}:=\nu y_{2} \in\left(0, b_{k}\right), y_{2} \in\left(x^{*}, b_{k} / \nu\right)
\end{aligned}
$$

According to the choice of $\delta_{0}$, the sets in (27) are nonempty, so $y_{1}, y_{2}, \hat{x}_{1}$ and $\hat{x}_{2}$ are well defined. Denote

$$
\begin{equation*}
y_{3}=\inf \left\{x \in\left[x^{*}, \infty\right) \mid g(x)-\delta_{0} \leq y_{1}\right\} \tag{28}
\end{equation*}
$$

where $y_{3}$ is assumed to be infinite if the set in the righthand side of (28) is empty. As stated in [4, Lemma 4.1], the numbers $y_{1}, y_{2}$ and $y_{3}$ defined by (27) and (28), respectively, exist.

Lemma 4.1 [4, Theorem 4.5] Let Assumptions 1 and 2 hold, $x^{*} \in(0, f(b))$ be an arbitrary point, $\nu=\nu\left(x^{*}\right)$ be chosen as in (12), $g(x)=f^{k}(\nu x)$ and $\delta_{0}$ satisfy (26). Suppose that $y_{1}, y_{2}, y_{3}$ are denoted in (27) and (28), respectively, and $z_{m}$ is a solution to equation (25) with an arbitrary $z_{0}>0$ and $\ell>0$ satisfying $\ell \leq \delta_{0}$. Then
(i) for each $\varepsilon_{1}>0$, there exists a random $\mathcal{M}(\omega)=$ $\mathcal{M}\left(\omega, x_{0}, \ell, x^{*}, \varepsilon_{1}\right)$ such that for $m \geq \mathcal{M}(\omega)$ we have, a.s. on $\Omega$,

$$
\begin{equation*}
y_{1} \leq z_{m} \leq y_{2}+\varepsilon_{1} \tag{29}
\end{equation*}
$$

(ii) for each $\varepsilon_{1}>0$ and $\gamma \in(0,1)$, there is a nonrandom number $M=M\left(\gamma, x_{0}, \ell, x^{*}, \varepsilon_{1}\right)$ such that

$$
\begin{equation*}
\mathbb{P}\left\{y_{1} \leq z_{m} \leq y_{2}+\varepsilon_{1}, \text { for } m \geq M\right\}>\gamma \tag{30}
\end{equation*}
$$

(iii) we have $\liminf _{m \rightarrow \infty} z_{m} \geq y_{1}, \limsup _{m \rightarrow \infty} z_{m} \leq y_{2}$, a.s.

Another result that will be used in future is also stated below. It illustrates that a solution will eventually be in any arbitrarily small neighborhood of $x^{*}$ with an arbitrarily close to one probability and will further be used in the proof of Theorem 4.4.

Lemma 4.2 [4, Theorem 4.6] Let Assumptions 1 and 2 hold, $z_{0}>0$ be an arbitrary initial value, $x^{*} \in(0, f(b))$ be an arbitrary point, $\nu=\nu\left(x^{*}\right)$ be chosen as in (12). Then, for each $\varepsilon>0$ and $\gamma \in(0,1)$, we can find $\delta_{0}$ such that for the solution $z_{m}$ to (25) with $\ell \leq \delta_{0}$, and for some nonrandom $M=M\left(\gamma, x_{0}, \ell, x^{*}, \varepsilon\right) \in \mathbb{N}$, we have $\mathbb{P}\left\{z_{m} \in\left(x^{*}-\varepsilon, x^{*}+\varepsilon\right) \forall m \geq M\right\} \geq \gamma$.

This leads to two main results for (4), Lemma 4.1 implying Theorem 4.3 on a.s. convergence to a blurred cycle, and Lemma 4.2 yielding Theorem 4.4 on the convergence with a prescribed close to one probability.

Theorem 4.3 Let Assumptions 1 and 2 hold, $\hat{x} \in\left(0, b_{k}\right)$ be an arbitrary point, $x^{*}=f^{k}(\hat{x}), \nu=\nu\left(x^{*}\right)$ be chosen as in (12), and $\delta_{0}$ satisfy (26). Suppose that $\hat{x}_{1}$ and $\hat{x}_{2}$ are defined as in (27), and $x_{n}$ is a solution to (4) with an arbitrary $x_{0}>0$ and $\ell>0$ satisfying $\ell \leq \delta_{0}$. Then
(i) For any $\varepsilon>0$, there exists a random $\mathcal{M}(\omega)=$ $\mathcal{M}\left(\omega, x_{0}, \ell, \hat{x}, \varepsilon\right)$ such that for $m \geq \mathcal{M}(\omega)$ we have, a.s. on $\Omega, f^{j}\left(\hat{x}_{1}\right) \leq x_{k m+j} \leq f^{j}\left(\hat{x}_{2}\right)+\varepsilon, j=1, \ldots k$.
(ii) For each $\varepsilon>0$ and $\gamma \in(0,1)$, there is a nonrandom number $M=M\left(\gamma, x_{0}, \ell, \hat{x}, \varepsilon\right)$ such that, for $j=1, \ldots k$,

$$
\begin{equation*}
\mathbb{P}\left\{f^{j}\left(\hat{x}_{1}\right) \leq x_{k m+j} \leq f^{j}\left(\hat{x}_{2}\right)+\varepsilon, m \geq M\right\}>\gamma \tag{31}
\end{equation*}
$$

(iii) For a solution $x_{n}$ of (4) we have, a.s., for $j=1, \ldots k$,

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} x_{k m+j} \geq f^{j}\left(\hat{x}_{1}\right), \limsup _{n \rightarrow \infty} x_{k m+j} \leq f^{j}\left(\hat{x}_{2}\right) \tag{32}
\end{equation*}
$$

Proof. Recall from (27) that $\nu y_{1}=\hat{x}_{1}, \nu y_{2}=\hat{x}_{2}$. From continuity and monotonicity of $f$, for any $\varepsilon>0$, there is a $\varepsilon_{1}>0$ such that (29) implies

$$
\begin{equation*}
f^{j}\left(\hat{x}_{1}\right) \leq f^{j}\left(\nu z_{m}\right) \leq f^{j}\left(\hat{x}_{2}\right)+\varepsilon, \quad j=1, \ldots, k \tag{33}
\end{equation*}
$$

We have

$$
\begin{equation*}
x_{m k}=z_{m}, \quad x_{m k+j}=f^{j}\left(\nu z_{m}\right), \quad j=1, \ldots, k \tag{34}
\end{equation*}
$$

(i) Choosing this $\varepsilon_{1}$ as in (i) of Lemma 4.1, we find $\mathcal{M}(\omega)=\mathcal{M}\left(\omega, x_{0}, \ell, x^{*}, \varepsilon_{1}\right)$ such that (29), and thus (31) are satisfied.
(ii) Further, (ii) in Lemma 4.1 implies for $M=$ $M\left(\gamma, x_{0}, \ell, x^{*}, \varepsilon_{1}\right)$ inequality (30). Thus by (33) and (34) we have $\mathbb{P}\left\{f^{j}\left(\hat{x}_{1}\right) \leq x_{k m+j} \leq f^{j}\left(\hat{x}_{2}\right)+\varepsilon\right.$, for $\left.m \geq M\right\}$ $\geq \mathbb{P}\left\{\hat{x}_{1} \leq z_{m} \leq y_{2}+\varepsilon\right.$, for $\left.m \geq M\right\}>\gamma$.
(iii) As $\bar{x}_{m k+j}$ and $z_{m}$ are connected with (34), application of Part (iii) in Lemma 4.1 immediately implies (32).

Theorem 4.4 Let Assumptions 1 and 2 hold, $x_{0}>0$, $\hat{x} \in\left(0, b_{k}\right)$ be an arbitrary point, $x^{*}=f^{k}(\hat{x}), \nu=\nu\left(x^{*}\right)$ be chosen as in (12). Then, for each $\varepsilon>0$ and $\gamma \in(0,1)$, we can find $\delta_{0}$ such that for the solution $x_{n}$ to (4) with $\ell \leq \delta_{0}$, and for some nonrandom $M=M\left(\gamma, x_{0}, \ell, \hat{x}, \varepsilon\right) \in$ $\mathbb{N}, j=1, \ldots, k$, we have
$\mathbb{P}\left\{x_{k m+j} \in\left(f^{j}(\hat{x})-\varepsilon, f^{j}(\hat{x})+\varepsilon\right) \forall m \geq M\right\} \geq \gamma$.
Proof. Let us choose $\varepsilon_{1}$ such that (33) is satisfied, fix $\gamma \in(0,1)$ and find $M=M\left(\gamma, x_{0}, \ell, \hat{x}, \varepsilon_{1}\right) \in \mathbb{N}$ as in Lemma 4.2. Then $\mathbb{P}\left\{\nu z_{m} \in\left(\hat{x}-\varepsilon_{1}, \hat{x}+\right.\right.$ $\varepsilon_{1}$ ) for all $\left.m \geq M\right\} \geq \gamma$, which by (34) implies the statement of the theorem.

## 5 Examples

We consider (7) combining multiplicative and additive noise. Similarly to the previous theorems, the following more general result can be obtained. However, the proof is long and technical and does not include any new ideas. Therefore we do not present it, but only illustrate stated below Proposition 1 with computer simulations.

Proposition 1 Let Assumptions 1 and 2 hold, $x_{0}>0$, $\hat{x} \in\left(0, b_{k}\right)$ be an arbitrary point, $x^{*}=f^{k}(\hat{x}), \nu=\nu\left(x^{*}\right)$ be chosen as in (12). Then, for each $\varepsilon>0$ and $\gamma \in(0,1)$, we can find $\delta_{1}$ and $\delta_{2}$ such that for the solution $x_{n}$ to (7) with $\ell_{1} \leq \delta_{1}, \ell_{2} \leq \delta_{2}$ and for some nonrandom $M=$ $M\left(\gamma, x_{0}, \ell_{1}, \ell_{2}, \hat{x}, \varepsilon\right) \in \mathbb{N}$, we have
$\mathbb{P}\left\{x_{k m+j} \in\left(f^{j}(\hat{x})-\varepsilon, f^{j}(\hat{x})+\varepsilon\right) \forall m \geq M\right\} \geq \gamma, j=$ $1, \ldots, k$.

Now we present examples of application of noisy PF control method to create a stable equilibrium or stable $k$-cycle in the neighborhood of nonzero point $K$. In all case noises $\chi$ are continuous uniformly distributed on $[-1,1]$. In all the simulations five runs with the same initial value are illustrated, with $n$ on the $x$-axis and $x_{n}$ (for all the five runs) on $y$-axis.

Example 1 Let us apply PF control to the Ricker model (5). For $r=2.8$, the non-controlled map is chaotic. We consider $\nu=0.002$, noise applied every third step. For (3) with $\ell_{1}=0.0001$ we observe a blurred stable 3-cycle, see Fig. 1, left. Next, we simulate additive noise as in (4). We observe a blurred stable 3-cycle with similar amplitudes for larger $\ell_{2}$, see Fig. 1, right. For the combined noise as in (7), the results of the runs are similar to Fig. 1, left.


Fig. 1. Solutions of the Ricker difference equation with $f=f_{1}$ as in (5), $r=2.8, x_{0}=0.5, k=3, \nu=0.002, n=0, \ldots, 1000$ and (left) (3) with $\ell_{1}=0.0001$, (right) (4) with $\ell_{2}=0.0005$.

Example 2 Consider a particular case of (6), see [3,4],

$$
\begin{equation*}
f(x)=x\left(0.55+\frac{3.45}{1+x^{9}}\right), \quad x \geq 0 \tag{35}
\end{equation*}
$$

We apply PF with $k=3$ to the three cases: the multiplicative noise, as in (3), the additive noise, as in (4), and the combined noise as in (7), see Fig. 2.


Fig. 2. Solutions $x_{n}$ vs. $n$ with $f$ as in (35), $k=3, \nu=0.02$, $x_{0}=0.5$, and (top left) (3) with $\ell=0.0005$, (top right) (4) with $\ell=0.005$, (bottom) (7) with $\ell_{1}=0.0005$ and $\ell_{2}=0.005$.

The standard PF control moves a positive equilibrium towards zero; applied at every $k$ th step, it leads to a stable cycle in a right neighbourhood of zero. Now we modify this method choosing a positive equilibrium $K_{1}$ instead of zero. We apply PF control method to create a stable equilibrium or $k$-cycle in the nighbourhood of nonzero point $K_{1}$. The non-shifted PF control brings the state variable $1 / \nu$ times closer to zero. We mimic this idea for a shifted version assuming that the state variable is proportionally moved to the fixed $K_{1}$. The controlled equation has the form $x_{n+1}=f\left(K_{1}+\nu\left(x_{n}-\right.\right.$ $\left.\left.K_{1}\right)\right)-K_{1}+K_{1}=f\left(\nu x_{n}+(1-\nu) K_{1}\right), x_{n} \geq K_{1}, x_{n+1}=$ $K_{1}-\left[K_{1}-f\left(K_{1}-\nu\left(K_{1}-x_{n}\right)\right)\right]=f\left(\nu x_{n}+(1-\nu) K_{1}\right)$, $x_{n} \in\left(0, K_{1}\right)$. Thus

$$
\begin{equation*}
x_{n+1}=f\left(\nu x_{n}+(1-\nu) K_{1}\right) \tag{36}
\end{equation*}
$$

Example 3 Define

$$
\begin{equation*}
f(x):=\frac{9}{2} x^{2}(1-x), \quad x \in[0,1] \tag{37}
\end{equation*}
$$

The maximum value of $f_{\max }$ is achieved at $x_{\max }=\frac{2}{3}$, $f\left(x_{\max }\right)=\frac{2}{3}$, the inflection point is $x^{*}=\frac{1}{3}, f^{\prime \prime}(x)>0$ for $x \in\left(0, \frac{1}{3}\right)$ and $f^{\prime \prime}(x)<0$ for $x \in\left(\frac{1}{3}, 1\right)$, $f$ has two positive equilibrium points $K_{1}=\frac{1}{3}, K_{2}=\frac{2}{3}$ and $f^{\prime}\left(\frac{1}{3}\right)=$ $\frac{3}{2}>1$.

Consider a modification of PF method "centered" at $K_{1}=1 / 3$, see (36). It can be shown that, for $\nu \in(2 / 3,1)$, equation (36) has two positive locally stable equilibrium points on both sides of $K_{1}$, each attracts a solution $x_{n}$ with corresponding position of $x_{0}$ around $K_{1}$, see bifurcation diagram on Fig 3.


Fig. 3. Bifurcation diagram for (36) with $f$ as in (37), $c=1-\nu$ changing from zero to 0.9 and $x_{0}$ changing from 0 to 1 . We get an upper branch if $x_{0}$ changes from $1 / 3$ to 1 and the lower branch if it changes from zero to $1 / 3$.

Note that (36) is a particular case of Target Oriented Control [6], sufficient conditions for stabilization of $K_{1}$ in (36) were obtained in [7]. A modification of PF method is responsible for the left part of the diagram (bistability) while [7] gives an exact bound $c^{*}$ such that for $c \in\left(c^{*}, 1\right)$, all solutions of (36) with $\nu:=1-c$ and $x_{0} \in(0,1)$ converge to $K_{1}=1 / 3$.

We introduce multiplicative noise in (36) to get for any $k \in \mathbb{N}, \nu \in(0,1]$,

$$
x_{n+1}=\left\{\begin{array}{l}
f\left(\left(\nu+\ell_{1} \chi_{m+1}\right) x_{n}\right.  \tag{38}\\
\left.+\left(1-\nu-\ell_{1} \chi_{m+1}\right) K_{1}\right), n \mid k \\
f\left(x_{n}\right), n \nmid k \\
n \in \mathbb{N}_{0}, x_{0}>0
\end{array}\right.
$$

A multiplicative noise with small $\ell_{1}$ does not change this type of behavior, as illustrated in Fig. 4. This also holds when coefficient $\ell_{2}$ of the additive noise is relatively small and $x_{0}$ is relatively far from $K_{1}$, see Fig 5, left and middle. However, when $\ell_{2}$ increases (in some limits), the solution started on the left of $K_{1}$ and close enough to $K_{1}$, is attracted to both equilibrium solutions, on the left and on the right of $K_{1}$, see Fig. 5, right. The same holds when $x_{0}>K_{1}$. Fig. 6 illustrates construction of stable threecycles when the initial value is taken on both sides of $K_{1}$.


Fig. 4. Five runs of the difference equation with $f$ as in (37), multiplicative noise with $\ell=0.0005, \nu=0.7$ and (left) $x_{0}=0.35$, (middle) $x_{0}=0.6,($ right $) x_{0}=0.2$.


Fig. 5. For difference equation (38) with $f$ as in (37), with additive noise (left) $\nu=0.7, \ell=0.001, x_{0}=0.4$, (middle) $\nu=0.7, \ell=0.001, x_{0}=0.3$, (right) $\nu=0.8, \ell=0.01$, $x_{0}=0.33$.

## Example 4 Define now

$$
\begin{equation*}
f(x):=6 x^{2}(1-x), \quad x \in[0,1] \tag{39}
\end{equation*}
$$

which has a positive equilibrium $K_{1} \approx 0.211<1 / 3$. Note that for $f$ as in (39), the results of Sections 3-4 can be applied for $x_{n}$ to the left of $K_{1}$, see the bifurcation diagram in Figure 6.


Fig. 6. Five runs of difference equation (38) with $f$ as in (37), PF control applied every third step, multiplicative noise with $\ell_{1}=0.0001$, additive noise with $\ell_{2}=0.001, \nu=0.7$ and (left) $x_{0}=0.3$, (right) $x_{0}=0.7$.


Fig. 7. Bifurcation diagram with $1-\nu$ changing from zero to 0.9 for the map $f$ as in (39).

Fig. 8 illustrates a construction of a stable 2-cycle with multiplicative and additive noise. The left-side pictures, where the initial value $x_{0}<K_{1}$, show a 2-cycle, while the right-side pictures, where $x_{0}>K_{1}$, produce a 3-cycle.



Fig. 8. Solutions of difference equation (38) with $f$ as in (39), PF control applied every second step, multiplicative noise with $\ell_{1}=0.001$, additive noise with $\ell_{2}=0.01, \nu=0.7$ and (left) $x_{0}=0.2$, (right) $x_{0}=0.3$.

## 6 Summary and discussion

First of all, numerical simulations show less restrictive conditions on $\nu$ in (36) than for classical (non-shifted) PF control. If we denote $c:=1-\nu$ in (36) then it becomes a particular case of Target Oriented Control with an unstable equilibrium $K_{1}$ as a target [6,7].

Possible generalizations and extensions of the present research include the following topics.
(a) Everywhere in simulations we assumed uniform continuous distribution, and all the estimates were dependent only on the noise amplitude. Specific estimates for particular types of noise distribution can be established.
(b) Everywhere we investigated asymptotic properties of solutions. However, analysis of so called transient behaviour, describing the speed of this convergence, starting from the initial point, maximal amplitudes for given initial values and noise characteristics, is interesting for applications.

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