

# Cluster Consensus on Matrix-weighted Switching Networks

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## Abstract

This paper examines the cluster consensus problem of multi-agent systems on matrix-weighted switching networks. Necessary and/or sufficient conditions under which cluster consensus can be achieved are obtained and quantitative characterization of the steady-state of the cluster consensus are provided as well. Specifically, if the underlying network switches amongst finite number of networks, a necessary condition for cluster consensus of multi-agent system on switching matrix-weighted networks is firstly presented, it is shown that the steady-state of the system lies in the intersection of the null space of matrix-valued Laplacians corresponding to all switching networks. Second, if the underlying network switches amongst infinite number of networks, the matrix-weighted integral network is employed to provide sufficient conditions for cluster consensus and the quantitative characterization of the corresponding steady-state of the multi-agent system, using null space analysis of matrix-valued Laplacian related of integral network associated with the switching networks. In particular, conditions for the bipartite consensus under the matrix-weighted switching networks are examined. Simulation results are finally provided to demonstrate the theoretical analysis.

*Keywords:* Matrix-weighted network, cluster consensus, switching network, integral network, bipartite consensus

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## 1. Introduction

Achieving consensus via local interactions amongst agents turns out to be an important paradigm in distributed control of multi-agent networks Mesbahi and Egerstedt [13], Olfati-Saber et al. [14], DeGroot [7], Zhang et al. [32]. However, it has long been assumed that the edges, representing the interaction between neighboring agents, are weighted by scalars, which apparently ignores the interdependencies among multi-dimensional states of neighboring agents. Recently, matrix-weighted network is introduced to characterize the complicated interactions amongst the high-dimensional states of agents Trinh et al. [24], Sun and Yu [22], Pan et al. [16], Tuna [25]. In fact, matrix-weighted networks naturally arise in scenarios such as graph effective resistance and its applications in distributed control and estimation Tuna [26], Baroah and Hespanha [3], opinion dynamics Friedkin et al. [9], Ye et al. [31], bearing-based formation control Zhao and Zelazo [33], coupled oscillators dynamics Tuna [27], and consensus and synchronization Trinh et al. [24], Tuna [25], Pan et al. [16], Su et al. [20].

As opposed to scalar-weighted networks, network connectivity does not translate to achieving consensus for matrix-weighted networks. Rather than achieving consensus, the expansion of the null space of the associated matrix-valued Laplacian enables the multi-agent system

on matrix-weighted networks to achieve cluster consensus (or clustering), even if the underlying network is connected Trinh et al. [24]. This elegant property enables one to design desired cluster structures of for multi-agent systems by elaborately investigating the connection between the matrix-valued edge weights and the null space of matrix-valued Laplacian. Notably, achieving the cluster synchronization in coupled oscillator systems has been shown to be closely related to memory process in human brain Skardal et al. [19], Ashwin et al. [2], Fell and Axmacher [8], Hipp et al. [10]. However, achieving desired cluster consensus is not trivial in the case of scalar-weighted networks, where network-wide information and specific control strategies have to be involved Wu et al. [29], Xia and Cao [30]. In contrast, cluster consensus can be naturally achieved under matrix-weighted networks. Recently, the conditions of achieving cluster consensus on fixed matrix-weighted networks were reported in Trinh et al. [24]. Nevertheless, the underlying network of a multi-agent system can vary over time in a great variety of situations Cao et al. [5], Olfati-Saber and Murray [15], Ren et al. [18], Cao et al. [4], Meng et al. [12], Anderson et al. [1], Cao et al. [6]. The continuous-time and discrete-time consensus problem on time-varying matrix-weighted networks are discussed Pan et al. [17], Van Tran et al. [28]. Nevertheless, to the best of our knowledge, the conditions for cluster consensus on dynamic matrix-weighted networks are still lacking.

This paper is intended to provide quantitative characterization for cluster consensus of multi-agent systems on

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matrix-weighted switching networks. The contribution of this paper is threefold. First, for the case that the underlying network switches amongst finite number of networks, necessary condition for the cluster consensus of matrix-weighted switching networks has been exploited and an essential connection between cluster consensus value and the Laplacian matrices of matrix weighted switching networks has been established. Second, for the case that the underlying network switches amongst infinite number of networks, sufficient conditions for cluster consensus are obtained by examining the structure of null spaces associated to the matrix-valued Laplacian of the associated integral network. Finally, we provide conditions for a class of specific cluster consensus, namely bipartite consensus, under the matrix-weighted switching networks, and graph-theoretic condition is obtained. The results obtained in this paper provide further insight into the collective behavior of multi-agent systems.

The remainder of the paper is organized as follows. Preliminaries and problem formulation are introduced in §2 and §3, respectively, followed by the cluster consensus and bipartite consensus conditions in §4. Simulation examples are presented in §5; we provide concluding remarks in §6.

## 2. Preliminaries

### 2.1. Notations

Let  $\mathbb{R}$ ,  $\mathbb{N}$  and  $\mathbb{Z}_+$  be the set of real numbers, natural numbers and positive integers, respectively. For  $n \in \mathbb{Z}_+$ , denote  $\underline{n} = \{1, 2, \dots, n\}$ . A symmetric matrix  $M \in \mathbb{R}^{n \times n}$  is positive definite (Resp. negative definite), denoted by  $M \succ 0$  (Resp.  $M \prec 0$ ), if  $\mathbf{z}^\top M \mathbf{z} > 0$  (Resp.  $\mathbf{z}^\top M \mathbf{z} < 0$ ) for all nonzero  $\mathbf{z} \in \mathbb{R}^n$ , and is positive (Resp. negative) semi-definite, denoted by  $M \succeq 0$  (Resp.  $M \preceq 0$ ), if  $\mathbf{z}^\top M \mathbf{z} \geq 0$  (Resp.  $\mathbf{z}^\top M \mathbf{z} \leq 0$ ) for all  $\mathbf{z} \in \mathbb{R}^n$ . The null space of a matrix  $M \in \mathbb{R}^{n \times n}$  is denoted by  $\mathbf{null}(M) = \{\mathbf{z} \in \mathbb{R}^n | M \mathbf{z} = \mathbf{0}\}$ .  $\mathbf{1}_n \in \mathbb{R}^n$  and  $0_{d \times d} \in \mathbb{R}^{d \times d}$  designate the vector whose components are all 1's and the matrix whose components are all 0's, respectively. Let  $\lfloor x \rfloor$  denote the greatest integer less than or equal to  $x \in \mathbb{R}$ . The sign function  $\mathbf{sgn}(\cdot) : \mathbb{R}^{n \times n} \mapsto \{0, -1, 1\}$  satisfies  $\mathbf{sgn}(M) = 1$  if  $M \succeq 0$  or  $M \succ 0$ ,  $\mathbf{sgn}(M) = -1$  if  $M \preceq 0$  or  $M \prec 0$ , and  $\mathbf{sgn}(M) = 0$  if  $M = 0_{d \times d}$ .

### 2.2. Graph Theory

A matrix-weighted switching graph is denoted by  $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$ , where  $t$  refers to time index. The node and edge sets of  $\mathcal{G}$  are denoted by  $\mathcal{V} = \{1, 2, \dots, n\}$  and  $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ , respectively. The weight on the edge  $(i, j) \in \mathcal{E}(t)$  is encoded by the symmetric matrix  $A_{ij}(t) \in \mathbb{R}^{d \times d}$  such that  $|A_{ij}(t)| \succeq 0$  or  $|A_{ij}(t)| \succ 0$ , and  $A_{ij}(t) = 0_{d \times d}$  for  $(i, j) \notin \mathcal{E}(t)$ . Thereby, the matrix-valued adjacency matrix  $A(t) = [A_{ij}(t)] \in \mathbb{R}^{dn \times dn}$  is a block matrix such that the block located in its  $i$ -th row and the  $j$ -th column is  $A_{ij}(t)$ . It is assumed that  $A_{ij}(t) = A_{ji}(t)$  for all  $i \neq j \in \mathcal{V}$  and  $A_{ii}(t) = 0_{d \times d}$  for all  $i \in \mathcal{V}$ . A bipartition of node set  $\mathcal{V}$

of matrix-weighted switching graph  $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$  at time  $t$  is two subsets of nodes  $\mathcal{V}_i \subset \mathcal{V}$  where  $i \in \underline{2}$  such that  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$  and  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ .

A path of  $\mathcal{G}(t)$  at time  $t$  is a sequence of edges of the form  $(i_1, i_2), (i_2, i_3), \dots, (i_{p-1}, i_p)$ , where nodes  $i_1, i_2, \dots, i_p \in \mathcal{V}$  are distinct; in this case we say that node  $i_p$  is reachable from  $i_1$ . The graph  $\mathcal{G}(t)$  is connected at time  $t$  if any two distinct nodes in  $\mathcal{G}(t)$  are reachable from each other. A tree is a connected graph with  $n \geq 2$  nodes and  $n - 1$  edges where  $n \in \mathbb{Z}_+$ . For matrix-weighted switching networks, we adopt the following terminology. An edge  $(i, j) \in \mathcal{E}(t)$  at time  $t$  is positive (respectively, negative) definite or positive (respectively, negative) semi-definite if the corresponding edge weight  $A_{ij}(t)$  is positive (respectively, negative) definite or positive (respectively, negative) semi-definite. A positive-negative path of  $\mathcal{G}(t)$  at time  $t$  is a path such that every edge in this path is either positive definite or negative definite. A positive-negative tree of  $\mathcal{G}(t)$  at time  $t$  is a tree such that every edge in this tree is either positive definite or negative definite. A positive-negative spanning tree of  $\mathcal{G}(t)$  at time  $t$  is a positive-negative tree containing all nodes in  $\mathcal{G}(t)$ .

## 3. Problem Formulation

Consider a multi-agent system consisting of  $n > 1$  ( $n \in \mathbb{Z}_+$ ) agents whose interaction network is characterized by a matrix-weighted switching graph  $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$ . Denote the state of an agent  $i \in \mathcal{V}$  as  $\mathbf{x}_i(t) = [x_{i1}(t), \dots, x_{id}(t)]^\top \in \mathbb{R}^d$ , evolving according to the protocol

$$\dot{\mathbf{x}}_i(t) = - \sum_{j \in \mathcal{N}_i(t)} |A_{ij}(t)| (\mathbf{x}_i(t) - \mathbf{sgn}(A_{ij}(t)) \mathbf{x}_j(t)), \quad i \in \mathcal{V}, \quad (1)$$

where  $\mathcal{N}_i(t) = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}(t)\}$  denotes the neighbor set of agent  $i \in \mathcal{V}$  at time  $t$ . Note that (1) degenerates into the scalar-weighted case when  $A_{ij}(t) = a_{ij}(t)I$ , where  $a_{ij}(t) \in \mathbb{R}$  and  $I$  denotes the  $d \times d$  identity matrix.

Let  $D(t) = \mathbf{diag}\{D_1(t), \dots, D_n(t)\} \in \mathbb{R}^{dn \times dn}$  be the matrix-valued degree matrix of  $\mathcal{G}(t)$ , where  $D_i(t) = \sum_{j \in \mathcal{N}_i} |A_{ij}(t)| \in \mathbb{R}^{d \times d}$  and  $i \in \mathcal{V}$ . The matrix-valued Laplacian is subsequently defined as  $L(t) = D(t) - A(t)$ . The dynamics of the overall multi-agent system now admits the form,

$$\dot{\mathbf{x}}(t) = -L(t)\mathbf{x}(t), \quad (2)$$

where  $\mathbf{x}(t) = [\mathbf{x}_1^\top(t), \dots, \mathbf{x}_n^\top(t)]^\top \in \mathbb{R}^{dn}$ .

*Remark 1.* It is well-known that network connectivity plays a central role in determining consensus for scalar-weighted time-invariant networks Olfati-Saber and Murray [15]. However, for the matrix-weighted time-invariant networks, network connectivity is only a necessary condition for consensus on matrix-weighted networks, it is possible to achieve cluster consensus even if the network is connected, which is related with the properties of the null space of matrix-valued Laplacian matrix for time-invariant network Trinh [23].

**Definition 1** (Cluster Consensus). The multi-agent system (2) admits cluster consensus, if there exists a partition of node set  $\mathcal{V}$ , say  $\mathcal{V}_1, \dots, \mathcal{V}_l$  where  $l \in \mathbb{Z}_+$  and  $l \leq n$ , such that all agents belonging to the same partition achieve consensus, while for any two agents  $i$  and  $j$  belonging to two different partitions,  $\lim_{t \rightarrow \infty} \mathbf{x}_i(t) \neq \lim_{t \rightarrow \infty} \mathbf{x}_j(t)$ . Each  $\mathcal{V}_i, i \in \underline{l}$  is referred to as a cluster. In particular, the cluster consensus is referred to as consensus and bipartite consensus if  $l = 1$  and  $l = 2$ , respectively.

The **Definition 1** implies that there exists a vector  $\mathbf{x}^* \in \mathbb{R}^{dn}$  such that  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$ , where  $\mathbf{x}^*$  is influenced by the initial states of the multi-agent system (2). This work aims to investigate conditions under which the cluster consensus state of multi-agent system (2) on matrix-weighted switching networks can be quantitatively characterized. We adopt the following assumptions on the underlying matrix-weighted switching network Olfati-Saber and Murray [15], Ren et al. [18], Cao et al. [4].

**Assumption 1.** There exists a sequence  $\{t_k | k \in \mathbb{N}\}$  such that  $\lim_{k \rightarrow \infty} t_k = \infty$  and the dwell time satisfies  $\Delta t_k = t_{k+1} - t_k \geq \alpha$  for all  $k \in \mathbb{N}$ , where  $\alpha > 0, t_0 = 0$ , and  $\mathcal{G}(t)$  is time-invariant for  $t \in [t_k, t_{k+1})$  for all  $k \in \mathbb{N}$ .

**Assumption 2.** In addition to **Assumption 1**, the switching networks  $\mathcal{G}(t)$  is chosen from a finite set  $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M\}$  for some  $M \in \mathbb{Z}_+$ , and  $\mathcal{G}_i, i \in \underline{M}$  appears in the sequence of  $\mathcal{G}(t)$  for infinitely times.

## 4. Main Results

In the following part, under the condition that the cluster consensus can be achieved for the multi-agent system (2), we shall exploit the connection between the cluster consensus and the null space of matrix-valued Laplacian matrices associated with a sequence of matrix-weighted networks. We shall start from the case that the underlying network of multi-agent system (2) switches amongst finite number of networks, as stated in the Assumption 2. Before showing the main result of this part, we first explore the properties of  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$ , which plays an important role in the proof of our main result.

**Lemma 1.** Consider the multi-agent system (2) on a matrix-weighted switching network  $\mathcal{G}(t)$  satisfying **Assumption 2**. If the multi-agent system (2) achieves the cluster consensus, namely, there exists  $\mathbf{x}^* \in \mathbb{R}^{dn}$  such that  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$ , then  $\lim_{t \rightarrow \infty} L(t)\mathbf{x}^* = \mathbf{0}$ .

*Proof.* Denote by  $\Phi(t, 0)$  as the state transition matrix of multi-agent system (2) over time interval  $[0, t]$ , then one has  $\|\Phi(t, 0)\| \leq 1$ , thus for any  $\mathbf{x}(0) \in \mathbb{R}^{dn}$  and  $t > 0$ ,

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\|.$$

In addition, based on  $\dot{\mathbf{x}}(t) = -L(t)\mathbf{x}(t)$ , one can derive  $\ddot{\mathbf{x}}(t) = (L(t))^2 \mathbf{x}(t)$ , therefore,

$$\|\ddot{\mathbf{x}}_i(t)\| \leq \|\ddot{\mathbf{x}}(t)\| \leq \beta^2 \|\mathbf{x}(t)\| \leq \beta^2 \|\mathbf{x}(0)\|,$$

where  $\beta = \max_{i \in \underline{M}} \|L_i\|$  and  $L_i$  is the matrix-valued Laplacian matrix corresponding to  $\mathcal{G}_i$ , where  $i \in \underline{M}$ . Therefore, according to **Lemma 6** in the Appendix, one has,

$$\lim_{t \rightarrow \infty} \dot{\mathbf{x}}_i(t) = \mathbf{0},$$

which imply that  $\lim_{t \rightarrow \infty} \dot{\mathbf{x}}(t) = \mathbf{0}$ . Due to

$$L(t)\mathbf{x}^* = (\dot{\mathbf{x}}(t) + L(t)\mathbf{x}^*) - \dot{\mathbf{x}}(t),$$

and

$$\begin{aligned} \|\dot{\mathbf{x}}(t) + L(t)\mathbf{x}^*\| &= \|-L(t)(\mathbf{x}(t) - \mathbf{x}^*)\| \\ &\leq \beta \|\mathbf{x}(t) - \mathbf{x}^*\|, \end{aligned}$$

thus,  $\lim_{t \rightarrow \infty} L(t)\mathbf{x}^* = \mathbf{0}$ .  $\square$

Denote the state transition matrix of multi-agent system (2) over time interval  $[t_0, t]$  as  $\Phi(t, t_0)$ , then  $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$ , the following lemma presents the properties of  $\Phi(t, t_0)$  which decides the convergence value of  $\mathbf{x}(t)$ .

**Lemma 2.** Consider the multi-agent system (2) on a matrix-weighted switching network  $\mathcal{G}(t)$  satisfying **Assumption 2**. Then,  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$  exists for any  $\mathbf{x}(t_0) \in \mathbb{R}^{dn}$  if and only if  $\lim_{t \rightarrow \infty} \Phi(t, t_0)$  exists. Moreover, denote by  $\lim_{t \rightarrow \infty} \Phi(t, t_0) = \Phi^*(t_0)$ , then  $[\Phi^*(t_0)]^i = \Phi^*(t_0)$  for any  $i \in \mathbb{Z}_+$ .

*Proof.* (Sufficiency) Due to  $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$  and  $\lim_{t \rightarrow \infty} \Phi(t, t_0)$  exists, thus,

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \lim_{t \rightarrow \infty} \Phi(t, t_0)\mathbf{x}(t_0) = \Phi^*(t_0)\mathbf{x}(t_0),$$

i.e.,  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$  exists.

(Necessity) If  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$  exists for any  $\mathbf{x}(t_0) \in \mathbb{R}^{dn}$ , without loss of generality, one can choose  $\mathbf{x}(t_0) = \mathbf{e}_i, i \in \underline{dn}$ , where  $\mathbf{e}_i \in \mathbb{R}^{dn}$  has its  $i$ -th component equal to one with others equal to zero. Then, one has,

$$\begin{aligned} &\lim_{t \rightarrow \infty} \Phi(t, t_0) \\ &= \left[ \lim_{t \rightarrow \infty} \Phi(t, t_0)\mathbf{e}_1, \dots, \lim_{t \rightarrow \infty} \Phi(t, t_0)\mathbf{e}_{dn} \right], \end{aligned}$$

due to  $\lim_{t \rightarrow \infty} \mathbf{x}(t)$  exists for any  $\mathbf{x}(t_0) \in \mathbb{R}^{dn}$  and  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \lim_{t \rightarrow \infty} \Phi(t, t_0)\mathbf{x}(t_0)$ , one can conclude that  $\lim_{t \rightarrow \infty} \Phi(t, t_0)$  exists.

Denote by  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$ , due to the fact  $\lim_{t \rightarrow \infty} L(t)\mathbf{x}^* = \mathbf{0}$  and  $\mathcal{G}_i, i \in \underline{M}$  appears in the sequence of  $\mathcal{G}(t)$  for infinitely times, one has  $L(t)\mathbf{x}^* = \mathbf{0}$ , therefore, for any  $\mathbf{x}(t_0) \in \mathbb{R}^{dn}$ ,

$$L(t)\mathbf{x}^* = L(t)\Phi^*(t_0)\mathbf{x}(t_0) = \mathbf{0},$$

then,

$$\begin{aligned} & L(t)\Phi^*(t_0) \\ &= [L(t)\Phi^*(t_0)\mathbf{e}_1, \dots, L(t)\Phi^*(t_0)\mathbf{e}_{dn}] \\ &= \mathbf{0}_{dn \times dn}. \end{aligned}$$

According to the Peano-Baker series form of  $\Phi(t, t_0)$ ,

$$\begin{aligned} & \Phi(t, t_0) \\ &= I_n + \sum_{k=1}^{\infty} \int_{t_0}^t [-L(\sigma_1)] \int_{t_0}^{\sigma_1} [-L(\sigma_2)] \cdots \\ & \quad \int_{t_0}^{\sigma_{k-1}} [-L(\sigma_k)] d\sigma_k \cdots d\sigma_2 d\sigma_1, \end{aligned}$$

thus,  $\Phi(t, t_0)\Phi^*(t_0) = \Phi^*(t_0)$ . Take the limit on both sides leads to  $[\Phi^*(t_0)]^2 = \Phi^*(t_0)$ , therefore,  $[\Phi^*(t_0)]^i = \Phi^*(t_0)$  for any  $i \in \mathbb{Z}_+$ .  $\square$

Based on the above established Lemmas, we shall show the relationship between the cluster consensus and the matrix-valued Laplacian matrix  $L(t)$  of the associated matrix-weighted networks, and further provide the explicit expression of the cluster consensus value.

**Theorem 1.** Consider the multi-agent system (2) on a matrix-weighted switching network  $\mathcal{G}(t)$  satisfying **Assumption 2**. If  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$ , then

$$\mathbf{x}^* \in \bigcap_{i \in \underline{M}} \mathbf{null}(L(\mathcal{G}_i)).$$

Moreover,

$$\mathbf{x}^* = \sum_{i=1}^r (\boldsymbol{\eta}_i^\top \mathbf{x}(t_0)) \boldsymbol{\eta}_i,$$

where  $\boldsymbol{\eta}_i \in \mathbb{R}^{dn}$  satisfies  $\mathbf{span}\{\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_r\} = \bigcap_{i \in \underline{M}} \mathbf{null}(L(\mathcal{G}_i))$  and

$$\boldsymbol{\eta}_i^\top \boldsymbol{\eta}_j = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}, \forall i, j \in \underline{r}.$$

*Proof.* Since  $[\Phi^*(t_0)]^2 = \Phi^*(t_0)$ , then  $\Phi^*(t_0)$  is idempotent and diagonalizable, and the eigenvalues of  $\Phi^*(t_0)$  are 0 or 1. We shall first prove that the eigenvector space corresponding to the eigenvalue 1 of  $\Phi^*(t_0)$  is  $\bigcap_{t \geq 0} \mathbf{null}(L(t))$ .

On the one hand, for any  $\boldsymbol{\alpha} \in \mathbf{span}\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_r\}$ , due to  $L(t)\boldsymbol{\alpha} = \mathbf{0}$  for any  $t \geq t_0$ , thus  $\Phi(t, t_0)\boldsymbol{\alpha} = \boldsymbol{\alpha}$ . By taking the limit of  $\Phi(t, t_0)$ , it is easy to derive  $\Phi^*(t_0)\boldsymbol{\alpha} = \boldsymbol{\alpha}$ . Conversely, for an arbitrary  $\boldsymbol{\alpha}$  such that  $\Phi^*(t_0)\boldsymbol{\alpha} = \boldsymbol{\alpha}$ ,  $L(t)\boldsymbol{\alpha} = L(t)\Phi^*(t_0)\boldsymbol{\alpha} = \mathbf{0}$ . Therefore, one has  $\Phi^*(t_0)\boldsymbol{\eta}_i =$

$\boldsymbol{\eta}_i$  and  $\boldsymbol{\eta}_i^\top \Phi^*(t_0) = \boldsymbol{\eta}_i^\top$  for any  $i \in \underline{r}$ . There exists a matrix  $P = [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_r, *, \dots, *]$  together with its inverse  $P^{-1} = [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_r, \star, \dots, \star]^\top$  such that

$$\begin{aligned} \Phi^*(t_0) &= P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= \sum_{i=1}^r \boldsymbol{\eta}_i \boldsymbol{\eta}_i^\top, \end{aligned}$$

and one can deduce that

$$\begin{aligned} \mathbf{x}^* &= \Phi^*(t_0)\mathbf{x}(0) \\ &= \sum_{i=1}^r \boldsymbol{\eta}_i \boldsymbol{\eta}_i^\top \mathbf{x}(0) \\ &= \sum_{i=1}^r (\boldsymbol{\eta}_i^\top \mathbf{x}(0)) \boldsymbol{\eta}_i. \end{aligned}$$

$\square$

*Remark 2.* In the **Theorem 1**, if  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$  exists for any initial state  $\mathbf{x}(t_0)$ , and  $\bigcap_{t \geq 0} \mathbf{null}(L(t)) = \{\mathbf{0}\}$ , then the system (2) achieves the asymptotic stability. Also, when  $\bigcap_{t \geq 0} \mathbf{null}(L(t)) = \{\mathbf{1}_n \otimes I_d\}$ , the average consensus will be achieved for the system (2).

*Remark 3.* In the **Theorem 1**, it is assumed that the switching network  $\mathcal{G}(t)$  is constructed from a finite set of graphs. Here, we shall ask whether or not the conclusion holds if the switching network  $\mathcal{G}(t)$  is constructed from an infinite set of graphs? To see this, let us choose, for instance, the multi-agent system  $\dot{\mathbf{x}}(t) = -\frac{1}{[t+1]^2} L\mathbf{x}(t)$ , where  $L$  is the matrix-valued Laplacian matrix of a time-invariant matrix-weighted network. Now, consider the underlying matrix-weighted switching network corresponding to the Laplacian matrix  $\frac{1}{[t+1]^2} L$ . One can see that  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = e^{-\frac{\pi^2}{6}} L\mathbf{x}(0)$ . Here, the convergence value is not only related to the null space of  $L(t)$ , but also to the other eigenvectors corresponding to the non-zero eigenvalues. However, if we choose the multi-agent system  $\dot{\mathbf{x}}(t) = -[t+1]L\mathbf{x}(t)$  and  $L$  is the same as the above example. Let  $\mathbf{null}(L) = \mathbf{span}\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m\}$ , where  $\boldsymbol{\xi}_i \in \mathbb{R}^{dn}$  satisfies

$$\boldsymbol{\xi}_i^\top \boldsymbol{\xi}_j = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}, \forall i, j \in \underline{m}.$$

Then,  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \sum_{i=1}^m (\boldsymbol{\xi}_i^\top \mathbf{x}(0)) \boldsymbol{\xi}_i$ . Therefore, the conclusion in **Theorem 1** does not always hold if the switching network  $\mathcal{G}(t)$  is constructed from an infinite set of graphs.

Then, we shall proceed to examine quantitative characterization of cluster consensus achieved on matrix-weighted switching networks. In particular, we are intended to establish quantitative connection between the cluster consensus value and specific properties of  $L(t)$ . To

this end, we introduce the notion of matrix-weighted integral network; this notion proves crucial in our subsequent analysis. In the following discussions, we also assume that the weight matrix associated with  $(i, j) \in \mathcal{E}(t)$  satisfies that either  $\mathbf{sgn}(A_{ij}(t)) \geq 0$  or  $\mathbf{sgn}(A_{ij}(t)) \leq 0$  for  $t \geq 0$ .

**Definition 2.** Pan et al. [17] Let  $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A(t))$  be a matrix-weighted switching network. The matrix-weighted integral network of  $\mathcal{G}(t)$  over time span  $[t_1, t_2] \subseteq [0, \infty)$  is defined as  $\tilde{\mathcal{G}}_{[t_1, t_2]} = (\mathcal{V}, \tilde{\mathcal{E}}, \tilde{A})$ , where

$$\tilde{\mathcal{E}} = \left\{ (i, j) \in \mathcal{V} \times \mathcal{V} \mid \int_{t_1}^{t_2} |A_{ij}(t)| dt \succ 0 \text{ or } \int_{t_1}^{t_2} |A_{ij}(t)| dt \succeq 0 \right\},$$

and

$$\tilde{A} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(t) dt.$$

According to **Definition 2**, let  $\tilde{D}$  denote the matrix-valued degree matrix of  $\tilde{\mathcal{G}}_{[t_1, t_2]}$ , that is,

$$\tilde{D} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} D(t) dt.$$

Furthermore, let  $\tilde{L}_{[t_1, t_2]}$  denote the matrix-valued Laplacian of  $\tilde{\mathcal{G}}_{[t_1, t_2]}$ . Thus,

$$\tilde{L}_{[t_1, t_2]} = \tilde{D} - \tilde{A} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} L(t) dt.$$

Under the definition of the integral network of matrix-weighted switching networks, we shall explore connections between the null space of the matrix-valued Laplacian matrices of a sequence of matrix-weighted networks and that of the corresponding integral network, which proves crucial in our subsequent analysis. With reference to **Assumption 1**, we denote  $\mathcal{G}(t)$  on dwell time  $t \in [t_k, t_{k+1})$  as  $\mathcal{G}_{[t_k, t_{k+1})}(t) = \mathcal{G}^k$  and denote the associated matrix-valued Laplacian as  $L^k$ , where  $k \in \mathbb{N}$ .

**Lemma 3.** Let  $\mathcal{G}(t)$  be a matrix-weighted switching network satisfying **Assumption 1**. Then

$$\mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]}) = \bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1}),$$

where  $k' < k'' \in \mathbb{N}$ .

*Proof.* On the one hand, we shall prove that  $\mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]}) \subseteq \bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1})$ , i.e., for any

$\eta \in \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})$ , one has  $\eta \in \mathbf{null}(L^{k' + i - 1})$  for all  $i \in \underline{k'' - k'}$ . Note that  $\eta^\top \tilde{L}_{[t_{k'}, t_{k''}]} \eta = 0$ , implying that,

$$\begin{aligned} & \eta^\top \tilde{L}_{[t_{k'}, t_{k''}]} \eta \\ &= \eta^\top \left( \frac{1}{t_{k''} - t_{k'}} \int_{t_{k'}}^{t_{k''}} L(t) dt \right) \eta \\ &= \frac{1}{t_{k''} - t_{k'}} \sum_{i=1}^{k'' - k'} \eta^\top L^{k' + i - 1} (t_{k' + i} - t_{k' + i - 1}) \eta \\ &= \mathbf{0}, \end{aligned}$$

due to the fact that  $L^{k' + i - 1}$  is positive semi-definite or positive definite for all  $i \in \underline{k'' - k'}$ , therefore,  $\eta^\top L^{k' + i - 1} \eta = 0$  for all  $i \in \underline{k'' - k'}$ , one has  $L^{k' + i - 1} \eta = \mathbf{0}$  and  $\eta \in \mathbf{null}(L^{k' + i - 1})$  for all  $i \in \underline{k'' - k'}$ , which would imply,

$$\mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]}) \subseteq \bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1}).$$

On the other hand, we shall prove that  $\bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1}) \subseteq \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})$ , i.e., for any  $\eta \in \bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1})$ , one has  $\eta \in \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})$ .

Considering the quantity  $\eta^\top \tilde{L}_{[t_{k'}, t_{k''}]} \eta$ ,

$$\begin{aligned} & \eta^\top \tilde{L}_{[t_{k'}, t_{k''}]} \eta \\ &= \eta^\top \left( \frac{1}{t_{k''} - t_{k'}} \int_{t_{k'}}^{t_{k''}} L(t) dt \right) \eta \\ &= \frac{1}{t_{k''} - t_{k'}} \sum_{i=1}^{k'' - k'} \eta^\top L^{k' + i - 1} (t_{k' + i} - t_{k' + i - 1}) \eta \\ &= \mathbf{0}, \end{aligned}$$

due to the fact that  $\tilde{L}_{[t_{k'}, t_{k''}]}$  is positive semi-definite or positive definite, therefore,  $\tilde{L}_{[t_{k'}, t_{k''}]} \eta = \mathbf{0}$ , which would imply,

$$\bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1}) \subseteq \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]}) .$$

Thus,

$$\mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]}) = \bigcap_{i \in \underline{k'' - k'}} \mathbf{null}(L^{k' + i - 1}).$$

□

**Lemma 3** indicates that the intersection of the null space of matrix-valued Laplacian matrices associated with a sequence of matrix-weighted networks is equal to the null space of the corresponding integral network. Using this fact, we proceed to explore the sufficient conditions

under which the multi-agent system (2) achieves cluster consensus; these conditions reveal the connection between the steady-state of the multi-agent system (2) and the null space of the related integral network.

Denote the state transition matrix of multi-agent system (2) over time interval  $[t_{k'}, t_{k''}]$  as

$$\Phi(t_{k''}, t_{k'}) = e^{-L^{k''-1} \Delta t_{k''-1}} \dots e^{-L^{k'} \Delta t_{k'}},$$

then  $\mathbf{x}(t_{k''}) = \Phi(t_{k''}, t_{k'}) \mathbf{x}(t_{k'})$ , where  $k' < k'' \in \mathbb{N}$ .

Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{dn}$  be the eigenvalues of the matrix-valued Laplacian matrix  $L$  corresponding to a time-invariant matrix-weighted network. Let  $\dim(\mathbf{null}(L)) = m$ , where  $m \in \underline{dn}$ , namely,

$$0 = \lambda_1 = \dots = \lambda_m \leq \lambda_{m+1} \leq \dots \leq \lambda_{dn}.$$

Denote by  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_{dn}$  as the eigenvalues of  $e^{-Lt}$ ; then  $\beta_i(e^{-Lt}) = e^{-\lambda_i(L)t}$ , i.e.,  $1 = \beta_1 = \dots = \beta_m \geq \beta_{m+1} \geq \dots \geq \beta_{dn}$ . In the meantime, the eigenvector corresponding to the eigenvalue  $\beta_i(e^{-Lt})$  is equal to that corresponding to  $\lambda_i(L)$ . Therefore, let

$$\dim\left(\bigcap_{i \in \underline{k''-k'}} \mathbf{null}(L^{k'+i-1})\right) = m,$$

where  $m \in \underline{dn}$ , then  $\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})$  has at least  $m$  eigenvalues at 1. Let  $\mu_j$  be the eigenvalues of  $\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})$ , where  $j \in \underline{dn}$  such that  $\mu_1 = \dots = \mu_m = 1$  and  $\mu_{m+1} \geq \mu_{m+2} \geq \dots \geq \mu_{dn}$ . Then applying the facts that  $\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'}) \geq 0$  and

$$\begin{aligned} & \max_{j \in \underline{dn}} \mu_j (\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})) \\ &= \|\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})\| \\ &\leq 1, \end{aligned}$$

one has  $\mu_{dn} \leq \dots \leq \mu_{m+2} \leq \mu_{m+1} \leq 1$ . The following lemma thereby provides the relationship between the null space of the matrix-valued Laplacian of  $\tilde{\mathcal{G}}_{[t_{k'}, t_{k''}]}$  and the eigenvalues of  $\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})$ , which is paramount in the subsequent analysis.

**Lemma 4.** *Let  $\mathcal{G}(t)$  be a matrix-weighted switching network satisfying Assumption 1. Let  $\dim(\mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})) = m$ , where  $m \in \underline{dn}$ . Then*

$$\mu_{m+1}(\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})) < 1,$$

where  $k' < k'' \in \mathbb{N}$ .

*Proof.* By contradiction, assume that

$$\mu_{m+1}(\Phi(t_{k''}, t_{k'})^\top \Phi(t_{k''}, t_{k'})) = 1$$

for  $k' < k'' \in \mathbb{N}$ . According to Lemma 5, there exists a non-zero  $\boldsymbol{\eta} \notin \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})$  such that

$$\|\boldsymbol{\eta}\| = \|\Phi(t_{k''}, t_{k'}) \boldsymbol{\eta}\|.$$

Denote  $\boldsymbol{\eta}_{k'} = \boldsymbol{\eta}$  and  $\boldsymbol{\eta}_{k'+i} = e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1}$  for all  $i \in \underline{k''-k'}$ . Moreover,  $\lambda_j(e^{-L^{k'+i-1} \Delta t_{k'+i-1}}) \leq 1$  for all  $j \in \underline{dn}$ , thereby,

$$\|e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1}\| \leq \|\boldsymbol{\eta}_{k'+i-1}\|,$$

and

$$\|\boldsymbol{\eta}\| = \|\boldsymbol{\eta}_{k''}\| \leq \|\boldsymbol{\eta}_{k''-1}\| \leq \dots \leq \|\boldsymbol{\eta}_{k'}\| = \|\boldsymbol{\eta}\|.$$

Hence,

$$\|e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1}\| = \|\boldsymbol{\eta}_{k'+i-1}\|.$$

Then

$$\begin{aligned} & \boldsymbol{\eta}_{k'+i-1}^\top e^{-L^{k'+i-1} \Delta t_{k'+i-1}} e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1} \\ &= \boldsymbol{\eta}_{k'+i-1}^\top \boldsymbol{\eta}_{k'+i-1}. \end{aligned}$$

By Lemma 5,

$$e^{-2L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1} = \boldsymbol{\eta}_{k'+i-1},$$

and thus,

$$L^{k'+i-1} \boldsymbol{\eta}_{k'+i-1} = \mathbf{0},$$

implying that  $\boldsymbol{\eta}_{k'+i-1} \in \mathbf{null}(L^{k'+i-1})$ . Using the fact

$$\begin{aligned} & \|\boldsymbol{\eta}_{k'+i} - \boldsymbol{\eta}_{k'+i-1}\| \\ &= \|e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \boldsymbol{\eta}_{k'+i-1} - \boldsymbol{\eta}_{k'+i-1}\| \\ &= \left\| \sum_{t=1}^{\infty} \frac{1}{t!} (-L^{k'+i-1} \Delta t_{k'+i-1})^t \boldsymbol{\eta}_{k'+i-1} \right\| \\ &= 0, \end{aligned}$$

one can conclude that  $\boldsymbol{\eta}_{k'+i-1} = \boldsymbol{\eta}_{k'+i}$  for all  $i \in \underline{k''-k'}$ , which implies that  $\boldsymbol{\eta} \in \bigcap_{i \in \underline{k''-k'}} \mathbf{null}(L^{k'+i-1})$ , i.e.,  $\boldsymbol{\eta} \in \mathbf{null}(\tilde{L}_{[t_{k'}, t_{k''}]})$ , leading to a contradiction.  $\square$

Based on the above established Lemmas, we shall show the main result of this part using null space analysis of matrix-valued Laplacian related of integral network associated with the switching networks.

**Theorem 2.** *Let  $\mathcal{G}(t)$  be a matrix-weighted switching network satisfying Assumption 1. If there exists a subsequence of  $\{t_k | k \in \mathbb{N}\}$ , denoted by*

$$\{t_{k_l} | t_{k_0} = t_0, \Delta t_{k_l} = t_{k_{l+1}} - t_{k_l} < \infty, l \in \mathbb{N}\},$$

and a scalar  $q \in (0, 1)$ , such that for all  $l \in \mathbb{N}$ ,

$$\mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathbf{null}(\tilde{L}_{[t_{k_{l+1}}, t_{k_{l+2}}]}),$$

and

$$\mu_{m+1}(\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l})) \leq q,$$

where  $m = \dim(\mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}))$ . Then the multi-agent network (2) admits the cluster consensus. Moreover, denote  $\mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathbf{span}\{\xi_1, \dots, \xi_m\}$  for all  $l \in \mathbb{N}$ , where  $\xi_i \in \mathbb{R}^{dn}$  satisfies

$$\xi_i^\top \xi_j = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}, \forall i, j \in \underline{m},$$

then the cluster consensus value is

$$\mathbf{x}^* = \sum_{i=1}^m (\xi_i^\top \mathbf{x}(0)) \xi_i.$$

*Proof.* Construct the error vector  $\omega(t) = \mathbf{x}(t) - \mathbf{x}^*$  which satisfies that  $\dot{\omega}(t) = -L(t)\omega(t)$ . Choose  $\omega(0) \notin \mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]})$  for any  $l \in \mathbb{N}$ , then  $\omega(0)^\top \xi_i = 0$  for all  $i \in \underline{m}$ . Thus,  $\omega(0) \perp \mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]})$  for any  $l \in \mathbb{N}$ . Applying **Lemma 5** yields

$$\mu_{m+1}(\Phi(t_{k_1}, t_{k_0})^\top \Phi(t_{k_1}, t_{k_0})) \geq \frac{\omega(t_{k_1})^\top \omega(t_{k_1})}{\omega(0)^\top \omega(0)},$$

implying that,

$$\|\omega(t_{k_1})\| \leq \mu_{m+1}(\Phi(t_{k_1}, t_{k_0})^\top \Phi(t_{k_1}, t_{k_0}))^{\frac{1}{2}} \|\omega(0)\|.$$

Therefore, for any  $l \in \mathbb{Z}_+$

$$\begin{aligned} & \|\omega(t_{k_{l+1}})\| \\ & \leq \left( \prod_{s=0}^l \mu_{m+1}(\Phi(t_{k_{s+1}}, t_{k_s})^\top \Phi(t_{k_{s+1}}, t_{k_s}))^{\frac{1}{2}} \right) \|\omega(0)\| \\ & \leq q^{\frac{1}{2}(l+1)} \|\omega(0)\|. \end{aligned}$$

Let

$$V(t) = \omega(t)^\top \omega(t) = \|\omega(t)\|^2;$$

then computing the derivative of  $V(t)$  along the trajectories of system  $\dot{\omega}(t) = -L(t)\omega(t)$  yields,

$$\dot{V}(t) = 2\omega(t)^\top (-L(t))\omega(t) \leq 0.$$

Thus

$$\|\omega(t)\| \leq \|\omega(t_{k_l})\| \leq q^{\frac{1}{2}l} \|\omega(0)\|,$$

for any  $t \in [t_{k_l}, t_{k_{l+1}}]$  and  $l \in \mathbb{N}$ . Note that  $0 < q < 1$ , and hence,

$$\lim_{t \rightarrow \infty} \|\omega(t)\| = 0.$$

As such, the multi-agent system (2) achieves cluster consensus and the cluster consensus value is  $\mathbf{x}^* = \sum_{i=1}^m (\xi_i^\top \mathbf{x}(0)) \xi_i$ .  $\square$

*Remark 4.* In the **Theorem 2**, for a matrix-weighted switching network  $\mathcal{G}(t)$ , if there exists a subsequence  $\{t_{h_l} | t_{h_0} = t_0, \Delta t_{h_l} = t_{h_{l+1}} - t_{h_l} < \infty, l \in \mathbb{N}\}$  of  $\{t_k | k \in \mathbb{N}\}$  such that  $\mathbf{null}(\tilde{L}_{[t_{h_l}, t_{h_{l+1}}]}) = \mathbf{null}(\tilde{L}_{[t_{h_{l+1}}, t_{h_{l+2}}]})$  for any  $l \in \mathbb{N}$ , then there does not exist another subsequence

$\{t_{q_l} | t_{q_0} = t_0, \Delta t_{q_l} = t_{q_{l+1}} - t_{q_l} < \infty, l \in \mathbb{N}\}$  of  $\{t_k | k \in \mathbb{N}\}$  such that  $\mathbf{null}(\tilde{L}_{[t_{q_l}, t_{q_{l+1}}]}) = \mathbf{null}(\tilde{L}_{[t_{q_{l+1}}, t_{q_{l+2}}]})$  for any  $l \in \mathbb{N}$  and  $\mathbf{null}(\tilde{L}_{[t_{h_l}, t_{h_{l+1}}]}) \neq \mathbf{null}(\tilde{L}_{[t_{q_l}, t_{q_{l+1}}]})$  for any  $l \in \mathbb{N}$ . We shall illustrate this point by contradiction. Choose one time interval  $[t_m, t_n)$  where  $m < n \in \mathbb{N}$ , such that there exist  $l_0 \in \mathbb{N}$  and  $h_0 \in \mathbb{N}$  satisfying  $[t_{k_{l_0}}, t_{k_{l_0+1}}] \subseteq [t_m, t_n)$  and  $[t_{k_{h_0}}, t_{k_{h_0+1}}] \subseteq [t_m, t_n)$ , then one has  $\mathbf{null}(\tilde{L}_{[t_m, t_n)}) = \mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]})$  for any  $l \in \mathbb{N}$  and  $\mathbf{null}(\tilde{L}_{[t_m, t_n)}) = \mathbf{null}(\tilde{L}_{[t_{k_h}, t_{k_{h+1}}]})$  for any  $h \in \mathbb{N}$ ; however,  $\mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) \neq \mathbf{null}(\tilde{L}_{[t_{k_h}, t_{k_{h+1}}]})$  for any  $l \in \mathbb{N}$  and  $h \in \mathbb{N}$ , which is a contradiction.

*Remark 5.* Consider a special class of switching networks, where  $\mathcal{G}(t)$  is periodic, i.e., there exists a  $T > 0$  such that  $\mathcal{G}(t+T) = \mathcal{G}(t)$  for any  $t \geq 0$ . One can see that it satisfies the condition of **Theorem 2**, therefore, one can apply **Theorem 3** in Trinh et al. [24] on the integral network of  $\mathcal{G}(t)$  over one period to derive the cluster situation for switching networks  $\mathcal{G}(t)$ .

Bipartite consensus is a special case of cluster consensus in the scalar-weighted time-invariant signed networks. Different from the scalar-weighted time-invariant signed networks where the connectivity and the structural balance of the network can completely guarantee the bipartite consensus, for the matrix-weighted time-invariant signed networks, even if the network is unbalanced, there may be a bipartite consensus solution. Recently, authors in Su et al. [20] provide a necessary and sufficient condition for achieving bipartite consensus from an algebraic perspective, that is, the null space of the matrix-valued Laplacian matrix corresponding to the matrix-weighted signed networks is in the form of  $C(1_n \otimes \Psi)$ , where  $\Psi = [\varphi_1, \varphi_2, \dots, \varphi_m]$ ,  $m \in \mathbb{Z}_+$  and  $\varphi_i \in \mathbb{R}^d$ ,  $i \in \underline{m}$ , are mutually perpendicular unit basis vectors,  $C = \mathbf{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\} \in \mathbb{R}^{dn \times dn}$  and  $\sigma_i = I_d$  or  $\sigma_i = -I_d$ . Based on these results, next we shall examine conditions for the bipartite consensus under the matrix-weighted switching networks.

**Corollary 1.** Let  $\mathcal{G}(t)$  be a matrix-weighted switching network satisfying **Assumption 1**; furthermore, suppose there exists a subsequence of  $\{t_k | k \in \mathbb{N}\}$ , denoted by  $\{t_{k_l} | t_{k_0} = t_0, \Delta t_{k_l} = t_{k_{l+1}} - t_{k_l} < \infty, l \in \mathbb{N}\}$ , and a scalar  $q \in (0, 1)$ , such that  $\mathbf{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = C(1_n \otimes \Psi)$  and  $\mu_{m+1}(\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l})) \leq q$  for all  $l \in \mathbb{N}$ , where  $\Psi = [\varphi_1, \varphi_2, \dots, \varphi_m]$ ,  $m \in \mathbb{Z}_+$  and  $\varphi_i \in \mathbb{R}^d$  is the unit basis vector and vertical to each other for all  $i \in \underline{m}$ . Then the multi-agent network (2) admits the bipartite consensus, and the bipartite consensus value is

$$\mathbf{x}^* = C \left( \mathbf{1}_n \otimes \left( \frac{1}{n} \Psi (\mathbf{1}_n^\top \otimes \Psi^\top) C \mathbf{x}(t_0) \right) \right).$$

*Proof.* The process is similar to the proof of **Theorem 2**, thus we omit here.  $\square$

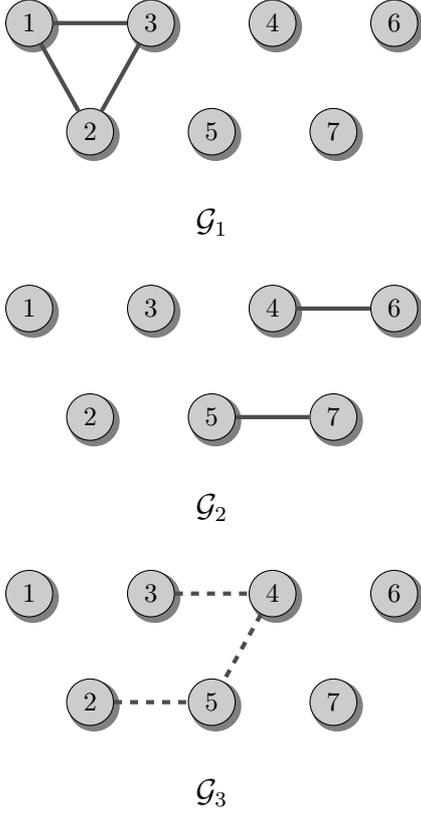


Figure 1: Three matrix-weighted networks  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  and  $\mathcal{G}_3$ . Those edges weighted by positive definite matrices are illustrated by solid lines and edges weighted by positive semi-definite matrices are illustrated by dotted lines.

*Remark 6.* Notably, the number of candidate networks for switching and dwell times in the aforementioned discussions can be infinite, which implies that  $\{\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l}) \mid l \in \mathbb{N}\}$  cannot be generated from a finite set. Therefore, in **Corollary 1**, condition  $\mu_{m+1}(\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l})) \leq q$  is used to ensure bipartite consensus. Subsequently, in order to remove this condition and obtain the analogous graph-theoretic condition for reaching bipartite consensus, we proceed to discuss the case where both the switching networks and the dwell times come from a finite set Cao et al. [5], Ren et al. [18].

**Assumption 3.** In addition to **Assumption 1** and **Assumption 2**, the dwell time  $\Delta t_k = t_{k+1} - t_k$  ( $k \in \mathbb{N}$ ) is chosen from a finite set of arbitrary positive numbers.

**Definition 3** (Simultaneously Structurally Balanced). A matrix-weighted switching network  $\mathcal{G}(t)$  is simultaneously structurally balanced if there exists a time-invariant bipartition of the node set  $\mathcal{V}$ , say  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , such that the matrix weights on the edges within each subset is positive definite or positive semi-definite, but negative definite or negative semi-definite for the edges between the two subsets. A matrix-weighted switching network is simultaneously structurally imbalanced if it is not simultaneously structurally balanced.

On the basis of the above discussions, an analogous graph-theoretic condition by use of simultaneously structurally balance is as follows.

**Corollary 2.** Let  $\mathcal{G}(t)$  be a matrix-weighted simultaneously structurally balanced switching network satisfying **Assumption 3** with a time-invariant node set bipartition  $\mathcal{V}_1$  and  $\mathcal{V}_2$ ; if there exists a subsequence of  $\{t_k \mid k \in \mathbb{N}\}$ , denoted by  $\{t_{k_l} \mid t_{k_0} = t_0, \forall l \in \mathbb{N}\}$ , and  $h > 0$  such that  $\Delta t_{k_l} = t_{k_{l+1}} - t_{k_l} \leq h$  and the integral graph of  $\mathcal{G}(t)$  over time span  $[t_{k_l}, t_{k_{l+1}})$  has a positive-negative spanning tree for all  $l \in \mathbb{N}$ , then the multi-agent network (2) admits the bipartite consensus, and the bipartite consensus value is

$$\mathbf{x}^* = C \left( \mathbf{1}_n \otimes \left( \frac{1}{n} (\mathbf{1}_n^\top \otimes I_d) C \mathbf{x}(0) \right) \right),$$

where  $C = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\} \in \mathbb{R}^{dn \times dn}$  satisfies  $\sigma_i = I_d$  if  $i \in \mathcal{V}_1$  and  $\sigma_i = -I_d$  if  $i \in \mathcal{V}_2$ .

*Proof.* Since  $\mathcal{G}(t)$  is simultaneously structurally balanced, then the integral graph of  $\mathcal{G}(t)$  over time span  $[t_{k_l}, t_{k_{l+1}})$  for any  $l \in \mathbb{N}$  is structurally balanced. In addition, the integral graph of  $\mathcal{G}(t)$  over time span  $[t_{k_l}, t_{k_{l+1}})$  has a positive-negative spanning tree for all  $l \in \mathbb{N}$ . Therefore, according to the **Theorem 2** in Pan et al. [16],  $\text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}})}) = C(\mathbf{1}_n \otimes I_d)$ , one can conclude that  $\mu_{d+1}(\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l})) < 1$  for all  $l \in \mathbb{N}$  by **Lemma 4**. Note that **Assumption 3** ensures that  $\{\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l}) \mid l \in \mathbb{N}\}$  can be generated from a finite set. Now choose

$$q = \max_{l \in \mathbb{N}} \{\mu_{d+1}(\Phi(t_{k_{l+1}}, t_{k_l})^\top \Phi(t_{k_{l+1}}, t_{k_l}))\};$$

hence according to the proof of **Theorem 2**, the multi-agent system (2) admits bipartite consensus.  $\square$

## 5. Simulation Results

Consider a sequence of matrix-weighted networks, consisting of (the same) seven agents, where their interaction networks are  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  and  $\mathcal{G}_3$ , respectively, as shown in Figure 1. Note that  $n = 7$  and  $d = 3$  in this example. The matrix-valued edge weights for each network are,

$$A_{12}(\mathcal{G}_1) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad A_{13}(\mathcal{G}_1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$A_{23}(\mathcal{G}_1) = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad A_{46}(\mathcal{G}_2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$A_{57}(\mathcal{G}_2) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad A_{34}(\mathcal{G}_3) = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix},$$

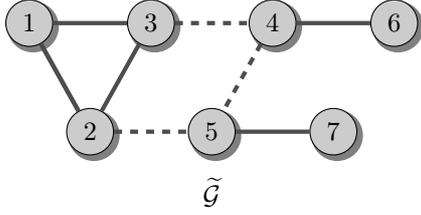


Figure 2: The integral graph of  $\mathcal{G}(t)$  over time span  $[t_{6l}, t_{6(l+1)})$  where  $l \in \mathbb{N}$ .

$$A_{25}(\mathcal{G}_3) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_{45}(\mathcal{G}_3) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 3 \end{bmatrix}.$$

Consider a time sequence  $\{t_k | k \in \mathbb{N}\}$  such that  $t_k = k\Delta t$  where  $\Delta t > 0$ . The coordination process is initiated from network  $\mathcal{G}_1$  (i.e.,  $\mathcal{G}(0) = \mathcal{G}_1$ ) with

$$\mathbf{x}_1(0) = [0.3922, 0.6555, 0.1712]^\top,$$

$$\mathbf{x}_2(0) = [0.7060, 0.0318, 0.5762]^\top,$$

$$\mathbf{x}_3(0) = [0.2688, 0.1592, 0.3266]^\top,$$

$$\mathbf{x}_4(0) = [0.6787, 0.7577, 0.7431]^\top,$$

$$\mathbf{x}_5(0) = [0.3830, 0.6112, 0.1212]^\top,$$

$$\mathbf{x}_6(0) = [0.3555, 0.9712, 0.8060]^\top,$$

and

$$\mathbf{x}_7(0) = [0.1318, 0.7762, 0.3688]^\top.$$

The switching among networks  $\mathcal{G}_1, \mathcal{G}_2$  and  $\mathcal{G}_3$  satisfies,

$$\mathcal{G}(t) = \begin{cases} \mathcal{G}_1, & t \in [t_{6l}, t_{6(l+2)}), \\ \mathcal{G}_2, & t \in [t_{6(l+2)}, t_{6(l+5)}), \\ \mathcal{G}_3, & t \in [t_{6(l+5)}, t_{6(l+1)}), \end{cases} \quad (3)$$

where  $l \in \mathbb{N}$ . The integral graph of  $\mathcal{G}_1, \mathcal{G}_2$  and  $\mathcal{G}_3$  over time span  $[t_{6l}, t_{6(l+1)})$ , where  $l \in \mathbb{N}$ , denoted by  $\tilde{\mathcal{G}}$ , is shown in Figure 2. One can see that it satisfies the conditions in **Theorem 2**, and the multi-agent system (2) on the switching networks admits cluster consensus as shown in Figure 3, which is the same as the system on the integral network; see Figure 4. The cluster conditions associated with the integral network of  $\mathcal{G}(t)$  over one period can therefore be employed to construct that applicable to switching networks  $\mathcal{G}(t)$  over  $[0, \infty)$ .

Consider a variant of the above Example by only changing the matrix weights on edges (3, 4), (2, 5) and (4, 5) into

$$A_{34}(\mathcal{G}_3) = - \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix},$$

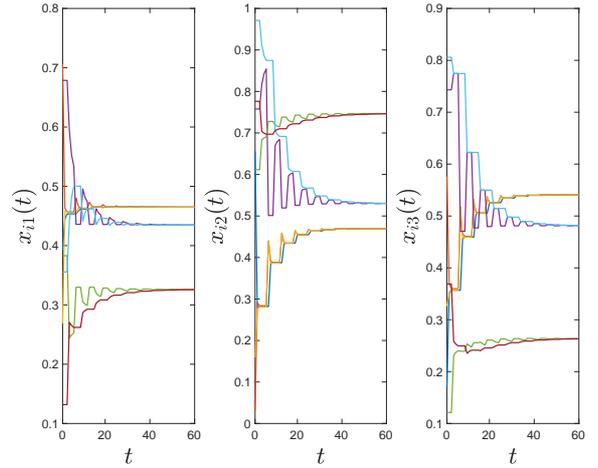


Figure 3: State evolution of the multi-agent system (2) on a sequence of networks in Figure 1 with switching sequences as (3).

$$A_{25}(\mathcal{G}_3) = - \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$A_{45}(\mathcal{G}_3) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 3 \end{bmatrix},$$

respectively. One can see that  $\mathcal{G}(t)$  is simultaneously structurally balanced and the integral graph of  $\mathcal{G}(t)$  over time span  $[t_{6l}, t_{6(l+1)})$ , where  $l \in \mathbb{N}$ , denoted by  $\tilde{\mathcal{G}}$ , has a positive-negative spanning tree  $\mathcal{T}(\tilde{\mathcal{G}})$ . Therefore, according to **Corollary 2**, the multi-agent system (2) admits bipartite consensus; see Figure 6.

## 6. Conclusion

This paper examines cluster consensus problems on matrix-weighted switching networks. For such networks, necessary and/or sufficient conditions for reaching cluster consensus that can be quantitatively characterized are provided. It is shown that if the matrix-weighted switching networks achieve the cluster consensus, then the cluster consensus value belongs to the intersection of the null space of all matrix-valued Laplacians. Furthermore, sufficient conditions for cluster consensus are obtained using the matrix-valued Laplacian of the associated integral network. In particular, conditions for bipartite consensus is further provided under the condition the matrix-weighted switching networks is simultaneously structurally balanced, as well as the explicit expression of convergence state.

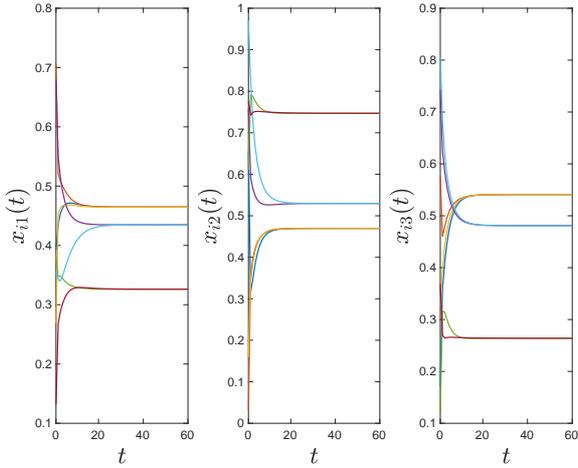


Figure 4: State evolution of the multi-agent system (2) on the integral network in Figure 2.

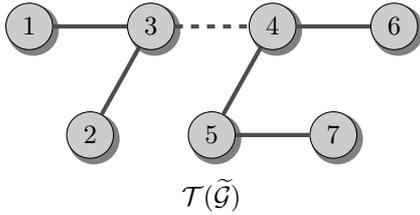


Figure 5: The positive-negative spanning tree  $\mathcal{T}(\tilde{\mathcal{G}})$  of the integral graph  $\mathcal{G}(t)$  over time span  $[t_{6l}, t_{6(l+1)})$  where  $l \in \mathbb{N}$ . Those edges weighted by positive definite matrices are illustrated by solid lines and edges weighted by negative definite matrices are illustrated by dotted lines.

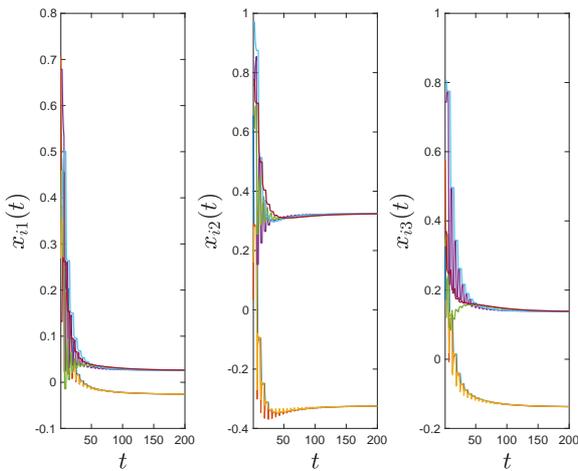


Figure 6: State evolution of the multi-agent system (2) on a sequence of networks in Figure 1 with switching sequences as (3).

## 7. Appendix

**Lemma 5.** *Horn and Johnson [11, p.235] (Rayleigh Theorem) Let  $M \in \mathbb{R}^{n \times n}$  be symmetric with eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$ . Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be corresponding mutually orthonormal vectors such that  $M\mathbf{x}_p = \lambda_p\mathbf{x}_p$ , where  $p \in \underline{n}$ . Then,*

$$\lambda_1 \leq \mathbf{x}^\top M \mathbf{x} \leq \lambda_n$$

for any unit vector  $\mathbf{x} \in \mathbb{R}^n$ , with equality in the right-hand (respectively, left-hand) inequality if and only if  $M\mathbf{x} = \lambda_n\mathbf{x}$  (respectively,  $M\mathbf{x} = \lambda_1\mathbf{x}$ ); moreover,

$$\lambda_n = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top M \mathbf{x}}{\mathbf{x}^\top \mathbf{x}},$$

and

$$\lambda_1 = \min_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top M \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}.$$

We also make the observation that when  $M \in \mathbb{R}^{n \times n}$  is positive semi-definite matrix,  $\mathbf{x}^\top M \mathbf{x} = 0$  if and only if  $M\mathbf{x} = \mathbf{0}$ .

**Lemma 6.** *Su and Huang [21] Let  $\{t_k | k \in \mathbb{N}\}$  be a sequence such that  $t_{k+1} - t_k \geq \alpha > 0$  for all  $k \in \mathbb{N}$  and  $t_0 = 0$ . Suppose  $F(t): [0, \infty) \rightarrow \mathbb{R}$  satisfies*

- 1)  $\lim_{t \rightarrow \infty} F(t)$  exists;
- 2)  $F(t)$  is twice differentiable on each interval  $[t_k, t_{k+1})$ ;
- 3)  $\ddot{F}(t)$  is bounded for  $t \geq 0$ .

Then  $\lim_{t \rightarrow \infty} \dot{F}(t) = 0$ .

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