Probabilistic feasibility guarantees for solution sets to uncertain variational inequalities

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Abstract

We develop a data-driven approach to the computation of a-posteriori feasibility certificates to the solution sets of variational inequalities affected by uncertainty. Specifically, we focus on instances of variational inequalities with a deterministic mapping and an uncertain feasibility set, and represent uncertainty by means of scenarios. Building upon recent advances in the scenario approach literature, we quantify the robustness properties of the entire set of solutions of a variational inequality, with feasibility set constructed using the scenario approach, against a new unseen realization of the uncertainty. Our results extend existing results that typically impose an assumption that the solution set is a singleton and require certain non-degeneracy properties, and thereby offer probabilistic feasibility guarantees to any feasible solution. We show that assessing the violation probability of an entire set of solutions, rather than of a singleton, requires enumeration of the support constraints that "shape" this set. Additionally, we propose a general procedure to enumerate the support constraints that does not require a closed form description of the solution set, which is unlikely to be available. We show that robust game theory problems can be modelling via uncertain variational inequalities, and illustrate our theoretical results through extensive numerical simulations on a case study involving an electric vehicle charging coordination problem.

1. Introduction

As a general purpose tool embracing a rich class of decision-making problems, variational inequalities (VIs) have been widely adopted in many scientific areas, from operation research and mathematical programming to optimization and game theory [30, 24]. Formally, a VI is defined by means of a feasibility set $\mathcal{X} \subseteq \mathbb{R}^n$, and a mapping $F : \mathcal{X} \to \mathbb{R}^n$. We denote by $\operatorname{VI}(\mathcal{X}, F)$ the problem of finding some vector $x^* \in \mathcal{X}$ such that

$$(y - x^{\star})^{\top} F(x^{\star}) \ge 0, \text{ for all } y \in \mathcal{X}.$$
 (1)

Finding a point x^* satisfying (1) amounts to solving a generalized nonlinear complementarity problem, and therefore encompasses a broad variety of equilibrium problems that appear in multiple engineering domains [27]. Prominent examples include network and traffic problems [15, 19, 29], optimal control [39, 28, 20], economics and demand-side management [53, 16, 40].

However, most of the analytical and algorithmic results in the literature on VIs are restricted to deterministic problems. Hence all of the aforementioned applications inherently neglect potential sources of uncertainty that strongly affect the data of the problem itself, i.e., the mapping F and the feasible set \mathcal{X} . To move beyond the limited scope of deterministic methods, we focus on stochastic approaches to uncertain VIs. The existing literature in this area is broadly split into two main directions for incorporating uncertainty into the model in (1) [49]: an expectation-based and a worst-case formulation.

Specifically, given a random variable $\delta \in \Delta$, we refer to the expected-value formulation of an uncertain VI, originally described in [35], as the problem of computing a deterministic vector $x^* \in \mathcal{X}$ such that

$$(y - x^{\star})^{\top} \mathbb{E}[F(x^{\star}, \delta)] \ge 0, \text{ for all } y \in \mathcal{X},$$

$$(2)$$

where, in this case, $F : \mathcal{X} \times \Delta \to \mathbb{R}^n$. Unfortunately, there are only a few cases in which the formulation in (2) is known to be computationally tractable, e.g., when δ takes values in a discrete set. In most stochastic regimes, the computation of the expected value requires multidimensional integration, which is a difficult task.

Moreover, for applications in which the decision arising from the solution to an uncertain VI is required to be robust to parametric uncertainties, the formulation in (2) may be inappropriate. In those cases a worst-case formulation may be more suitable, allowing one to incorporate both uncertainty in the mapping $F(\cdot)$ and in the constraints \mathcal{X} . Specifically, the feasible set can be modelled as the (infinite) intersection of sets \mathcal{X}_{δ} generated by every possible realization of the random variable δ , namely $\cap_{\delta \in \Delta} \mathcal{X}_{\delta}$. Thus, a vector $x^* \in \cap_{\delta \in \Delta} \mathcal{X}_{\delta}$ is considered a solution to the uncertain VI if

$$(y - x^*)^\top F(x^*, \delta) \ge 0$$
, for all $y \in \bigcap_{\delta \in \Delta} \mathcal{X}_{\delta}$, $\delta \in \Delta$. (3)

However, such a worst-case formulation imposes two main challenges: i) the set Δ may be unknown and the only information available may come via data/scenarios for δ ; ii) even if Δ is known, it might be a continuous set (more generally, a set with infinite cardinality), thereby giving rise to an infinite set of constraints in (3).

To address these challenges, we adopt the data-driven approach proposed in [10] to quantify a-posteriori the feasibility of the entire set of solutions to the VI against previously unseen realizations of the uncertainty. Specifically, in this paper we focus on a subclass of uncertain VIs defined as in (3) – namely those instances characterized by a deterministic mapping $F(\cdot)$ and an uncertain feasible set \mathcal{X} . Using a set-oriented perspective, we recast our problem to the form of the abstract decision-making problems considered in [10]. This enables us to inherit the probabilistic feasibility results established in [10, Th. 1], and thereby characterize the robustness properties of the entire solution set to an uncertain VI.

Remarkably, the family of VIs we investigate is quite large, encompassing applications across several domains:

- A wide class of Nash equilibrium problems (NEPs) and generalized Nash equilibrium problems (GNEPs) can be characterized as VIs [24, 23] in several applications of practical interest. We show that the robust variant of NEPs/GNEPs falls directly into the class of VIs investigated in this paper [1, 44, 22].
- As a static assignment problem, an optimal network flow in traffic networks can also be computed via solution to an associated VI [45, Th. 3.14]. In this domain of application, it is quite common to model the overall traffic demand as an uncertain variable [32, 50, 13, 17, 34], thus randomly constraining the traffic flow over admissible paths of the network.

• Finite horizon control problems are typically formalized as constrained optimization problems, which can be modelled as VIs [24, §1.3.1]. Specifically, the optimal sequence of control inputs is required to minimize some predefined cost function, while being subject to operational and dynamic constraints, both of which may be affected by uncertainty [47, 48, 52].

1.1. Literature review and main contributions

To the best of our knowledge, this work is the first to address the problem of evaluating the robustness of the entire set of solutions to an uncertain VI in a distribution-free fashion. Compared to existing results on data-driven approaches to assess the robustness of solutions to general VIs (or to particular cases thereof), we consider a broad family of uncertain VIs in (3) rather than just VI problems arising from the computation of variational generalized Nash equilibria (v-GNE), a subset of generalized Nash equilibria (GNE) in GNEPs [25, 26, 44, 22].

Specifically, a NEP is considered in [25, 26] where the uncertain parameter is encoded as a common term in the agents' cost function, while the constraint set of each player is deterministic. In this context, [25, 26] provide an a-posteriori certificate on the probability that a (non-unique) variational solution to the NEP remains unaltered upon a new realization of the uncertainty. The present work is complementary to the one in [25, 26], as they (indirectly) investigate VIs with an uncertain mapping and a deterministic feasible set.

Conversely, a NEP with uncertain, yet affine, local constraints is considered in [44]. By assuming deterministic cost functions, a contribution of [44] is to provide robustness certificates for the constraint violation of any feasible point (thus including variational equilibria as special case) of the game considered. In contrast, we show in §3 that assessing the robustness of an equilibrium at a point inside the feasible set may lead to an overconservative bound compared to the one derived in this paper, which is tailored for the entire set of equilibria.

The approach proposed in [22], instead, paves the way to the set-oriented perspective investigated in this paper. Specifically, [22] leverages the specific structure of the game in question, i.e., a GNEP in aggregative form, to design probabilistic bounds on the feasibility of the entire set of variational equilibria.

Finally, [42] has addressed robustness questions for uncertain VIs, providing a-posteriori robustness certificates for the solution to uncertain (quasi-)VI in (3). However, the perspective and proof line is substantially different from the one adopted in this paper. Specifically, in [42] it is postulated that the VI admits a *unique* solution, while certain non-degeneracy assumptions are imposed. Uniqueness restricts the class of VIs that can be captured by problems of the type (3), while non-degeneracy is in general hard to verify even in optimization problems, and even moreso in VIs and games [9, 26]. By considering a specific instance of the family of uncertain VIs in (3), focusing on the entire set of solutions allow us to bypass both assumptions.

To conclude, we summarize our main contributions as follows:

- We consider a broad family of uncertain VIs in (3) rather than just VI problems arising in computing v-GNE, thus complementing the results in [44, 22];
- In the spirit of [22], by relying on the data-driven tools given in [10] we provide a-posteriori robustness certificates for the entire set of solutions to an uncertain VIs; Note that our set-oriented perspective is crucial for two reasons:

- 1. We are able to bypass the uniqueness and non-degeneracy assumptions postulated in [42];
- 2. Compared to [44], we show that our bounds are, in general, less conservative;
- 3. We offer guarantees for any feasible solution; hence we can support possibly suboptimal solutions returned by a generic algorithm.
- Our robustness certificates depend strongly on the number of support subsamples characterizing the set of solutions to the uncertain VI. We show that computing these support subsamples requires only an enumeration of the constraints that "shape" the solution set. An explicit representation of the unknown set of solutions is therefore not needed. In the case of affine constraints, we design a procedure to compute these samples that, in general, requires fewer iterations compared to the one in [42, 10].

Finally, we show that problems in robust game theory falls within the class of uncertain VIs we consider. Our theoretical results are supported through extensive numerical simulations on a GNEP modelling the charging coordination of a fleet of plug-in electric vehicles (PEVs).

1.2. Paper organization

We formalize the data-driven problem addressed and state the main result of the paper, i.e., Theorem 1, in §2. In §3 we discuss how the set-oriented problem we consider can be recast in the framework proposed in [10], thereby paving the way for a formal proof of Theorem 1. We then describe a systematic procedure to compute the number of support subsamples in the case of affine constraints in §4, also discussing the computational aspects associated with the proposed approach. Finally, in §5 we demonstrate our theoretical results through a numerical simulations on a GNEP.

1.3. Notation

Basic notation: \mathbb{N} , \mathbb{R} , and $\mathbb{R}_{\geq 0}$ denote the set of natural, real, and nonnegative real numbers, respectively, with $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$. Given some $x \in \mathbb{R}^n$, ||x|| is the Euclidean norm. Denote vectors of appropriate dimensions with elements all equal to 1 (0) as **1** (0). Given a matrix $A \in \mathbb{R}^{m \times n}$, its (i, j) entry is denoted by $a_{i,j}$, A^{\top} denotes its transpose, while for $A \in \mathbb{R}^{n \times n}$, $A \succ 0 (\succeq 0)$ implies that A is symmetric and positive (semi)-definite. For $A \succ 0$, $||x||_A := \sqrt{x^{\top}Ax}$. C^1 is the class of continuously differentiable functions. For a given set $S \subseteq \mathbb{R}^n$, |S| represents its cardinality, and $\operatorname{int}(S)$, $\operatorname{relint}(S)$ and $\operatorname{bdry}(S)$ denote its topological interior, relative interior and boundary, respectively. The set $\operatorname{aff}(S)$ denotes its affine hull, i.e., the smallest affine set containing S. The operator \otimes denotes the Kronecker product, while $\operatorname{col}(\cdot)$ stacks its arguments in column vectors or matrices of compatible dimensions. For vectors $v_1, \ldots, v_N \in \mathbb{R}^n$ and $\mathcal{I} = \{1, \ldots, N\}$, we denote $\boldsymbol{v} \coloneqq (v_1^{\top}, \ldots, v_N^{\top})^{\top} = \operatorname{col}((v_i)_{i\in\mathcal{I}})$ and $\boldsymbol{v}_{-i} \coloneqq \operatorname{col}((v_j)_{j\in\mathcal{I}\setminus\{i\}})$. With a slight abuse of notation, we sometimes use $\boldsymbol{v} = (v_i, \boldsymbol{v}_{-i})$.

Operator-theoretic definitions: Given a function $\phi : \mathbb{R}^n \to \mathbb{R}$, dom $(\phi) := \{x \in \mathbb{R}^n \mid \phi(x) < \infty\}$ is the domain of ϕ ; $\partial \phi : \operatorname{dom}(\phi) \Rightarrow \mathbb{R}^n$ denotes the subdifferential set-valued mapping of ϕ , defined as $\partial \phi(x) := \{d \in \mathbb{R}^n \mid \phi(z) \ge \phi(x) + d^\top(z - x), \forall z \in \operatorname{dom}(\phi)\}$, for all $x \in \operatorname{dom}(\phi)$. For a given set $S \subseteq \mathbb{R}^n$, the mapping $T : \mathcal{X} \to \mathbb{R}^n$ is pseudomonotone on S if for all $x, y \in S$, $(x - y)^\top T(y) \ge 0 \implies (x - y)^\top T(x) \ge 0$; (strictly) monotone if $(T(x) - T(y))^\top (x - y) (>) \ge 0$ for all $x, y \in S$ (and $x \ne y$); strongly monotone if there exists a constant c > 0 such that $(T(x) - T(y))^\top (x - y) \ge c ||x - y||^2$ for all $x, y \in S$. If S is nonempty and convex, the normal cone of S evaluated at x is the set-valued mapping

 $\mathcal{N}_{\mathcal{S}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined as $\mathcal{N}_{\mathcal{S}}(x) \coloneqq \{ d \in \mathbb{R}^n \mid d^\top(y-x) \le 0, \forall y \in \mathcal{S} \}$ if $x \in \mathcal{S}, \mathcal{N}_{\mathcal{S}}(x) \coloneqq \emptyset$ otherwise.

2. Problem statement and main result

We start by recalling some key concepts about variational inequalities (VIs) [24], and then describe the data-driven problem we consider throughout the paper. We also state the main result of the paper, i.e., Theorem 1, in this section, but will defer the proof of this result to §3. Unless otherwise specified, we assume measurability of all the quantities introduced hereafter.

2.1. Background on VIs

Let us consider the deterministic VI formally introduced in (1), and let $\Omega \subseteq \mathcal{X}$ be the set of solutions to $\operatorname{VI}(\mathcal{X}, F)$. The relation in (1) has a strong geometric interpretation that relies on the definition of the normal cone [24, Ch. 1.1]. If \mathcal{X} is nonempty and convex, a vector $x^* \in \mathcal{X}$ solves $\operatorname{VI}(\mathcal{X}, F)$ if and only if $-F(x^*) \in \mathcal{N}_{\mathcal{X}}(x^*)$. For example, in the specific case of an optimization problem $\min_{x \in \mathcal{X}} f(x)$, we have $F = \nabla f$ and the inclusion $-\nabla f(x^*) \in \mathcal{N}_{\mathcal{X}}(x^*)$ corresponds to satisfaction of the KKT conditions at some x^* . Thus, in view of the definition of the normal cone, any point belonging to $\operatorname{int}(\mathcal{X})$ solves $\operatorname{VI}(\mathcal{X}, F)$ if and only if $F(x^*) = 0$.

The structural properties of both the feasible set \mathcal{X} and the mapping $F(\cdot)$ allow one to establish the existence and uniqueness of the solution to $VI(\mathcal{X}, F)$, as well as to provide the minimal conditions that enable one to design suitable solution algorithms with convergence guarantees. By combining [24, Cor. 2.2.5, Th. 2.3.5], we can characterize the solution set to $VI(\mathcal{X}, F)$ as follows:

Lemma 1. Let \mathcal{X} be a compact and convex set, and let $F(\cdot)$ be a continuous mapping. Then, the following statements hold true:

- (i) Ω is a nonempty and compact set;
- (ii) If $F(\cdot)$ is also pseudomonotone, then Ω is also a convex set.

Note that assuming strong monotonicity of $F(\cdot)$ would guarantee the existence of a unique solution to VI(\mathcal{X}, F) [24, Th. 2.3.3]. Requiring the mapping $F(\cdot)$ to be only pseudomonotone is clearly weaker than assuming (strong) monotonicity. However, pseudomonotonicity is not always a trivial condition to verify, while monotonicity is naturally satisfied in several practical applications that involve, e.g., the subdifferential ∂f of a proper, closed, convex function $f : \mathbb{R}^n \to \mathbb{R}$ (or its conjugate, see [3, §20]). Both conditions, however, individually represent one of the weakest assumptions that guarantee convergence of many efficient solution algorithms, see, e.g., [21], [33], [37], [41], [51], or the dedicated sections in [24, §7, §12], [43, Ch. 12] and references therein.

2.2. Uncertain VIs and scenario-based formulation

We aim to provide out-of-sample feasibility certificates for the entire set of solutions to a given uncertain VI by exploiting some observed realizations, i.e., *scenarios*, of the uncertain parameter δ . Formally, let us consider a probability space $(\Delta, \mathcal{D}, \mathbb{P})$, where $\Delta \subseteq \mathbb{R}^{\ell}$ represents the set of values that δ can take, \mathcal{D} is a σ -algebra and \mathbb{P} is a (possibly unknown) probability measure over \mathcal{D} . Given a mapping $F : \mathcal{X} \to \mathbb{R}^n$ and a deterministic feasible set $\mathcal{X} \subseteq \mathbb{R}^n$, let $\mathcal{X}_{\delta} \subseteq \mathbb{R}^n$ be an additional set of constraints associated with the uncertain

parameter δ . We define the worst-case VI problem, denoted as $VI(\mathcal{X} \cap \mathcal{X}_{\delta}, F)$, as the problem of finding some $x^* \in \mathcal{X} \cap \mathcal{X}_{\delta}$ that satisfies

$$(y - x^{\star})^{\top} F(x^{\star}) \ge 0$$
, for all $y \in \mathcal{X} \cap \mathcal{X}_{\delta}, \ \delta \in \Delta$. (4)

However, given the possibly infinite cardinality of Δ in (4) and motivated by the increasing availability of data, we investigate a data-driven approach. Specifically, let us consider $\delta_K \coloneqq \{\delta^{(i)}\}_{i \in \mathcal{K}} = \{\delta^{(1)}, \ldots, \delta^{(K)}\} \in \Delta^K, \ \mathcal{K} \coloneqq \{1, 2, \ldots, K\}$, hereafter also called a *K*multisample, as a finite collection of $K \in \mathbb{N}$ independent and identically distributed (i.i.d.) observed realizations of δ . Here, every *K*-multisample is defined over the probability space $(\Delta^K, \mathcal{D}^K, \mathbb{P}^K)$, resulting from the *K*-fold Cartesian product of the original probability space $(\Delta, \mathcal{D}, \mathbb{P})$. Let $\mathcal{X}_{\delta^{(i)}}$ be a constraint set associated with the *i*-th sample, which constrains the decisions that are admissible for the situation represented by $\delta^{(i)}$. The scenario-based VI problem $\operatorname{VI}(\mathcal{X}_{\delta_K}, F)$, with $\mathcal{X}_{\delta_K} \coloneqq \bigcap_{i \in \mathcal{K}} \mathcal{X}_{\delta^{(i)}} \cap \mathcal{X}$, is then the problem of finding an $x^* \in \mathcal{X}_{\delta_K}$ such that

$$(y - x^{\star})^{\top} F(x^{\star}) \ge 0$$
, for all $y \in \mathcal{X}_{\delta_K}$. (5)

Let us define the set of solutions to (5) as

$$\Omega_{\delta_K} \coloneqq \{ x \in \mathcal{X}_{\delta_K} \mid (y - x)^\top F(x) \ge 0, \, \forall y \in \mathcal{X}_{\delta_K} \}.$$
(6)

Note that given the dependency on the set of K realizations δ_K , the set Ω_{δ_K} is itself a random variable. For the case K = 0 our problem reduces to a deterministic VI problem, i.e., $VI(\mathcal{X}, F)$, where no uncertainty is present. Let Ω_{δ_0} be the solution set for this case. In light of the results of Lemma 1, we will make the following assumptions throughout the remainder of the paper:

Standing Assumption 1. For any $K \in \mathbb{N}_0$, the set \mathcal{X}_{δ_K} is a nonempty, compact and convex set for all $\delta_K \in \Delta^K$.

Standing Assumption 2. The mapping $F : \mathcal{X} \to \mathbb{R}^n$ is continuous and pseudomonotone.

These assumptions ensure that our scenario-based VI (5) has a non-empty solution set:

Lemma 2. For all $K \in \mathbb{N}_0$, Ω_{δ_K} is a nonempty, compact and convex set.

Proof. It follows immediately from Lemma 1(ii), as $F(\cdot)$ is continuous and pseudomonotone and \mathcal{X}_{δ_K} is a finite intersection (due to Standing Assumption 1) of nonempty, compact and convex sets.

In the spirit of [10], we introduce $\Theta_K : \Delta^K \Rightarrow \mathcal{X}$ as the mapping that, given a set of realizations δ_K , returns the solution set to VI($\mathcal{X}_{\delta_K}, F$), namely

$$\Theta_K(\delta^{(1)}, \dots, \delta^{(K)}) = \Theta_K(\delta_K) \coloneqq \Omega_{\delta_K}.$$
(7)

When K = 0, we assume that Θ_0 returns the solution set to $VI(\mathcal{X}, F)$, Ω_{δ_0} .

2.3. Robustness certificates for solution sets to VIs

Given any K-multisample δ_K , we are interested in evaluating the robustness of the entire set of solutions Ω_{δ_K} in (6) to a previously unseen realization of the uncertain parameter δ . To this end, let Ω_{δ} be the set of solutions induced by a certain realization $\delta \in \Delta$. We now introduce the following definition of violation probability of a generic set of solutions, Ω . **Definition 1.** (Violation Probability of a Set) The violation probability associated with a set of solutions Ω is defined as

$$V(\Omega) \coloneqq \mathbb{P}\{\delta \in \Delta \mid \Omega \not\subseteq \Omega_{\delta}\}.$$
(8)

Informally speaking, the condition $\Omega \not\subseteq \Omega_{\delta}$ implies that, once δ is drawn, at least one element in Ω ceases to be a solution. Note that the set Ω_{δ_K} is itself a random variable, and hence so is the violation probability $V(\Omega_{\delta_K})$. We therefore wish to characterise our confidence that $V(\Omega_{\delta_K})$ is below some violation level. Before stating the main result of this section, we recall the following definition that will be crucial for the remainder of the paper:

Definition 2. (Support Subsample) [10, Def. 2] Given any $\delta_K \in \Delta^K$, a support subsample $S \subseteq \delta_K$ is a p-tuple of unique elements of δ_K , i.e., $S \coloneqq \{\delta^{(i_1)}, \ldots, \delta^{(i_p)}\}, i_1 < \ldots < i_p$, that gives the same solution as the original sample, i.e.,

$$\Theta_p(\delta^{(i_1)},\ldots,\delta^{(i_p)}) = \Theta_K(\delta^{(1)},\ldots,\delta^{(K)}).$$

Here, let $\Upsilon_K : \delta_K \rightrightarrows \mathcal{K}$ be any algorithm returning a *p*-tuple $\{i_1, \ldots, i_p\}, i_1 < \ldots < i_p$, such that $\{\delta^{(i_1)}, \ldots, \delta^{(i_p)}\}$ is a support subsample for δ_K , and let $s_K \coloneqq |\Upsilon_K(\delta_K)|$. Note that s_K is itself a random variable since it depends on δ_K . We discuss the construction of such an algorithm in §4. Our main result characterizes the violation probability of Ω_{δ_K} , i.e., the solution set to the scenario-based VI in (5), as follows:

Theorem 1. Fix $\beta \in (0,1)$, and let $\varepsilon : \mathcal{K} \cup \{0\} \to [0,1]$ be a function such that

$$\begin{cases} \varepsilon(K) = 1, \\ \sum_{h=0}^{K-1} \binom{K}{h} (1 - \varepsilon(h))^{K-h} = \beta. \end{cases}$$
(9)

Then, for any mappings Θ_K , Υ_K and distribution \mathbb{P} , it holds that

$$\mathbb{P}^{K}\{\delta_{K} \in \Delta^{K} \mid V(\Omega_{\delta_{K}}) > \varepsilon(s_{K})\} \le \beta.$$
(10)

Note that the bound in (10) is an a-posteriori statement since s_K depends on the multisample extracted. In words, Theorem 1 implies that the probability that $\Omega_{\delta_K \cup \{\delta\}}$ differs from Ω_{δ_K} (as $\Omega_{\delta_K} \subseteq \Omega_{\delta_K \cup \{\delta\}}$ necessarily implies that $\Omega_{\delta_K} = \Omega_{\delta_K \cup \{\delta\}}$ – see also Lemma 4) is at most equal to $\varepsilon(s_K)$, with confidence at least $1 - \beta$, for an arbitrarily small $\beta \in (0, 1)$. We give the proof of Theorem 1 in the next section, after first stating and proving some ancillary results.

3. The scenario approach to uncertain VIs

In this section, we first recall some key notions of the scenario approach theory, and then we show how they can be extended to the context of solution sets to VIs. We finally conclude by proving and discussing Theorem 1.

3.1. Scenario approach for decision-making problems

The scenario approach was initially conceived to provide a-priori out-of-sample feasibility guarantees associated with the solution to an uncertain convex optimization problem [6, 5, 8]. It has recently been extended to abstract decision-making problems through an a-posteriori assessment of the feasibility risk [10, 9].

With a slight abuse of notation, we assume here that $\Theta_K : \Delta^K \to \mathcal{X}$ represents a function leading to a single scenario decision $\theta^*_{\delta_K}$ for some generic abstract decision-making problem, rather than as a set of solutions specific to a VI as in (6)–(7). In accordance with [10], $\theta^*_{\delta_K}$ is assumed to be unique, otherwise any convex tie-break rule may be employed [7]. Then, we recall the following assumption that is crucial to prove [10, Th. 1].

Assumption 3. [10, Ass. 1] For all $K \in \mathbb{N}$ and for all $\delta_K \in \Delta^K$, it holds that $\Theta_K(\delta_K) \in \mathcal{X}_{\delta^{(i)}}$, for all $i \in \mathcal{K}$.

Assumption 3 implies that the decision taken while observing K realizations of the uncertainty δ is consistent with respect to (w.r.t.) all the extracted scenarios. The goal in [10] was then to assess the violation probability of the scenario decision $\theta_{\delta_K}^{\star}$, as formalized next.

Definition 3. (Violation Probability of a Singleton) The violation probability of a decision $\theta \in \mathcal{X}$ is given by

$$V(\theta) \coloneqq \mathbb{P}\{\delta \in \Delta \mid \theta \notin \mathcal{X}_{\delta}\}.$$

Notice again the slight abuse of notation, where we use V to denote both the violation of a singleton θ in Definition 3 and of a set in Definition 1, while Θ_K in this subsection returns an element of \mathcal{X} (the solution $\theta_{\delta_K}^*$) rather than a set as in §2.2. The results in [10] hold for generic decisions, as long as Assumption 3 is satisfied. In the next subsection, we show how we can employ those results and adapt the sequence of inclusions in Assumption 3 when our decision is a set. With the set-oriented perspective introduced in §2, for the uncertain VI in (4) we let the admissible decision for the situation represented by δ coincide with the solution set Ω_{δ} , which is clearly a subset of the feasible set shaped by the uncertain parameter, i.e., the set $\mathcal{X} \cap \mathcal{X}_{\delta}$. This clarifies the analogy between Definition 1 and 3. For completeness, we restate as a lemma the crucial result provided in [10] to bound the violation probability of $\theta_{\delta_K}^*$.

Lemma 3. [10, Th. 1] Let Assumption 3 hold true and fix $\beta \in (0, 1)$. Let $\varepsilon : \mathcal{K} \cup \{0\} \rightarrow [0, 1]$ be the function defined in (9). Then, for the Θ_K defined in this subsection, and for any Υ_K as defined below Definition 2, we have that $\mathbb{P}^K \{\delta_K \in \Delta^K \mid V(\theta^*_{\delta_K}) > \varepsilon(s_K)\} \leq \beta$.

Notice that, in this case, s_K would be the number of samples such that, by feeding Θ_K only with those samples, would return the same optimal solution $\theta^{\star}_{\delta_K}$ that would have been obtained if all samples were employed. We will use Lemma 3 to prove our main result in Theorem 1, which characterizes the entire set of solutions to VI($\mathcal{X}_{\delta_K}, F$), but require some preliminary results first.

3.2. Scenario-based VI solution sets as nested sets of decisions

In view of the analogy between Definition 1 and 3, we aim to follow the approach of [10] by focusing on a set of decisions, extending the conditions in §3.1, and in particular the sequence of inclusions in Assumption 3, to the solution set for the uncertain VI in (4). Thus, returning to the more general case where Θ_K is a set-valued mapping as defined in (7), i.e. $\Theta_K : \Delta^K \Rightarrow \mathcal{X}$, since we focus on the entire set of solutions, we remark



Figure 1: Compared to Ω_{δ_0} (red line), every realization of δ (dashed blue lines, while the shaded cyan area denotes a region excluded by any $\mathcal{X}_{\delta^{(i)}}$, i = 1, 2, 3) results in a solution set Ω_{δ_i} , i = 1, 2, 3, that belongs to a different affine hull and/or on a space of lower dimension (green dots or line). Standing Assumption 4 allows us to rule out such cases.

that, for any K-multisample $\delta_K \in \Delta^K$, the uniqueness of the solution returned by Θ_K holds by definition. Then, in the spirit of [38, Def. 2], we envision that the set-oriented counterpart of the sequence of inclusions in Assumption 3 shall be naturally translated into a consistency property of Ω_{δ_K} , as defined next.

Definition 4. (Consistency of Solution Sets) Given some $K \in \mathbb{N}$ and $\delta_K \in \Delta^K$, the solution set to $\operatorname{VI}(\mathcal{X}_{\delta_K}, F)$ is consistent with the collected scenarios if $\Theta_K(\delta_K) = \Omega_{\delta_K} \subseteq \mathcal{X}_{\delta^{(i)}}$, for all $i \in \mathcal{K}$.

In analogy with Assumption 3, Definition 4 establishes that the set of solutions to $\operatorname{VI}(\mathcal{X}_{\delta_K}, F)$, Ω_{δ_K} , which is based on K scenarios, should be feasible for each of the sets $\mathcal{X}_{\delta^{(i)}}$, $i \in \mathcal{K}$, corresponding to each of the K realizations of the uncertain parameter. Thus, a first step towards applying the bound in Lemma 3 is to show that the mapping $\Theta_K(\cdot)$ introduced in (7) is consistent with the realizations observed in the scenario-based VI in (5). This fact, however, follows by definition. For any $K \in \mathbb{N}$ and associated K-multisample $\delta_K \in \Delta^K$, indeed, it holds that $\Theta_K(\delta_K) \coloneqq \Omega_{\delta_K} \subseteq \bigcap_{i \in \mathcal{K}} \mathcal{X}_{\delta^{(i)}} \cap \mathcal{X}$, which on the other hand implies that $\Theta_K(\delta_K) \subseteq \mathcal{X}_{\delta^{(i)}}$, for all $i \in \mathcal{K}$, thus directly falling within Definition 4. We will make use of these considerations to rely on the bound in Lemma 3 in the proof of Theorem 1, along with of the assumption on the solution set Ω_{δ_K} introduced next.

Standing Assumption 4. For all $K \in \mathbb{N}$ and $\delta_K \in \Delta^K$, $\operatorname{aff}(\Omega_{\delta_K}) = \operatorname{aff}(\Omega_{\delta_0})$.

Specifically, if the uncertain VI in (4) is defined in \mathbb{R}^n and Ω_{δ_K} is a convex, *m*-dimensional set, then Standing Assumption 4 allows for m < n. In this sense, assuming $\operatorname{aff}(\Omega_{\delta_K}) = \operatorname{aff}(\Omega_{\delta_0})$ for any $\delta_K \in \Delta^K$, $K \in \mathbb{N}$, is weaker than, e.g., assuming the nonemptiness of int (Ω_{δ_K}) for every possible realization of δ_K . Standing Assumption 4 rules out the scenario that a given realization of the uncertainty δ_K reduces the solution set Ω_{δ_K} to one of lower dimension with a different affine hull compared to the one of the deterministic VI. To clarify its role, we introduce and discuss the following illustrative example.

Example 1. Let us consider a two-dimensional case as shown in Fig. 1, where F =

 $\operatorname{col}(0,-1)$, is monotone and \mathcal{X} has a triangular shape. Here, $\Omega_{\delta_0} = \{x \in \mathbb{R}^2 \mid x_1 \in [0,1], x_2 = 0\}$, and its affine hull corresponds to the entire x_1 -axis. After observing the first realization of δ , i.e., $\delta^{(1)}$, which introduces the set $\mathcal{X}_{\delta^{(1)}} = \{x \in \mathbb{R}^2 \mid -[1/3 \ 1]^\top x \leq -1/3\}$, the solution set reduces to a singleton $\Omega_{\delta_1} = \{x \in \mathbb{R}^2 \mid x = \operatorname{col}(1,0)\}$. Here, Ω_1 has a smaller dimension compared to Ω_{δ_0} , despite its affine hull, i.e., the singleton itself, being a subset of the x_1 -axis. Then, drawing a new sample $\delta^{(2)}$, which introduces the set $\mathcal{X}_{\delta^{(2)}} = \{x \in \mathbb{R}^2 \mid [1/3 \ -1]^\top x \leq 1/15\}$, we have $\Omega_{\delta_2} = \{x \in \mathbb{R}^2 \mid x = \operatorname{col}(3/5, 2/15)\}$, which has the same dimension as Ω_{δ_1} but its affine hull is not a subset of $\operatorname{aff}(\Omega_{\delta_0})$. Finally, the third sample, $\delta^{(3)}$, introduces the set $\mathcal{X}_{\delta^{(3)}} = \{x \in \mathbb{R}^2 \mid [0 \ -1]^\top x \leq -1/2\}$, and hence we have $\Omega_{\delta_3} = \{x \in \mathbb{R}^2 \mid x_1 \in [1/4, 3/4], x_2 = 1/2\}$. Here, Ω_{δ_3} has the same dimension of Ω_{δ_0} but a different affine hull, i.e., the x_1 -axis translated to $x_2 = 1/2$. Standing Assumption 4 is meant to rule out all these possible scenarios, allowing only for samples that "shape" aff(\Omega_{\delta_0}) without altering its dimension.

As we investigate uncertain VIs of the form (4) where the uncertainty δ affects the feasible set only, Example 1 provides insight on translating Standing Assumption 4 to a condition on the probability space Δ . In fact, it represents situations that can generally happen with non-zero probability. Specifically, let Δ in Example 1 be a subset of \mathbb{R}^2 , namely the uncertainty $\delta = \operatorname{col}(a, b)$ has two components, and let $a \in \mathbb{R}$ parametrize the slope and $b \in \mathbb{R}$ the offset of the halfspaces introduced by every scenario, i.e., $\mathcal{X}_{\delta} = \{x \in \mathcal{X} \in \mathcal{X} \}$ $\mathbb{R}^2 \mid [a \ 1] x \leq b$, for every $\delta \in \Delta$. Then, for any distribution that admits a density, we can find non-zero intervals for a and b such that the i.i.d. scenarios δ can be extracted from a restricted subset of Δ , determined by the values of a and b themselves, in order to meet Standing Assumption 4, and hence ruling out the pathological cases shown in Example 1. In the case the samples are extracted from a restricted subset of Δ , note that the guarantees would hold for the probability measure that is induced by this restriction, and not for the original uncertainty measure. Alternatively, if Standing Assumption 4 is not satisfied for all multisamples, then we can still claim that with confidence at most β , if Standing Assumption 4 is satisfied, then the probability of violation is greater than $\varepsilon(s_K)$. To achieve this, in the statement of Theorem 1, rather than Θ_K we can restrict the space of multisamples to the ones for which Standing Assumption 4 is satisfied. This is analogous to the way infeasible problem instances are accounted for in [5, 6]. Moreover, note that by adopting restrictions on Δ as described above, Standing Assumption 4 allows us to address the strongly monotone case, where $VI(\mathcal{X} \cap \mathcal{X}_{\delta}, F)$ has a unique solution, for all $\delta \in \Delta$.

Remark 1. In view of [24, Th. 2.4.15], Standing Assumption 4 can be replaced by a simpler one, which is easier to verify, in all problems that involve a (monotone) affine VI with polyhedral feasible set – see, e.g., Assumption 2 in [22].

Given some $K \in \mathbb{N}$, let $\Omega_{\delta_{K+1}} \coloneqq \Omega_{\delta_K \cup \{\delta^{(K+1)}\}}$ be the solution set to the scenario-based VI in (5) after observing the (K+1)-th realization of δ , i.e., the feasible set of the VI shrinks to $\mathcal{X}_{\delta_{K+1}} \coloneqq \mathcal{X}_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}$, for some $\delta^{(K+1)} \in \Delta$. We have the following preliminary result.

Lemma 4. For all $K \in \mathbb{N}_0$ and for all $\delta_K \in \Delta^K$, $\Omega_{\delta_{K+1}} = \Omega_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}$.

Proof. We split the proof into two different inclusions. Specifically, we first prove (i) that $\Omega_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}} \subseteq \Omega_{\delta_{K+1}}$, and then (ii) that $\Omega_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}} \supseteq \Omega_{\delta_{K+1}}$.

(i) We will show that if $x^* \in \Omega_{\delta_K}$ and $x^* \in \mathcal{X}_{\delta^{(K+1)}}$, then $x^* \in \Omega_{\delta_{K+1}}$. Note that, in view of Standing Assumptions 1–2, given an arbitrary $K \in \mathbb{N}_0$ and related K-multisample $\delta_K \in \Delta^K$, \mathcal{X}_{δ_K} is a compact and convex set, as it is finite intersection of convex sets. Then,



Figure 2: Schematic two-dimensional construction of the proof of Lemma 4, part (ii). Due to the convexity, there always exists some $\tilde{y} \in \mathcal{X}_{\delta_{K+1}}$, but $\tilde{y} \notin \operatorname{int}(\Omega_{\delta_{K+1}})$, that allows to construct a contradiction. In this case, $\tilde{y} \in \operatorname{bdry}(\Omega_{\delta_{K+1}})$.

a vector $x^* \in \mathcal{X}_{\delta_K}$ is a solution to $\operatorname{VI}(\mathcal{X}_{\delta_K}, F)$ if and only if $x^* \in \operatorname{argmin}_{y \in \mathcal{X}_{\delta_K}} y^\top F(x^*)$. Since the uncertain parameter does not affect the mapping $F(\cdot)$, but enters in the constraints only, every sample $\delta^{(K+1)} \in \Delta$ introduces an additional set of convex constraints, i.e., $\mathcal{X}_{\delta_{K+1}} = \mathcal{X}_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}} \subseteq \mathcal{X}_{\delta_K}$, which is compact and convex as well. Thus, it follows immediately that, if $x^* \in \mathcal{X}_{\delta^{(K+1)}}$, then $x^* \in \mathcal{X}_{\delta_{K+1}}$. Therefore, $x^* \in \operatorname{argmin}_{y \in \mathcal{X}_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}} y^\top F(x^*)$, which by definition implies that $x^* \in \Omega_{\delta_{K+1}}$.

(ii) We first prove that, if $x^* \in \operatorname{relint}(\Omega_{\delta_{K+1}})$, then $x^* \in \Omega_{\delta_K}$. The case where $x^* \in \operatorname{bdry}(\Omega_{\delta_{K+1}})$ will be treated in the sequel. Let us recall that, in view of [46, Cor. 1.6.1], for any given *m*-dimensional convex set S in \mathbb{R}^n , $m \leq n$, there always exists an affine transformation which carries $\operatorname{aff}(S)$ onto the subspace

$$V := \{ x = (z_1, \dots, z_m, z_{m+1}, \dots, z_n)^\top \in \mathbb{R}^n \mid z_{m+1} = \dots = z_n = 0 \}.$$

Therefore, as closures and relative interiors are preserved under one-to-one affine transformations of \mathbb{R}^n onto itself, we can limit our attention to the case where $\Omega_{\delta_{K+1}}$, and hence Ω_{δ_K} (since $\operatorname{aff}(\Omega_{\delta_{K+1}}) = \operatorname{aff}(\Omega_{\delta_K}) = \operatorname{aff}(\Omega_{\delta_0})$ from Standing Assumption 4), is *n*dimensional so that $\operatorname{relint}(\Omega_{\delta_{K+1}}) = \operatorname{int}(\Omega_{\delta_{K+1}})$.

Now, for the sake of contradiction, let $x^* \in \mathcal{X}_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}$ be any point such that $x^* \in \inf(\Omega_{\delta_{K+1}})$, but $x^* \notin \Omega_{\delta_K}$. Since $x^* \in \mathcal{X}_{\delta_K}$, $x^* \notin \Omega_{\delta_K}$ implies that there exists some $\bar{y} \in \mathcal{X}_{\delta_K}$, with $\bar{y} \neq x^*$, such that the VI is not satisfied, i.e., $(\bar{y} - x^*)^\top F(x^*) < 0$. Given the convexity of the sets involved, there must exist some $\lambda \in (0, 1)$ that allows one to construct some $\tilde{y} = \lambda x^* + (1 - \lambda)\bar{y}$ such that $\tilde{y} \in \mathcal{X}_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}$, but $\tilde{y} \notin \operatorname{int}(\Omega_{\delta_{K+1}})$ (see Fig. 2 for a graphical representation). Therefore, since $x^* \in \operatorname{int}(\Omega_{\delta_{K+1}})$, it shall satisfy $(\tilde{y} - x^*)^\top F(x^*) \ge 0$, which leads to $(1 - \lambda)(\bar{y} - x^*)^\top F(x^*) \ge 0$ that clearly generates a contradiction, since $(1 - \lambda) > 0$.

It remains to show the claim for the case where $x^* \in \text{bdry}(\Omega_{\delta_{K+1}})$. Notice that, since $\operatorname{relint}(\Omega_{\delta_{K+1}}) \neq \emptyset$ as $\Omega_{\delta_{K+1}}$ is nonempty, and since the involved sets are closed and convex, for any $x^* \in \operatorname{bdry}(\Omega_{\delta_{K+1}})$ we can always construct a convergent sequence of points $\{x_t\}_{t\in\mathbb{N}}$ such that, for all $t \in \mathbb{N}$, $x_t \in \operatorname{relint}(\Omega_{\delta_{K+1}}) \subseteq \Omega_{\delta_K}$, and $\{x_t\}_{t\in\mathbb{N}} \to x^*$, implying that $x^* \in \Omega_{\delta_K}$. Specifically, given any $\bar{x} \in \operatorname{relint}(\Omega_{\delta_{K+1}})$, in view of [46, Th. 6.1], for all $t \geq 1$, any term of the sequence $x_t \coloneqq \frac{1}{t}\bar{x} + (1 - \frac{1}{t})x^* \in \Omega_{\delta_K} \cap \mathcal{X}_{\delta^{(K+1)}}$ belongs to $\operatorname{relint}(\Omega_{\delta_{K+1}})$. Therefore, we have the inclusion $\Omega_{\delta_{K+1}} \subseteq \Omega_{\delta_K}$ as desired.



Figure 3: The solution set to $VI(\mathcal{X}_{\delta_K}, F)$, Ω_{δ_K} (green region), can be "shaped" by the set of constraints, $\mathcal{X}_{\delta^{(i)}}, i \in \mathcal{K}$ (dashed blue lines). According to Definition 2, the number of support subsamples for δ_K w.r.t. to Ω_{δ_K} is, in general, smaller compared to $|\mathcal{X}_{\delta_K}|$ (dashed orange lines, whose intersections are defined by orange dots).

Note that a direct consequence of Lemma 4 is that $\Theta_0 \cong \Omega_{\delta_0} \supseteq \Omega_{\delta_1} \supseteq \ldots \supseteq \Omega_{\delta_K} \cong \Theta_K(\delta_K)$. Moreover, the intrinsic consistency of the set Ω_{δ_K} implies that by introducing additional constraints, the effect of the uncertain parameter is to (possibly) shrink the feasible set on which the scenario-based VI in (5) is defined, and therefore the set of solutions can only shrink, accordingly (see Fig. 3 for a graphical interpretation).

3.3. Proof of Theorem 1 and discussion

We are now ready to prove Theorem 1.

Proof. For any $K \in \mathbb{N}$, $\delta_K \in \Delta^K$, we know that Ω_{δ_K} is consistent w.r.t. the collected scenarios, δ_K . In view of the definition in (7), indeed, we have that $\Omega_{\delta_K} \subseteq \cap_{i \in \mathcal{K}} \mathcal{X}_{\delta^{(i)}} \cap \mathcal{X}$, which implies that $\Omega_{\delta_K} \subseteq \mathcal{X}_{\delta^{(i)}}$, for all $i \in \mathcal{K}$, thus directly satisfying Definition 4. Then, we can apply Lemma 3 with $\theta^*_{\delta_K}$ being replaced by Ω_{δ_K} . We thus have that $\mathbb{P}^K \{\delta_K \in \Delta^K \mid \mathbb{P}\{\delta \in \Delta \mid \Omega_{\delta_K} \not\subseteq \mathcal{X}_{\delta}\} > \varepsilon(s_K)\} \leq \beta$. However, by Lemma 4, $\Omega_{\delta} = \Omega_{\delta_K} \cap \mathcal{X}_{\delta}$. Therefore, $\Omega_{\delta_K} \not\subseteq \mathcal{X}_{\delta}$ is equivalent to $\Omega_{\delta} \neq \Omega_{\delta_K}$, and since the set of solutions can only shrink once a new scenario is added, this is in turn equivalent to $\Omega_{\delta} \not\subseteq \Omega_{\delta_K}$. Thus, in view of the definition of V in (8), we finally have that $\mathbb{P}^K \{\delta_K \in \Delta^K \mid V(\Omega_{\delta_K}) > \varepsilon(s_K)\} \leq \beta$. \Box

A more direct expression of the critical parameter $\varepsilon(\cdot)$ can be obtained by splitting the confidence parameter β evenly among the K terms within the summation, i.e.,

$$\varepsilon(h) = \begin{cases} 1 & \text{if } h = K, \\ 1 - \sqrt[K-h]{\frac{\beta}{K\binom{K}{h}}} & \text{otherwise.} \end{cases}$$
(11)

Remark 2. In the case of a non-degenerate VI, the bound in (9) could be improved by means of the wait-and-judge analysis in [9]. Specifically, in view of [9, Th. 2], we can replace the expression for $\varepsilon(\cdot)$ in (11) with $\varepsilon(h) = 1 - t(h)$, where t(h) is shown to be the unique solution in (0, 1) to

$$\frac{\beta}{K+1}\sum_{m=h}^{K} \binom{m}{h} t^{m-h} - \binom{K}{h} t^{K-h} = 0.$$

However, note that the non-degeneracy condition is in general difficult to verify even in convex optimization settings, a challenge that becomes more involved for VIs.

Similarly to $\Upsilon(\cdot)$, let us suppose to have available an algorithm that allows us to compute a support subsamples for δ_K associated with the feasible set \mathcal{X}_{δ_K} , i.e., the subset of samples such that by using only this subset leads to \mathcal{X}_{δ_K} (as opposed to Ω_{δ_K} , if $\Upsilon(\cdot)$ is employed). By comparing the bound in (10) with the certificates in [44], we provide an upper bound for $V(\Omega_{\delta_K})$.

Proposition 1. Given any $K \in \mathbb{N}_0$ and $\delta_K \in \Delta^K$, let s_K and v_K be the cardinality of the support subsample for δ_K , evaluated w.r.t. Ω_{δ_K} and \mathcal{X}_{δ_K} , respectively. Then, $s_K \leq v_K$.

Proof. For every $K \in \mathbb{N}_0$ and $\delta_K \in \Delta^K$, Definition 2 suggests that some sample $\delta^{(k)}$ is of support for δ_K w.r.t. \mathcal{X}_{δ_K} if $\mathcal{X}_{\delta^{(k)}}$ is active on $\mathrm{bdry}(\mathcal{X}_{\delta_K})$, i.e., $\mathrm{bdry}(\mathcal{X}_{\delta^{(k)}}) \cap \mathrm{bdry}(\mathcal{X}_{\delta_K}) \neq \emptyset$. On the other hand, $\delta^{(k)}$ is of support w.r.t. Ω_{δ_K} if $\mathrm{bdry}(\mathcal{X}_{\delta^{(k)}}) \cap \Omega_{\delta_K} \neq \emptyset$ (see Fig. 3 for a graphic illustration). Since $\Omega_{\delta_K} \subseteq \mathcal{X}_{\delta_K} \coloneqq \bigcap_{k \in \mathcal{K}} \mathcal{X}_{\delta^{(k)}} \cap \mathcal{X}$, those samples that are of support for δ_K w.r.t. Ω_{δ_K} , are necessarily of support w.r.t. \mathcal{X}_{δ_K} , but not vice-versa, and hence $s_K \leq v_K$.

Under Proposition 1, Theorem 1 improves over [44], where $V(\Omega_{\delta_K}) > \varepsilon(v_K)$ was claimed with confidence at most β . The latter is since $\varepsilon(s_K) \leq \varepsilon(v_K)$ as $\varepsilon(\cdot)$ is non-decreasing. Moreover, within the set-oriented framework proposed in §2, as evident from (10), to bound the feasibility risk (8) of the entire set of solutions Ω_{δ_K} , one does not need an explicit characterization of Ω_{δ_K} , namely some mapping $\Theta_K(\cdot)$, but rather the number of support subsamples s_K , computed through an algorithm $\Upsilon(\cdot)$. We investigate the computation of s_K in the next section.

4. The case of affine constraints: computation of the support subsample

In this section we first propose an iterative procedure that, in case of affine constraints, allows one to compute a support subsample for the unknown Ω_{δ_K} . We also discuss the related computational complexity.

4.1. Support subsample computation

We start by noting that the bound in (10) depends only on the support subsample that characterizes the solution set Ω_{δ_K} , and not on its actual shape. Except in some particular cases, e.g., monotone affine mapping $F(\cdot)$ (see [24, Th. 2.4.15]), an explicit representation of Ω_{δ_K} is either unavailable, or difficult to compute in advance. However, the general setting considered so far, i.e., pseudomonotone mapping F and convex constraint set \mathcal{X}_{δ_K} , for any $\delta_K \in \Delta^K$, poses several challenges in designing an efficient procedure to compute the number of support subsamples w.r.t. Ω_{δ_K} . We therefore introduce the following additional assumption that restricts attention to the class of linearly constrained, pseudomonotone VIs.

Assumption 5. Let $\mathcal{X} \coloneqq \{ \boldsymbol{x} \in \mathbb{R}^n \mid C\boldsymbol{x} \leq d \}$, $C \in \mathbb{R}^{m \times n}$ and $d \in \mathbb{R}^m$, with $\operatorname{rank}(C) = n$, and, for all $\delta \in \Delta$, $\mathcal{X}_{\delta} \coloneqq \{ \boldsymbol{x} \in \mathbb{R}^n \mid A(\delta)\boldsymbol{x} \leq b(\delta) \}$, $A : \Delta \to \mathbb{R}^{p \times n}$ and $b : \Delta \to \mathbb{R}^p$.

Then, given any K-multisample δ_K , let $\Phi : \Delta^K \Rightarrow \Omega_{\delta_K}$ be any mapping that returns a (set of) solution(s) to VI($\mathcal{X}_{\delta_K}, F$). The procedure $\Phi(\cdot)$ is run in (S1) to verify whether (at least) one solution to the VI with constraints involving $\mathcal{X}_{\delta_K} \cap \text{bdry}(\mathcal{X}_{\delta^{(i)}})$ exists, thus increasing the counter s_K in case of affirmative answer in (S2). The next result states

Algorithm 1 Computation of the cardinality of the support subsample w.r.t. Ω_{δ_K}

Initialization:

Set $s_K := 0$, identify $\mathcal{A}_K := \{k \in \mathcal{K} \mid bdry(\mathcal{X}_{\delta^{(k)}}) \cap bdry(\mathcal{X}_{\delta_K}) \neq \emptyset\}$ **Iteration** $(i \in \mathcal{A}_K)$: (S1) Run $\Phi(\delta_{K,i})$ to solve $VI(\mathcal{X}_{\delta_K} \cap bdry(\mathcal{X}_{\delta^{(i)}}), F)$ (S2) If $\Phi(\delta_{K,i}) \neq \emptyset$, set $s_K := s_K + 1$

that, even without knowing the set Ω_{δ_K} , Algorithm 1 returns the cardinality of a support subsample for δ_K w.r.t. Ω_{δ_K} .

Proposition 2. Let Assumption 5 hold true. Given any $K \in \mathbb{N}$ and $\delta_K \in \Delta^K$, Algorithm 1 returns s_K^* , the cardinality of a support subsample δ_K w.r.t. Ω_{δ_K} .

Proof. First note that, in view of Assumption 5, \mathcal{A}_K forms a support subsample for δ_K w.r.t. \mathcal{X}_{δ_K} . Then, by following the considerations adopted within the proof of Proposition 1, every $\delta^{(k)}$, $k \in \mathcal{A}_K$, is of support also w.r.t. to Ω_{δ_K} if and only if $\mathrm{bdry}(\mathcal{X}_{\delta^{(k)}}) \cap \Omega_{\delta_K} \neq \emptyset$. To determine this, it is sufficient to compute a solution (if any) on the active region of \mathcal{X}_{δ_K} associated with $\mathcal{X}_{\delta^{(k)}}$. Then, s_K increases only if $\Phi(\delta_{K,k}) \neq \emptyset$, excluding all those samples such that $\mathcal{X}_{\delta^{(k)}}$ does not intersect Ω_{δ_K} .

Remark 3. Algorithm 1 requires one to run $|\mathcal{A}_K|$ -times the adopted solution algorithm $\Phi(\delta_K)$, with $|\mathcal{A}_K| \leq K$. This clearly improves w.r.t. the greedy algorithms proposed in [10, §II] and [42, §III], which would require one to run $\Phi(\delta_K)$ K-times. On the other hand, we remark that the greedy algorithm applies more generally and, in particular, if one assumes uniqueness of the solution as in [10, 42], it can be employed (not necessarily only in the case of affine constraints) as long as this solution is computable.

We remark that the design of an analogous procedure to Algorithm 1 involving general convex constraints constitutes a topic of current investigation, as discussed below.

4.2. Computational aspects

From a computational point of view, we note that Assumption 5 is needed for two main reasons:

- i) Evaluating a solution to the VI on the boundary of a convex set, i.e., (S1), may require solution of a VI defined over a nonconvex domain. Unlike the convex case, the literature on solution algorithms with suitable convergence guarantees or performance for the nonconvex case is not extensive.
- ii) The initialization step becomes trivial, since it requires one to identify the minimal number of active constraints, a task that is closely related to enumerating the number of facets of the polytope $\mathcal{X}_{\delta_{K}}$.

While item i) prevents us from trivially extending Algorithm 1 to the case of general convex constraints, item ii) can be equivalently seen as a problem of removing redundant halfspaces. Specifically, in view of the affine structure of both \mathcal{X}_{δ} and \mathcal{X} , the "offline" initialization step amounts to solve a family of linear programs (LPs), a task that can be efficiently accomplished in polynomial time by means of available solvers. In fact, given any K-multisample $\delta_K \in \Delta^K$, the convex polytope \mathcal{X}_{δ_K} is described by the system of (m + Kp) linear inequalities $A_{\delta_K} x \leq b_{\delta_K}$, with $A_{\delta_K} \coloneqq \operatorname{col}(C, \operatorname{col}((A(\delta_k))_{k \in \mathcal{K}}))$ and

 $b_{\delta_K} \coloneqq \operatorname{col}((b(\delta_k))_{k \in \mathcal{K}}))$. Here, let a_i^{\top} (respectively, b_i) be the *i*-th row (*i*-th element) of A_{δ_K} (b_{δ_K}), and let $\mathcal{L} \coloneqq \{1, \ldots, m + Kp\}$ be the associated set of row indices. Then, it turns out that a particular $i \in \mathcal{L}$ is not redundant for $A_{\delta_K} x \leq b_{\delta_K}$ if and only if the optimal value of the following LP

$$\begin{cases} \max_{x \in \mathbb{R}^n} & a_i^\top x \\ \text{s.t.} & a_j^\top x \le b_j, \text{ for all } j \in \mathcal{L} \setminus \{i\}, \\ & a_i^\top x \le b_i + 1, \end{cases}$$

is strictly greater than b_i (notice that the *i*-th constraint has been relaxed). However, particularly when the dimension of the VI problem *n* is large, arbitrarily removing a single inequality constraint at time might prove highly inefficient, or even prohibitive for large amounts of data. For this reason, one may rely on tailored algorithms available in the literature, e.g., [2, 4, 14, 54]. Successively, once the set of constraints that determines $bdry(\mathcal{X}_{\delta_K})$ has been identified, Algorithm 1 requires one to run some $\Phi(\cdot)$ to verify whether a solution to the VI on each facet of the convex polytope \mathcal{X}_{δ_K} exists (S1). Since the feasible set of $VI(\mathcal{X}_{\delta_K} \cap bdry(\mathcal{X}_{\delta^{(i)}}), F)$ is affine, we remark that the computational complexity of solving (S1) $|\mathcal{A}_K|$ -times is directly driven by the class of (pseudo)monotone mapping Fwe are tackling, as well as by the family of solution algorithms to solve $VI(\mathcal{X}_{\delta_K} \cap$ $bdry(\mathcal{X}_{\delta^{(i)}}), F)$ is vast, and therefore an a-priori estimate of the computational burden appears hard to quantify. In §5.2, we detail the computational effort when applying this procedure on our case study.

5. Application to robust game theory

In this section we first discuss how robust game theory and GNEPs can be modelled as uncertain VIs of the form (4), and then we present a numerical case study modelling the charging coordination of a fleet of PEVs. Note that an explicit characterization of the set of solutions to the scenario-based VI, Ω_{δ_K} , is unlikely to exist in this case. Therefore, under appropriate assumptions, i.e., a suitable counterpart of Assumption 5, the systematic procedure in Algorithm 1 allows us to compute the cardinality of the support subsample.

5.1. Uncertain GNEPs and scenario-based formulation

We consider a finite population of N agents taking part in a noncooperative game. The agents, indexed by the set $\mathcal{I} := \{1, \ldots, N\}$, control a decision vector $x_i \in \mathbb{R}^{n_i}$, which is locally constrained to a deterministic set $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, which may include, e.g., operational and dynamic constraints over a certain prediction/control horizon. Each agent aims to minimize a cost function $J_i : \mathbb{R}^n \to \mathbb{R}$, $n := \sum_{i \in \mathcal{I}} n_i$, while satisfying a set of coupling constraints affected by uncertainty δ . The uncertain GNEP is hence formalized by the following collection of coupled optimization problems:

$$\forall i \in \mathcal{I} : \begin{cases} \min_{x_i \in \mathcal{X}_i} & J_i(x_i, \boldsymbol{x}_{-i}) \\ \text{s.t.} & (x_i, \boldsymbol{x}_{-i}) \in \mathcal{X}_{\delta}, \, \delta \in \Delta. \end{cases}$$
(12)

We then introduce the following standard assumptions.

Assumption 6. For all $i \in \mathcal{I}$, for all $\mathbf{x}_{-i} \in \mathbb{R}^{n-n_i}$, $J_i(\cdot, \mathbf{x}_{-i})$ is a convex function of class C^1 .

Assumption 7. For all $i \in \mathcal{I}$, \mathcal{X}_i is a nonempty compact, convex set. For all $\delta \in \Delta$, \mathcal{X}_{δ} is a nonempty closed, convex set.

Let δ_K be the K-multisample introduced in §2, for some $K \in \mathbb{N}_0$. The scenario-based GNEP Γ is defined as the tuple $\Gamma := (\mathcal{I}, (\mathcal{X}_i)_{i \in \mathcal{I}}, (J_i)_{i \in \mathcal{I}}, \delta_K)$, formally represented by the following family of optimization problems:

$$\forall i \in \mathcal{I} : \begin{cases} \min_{x_i \in \mathcal{X}_i} & J_i(x_i, \boldsymbol{x}_{-i}) \\ \text{s.t.} & (x_i, \boldsymbol{x}_{-i}) \in \mathcal{X}_{\delta^{(k)}}, \, \forall k \in \mathcal{K}. \end{cases}$$
(13)

Given its deterministic nature, in view of Assumption 6 and 7, the game Γ is a jointly convex GNEPs [23, Def. 2]. Then, let us define the sets $\mathcal{X} := \prod_{i \in \mathcal{I}} \mathcal{X}_i, \mathcal{X}_i^{\delta_K}(\boldsymbol{x}_{-i}) := \{x_i \in \mathcal{X}_i \mid (x_i, \boldsymbol{x}_{-i}) \in \bigcap_{k \in \mathcal{K}} \mathcal{X}_{\delta^{(k)}}\}$, and $\mathcal{X}_{\delta_K} := \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{x} \in \bigcap_{k \in \mathcal{K}} \mathcal{X}_{\delta^{(k)}}\} \subseteq \mathcal{X}$. We recall now the following key notion of a Nash equilibrium for Γ :

Definition 5. (Generalized Nash Equilibrium) Let $\delta_K \in \Delta^K$ be a given K-multisample. The collective strategy $\mathbf{x}^* \in \mathcal{X}_{\delta_K}$ is a GNE of the scenario-based GNEP Γ in (13) if, for all $i \in \mathcal{I}$,

$$J_i(x_i^{\star}, \boldsymbol{x}_{-i}^{\star}) \leq \min_{y_i \in \mathcal{X}_i^{\delta_K}(\boldsymbol{x}_{-i}^{\star})} J_i(y_i, \boldsymbol{x}_{-i}^{\star}).$$

Clearly, given the dependency on the set of K realizations δ_K , any equilibrium of Γ is a random variable itself.

According to Definition 5, a popular subset of GNE of a GNEP is the one of v-GNE [23], which coincides with the set of collective strategies that solve the VI associated with the GNEP in (12). Specifically, this type of equilibrium problem has certain advantageous structural properties and can be modelled as an uncertain VI problem of the type we have considered in this paper. Moreover, it is significant per se as it provides "larger social stability" and "economic fairness" [11, §5],[23, Th. 4.8]. Thus, by defining the game mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ as $F(\mathbf{x}) := \operatorname{col}((\nabla_{x_i} J_i(x_i, \mathbf{x}_{-i}))_{i \in \mathcal{I}})$, we formally introduce the class of v-GNE as follows.

Definition 6. (Variational Generalized Nash Equilibrium) [23, Def. 3] Let $\delta_K \in \Delta^K$ be a given K-multisample defining a jointly convex GNEP. A v-GNE is any solution $\mathbf{x}^* \in \mathcal{X}_{\delta_K}$ to the GNEP Γ that is also a solution to $VI(\mathcal{X}_{\delta_K}, F)$.

In summary, any vector that solves $VI(\mathcal{X} \cap \mathcal{X}_{\delta}, F)$, $\delta \in \Delta$, belongs to the set of v-GNE of the GNEP in (12). Thus, analogously to (6), the set of v-GNE of the scenario-based GNEP in (13) is

$$\Omega_{\delta_K} \coloneqq \{ \boldsymbol{x} \in \mathcal{X}_{\delta_K} | (\boldsymbol{y} - \boldsymbol{x})^\top F(\boldsymbol{x}) \ge 0, \ \forall \boldsymbol{y} \in \mathcal{X}_{\delta_K} \}.$$
(14)

Assume that the set of GNE coincides with the one of v-GNE. To invoke Lemma 2, we assume the following:

Assumption 8. The game mapping $F(\cdot)$ is pseudomonotone.

Thus, in the spirit of Definition 1, by relying on the K-multisample and the associated scenario-based GNEP, we can now employ Theorem 1 to provide a-posteriori feasibility certificates to the equilibrium set Ω_{δ_K} in (14).

5.2. Case study: Charging coordination of plug-in electric vehicles

The problem of coordinating the day-ahead charging of a fleet of PEVs, originally introduced in [36], can be modelled as a noncooperative GNEP [18, 12]. Specifically, in the spirit of the previous section, for each PEV $i \in \mathcal{I}$, we consider a discrete-time linear dynamical system $s_i(t+1) = s_i(t) + b_i x_i(t), t \in \mathbb{N}$, where $s_i \in [0,1]$ is the State of Charge (SoC), i.e., $s_i = 1$ represents a fully charged battery, while $s_i = 0$ a completely discharged one; $x_i(t) \in [0, 1]$ is the charging control input at the specific time instant t, and $b_i > 0$ denotes the charging efficiency. According to a desired level of SoC that has to be achieved, the goal of each PEV is to acquire (at least) a charge amount γ_i within a finite charging horizon $T \in \mathbb{N}$, in order to satisfy the charging constraint $\sum_{t \in \mathcal{T}} x_i(t) = \mathbf{1}_T^\top x_i \geq \gamma_i$, with $\mathcal{T} \coloneqq \{0, \ldots, T-1\}$ and $x_i \coloneqq \operatorname{col}((x_i(t))_{t \in \mathcal{T}}) \in \mathbb{R}^T$, while, minimizing its charg-ing cost, $J_i(x_i, \mathbf{x}_{-i}) \coloneqq p(\mathbf{x})^\top x_i$. Here, $p : \mathbb{R}_{\geq 0}^T \to \mathbb{R}_{\geq 0}^T$, denotes the electricity price function over the charging horizon, which for simplicity we assume to be affine in the aggregate demand of energy associated with the set of PEVs, namely $p(\mathbf{x}) \coloneqq \alpha \sigma(\mathbf{x}) + \eta$, with $\sigma(\boldsymbol{x}) \coloneqq \sum_{j \in \mathcal{I}} x_j \in \mathbb{R}^T$, for some $\alpha > 0$ and $\eta \in \mathbb{R}_{\geq 0}^T$. Moreover, due to the intrinsic limitations of the grid capacity $d_{\text{max}} > 0$, we assume that the amount of energy required in each single time period by both the PEVs and non-PEV loads should not be greater than d_{max} . This translates into a constraint on the PEVs total demand, i.e., $d(t) + \sum_{j \in \mathcal{I}} x_j(t) \in [0, d_{\max}], \text{ for all } t \in \mathcal{T}.$

The inflexible non-PEV demand $d \in \mathbb{R}_{\geq 0}^{T}$ is subject to uncertainty and therefore is modelled as $d \coloneqq d_{\text{nom}} + \delta$. Here, $d_{\text{nom}} \in \mathbb{R}_{\geq 0}^{T}$ is the nominal non-PEV daily energy demand, which can be extracted, e.g., from data (see [31] for typical daily energy profiles in the UK), while δ is a random variable that follows a uniform probability distribution on $\Delta \subseteq \mathbb{R}^{T}$. The (uncertain) GNEP coincides with the following collection of optimization problems

$$\forall i \in \mathcal{I} : \begin{cases} \min_{\substack{x_i \in [0,1]^T \\ \text{s.t.} \\ d_{\text{nom}} + \delta \end{pmatrix} + \sigma(\boldsymbol{x}) \leq \mathbf{1}_T d_{\max}, \forall \delta \in \Delta, \\ A_i x_i \leq c_i, \end{cases}$$
(15)

where $A_i \coloneqq \operatorname{col}(-B_i, B_i, -\mathbf{1}_T^{\top}) \in \mathbb{R}^{(2T+1)\times T}$, $B_i \in \mathbb{R}^{T\times T}$ is matrix with all entries in the lower triangular part equal to b_i , $c_i \coloneqq \operatorname{col}(\mathbf{1}_T s_i(0), \mathbf{1}_T(1 - s_i(0)), -\gamma_i) \in \mathbb{R}^{2T+1}$, and $s_i(0) \in [0, 1]$ is a given initial SoC. Problem (15) is in the form of (12). We note that the game mapping $F(\mathbf{x}) \coloneqq \operatorname{col}(\nabla_{x_i}((\alpha\sigma(\mathbf{x}) + \eta)^{\top}x_i)_{i\in\mathcal{I}}))$, which allows us to define the VI whose solution set determines the v-GNE of the game, turns out to be affine in \mathbf{x} . Specifically, $F(\mathbf{x}) = M\mathbf{x} + q$, where $M \in \mathbb{R}^{NT \times NT}$ has entries all equal to α , while $q \coloneqq \mathbf{1}_N \otimes \eta \in \mathbb{R}^{NT}$. Note that, for any $\alpha > 0$, $F(\cdot)$ is a monotone mapping (or, equivalently, $M + M^{\top} \succeq 0$).

Thus, based on K observations of historical data, the GNEP in (15) admits a scenariobased counterpart as in (13), for which we quantify the robustness of Ω_{δ_K} , the solution set to $\operatorname{VI}(\mathcal{X}_{\delta_K}, F)$. Here, $\mathcal{X}_{\delta_K} \coloneqq \prod_{i \in \mathcal{I}} \mathcal{X}_i \cap_{k \in \mathcal{K}} \mathcal{X}_{\delta^{(k)}}$, with $\mathcal{X}_i \coloneqq \{x_i \in [0, 1]^T \mid A_i x_i \leq c_i\}$, and $\mathcal{X}_{\delta^{(k)}} \coloneqq \{x \in \mathbb{R}^{NT} \mid (d_{\operatorname{nom}} + \delta^{(k)}) + \sigma(x) \leq \mathbf{1}_T d_{\max}\}, k \in \mathcal{K}.$

5.2.1. Numerical simulations

To numerically test the theoretical results provided in the paper, in this section we fit a multidimensional Gaussian distribution to 10^2 daily profiles from [31] to generate large sets of realistic samples. Thus, we first support the consistency of Ω_{δ_K} numerically. Specifically, we estimate Ω_{δ_0} by computing 10^3 different solutions to VI($\mathcal{X}_{\delta_0}, F$), hence obtaining

Table 1: Simulation parameters					
Name	Description	Value			
N	PEVs number	20			
T	Time intervals	24			
b_i	Charging efficiency	[0.075, 0.25]			
$s_i(0)$	Initial SoC of battery	[0.1, 0.4]			
$s_i(T)$	Desired SoC of bat-	[0.7, 1]			
	tery				
γ_i	Required charge	[1.62, 7.49]			
	amount				
α	Inverse of price elas-	0.01			
	ticity				
η	Baseline price	0_T			
$d_{ m nom}$	Non-PEV demand	Average over			
		10^2 daily pro-			
		files $[31]$			
d_{\max}	Grid power capacity	$2 \cdot \max_{t \in \mathcal{T}} d_{\text{nom}}(t)$			
Δ	Uncertainty support	$d_{ m nom}$.			
		[-0.05, 0.05]			

Table 2: Robustness certificate (10) and empirical violation probability

K	$ \mathcal{A}_K $	s_K^\star	$\varepsilon(s_K^\star)$	$V_{\max}(ilde{\Omega}_{\delta_K})$	$\operatorname{avg}(V_{\max}(\tilde{\Omega}_{\delta_K}))$
10^{2}	28	4	0.29	0.018	0.012
10^{3}	381	7	0.06	$1.2 \cdot 10^{-3}$	$0.8\cdot 10^{-3}$
10^{4}	469	9	0.01	$0.9 \cdot 10^{-3}$	$0.4 \cdot 10^{-3}$

 $\tilde{\Omega}_{\delta_0}$, with the numerical parameters reported in Table 1. Every solution is computed by means of a typical extragradient algorithm [41], initialized with a different condition and fixed step size, whose convergence is guaranteed as F is monotone and Lipschitz continuous with constant αNT , for any step size $(0, \alpha^{-1}/NT)$. We emphasize that we are interested in computing a set of solutions to $VI(\mathcal{X}_{\delta_{K}}, F)$, and this motivates us to partially neglect the multi-agent nature of the problem addressed by adopting an extragradient method¹. Moreover, to compute a solution to VI($\mathcal{X}_{\delta_0}, F$) with a precision in norm of 10⁻⁶, the extragradient algorithm in [41] takes around 21.47[s] on average, resulting in 18–24 iterations. Given the linearity of the constraints, this value is representative for solving (S1) in Algorithm 1. Thus, as illustrated in Fig. 4, the average number of solutions contained in Ω_{δ_K} over 100 numerical experiments, normalized w.r.t. Ω_{δ_0} , shrinks as K grows. Note that, in view of the structure of Δ , as the number of samples K increases, the standard deviation of δ narrows around the average. An example of aggregate behaviour of the fleet of PEVs is reported in Fig. 5, where $avg(\cdot)$ returns the average among the solutions lying in $\tilde{\Omega}_{\delta_K}$, estimated after observing 10³ realizations of the uncertainty. Note that $\sigma(\boldsymbol{x})$ exhibits the so-called "valley filling" property, which is desirable since the overall demand has no peaks.

For any $K \in \mathbb{N}_0$, the feasible set of the scenario-based counterpart of (15) satisfies As-

¹Decentralized equilibrium computation is, indeed, outside the scope of the current paper.



Figure 4: Number of solutions contained in $\tilde{\Omega}_{\delta_K} = \tilde{\Omega}_{\delta_0} \cap \mathcal{X}_{\delta_K}$, normalized with the ones lying in $\tilde{\Omega}_{\delta_0}$, as a function of the number of samples K. The solid blue line represents the average of $|\tilde{\Omega}_{\delta_0} \cap \mathcal{X}_{\delta_K}|/|\tilde{\Omega}_{\delta_0}|$ over 100 numerical experiments, while the vertical blue lines the standard deviation.



Figure 5: Average behaviour of the fleet of PEVs, computed across the estimated set of solutions $\tilde{\Omega}_{\delta_K}$ after observing 10^3 realizations of the uncertainty. The overall demand, affected by the uncertainty δ , meets the grid capacity limitations.

sumption 5. Thus, in Table 2 we compare the output of the procedure summarized in Algorithm 1 to compute the cardinality s_K^* of the support subsample w.r.t. Ω_{δ_K} , for different values of K. The bound on the violation probability is then computed by the function $\varepsilon(\cdot)$ in (11) with $\beta = 10^{-6}$. Note that Algorithm 1 requires us to run $\Phi(\cdot)$ only $|\mathcal{A}_K|$ -times, which represents a noticeable improvement compared to the greedy algorithm proposed in [10, 42], which would require running $\Phi(\cdot)$ K-times. On the other hand, the "offline" initialization step with $K = 10^4$, which translates into 241940 linear inequalities, takes around 11600[s] to identify the set of constraints defining $\mathcal{X}_{\delta_{104}}$. Finally, the last two columns in Table 2 report both the maximum and the average value of the empirical violation probability of $\tilde{\Omega}_{\delta_0} \cap \mathcal{X}_{\delta_K}$ computed against 10^2 , 10^3 and 10^4 new realizations. The empirical probability, as expected, is lower than the theoretical bound in Theorem 1.

6. Concluding remarks

The scenario approach paradigm applied to uncertain VIs provides a numerically tractable framework to compute solutions with quantifiable robustness properties in a distributionfree fashion. In the specific family of uncertain VIs considered, which encompasses a broad class of practical applications belonging to different domains, we are able to evaluate the robustness properties of the entire set of solutions, thereby relaxing the requirement of a unique solution as often imposed in the literature. We have shown that this only requires us to enumerate the active coupling constraints that "shape" that set.

Future research directions involve synthesizing algorithms to enumerate the number of support subsamples in a general convex setting, as well as investigating extensions of the approach we have developed to quasi-variational inequalities. This would enable us to incorporate the uncertainty within the mapping defining the VI, thus extending the results of [25, 26] to the entire set of solutions to VIs.

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