Advances in Stabilization of Highly Nonlinear Hybrid Delay Systems ⋆

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Abstract

Given an unstable highly nonlinear hybrid stochastic differential delay equation (SDDE, also known as an SDDE with Markovian switching), can we design a delay feedback control to make the controlled hybrid SDDE become exponentially stable? Recent work by Li and Mao in 2020 gave a positive answer when the delay in the given SDDE is a positive constant. It is also noted that in their paper the time lag in the feedback control is another constant. However, time delay in a real-world system is often a variable of time while it is difficult to implement the feedback control in practice if the time lag involved is a strict constant. Mathematically speaking, the stabilization problem becomes much harder if these delays are time-varying, in particular, if they are not differentiable. The aim of this paper is to tackle the stabilization problem under non-differentiable time delays. One more new feature in this paper is that the feedback control function used is bounded.

Key words: Brownian motion, Markov chain, Hybrid SDDE, Bounded feedback control, Exponential stability, Lyapunov functional.

1 Introduction

Recently, Li and Mao [9] successfully investigated the following stabilization problem: given an unstable highly nonlinear hybrid SDDE

$$\begin{aligned} dx(t) = {} & f(x(t), x(t-\delta), r(t), t)dt \\ & + g(x(t), x(t-\delta), r(t), t)dB(t), \end{aligned} \tag{1.1}$$

how could we design a delay feedback control $u(x(t-\tau), r(t), t)$ to make the controlled system

$$\begin{aligned} dx(t) = {} & [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ & + g(x(t), x(t-\delta), r(t), t)dB(t) \end{aligned} \tag{1.2}$$

to be stable? Here the state $x(t)$ takes values in R^d and the mode $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \cdots, N\}$, $B(t)$ is a Brownian motion, f and g are referred to as the drift and diffusion coefficient, respectively, while u is the feedback control function, δ is a positive constant which stands for the time delay of the given system, and τ another positive constant which is the time lag between the time when the state is observed and that when the corresponding feedback control acts in the system. The high nonlinearity means that the coefficients f and g do not satisfy the linear growth condition (see, e.g., [7,12]). For the general theory of hybrid SDDEs we refer the reader to, for example, [17,21,23], while for the stability theory to [2,6,14,24,25] and the references therein. In particular, the reader can find more information on stabilization of hybrid systems by delay feedback controls in [14,18,19,27].

Although the theory established in [9] is useful in the area of feedback controls for highly nonlinear hybrid SDDEs, there are at least three issues to be addressed in order to make the theory more useful and applicable:

- Q1 The time delay δ in the given SDDE (1.1) is a constant.
- Q2 The time lag τ in the feedback control is a (different) constant.
- Q3 The control function $u(x, i, t)$ could be unbounded.

The reasons why we need to address these issues are because: (1) the time delay is in general a variable of time in many real-world SDDE models (see, e.g., [4,23,26,28]);

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(2) it is hard and costs more to design a feedback control with a strict constant time lag but will be much easier and cost less if the time lag is within a certain time interval; (3) the cost of feedback control is in general proportional to $|u(x(t-\tau), r(t), t)|$ and hence the cost will be reduced if a bounded control function u could be designed. The aim of this paper is to address all these issues positively. Our new theory will not only applicable to much wider class of hybrid SDDE systems in the real world but the delay feedback controls could also be more easily implemented while the corresponding costs might be reduced.

Mathematically speaking, it is no trivial to investigate any of these issues (see, e.g., the technical proof of Theorem 2.4 below). Let us discuss Q1 a bit more to see this point of view. Replacing the time delay δ in the SDDE (1.1) with time-varying delay δ_t, we recall a frequently imposed condition in the stability study is that δ_t is differentiable with its derivative being bounded by a positive number less than 1 (see, e.g., [4,15]). This condition has been imposed only because of the mathematical technique used—the technique of time change but might not be a natural feature of SDDE models in the real world (see, e.g., [5,11,20,22]). For example, discontinuous or sawtooth delays occur frequently in sampled-data controls or network-based controls where delays are commonly referred to as fast varying delays (no assumptions on the delay-derivatives). Also, data are usually buffered and sent through a network in packets traveling independently from each other, and the delay changes abruptly when processing proceeds from a packet to the subsequent one (see, e.g., [3,29]). A simplest case is that the time delay in a network is larger during business hours than other time. Such a time delay can be described by a piecewise constant function, e.g.,

$$\delta_t = \sum_{k=0}^{\infty} \big(hI_{[k,k+1/3)}(t) + h_1 I_{[k+1/3,k+1)}(t)\big), \qquad (1.3)$$

where $h > h_1 > 0$, the time unit is one day and $[0, 1/3)$ and $[1/3, 1)$ are business and no business period per day, respectively. But, even such a simple function is not differentiable. These show clearly that to make the stabilization theory more useful, we should avoid imposing the differentiability on the time delay δ_t. Of course, Q1 will become more challenged as the stability study of systems with non-differentiable time-varying delay is much harder than constant or differentiable delay. Similarly, we see another challenge if we replace the constant time lag τ in the controlled SDDE (1.2) with time-varying lag τ_t which is only Borel measurable and takes values in a time interval, say $(0, \bar{\tau}]$. To see the challenge due to Q3, we only briefly point out here that the class of bounded control functions is much smaller than unbounded functions as in [9] and hence the design of a feasible control function becomes much harder. We will explain much more in Section 3 below. Tackling these new challenge makes the mathematics presented in this paper to be significantly different from [9].

In summary, it is necessary to investigate three issues listed above in order to make the stabilization theory more useful in applications. New mathematics needs to be developed to tackle these issues. Let us begin to develop our new theory on the stabilization problem.

2 Uncontrolled SDDE

2.1 Notation and assumptions

Throughout this paper, unless otherwise specified, we use the following notation. Let R^d be the d-dimensional Euclidean space and $|x|$ denotes the Euclidean norm of $x \in R^d$. Let $R_+ = [0, \infty)$. Let A^T denote the transpose of a vector or matrix A. Let $|A| = \sqrt{\text{trace}(A^T A)}$ be the trace norm of a matrix A. If A is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. For $h > 0$, denote by $C([-h, 0]; R^d)$ the family of continuous functions φ from $[-h, 0] \to R^d$ with the norm $\|\varphi\| = \sup_{-h \le u \le 0} |\varphi(u)|$. Denote by $C(R^d; R_+)$ the family of continuous functions from R^d to R_+. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a set, denote by I_A its indicator function; that is, $I_A(z) = 1$ if $z \in A$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with its filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the same probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j \ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$ under $\mathbb{P}$.

As mentioned in Section 1, one of the key features in this paper is that the time delay in the given unstable SDDE is a non-differentiable function of time. To be precise, we state it as an assumption.

Assumption 2.1 *The time-varying delay δ_t is a Borel measurable function from R_+ to $[h_1, h]$ and has the prop-*

erty that

$$\bar{h} := \limsup_{\Delta \to 0+} \Big(\sup_{s \ge -h} \frac{\mu(M_{s,\Delta})}{\Delta} \Big) < \infty, \qquad (2.1)$$

where h_1 and h are both positive constants with $h_1 < h$, $M_{s,\Delta} = \{t \in R_+ : t - \delta_t \in [s, s+\Delta)\}$ and $\mu(\cdot)$ denotes the Lebesgue measure on R_+.

It is worth noting that many time-varying delay functions in practice satisfy this assumption. For example, the left-limited-right-continuous piecewise constant function

$$\delta_t = \sum_{k=0}^{\infty} m_k I_{[t_k, t_{k-1})}(t), \quad t \ge 0, \qquad (2.2)$$

satisfies Assumption 2.1 with $\bar{h} = [(h - h_1)/\Delta^*] + 2$, where $m_k \in [h_1, h]$ and $0 = t_0 < t_1 < \cdots < t_k \to \infty$ with $\Delta^* := \inf_{k \ge 0}(t_{k+1} - t_k) > 0$ while $[(h - h_1)/\Delta^*]$ is the integer part of $(h - h_1)/\Delta^*$. Moreover, if δ_t is a Lipschitz continuous function with its Lipscitiz coefficient $h_2 \in (0, 1)$, namely

$$|\delta_t - \delta_s| \le h_2(t - s), \ \forall\, 0 \le s < t < \infty, \qquad (2.3)$$

the δ_t satisfies Assumption 2.1 with $\bar{h} = 1/(1 - h_2)$. Specially, if δ_t is differentiable and its derivative is bounded by $h_2 \in (0, 1)$, then δ_t satisfies Assumption 2.1. These examples show that there is a rich class of functions δ_t. It should also be pointed out that we must have $\bar{h} \ge 1$. This can be seen from the following useful lemma.

Lemma 2.2 *Let Assumption 2.1 hold. Let $T > 0$ and $\varphi : [-h, T - h_1] \to R_+$ be a continuous function. Then*

$$\int_0^T \varphi(t - \delta_t)dt \le \bar{h} \int_{-h}^{T-h_1} \varphi(t)dt. \qquad (2.4)$$

Proof. By Assumption 2.1, for any $\varepsilon > 0$, there is a positive number $\bar{\Delta}$ such that

$$\sup_{s \ge -h} \frac{\mu(M_{s,\Delta})}{\Delta} \le \bar{h} + \varepsilon, \ \ \forall \Delta \in (0, \bar{\Delta}). \qquad (2.5)$$

Note that $-h \le t - \delta_t \le T - h_1$ for $t \in [0, T]$. Let n be any large integer such that $\Delta := (T - h_1 + h)/n < \bar{\Delta}$. Set $t_u = -h + u\Delta$ for $u = 0, 1, \cdots, n$. By the definition of the Riemann-Lebesgue integral, we have

$$\int_0^T \varphi(t - \delta_t)dt = \lim_{n \to \infty} \sum_{u=0}^{n-1} \mu(M_{t_u,\Delta})\varphi(t_u).$$

But, by (2.5), $\mu(M_{t_u,\Delta}) \le (\bar{h} + \varepsilon)\Delta$. Hence

$$\begin{aligned} \int_0^T \varphi(t - \delta_t)ds &\le \lim_{n \to \infty} \sum_{u=0}^{n-1} (\bar{h} + \varepsilon)\Delta\varphi(t_u) \\ &= (\bar{h} + \varepsilon) \int_{-h}^{T-h_1} \varphi(t)dt. \end{aligned} \qquad (2.6)$$

Letting $\varepsilon \to 0$ yields the required assertion (2.4). □

If we let $\varphi(t) = 1$ for all $t \ge -h$, the lemma shows that $T \le \bar{h}(T - h_1 + h)$ for any $T > 0$, which implies $\bar{h} \ge \lim_{T \to \infty} T/(T - h_1 + h) = 1$. In other words, Assumption 2.1 forces $\bar{h} \ge 1$ inexplicitly.

As explained in Section 1, the constant delay in the SDDE (1.1) is replaced by δ_t in this paper. To be precise, the given unstable system discussed in this paper is described by the nonlinear hybrid SDDE

$$\begin{aligned} dx(t) = &\, f(x(t), x(t - \delta_t), r(t), t)dt \\ &+ g(x(t), x(t - \delta_t), r(t), t)dB(t), \end{aligned} \qquad (2.7)$$

with the initial data

$$\{x(t) : -h \le t \le 0\} = \xi \in C([-h, 0]; R^d), \qquad (2.8)$$

where the coefficients $f : R^d \times R^d \times S \times R_+ \to R^d$ and $g : R^d \times R^d \times S \times R_+ \to R^{d \times m}$ are Borel measurable functions. We impose the following assumption on the coefficients.

Assumption 2.3 *Both coefficients f and g are locally Lipschitz continuous. Moreover, there exist positive constants $p, q, \alpha_1, \alpha_2, \alpha_3$ with $p \wedge q > 2$ such that*

$$\begin{aligned} x^T f(x, y, i, t) &+ \frac{q-1}{2}|g(x, y, i, t)|^2 \\ &\le \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p \end{aligned} \qquad (2.9)$$

for all $(x, i, t) \in R^d \times S \times R_+$.

It should be emphasized that we do NOT require $\alpha_2 > \alpha_3$, which differs significantly from many existing papers, e.g., [4,9]. Theorem 2.4 below hence covers much more general hybrid SDDEs.

2.2 Global solution

The following theorem does not only show the existence and uniqueness of the global solution but also the finiteness of the moments of the solution.

Theorem 2.4 *Under Assumptions 2.1 and 2.3, equation (2.7) with the initial data (2.8) has a unique global* *solution $x(t)$ on $[-h,\infty)$ and the solution has the properties that for all $t \ge 0$*

$$\mathbb{E}|x(t)|^q < \infty \tag{2.10}$$

and

$$\mathbb{E}\int_0^t |x(s)|^{p+q-2} ds < \infty. \tag{2.11}$$

Proof. The local Lipschitz condition guarantees that the hybrid SDDE (2.7) with the initial data (2.8) has a unique maximal local solution, denoted by $x(t)$ on $[-h, e_\infty)$, where e_∞ is the explosion time (see, e.g., [23]). We need to show $e_\infty = \infty$ a.s. For each integer $k \ge \|\xi\|$, define the stopping time

$$\sigma_k = e_\infty \wedge \inf\{t \in [0, e_\infty) : |x(t)| \ge k\},$$

where throughout this paper we set $\inf \emptyset = \infty$. As σ_k is increasing, it has a limit and we set $\sigma_\infty = \lim_{k\to\infty} \sigma_k$. Obviously, $\sigma_\infty \le e_\infty$ a.s. We divide the whole proof into two steps.

Step 1. Restrict $t \in [0, h_1]$. Noting that $-h \le t - \delta_t \le 0$ we see $x(t-\delta_t) = \xi(t-\delta_t)$ so is already known. By the Itô formula and Assumption 2.3, it is easy to show that

$$\begin{aligned}&\mathbb{E}|x(t\wedge\sigma_k)|^q - |\xi(0)|^q \\ &\le \mathbb{E}\int_0^{t\wedge\sigma_k} q|x(s)|^{q-2}\big[\alpha_1(|x(s)|^2 + |x(s-\delta_s)|^2) \\ &- \alpha_2|x(s)|^p + \alpha_3|x(s-\delta_s)|^p\big] ds.\end{aligned} \tag{2.12}$$

But, by the well-known Young inequality,

$$|x(s)|^{q-2}|x(s-\delta_s)|^2 \le |x(s)|^q + |x(s-\delta_s)|^q$$

and

$$\begin{aligned}&\alpha_3|x(s)|^{q-2}|x(s-\delta_s)|^p \\ &\le 0.5\alpha_2|x(s)|^{p+q-2} + \alpha_4|x(s-\delta_s)|^{p+q-2},\end{aligned}$$

where

$$\alpha_4 = \frac{p}{p+q-2}\alpha_3^{\frac{p+q-2}{p}}\Big(\frac{2(q-2)}{\alpha_2(p+q-2)}\Big)^{\frac{q-2}{p+q-2}}.$$

Hence

$$\begin{aligned}&\mathbb{E}|x(t\wedge\sigma_k)|^q + 0.5q\alpha_2\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s)|^{p+q-2} ds \\ &\le |\xi(0)|^q + \alpha_5 + 2q\alpha_1\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)|^q ds,\end{aligned} \tag{2.13}$$

where

$$\alpha_5 = q\int_0^{h_1}\big[\alpha_1|x(s-\delta_s)|^q + \alpha_4|x(s-\delta_s)|^{p+q-2}\big] ds,$$

which is finite clearly. It follows from (2.13) that

$$\mathbb{E}|x(t\wedge\sigma_k)|^q \le |\xi(0)|^q + \alpha_5 + 2q\alpha_1\int_0^t \mathbb{E}|x(s\wedge\sigma_k)|^q ds.$$

An application of the well-known Gronwall inequality yields

$$\mathbb{E}|x(t\wedge\sigma_k)|^q \le (|\xi(0)|^q + \alpha_5)e^{2q\alpha_1 h_1} =: \alpha_6 \tag{2.14}$$

for all $t \in [0, h_1]$, where throughout this paper we use $=:$ to stand for 'denoted by'. This implies

$$k^q\mathbb{P}(\sigma_k \le h_1) \le \mathbb{E}|x(h_1\wedge\sigma_k)|^q \le \alpha_6.$$

Letting $k\to\infty$ we see that $\mathbb{P}(\sigma_\infty \le h_1) = 0$ and hence $\sigma_\infty \ge h_1$ a.s. We can now letting $k\to\infty$ in (2.14) to obtain

$$\mathbb{E}|x(t)|^q \le \alpha_6, \quad \forall t \in [0, h_1]. \tag{2.15}$$

Setting $t = h_1$ in (2.13) and then letting $k\to\infty$ we also get

$$\mathbb{E}\int_0^{h_1}|x(s)|^{p+q-2} ds \le \alpha_7, \tag{2.16}$$

where $\alpha_7 = (|\xi(0)|^q + \alpha_5 + 2q\alpha_1\alpha_6 h_1)/(0.5q\alpha_2)$.

Step 2. Restrict $t \in [0, 2h_1]$. We have just shown that up to time h_1, $x(t)$ has properties (2.15) and (2.16). We also observe that $-h \le t - \delta_t \le h_1$ whenever $t \in [0, 2h_1]$. In other words, we already have $x(t-\delta_t)$ from Step 1. By Lemma 2.2,

$$\int_0^{2h_1}|x(s-\delta_s)|^{p+q-2} ds \le \bar{h}\int_{-h}^{h_1}|x(s)|^{p+q-2} ds.$$

Consequently, using (2.16), we have

$$\begin{aligned}&\mathbb{E}\int_0^{2h_1}|x(s-\delta_s)|^{p+q-2} ds \\ \le&\bar{h}h\|\xi\|^{p+q-2} + \bar{h}\alpha_7 < \infty.\end{aligned} \tag{2.17}$$

It is easy to see that (2.12) still holds for $t \in [0, 2h_1]$. In the same way as (2.13) was proven, we can show that

$$\begin{aligned}&\mathbb{E}|x(t\wedge\sigma_k)|^q + 0.5q\alpha_2\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)|^{p+q-2} ds \\ &\le |\xi(0)|^q + \alpha_8 + 2q\alpha_1\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)|^q ds,\end{aligned} \tag{2.18}$$

where

$$\alpha_8 = q\int_0^{2h_1}\big[\alpha_1|x(s-\delta_s)|^q + \alpha_4|x(s-\delta_s)|^{p+q-2}\big] ds,$$

which is finite by (2.15) and (2.17). From (2.18), we can show in the similar fashion as in Step 1 that $\sigma_\infty \geq 2h_1$ a.s.,

$$\mathbb{E}|x(t)|^q < \infty, \quad \forall\, t \in [0, 2h_1]$$

and

$$\mathbb{E}\int_0^{2h_1} |x(s)|^{p+q-2} ds < \infty.$$

Repeating Step 2 for $t \in [0, 3h_1]$ and then $[0, 4h_1]$ etc., we can show that $\sigma_\infty = \infty$ a.s. and assertions (2.10) and (2.11) hold. The proof is therefore complete. □

2.3 Boundedness

Theorem 2.4 shows the finiteness of the moments of the solution. In this subsection we are going to show the boundedness of the moments. For this purpose, we need to strengthen Assumption 2.3 slightly.

Assumption 2.5 *Assumption 2.3 holds and $\bar\alpha_1 > \bar\alpha_2 \bar h$, where*

$$\bar\alpha_1 = q\alpha_2 - \frac{\alpha_3 q(q-2)}{p+q-2} \text{ and } \bar\alpha_2 = \frac{\alpha_3 qp}{p+q-2}. \quad (2.19)$$

Theorem 2.6 *If Assumptions 2.1 and 2.5 hold, then the solution of the SDDE (2.7) with the initial data (2.8) has the properties that*

$$\sup_{0\le t<\infty} \mathbb{E}|x(t)|^q < \infty \quad (2.20)$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}|x(s)|^{p+q-2} ds < \infty. \quad (2.21)$$

Proof. Let $\varepsilon_1 > 0$ be the unique root to the equation

$$\bar\alpha_1 - \varepsilon_1 = \bar h(\bar\alpha_2 + \varepsilon_1)e^{\varepsilon_1 h}. \quad (2.22)$$

By the Itô formula and Assumption 2.3, it is easy to show that

$$\begin{aligned} &e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \le \mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\varepsilon_1|x(s)|^q \\ &+ q|x(s)|^{q-2}\big[\alpha_1(|x(s)|^2 + |x(s-\delta_s)|^2) \\ &- \alpha_2|x(s)|^p + \alpha_3|x(s-\delta_s)|^p\big]\Big)ds. \quad (2.23)\end{aligned}$$

It should be mentioned that based on Theorem 2.4, we no longer need to use the technique of stopping times. A simple application of the Young inequality shows

$$\begin{aligned} &e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \\ &\le \mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\bar\alpha_3|x(s)|^q + \bar\alpha_4|x(s-\delta_s)|^q \\ &- \bar\alpha_1|x(s)|^{p+q-2} + \bar\alpha_2|x(s-\delta_s)|^{p+q-2}\Big)ds, \quad (2.24)\end{aligned}$$

where $\bar\alpha_3 = \varepsilon_1 + 2\alpha_1(q-1)$, $\bar\alpha_4 = 2\alpha_1$ but $\bar\alpha_1$ and $\bar\alpha_2$ have been defined in the statement of the theorem. It then follows

$$\begin{aligned} &e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \\ &\le \mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\bar\alpha_5 - (\bar\alpha_1 - \varepsilon_1)|x(s)|^{p+q-2} \\ &+ (\bar\alpha_2 + \varepsilon_1)|x(s-\delta_s)|^{p+q-2}\Big)ds, \quad (2.25)\end{aligned}$$

where

$$\bar\alpha_5 = 2\sup_{u\ge 0}\big[\bar\alpha_3 u^q - \varepsilon_1 u^{p+q-2}\big].$$

But, by Lemma 2.2, we can show that

$$\begin{aligned} &\int_0^t e^{\varepsilon_1 s}|x(s-\delta_s)|^{p+q-2}ds \\ \le& e^{\varepsilon_1 h}\int_0^t e^{\varepsilon_1(s-\delta_s)}|x(s-\delta_s)|^{p+q-2}ds \\ \le& \bar h e^{\varepsilon_1 h}\Big(h\|\xi\|^{p+q-2} + \int_0^t e^{\varepsilon_1 s}|x(s)|^{p+q-2}ds\Big).\end{aligned}$$

Substituting this into (2.25) and making use of (2.22), we get

$$e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \le \frac{\bar\alpha_5}{\varepsilon_1}e^{\varepsilon_1 t} + h\bar h(\bar\alpha_2 + \varepsilon_1)e^{\varepsilon_1 h}\|\xi\|^{p+q-2}.$$

This implies

$$\mathbb{E}|x(t)|^q \le |\xi(0)|^q + h\bar h(\bar\alpha_2+\varepsilon_1)e^{\varepsilon_1 h}\|\xi\|^{p+q-2} + \frac{\bar\alpha_5}{\varepsilon_1}$$

for all $t \ge 0$, which is the first assertion (2.20).

To show the second assertion, we let $\varepsilon_2 > 0$ be the unique root to the equation $\bar\alpha_1 - 2\varepsilon_2 = (\bar\alpha_2 + \varepsilon_2)\bar h$. In a similar fashion as (2.25) was proven, we can show that

$$\begin{aligned} &\mathbb{E}|x(t)|^q - |\xi(0)|^q \\ &\le \mathbb{E}\int_0^t \Big(\bar\alpha_6 - (\bar\alpha_1 - \varepsilon_2)|x(s)|^{p+q-2} \\ &+ (\bar\alpha_2 + \varepsilon_2)|x(s-\delta_s)|^{p+q-2}\Big)ds, \quad (2.26)\end{aligned}$$

where

$$\bar\alpha_6 = 2\sup_{u\ge 0}\big[2\alpha_1(q-1)u^q - \varepsilon_2 u^{p+q-2}\big].$$

But, by Lemma 2.2,

$$\int_0^t |x(s-\delta_s)|^{p+q-2}ds \le \bar{h}\Big(h\|\xi\|^{p+q-2} + \int_0^t |x(s)|^{p+q-2}ds\Big).$$

Hence

$$0 \le |\xi(0)|^q + \bar{\alpha}_6 t + \bar{h}h(\bar{\alpha}_2 + \varepsilon_2)\|\xi\|^{p+q-2} - \varepsilon_2 \mathbb{E}\int_0^t |x(s)|^{p+q-2}ds. \quad (2.27)$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}|x(s)|^{p+q-2}ds \le \bar{\alpha}_6/\varepsilon_2 < \infty$$

as required. The proof is therefore complete. □

It is worth noting that if $\bar{h} = 1$ (e.g., in the case of constant delay), $\bar{\alpha}_2 > \bar{\alpha}_3\bar{h}$ is equivalent to $\alpha_2 > \alpha_3$, and the latter was imposed in [9]. In other words, Theorem 2.6 is a generalization of Theorem 2.3 in [9].

3 Controlled SDDE

3.1 Solution

The solution of the given hybrid SDDE (2.7) has finite or bounded moments under Assumption 2.3 or 2.5, respectively, but it may not tend to 0 exponentially in moment or with probability 1 (i.e., exponentially stable). In this case, we may required to design a delay feedback control $u(x(t-\tau_t), r(t), t)$ for the controlled SDDE

$$dx(t) = [f(x(t), x(t-\delta_t), r(t), t) + u(x(t-\tau_t), r(t), t)]dt + g(x(t), x(t-\delta_t), r(t), t)dB(t) \quad (3.1)$$

to become stable. Here the control function $u : R^d \times S \times R_+ \to R^d$ is Borel measurable, while τ_t is a function on R_+ which stands for the time lag between the time when the state observation is made and the time when the corresponding control reaches the system. We impose an assumption on τ_t.

Assumption 3.1 *The control time lag τ_t is a Borel measurable function from R_+ to $[0, \bar{\tau}]$, where $\bar{\tau}$ is a positive number.*

It is worth noting that we impose a much weaker condition on the control time lag τ_t than the time delay δ_t in the given SDDE. This enables the feedback control to be implemented more easily. Naturally we need slightly adjust the initial data by imposing

$$\{x(t) : -h_0 \le t \le 0\} = \xi \in C([-h_0, 0]; R^d), \quad (3.2)$$

where $h_0 = h \vee \bar{\tau}$. The class of feasible control functions to be used in this paper is described in the following assumption.

Assumption 3.2 *The control function $u : R^d \times S \times R_+ \to R^d$ is bounded and, moreover, there exists a positive number β such that*

$$|u(x, i, t) - u(y, i, t)| \le \beta|x - y| \quad (3.3)$$

and $u(0, i, t) \equiv 0$ for all $(x, y, i, t) \in R^d \times R^d \times S \times R_+$.

As explained in Section 1, the reason why we ask the control function to be bounded is because that, in general, the control cost is proportional to $|u(x(t-\tau_t), r(t), t)|$. The bounded condition on u is not imposed in [9]. This means the class of feasible control functions is smaller than that in [9]. However, there are still lots of such control functions. For example, $u(x, i, t) = F_iG_i\pi_r(x)$, where $F_i \in R^{d\times l}$ and $G_i \in R^{l\times d}$ for some $1 \le l \le d$ with either G_i or F_i being known but F_i or G_i to be designed, respectively; while $r > 0$ and $\pi_r : R^d \to S_r := \{x \in R^d : |x| \le r\}$ defined by

$$\pi_r(x) = (|x| \wedge r)x/|x|, \quad x \in R^d, \quad (3.4)$$

where throughout this paper we set $x/|x| = 0$ when $x = 0$. Another example is

$$u(x, i, t) = F_iG_i(\sin(x_1/r), \cdots, \sin(x_d/r))^T.$$

The following theorem shows that any such a feasible control function preserves the property of the unique global solution.

Theorem 3.3 *If Assumptions 2.1, 2.3, 3.1 and 3.2 hold, then the controlled SDDE (3.1) with the initial data (3.2) has a unique global solution $x(t)$ on $[-h_0, \infty)$ which has properties (2.10) and (2.11). If, moreover, Assumption 2.3 is strengthened to Assumption 2.5, then the solution has properties (2.20) and (2.21).*

This theorem can be proven in a similar fashion as Theorems 2.4 and 2.6 were proven. The reason why we can impose a weaker condition on τ_t than we did on δ_t is because u is globally Lipschitz continuous. We omit the proof here. To close this subsection, we introduce a technical assumption.

Assumption 3.4 *There exist constants $K > 0$, $q_1 > 1$ and $q_i \ge 1$ $(2 \le i \le 4)$ such that*

$$\begin{aligned} |f(x, y, i, t)| &\le K(|x| + |y| + |x|^{q_1} + |y|^{q_2}), \\ |g(x, y, i, t)| &\le K(|x| + |y| + |x|^{q_3} + |y|^{q_4}) \end{aligned} \quad (3.5)$$

for all $(x, y, i, t) \in R^d \times R^d \times S \times R_+$. Moreover, p and q in Assumption 2.3 satisfy

$$q > (p + q_1 - 1) \vee (2(q_1 \vee q_2 \vee q_3 \vee q_4)), \tag{3.6}$$
$$p \geq 2(q_1 \vee q_2 \vee q_3 \vee q_4) - q_1 + 1. \tag{3.7}$$

3.2 Rules for the control function

In this subsection, we shall propose a couple of rules. We will then show in the next subsection that if the control function u to be designed can meet these rules, then the controlled SDDE (3.1) will be stable. Our first rule is:

Rule 3.5 *Design the control function $u : R^d \times S \times R_+ \to R^d$ so that we can find real numbers a_i, $\bar{a}_i$, positive numbers c_i, $\bar{c}_i$ and nonnegative numbers b_i, $\bar{b}_i$, d_i, $\bar{d}_i$ $(i \in S)$ such that*

$$\begin{aligned} &x^T[f(x, y, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, y, i, t)|^2 \\ &\quad \leq a_i|x|^2 + b_i|y|^2 - c_i|x|^p + d_i|y|^p \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &x^T[f(x, y, i, t) + u(x, i, t)] + \frac{q_1}{2}|g(x, y, i, t)|^2 \\ &\quad \leq \bar{a}_i|x|^2 + \bar{b}_i|y|^2 - \bar{c}_i|x|^p + \bar{d}_i|y|^p \end{aligned} \tag{3.9}$$

for all $(x, y, i, t) \in R^d \times R^d \times S \times R_+$; while both

$$\begin{aligned} &\mathcal{A}_1 := -2\text{diag}(a_1, \cdots, a_N) - \Gamma, \\ \text{and } &\mathcal{A}_2 := -(q_1 + 1)\text{diag}(\bar{a}_1, \cdots, \bar{a}_N) - \Gamma \end{aligned} \tag{3.10}$$

are nonsingular M-matrices; and moreover,

$$\begin{cases} 1 > \bar{h}\zeta_1, \ \zeta_2 > \bar{h}\zeta_3, \\ 1 > \frac{\zeta_4[q_1 - 1 + 2\bar{h}]}{q_1 + 1}, \\ \zeta_5 > \frac{\zeta_6[q_1 - 1 + p\bar{h}]}{p + q_1 - 1}, \end{cases} \tag{3.11}$$

where q_1 is the same as in Assumption 3.4,

$$\begin{aligned} (\theta_1, \cdots, \theta_N)^T &= \mathcal{A}_1^{-1}(1, \cdots, 1)^T, \\ (\bar{\theta}_1, \cdots, \bar{\theta}_N)^T &= \mathcal{A}_2^{-1}(1, \cdots, 1)^T, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\zeta_1 = \max_{i \in S} 2\theta_i b_i, &&\zeta_2 = \min_{i \in S} 2\theta_i c_i, \\ &\zeta_3 = \max_{i \in S} 2\theta_i d_i, &&\zeta_4 = \max_{i \in S}(q_1 + 1)\bar{\theta}_i \bar{b}_i, \\ &\zeta_5 = \min_{i \in S}(q_1 + 1)\bar{\theta}_i \bar{c}_i, &&\zeta_6 = \max_{i \in S}(q_1 + 1)\bar{\theta}_i \bar{d}_i. \end{aligned} \tag{3.13}$$

In the definition above we use the theory of nonsingular M-matrices (see, e.g., [23, Section 2.6]), by which we see that all θ_i and $\bar{\theta}_i$ defined by (3.12) are positive.

The control functions used in this paper are required to be *bounded* in addition to the global Lipschitz continuity. These functions are very much different from those used in [9] and we have explained why we need to impose the bounded condition in this paper.

Let us explain why we propose Rule 3.5. Assuming that the feedback control acts instantly, namely $\tau_t = 0$, we observe that the controlled SDDE (3.1) becomes

$$\begin{aligned} dx(t) = &[f(x(t), x(t - \delta_t), r(t), t) + u(x(t), r(t), t)]dt \\ &+ g(x(t), x(t - \delta_t), r(t), t)dB(t). \end{aligned} \tag{3.14}$$

Define a function $U : R^d \times S \to R_+$ by

$$U(x, i) = \theta_i|x|^2 + \bar{\theta}_i|x|^{q_1+1}, \quad (x, i) \in R^d \times S \tag{3.15}$$

while define a function $\mathcal{L}U : R^d \times R^d \times S \times R_+ \to R$ by

$$\begin{aligned} &\mathcal{L}U(x, y, i, t) \\ &= 2\theta_i\Big[x^T[f(x, y, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, y, i, t)|^2\Big] \\ &+ (q_1 + 1)\bar{\theta}_i\Big(|x|^{q_1-1}x^T[f(x, y, i, t) + u(x, i, t)] \\ &+ 0.5|x|^{q_1-1}|g(x, y, i, t)|^2 \\ &+ 0.5(q_1 - 1)|x|^{q_1-3}|x^T g(x, y, i, t)|^2\Big) \\ &+ \sum_{j=1}^N \gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{q_1+1}), \end{aligned} \tag{3.16}$$

where θ_i and $\bar{\theta}_i$ have been defined by (3.12). It is worth mentioning that $\mathcal{L}$ is in fact the diffusion operator acting on $C^{2,1}$ functions with respect to (3.14) but we here prefer to treat $\mathcal{L}U$ as a single function. By (3.8), (3.9) and (3.12), (3.13), we have

$$\begin{aligned} &\mathcal{L}U(x, y, i, t) \\ &\leq -|x|^2 + \zeta_1|y|^2 - \zeta_2|x|^p + \zeta_3|y|^p \\ &- |x|^{q_1+1} + \zeta_4|x|^{q_1-1}|y|^2 - \zeta_5|x|^{p+q_1-1} + \zeta_6|x|^{q_1-1}|y|^p \\ &\leq -|x|^2 + \zeta_1|y|^2 - \zeta_2|x|^p + \zeta_3|y|^p + \frac{2\zeta_4}{q_1 + 1}|y|^{q_1+1} \\ &- \left(1 - \frac{\zeta_4(q_1 - 1)}{q_1 + 1}\right)|x|^{q_1+1} + \frac{\zeta_6 p}{p + q_1 - 1}|y|^{p+q_1-1} \\ &- \left(\zeta_5 - \frac{\zeta_6(q_1 - 1)}{p + q_1 - 1}\right)|x|^{p+q_1-1}. \end{aligned} \tag{3.17}$$

In a similar way as [4, Theorem 3.1] was proven but using condition (3.11) and Lemma 2.2, we can show that the SDDE (3.14) is exponentially stable. In other words, the control function $u(x, i, t)$ satisfying Rule 3.5 will stabilize the given SDDE if the feedback control acts instantly, namely the feedback control has the form of $u(x(t), r(t), t)$. However, as explained in Section 1, we should use the delay state feedback control $u(x(t -$

$\tau_t), r(t), t)$ in practice. That is, the controlled SDDE should be of the form (3.1) instead of (3.14). Comparing (3.1) with (3.14), we observe that if τ_t, the time lag between the time when the state is observed and that when the feedback control reaches the system, is sufficiently small, equation (3.1) should behave similarly to what equation (3.14) performs (i.e., stable). As τ_t is bounded by $\bar{\tau}$, this means $\bar{\tau}$ needs to be sufficiently small. To describe "sufficiently small" precisely while to cope with the polynomial growth of the coefficients f and g, we now propose the second rule.

Rule 3.6 *Find eight positive constants υ_j $(1 \le j \le 8)$ with $\upsilon_4 > \upsilon_5 \bar{h}$ and $\upsilon_6 \in (0, 1/\bar{h})$, and a function $W \in C(R^d; R_+)$, such that*

$$\begin{aligned}&\mathcal{L}U(x, y, i, t) + \upsilon_1 \big(2\theta_i |x| + (q_1+1)\bar{\theta}_i |x|^{q_1}\big)^2 \\ &\quad + \upsilon_2 |f(x, y, i, t)|^2 + \upsilon_3 |g(x, y, i, t)|^2 \\ &\quad \le -\upsilon_4 |x|^2 + \upsilon_5 |y|^2 - W(x) + \upsilon_6 W(y), \qquad (3.18)\end{aligned}$$

and

$$\upsilon_7 |x|^{p+q_1-1} \le W(x) \le \upsilon_8 (1 + |x|^{p+q_1-1}) \qquad (3.19)$$

for all $(x, y, i, t) \in R^d \times R^d \times S \times R_+$.

It is not very difficult to show that this rule can always be met under Assumptions 2.3 and 3.4.

3.3 Exponential stabilization

We can finally establish our theory on the exponential stabilization.

Theorem 3.7 *Let Assumptions 2.1, 2.3, 3.1 and 3.4 hold. Design a control function u in the feasible class (i.e., satisfying Assumption 3.2) to meet Rule 3.5 and then find eight positive constants υ_j $(1 \le j \le 8)$ and a function $W \in C(R^d; R_+)$ to meet Rule 3.6. If we further make sure*

$$\bar{\tau} < \frac{\sqrt{(\upsilon_4 - \upsilon_5 \bar{h})\upsilon_1}}{2\beta^2} \wedge \frac{\sqrt{\upsilon_1 \upsilon_2}}{\sqrt{2}\beta} \wedge \frac{\upsilon_1 \upsilon_3}{\beta^2} \wedge \frac{1}{4\sqrt{2}\beta}, \qquad (3.20)$$

then the solution of the controlled SDDE (3.1) with the initial data (3.2) has the property

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) < 0. \qquad (3.21)$$

That is, the controlled SDDE (3.1) is exponentially stable in mean square.

Proof. We will use the method of Lyapunov functionals to prove the theorem (please see, e.g., [2,8,9] for more details on the method). For this purpose, we define two segments $\tilde{x}_t := \{x(t+s) : -2h_0 \le s \le 0\}$ and $\tilde{r}_t := \{r(t+s) : -h_0 \le s \le 0\}$ for $t \ge 2h_0$. For $\tilde{x}_t$ and $\tilde{r}_t$ to be well defined for $0 \le t < 2h_0$, we set $x(s) = \xi(-h_0)$ for $s \in [-2h_0, -h_0)$ and $r(s) = r(0)$ for $s \in [-2h_0, 0)$. The Lyapunov functional is defined by

$$V(\tilde{x}_t, \tilde{r}_t, t) = U(x(t), r(t)) + (\beta^2/\upsilon_1)\Psi(t) \qquad (3.22)$$

for $t \ge 2h_0$, where U has been defined by (3.15) and

$$\Psi(t) = \int_{-\bar{\tau}}^{0} \int_{t+s}^{t} \big[\bar{\tau}|f_v + u_v|^2 + |g_v|^2\big] dv ds. \qquad (3.23)$$

Here throughout this proof we use the following simplified notations $f_v = f(x(v), x(v-\delta_v), r(v), v)$, $u_v = u(x(v-\tau_v), r(v), v)$, $g_v = g(x(v), x(v-\delta_v), r(v), v)$ for $v \ge 0$. By the generalized Itô formula (see, e.g., [23]) and the fundamental theory of calculus, it is straightforward to show that

$$dV(\tilde{x}_t, \tilde{r}_t, t) \le \mathbb{L}V(\tilde{x}_t, \tilde{r}_t, t)dt + dM(t) \qquad (3.24)$$

on $t \ge 2h_0$, where

$$\begin{aligned}&\mathbb{L}V(\tilde{x}_t, \tilde{r}_t, t) = \mathcal{L}U(x(t), x(t-\delta_t), r(t), t) \\ &- [2\theta_{r(t)} + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1-1}]x^T(t) \\ &\quad \times [u(x(t), r(t), t) - u(x(t-\tau_t), r(t), t)] \\ &+ (\beta^2 \bar{\tau}/\upsilon_1)\big[\bar{\tau}|f_t + u_t|^2 + |g_t|^2\big] \\ &- (\beta^2/\upsilon_1)\int_{t-\bar{\tau}}^{t} \big[\bar{\tau}|f_v + u_v|^2 + |g_v|^2\big] dv, \qquad (3.25)\end{aligned}$$

in which $\mathcal{L}U$ has been defined by (3.16). By Theorem 3.3 and Assumptions 3.2 and 3.4 as well as Rule 3.5, it is obvious that

$$\mathbb{E}|\mathbb{L}V(\tilde{x}_t, \tilde{r}_t, t)| < \infty, \ \forall t \ge 2h_0. \qquad (3.26)$$

Let ε be a sufficiently small positive number, which will be determined later. It is standard to show that for $t \ge 2h_0$,

$$\begin{aligned}&e^{\varepsilon t}\mathbb{E}V(\tilde{x}_t, \tilde{r}_t, t) - C_1 \\ &\le \int_{2h_0}^{t} e^{\varepsilon s}\Big(\varepsilon \mathbb{E}V(\tilde{x}_s, \tilde{r}_s, s) + \mathbb{E}\mathbb{L}V(\tilde{x}_s, \tilde{r}_s, s)\Big)ds, \qquad (3.27)\end{aligned}$$

where $C_1 = e^{2\varepsilon h_0}\mathbb{E}V(\tilde{x}_{2h_0}, \tilde{r}_{2h_0}, 2h_0)$. Set $\eta_1 = \min_{i\in S}\theta_i$, $\eta_2 = \max_{i\in S}\theta_i$ and $\eta_3 = \max_{i\in S}\bar{\theta}_i$. Then

$$\begin{aligned}&\eta_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le C_1 + \frac{\varepsilon\beta^2}{\upsilon_1}\Phi_1(t) \\ &+ \int_{2h_0}^{t} \varepsilon e^{\varepsilon s}\Big(\eta_2 \mathbb{E}|x(s)|^2 + \eta_3 \mathbb{E}|x(s)|^{q_1+1}\Big)ds \\ &+ \int_{2h_0}^{t} e^{\varepsilon s}\mathbb{E}\mathbb{L}V(\tilde{x}_s, \tilde{r}_s, s)ds, \qquad (3.28)\end{aligned}$$

where

$$\Phi_1(t) = \mathbb{E}\int_{2h_0}^t e^{\varepsilon s}\Big(\int_{-\bar\tau}^0 \int_{s+u}^s \big[\bar\tau|f_v+u_v|^2+|g_v|^2\big]dvdu\Big)ds.$$

In a similar fashion as in [9], we can show that

$$\begin{aligned}&\mathbb{E}\mathbb{L}V(\tilde x_s,\tilde r_s,s)\\ &\le -\Big(\upsilon_4-\frac{4\bar\tau^2\beta^4}{\upsilon_1}\Big)\mathbb{E}|x(s)|^2+\upsilon_5\mathbb{E}|x(s-\delta_s)|^2\\ &\quad-\mathbb{E}W(x(s))+\upsilon_6\mathbb{E}W(x(s-\delta_s))\\ &\quad-\frac{\beta^2}{4\upsilon_1}\mathbb{E}\int_{s-\bar\tau}^s\big[\bar\tau|f_v+u_v|^2+|g_v|^2\big]dv. \qquad (3.29)\end{aligned}$$

On the other hand, we obviously have

$$\begin{aligned}\mathbb{E}|x(s)|^{q_1+1}&\le \mathbb{E}|x(s)|^2+\mathbb{E}|x(s)|^{p+q_1-1}\\ &\le \mathbb{E}|x(s)|^2+\upsilon_7^{-1}\mathbb{E}W(x(s)). \qquad (3.30)\end{aligned}$$

Substituting (3.29) and (3.30) into (3.28) while noting by Lemma 2.2 that

$$\int_{2h_0}^t e^{\varepsilon s}\mathbb{E}|x(s-\delta_s)|^2ds \le \bar h e^{\varepsilon h}\int_{-h}^t e^{\varepsilon s}\mathbb{E}|x(s)|^2ds$$

and

$$\int_{2h_0}^t e^{\varepsilon s}\mathbb{E}W(x(s-\delta_s))ds \le \bar h e^{\varepsilon h}\int_{-h}^t e^{\varepsilon s}\mathbb{E}W(x(s))ds,$$

we get

$$\begin{aligned}&\eta_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2\le C_2+\frac{\varepsilon\beta^2}{\upsilon_1}\Phi_1(t)-\frac{\beta^2}{4\upsilon_1}\Phi_2(t)\\ &-\Big(\upsilon_4-\frac{4\bar\tau^2\beta^4}{\upsilon_1}-\varepsilon\eta_2-\varepsilon\eta_3-\upsilon_5\bar h e^{\varepsilon h}\Big)\int_{2h_0}^t e^{\varepsilon s}\mathbb{E}|x(s)|^2ds\\ &-\Big(1-\frac{\varepsilon\eta_3}{\upsilon_7}-\upsilon_6\bar h e^{\varepsilon h}\Big)\int_{2h_0}^t e^{\varepsilon s}\mathbb{E}W(x(s))ds \qquad (3.31)\end{aligned}$$

for $t\ge 2h_0$, where $C_2=C_1+\bar h e^{\varepsilon h}\int_{-h}^{2h_0}e^{\varepsilon s}[\upsilon_5\mathbb{E}|x(s)|^2+\upsilon_6\mathbb{E}W(x(s))]ds$ and

$$\Phi_2(t)=\mathbb{E}\int_{2h_0}^t e^{\varepsilon s}\Big(\int_{s-\bar\tau}^s\big[\bar\tau|f_v+u_v|^2+|g_v|^2\big]dv\Big)ds.$$

On the other hand, it is easy to see that

$$\Phi_1(t)\le\bar\tau\Phi_2(t).$$

Recalling that $1>\upsilon_6\bar h$ and (3.20), we can now choose a sufficiently small $\varepsilon\in(0,1/(4h_0))$ such that

$$\upsilon_4-\frac{4\bar\tau^2\beta^4}{\upsilon_1}-\varepsilon\eta_2-\varepsilon\eta_3-\upsilon_5\bar h e^{\varepsilon h}\ge 0$$

and

$$1-\frac{\varepsilon\eta_3}{\upsilon_7}-\upsilon_6\bar h e^{\varepsilon h}\ge 0.$$

Consequently, it follows from (3.31) that

$$\mathbb{E}|x(t)|^2\le (C_2/\eta_1)e^{-\varepsilon t},\quad \forall t\ge 2h_0, \qquad (3.32)$$

which implies the required assertion (3.21). The proof is hence complete. □

The following theorem shows that if Assumption 2.3 is strengthened into Assumption 2.5, stronger results can be obtained.

Theorem 3.8 *If all conditions in Theorem 3.7 hold except Assumption 2.3 is replaced with Assumption 2.5, then the solution of the controlled SDDE (3.1) with the initial data (3.2) has the properties that*

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^{\bar q})<0,\quad \forall\bar q\in[2,q) \qquad (3.33)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)<0\quad a.s. \qquad (3.34)$$

That is, the controlled SDDE (3.1) is not only exponentially stable in $L^{\bar q}$ but also almost surely exponentially stable.

The proof is standard (see, e.g., [9]) so is omitted.

4 Example

Due to the page limit, we will only discuss one example but the theoretical results established in this paper are illustrated fully.

Example 4.1 Consider the scalar hybrid SDDE

$$\begin{aligned}dx(t)&=f(x(t),x(t-\delta_t),r(t))dt\\ &\quad+g(x(t),x(t-\delta_t),r(t))dB(t)\end{aligned} \qquad (4.1)$$

on $t\ge 0$, but we will omit mentioning the initial data. Here the coefficients f and g are defined by

$$\begin{aligned}f(x,y,1)&=x(-1.5x^2+2y),\quad g(x,y,1)=\sigma_1 xy,\\ f(x,y,2)&=x(-2.1x^2+y),\quad g(x,y,2)=\sigma_2 xy\end{aligned}$$

for $x,y\in R$, where σ_1, σ_2 are two arbitrary numbers, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S=\{1,2\}$ with its generator $\Gamma=$

$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$, and

$$\delta_t = \sum_{k=0}^{\infty} \big[(0.1 + 0.1(t-2k))I_{[2k,2k+1)}(t) + (0.2 - 0.1(t-2k-1))I_{[2k+1,2(k+1))}(t)\big].$$

This is a simple version of the hybrid SDDE food chain model (see, e.g., [1,15]). It is easy to see that δ_t satisfies Assumption 2.1 with $h_1 = 0.1$, $h = 0.2$ and $\bar{h} = 1.1111$. It is also easy to show that Assumption 2.3 holds for $p = 4$, $\alpha_1 = 4$, $\alpha_2 = 1$, $\alpha_3 = (q-1)^2[(\sigma_1^4/4) \vee (\sigma_2^4/16)]$ and any $q > 2$. (It is possible to do better but we do not want to make it more complicated.) By Theorem 2.4, the SDDE (4.1) has a unique global solution $x(t)$ which has properties (2.10) and (2.11).

To make Assumption 2.5 hold, it is sufficient if $1 > \bar{h}\alpha_3$, namely $4\sigma_1^4 \vee \sigma_2^4 < 14.4/(q-1)^2$. In this case, Theorem 2.6 shows the solution $x(t)$ has properties (2.20) and (2.21).

We can also check Assumption 3.4 hold with $q_1 = 3$, $q_2 = q_3 = q_4 = 2$, $p = 4$ and any $q > 6$. In the remaining part of this example, we will fix $q = 7$ and $\sigma_1 = 0.2$, $\sigma_2 = 0.5$. By Theorem 2.6, the solution is bounded up to the 7th moment but the SDDE (4.1) may not be stable. To stabilize it, we use the delay feedback control to form the controlled system

$$dx(t) = [f(x(t), x(t-\delta_t), r(t)) + u(x(t-\tau_t), r(t))]dt + g(x(t), x(t-\delta_t), r(t))dB(t), \tag{4.2}$$

where

$$u(x,i) = (-3I_{\{1\}}(i) - 2I_{\{2\}}(i))\pi_{r_i}(x) \tag{4.3}$$

with $r_1 = 2.2$ and $r_2 = 1.8$ (please recall (3.4) for the definition of $\pi_{r_i}(x)$), and τ_t satisfies Assumption 3.1. Clearly, Assumption 3.2 is satisfied with $\beta = 3$. By Theorem 3.3, this controlled system has the unique solution $x(t)$ which has properties (2.20) and (2.21). Making use of the property that

$$xu(x,i) - 0.1x^4 \le (-3I_{\{1\}}(i) - 2I_{\{2\}}(i))x^2,$$

we can easily show that, for $(x, y, i) \in R \times R \times S \times R_+$,

$$x[f(x,y,i) + u(x,i)] + \frac{1}{2}|g(x,y,i)|^2 \le \begin{cases} -3x^2 + 2y^2 - 1.14x^4 + 0.01y^4, & i = 1, \\ -2x^4 + y^2 - 1.6875x^4 + 0.0625y^4, & i = 2, \end{cases}$$

and

$$x[f(x,y,i) + u(x,i)] + \frac{q_1}{2}|g(x,y,i)|^2 \le \begin{cases} -3x^2 + 2y^2 - 1.12x^4 + 0.03y^4, & i = 1, \\ -2x^2 + y^2 - 1.563x^4 + 0.1875y^4, & i = 2. \end{cases}$$

Namely, (3.8) and (3.9) hold with $a_1 = -3$, $b_1 = 2$, $c_1 = 1.14$, $d_1 = 0.01$, $a_2 = -2$, $b_2 = 1$, $c_2 = 1.6875$, $d_2 = 0.0625$, $\bar{a}_1 = -3$, $\bar{b}_1 = 2$, $\bar{c}_1 = 1.12$, $\bar{d}_1 = 0.03$, $\bar{a}_2 = -2$, $\bar{b}_2 = 1$, $\bar{c}_2 = 1.563$, $\bar{d}_2 = 0.1875$. Consequently, $\mathcal{A}_1 = \begin{pmatrix} 8 & -2 \\ -1 & 5 \end{pmatrix}$ and $\mathcal{A}_2 = \begin{pmatrix} 14 & -2 \\ -1 & 9 \end{pmatrix}$, which are both M-matrices. By (3.12) and (3.13), we then have $\theta_1 = 0.1842$, $\theta_2 = 0.2368$, $\bar{\theta}_1 = 0.0887$, $\bar{\theta}_2 = 0.1210$, and $\zeta_1 = 0.7368$, $\zeta_2 = 0.4200$, $\zeta_3 = 0.0296$, $\zeta_4 = 0.7096$, $\zeta_5 = 0.3974$, $\zeta_6 = 0.0908$, which satisfy (3.11). In other words, the control function $u(x,i)$ satisfies Rule 3.5. To verify Rule 3.6, we note that the function U defined by (3.15) has the form

$$U(x,i) = \begin{cases} 0.18428x^2 + 0.0887x^4 & \text{if } i = 1, \\ 0.2368x^2 + 0.1210x^4 & \text{if } i = 2. \end{cases}$$

By (3.17), we get

$$\mathcal{L}U(x,y,i,t) \le -x^2 + 0.7368y^2 - 1.0652x^4 + 0.3844y^4 - 0.3671x^6 + 0.0605y^6.$$

Also

$$\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \le 0.2243x^2 + 0.4584x^4 + 0.2343x^6,$$

$$|f(x,y,i)|^2 \le 6x^4 + 7.91x^6 + 4y^4 + y^6,$$

$$|g(x,y,i)|^2 \le 0.125x^4 + 0.125y^4.$$

Choosing $\upsilon_1 = 0.3$, $\upsilon_2 = 0.02$ and $\upsilon_3 = 0.31$, we then obtain

$$\mathcal{L}U(x,y,i,t) + \upsilon_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 + \upsilon_2|f(x,y,i,t)|^2 + \upsilon_3|g(x,y,i,t)|^2 \le -0.9327x^2 + 0.7368y^2 - W(x) + 0.6544W(y), \tag{4.4}$$

where $W(x) = 0.7689x^4 + 0.1386x^6$. That is, Rule 3.6 is satisfied with additional $\upsilon_4 = 0.9327$, $\upsilon_5 = 0.7368$, $\upsilon_6 = 0.6544$, $\upsilon_7 = 0.1386$ and $\upsilon_8 = 0.9075$. Consequently, condition (3.20) becomes $\bar{\tau} < 0.0103$. By Theorems 3.8, we can therefore conclude that the controlled SDDE (4.2) with $\sigma_1 = 0.2$, $\sigma_2 = 0.5$ and the control function (4.3) is not only exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \in [2, 7)$ but also almost surely exponentially stable provided $\bar{\tau} < 0.0103$.

5 Conclusion

In this paper we have made some advances in the theory on stabilization by delay feedback controls for highly nonlinear hybrid SDDEs. In particular, comparing with the results in the recent paper [9], we have advanced in three aspects: (a) The time delay in the given unstable SDDE is no longer needed to be a constant but a variable of time which may not have to be differentiable. Our new theory hence covers a much wider class of SDDEs. (b) The time lag in the feedback control is now allowed to take values in an interval but not a constant. Our new theory can therefore be implemented much more easily. (c) The control function used is bounded hence our new theory could reduce the control cost.

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