Advances in Nonlinear Hybrid Stochastic Differential Delay Equations: Existence, Boundedness and Stability ⋆

Junhao Hu [a], Wei Mao [b], Xuerong Mao [c]

[a] *School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan, 430074, China.*

[b] *School of Mathematical Sciences, Jiangsu Second Normal University, Nanjing, 210013, China.*

[c] *Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK.*

Abstract

This paper is concerned with a class of highly nonlinear hybrid stochastic differential delay equations (SDDEs). Different from the most existing papers, the time delay functions in the SDDEs are no longer required to be differentiable, not to mention their derivatives are less than 1. The generalized Hasminskii-type theorems are established for the existence and uniqueness of the global solutions. Comparing with the existing results, we show our new theorems are much more general and can be applied to a much wider class of highly nonlinear SDDEs. Further sufficient conditions are also obtained for the asymptotic boundedness and stability.

Key words: Stochastic differential delay equations, non-differentiable time delay functions, Hasminskii-type theorems, boundedness, stability.

1 Introduction

This paper is concerned with a class of highly nonlinear hybrid stochastic differential delay equations (SDDEs)

$$\begin{aligned} dx(t) &= f(x(t), x(t-\delta_t), r(t), t)dt \\ &\quad + g(x(t), x(t-\delta_t), r(t), t)dB(t) \end{aligned} \tag{1.1}$$

on $t \geq 0$. Here the state $x(t)$ takes values in $\mathbb{R}^d$ and the mode $r(t)$ is a Markov chain taking values in a finite space $\mathbb{S} = \{1, 2, \cdots, N\}$, $B(t)$ is an m-dimensional Brownian motion, $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ are referred to as the drift and diffusion coefficient, respectively, while δ_t is a mapping from $\mathbb{R}_+$ to itself and stands for the time delay at time t. Further details on the notations will be explained in Section 2. The high non-linearity means that the coefficients f and g do not satisfy the linear growth condition (see, e.g., [10,11,16,18]). The SDDEs have been studied by many researchers. For the general theory of hybrid SDDEs, including the stability theory, we refer the reader to, for example, [5,8,9,17,21–24,29]. More references can be found in the book [20].

In general, the time delay is a variable of time in many real-world SDDE models (see, e.g., [6,20,26,30–33]) and that is why a time-varying function δ_t is used to stand for it in the SDDE (1.1). There are already many results on the existence-and-uniqueness theorems and asymptotic properties. For example, the Hasminskii-type theorem was established in [7] (i.e., the cited Theorem 2.9 in this paper), which forms the foundation for several recent papers on stochastic stabilization [3,9,13,15,29]. However, a condition which was frequently imposed in many existing papers is that the delay function δ_t is differentiable with its derivative being bounded by a positive number less than 1 (i.e., Assumption 2.1 below). This condition has been imposed only because of the mathematical technique used—the technique of time change but might not be a natural feature of SDDE models in the real world (see, e.g., [8,19]). For example, piece-wise constant delays (e.g., (1.2)) or sawtooth delays (e.g., (5.16)) occur frequently in sampled-data controls or network-based controls where delays are commonly referred to as fast varying delays (no assumptions on the delay-derivatives) (see, e.g., [6,31]). A simplest example for

⋆ This paper was not presented at any IFAC meeting. This work is entirely theoretical and the results can be reproduced using the methods described in this paper. Corresponding author W. Mao. E-mail: jsjysxx365@126.com

Email addresses: `junhaohu74@163.com` (Junhao Hu), `jsjysxx365@126.com` (Wei Mao), `x.mao@strath.ac.uk` (Xuerong Mao).

This is a peer reviewed, accepted author manuscript of the following research article: Hu, J., Mao, W., & Mao, X. (Accepted/In press - 23 August 2022). Advances in nonlinear hybrid stochastic differential delay equations : existence, boundedness and stability. *Automatica.*

the piece-wise constant delays is the case when the time delay in a network is larger during business hours than other time. Such a time delay can be described by a piecewise constant function

$$\delta_t = \sum_{k=0}^{\infty} \big(d_1 I_{[k,k+1/3)}(t) + d_2 I_{[k+1/3,k+1)}(t) \big), \quad (1.2)$$

where $d_1 > d_2 > 0$ are two numbers, the time unit is one day and $[0,1/3)$ and $[1/3,1)$ stands for the business and no business period per day, respectively. But, even such a simple delay function is not differentiable. These show clearly that there is a need to replace the differentiability condition on the time delay function δ_t with a weaker condition in the study of SDDEs. One of our key aims in this paper is to establish such a weaker condition (namely, Assumption 2.2). We will demonstrate in Section 2 that this proposed weaker condition covers many discontinuous or sawtooth delays. The study becomes more challenged when both coefficients f and g do not satisfy the classical linear growth condition. The key contributions of this paper are:

- There is few result on the SDDE (1.1) when the delay function only satisfies the proposed weaker condition while both coefficients f and g are highly nonlinear. The conditions in the recent paper [3] are still stronger than ours (see the comparisons in Section 2.3).
- The existence-and-uniqueness theorems are established in terms of general Lyapunov functions and are much more general than that in [3]. They will form a foundation for further study of SDDEs when the delay functions are not differentiable.
- The study of the long-time properties of the solutions, including whether the LaSalle-type asymptotic stability (Theorem 4.1) still holds under the proposed weaker conditions, presents a real challenge in mathematics. There is no LaSalle-type result in [3].

2 Existence and Uniqueness

2.1 Notation and assumptions

Throughout this paper, unless otherwise specified, we use the following notation. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. Let $\mathbb{R}_+ = [0,\infty)$. Let A^T denote the transpose of a vector or matrix A. Let $|A| = \sqrt{\text{trace}(A^T A)}$ be the trace norm of a matrix A. For $h > 0$, denote by $C([-h,0];\mathbb{R}^d)$ the family of continuous functions φ from $[-h,0] \to \mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{-h\le u\le 0} |\varphi(u)|$. Denote by $C(\mathbb{R}^d;\mathbb{R}_+)$ the family of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}_+$. If both a, b are real numbers, then $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. If A is a set, denote by I_A its indicator function; that is, $I_A(z) = 1$ if $z \in A$ and 0 otherwise. Moreover, $a := x$ means that denote x by a while $x =: a$ means x is denoted by a.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with its filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the same probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$ under $\mathbb{P}$.

Consider the nonlinear hybrid SDDE (1.1). As pointed out in Section 1, the following differentiability of the delay function δ_t has been imposed in many existing papers.

Assumption 2.1 *The delay function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and its derivative is less than 1. That is*

$$\frac{d\delta_t}{dt} \le \bar{\delta} < 1, \quad \forall t \ge 0.$$

One of our key contributions in this paper is to replace this assumption by a much weaker one.

Assumption 2.2 *Let h_1 be a non-negative constant. The time-varying delay δ_t is a Borel measurable function from $\mathbb{R}_+$ to $[h_1,\infty)$ and has the properties that*

$$-h := \inf_{0\le t<\infty} (t - \delta_t) > -\infty \quad (2.1)$$

and

$$\bar{h} := \limsup_{\Delta\to 0+} \Big(\sup_{s\ge -h} \frac{\mu(M_{s,\Delta})}{\Delta} \Big) < \infty, \quad (2.2)$$

where $M_{s,\Delta} = \{t \in \mathbb{R}_+ : t - \delta_t \in [s, s+\Delta)\}$ and $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_+$.

Remark 2.3 Let us make some useful remarks. We first point out that under Assumption 2.2 we always have $h \ge h_1$ and $\bar{h} \ge 1$. In fact, condition (2.1) implies that $-h \le 0 - \delta_0 \le -h_1$ and hence $h \ge h_1$. But we will explain why $\bar{h} \ge 1$ after the proof of Lemma 7.1 in the Appendix. We next highlight that Lemma 7.2 in the Appendix shows that our new Assumption 2.2 is indeed weaker than Assumption 2.1. Moreover, we are going to demonstrate that many time-varying delay functions in practice satisfy Assumption 2.2 but not Assumption 2.1. Due to the page limit, we only discuss two cases.

Case 1. Consider the left-limited-right-continuous piecewise constant function

$$\delta_t = \sum_{k=0}^{\infty} m_k I_{[t_k, t_{k+1})}(t), \quad t \geq 0, \tag{2.3}$$

where $\{t_k\}_{k\geq 0}$ and $\{m_k\}_{k\geq 0}$ are two sequence of numbers such that $t_0 = 0$, $\inf_{k\geq 0}(t_{k+1} - t_k) > 0$ and

$$0 < h_1 := \inf_{k\geq 0} m_k < \sup_{k\geq 0} m_k =: h_2 < \infty.$$

Letting $\bar{k} = \inf\{k \geq 0 : t_k \geq h_2\}$, we have

$$-h := \inf_{0\leq t<\infty}(t - \delta_t) = \min_{0\leq k\leq \bar{k}}(t_k - m_k) \geq -h_2 > -\infty,$$

namely, (2.1) is satisfied. We claim that $\bar{h}$ defined by (2.2) obeys $\bar{h} \leq [(h_2 - h_1)/\Delta^*] + 2$, in which $\Delta^* = \inf_{k\geq 0}(t_{k+1} - t_k) > 0$ while $[(h_2 - h_1)/\Delta^*]$ is the integer part of $(h_2 - h_1)/\Delta^*$. To show this, let $s \geq -h$ and $\Delta \in (0, \Delta^*)$ be arbitrary. We need only consider the case when $M_{s,\Delta} \neq \emptyset$; otherwise $\mu(M_{s,\Delta}) = 0$. Let $\bar{a} = \inf\{t \in M_{s,\Delta}\}$. It is easy to see that $\bar{a} \in M_{s,\Delta}$. Identify the unique $\hat{k} \geq 0$ such that $t_{\hat{k}} \leq \bar{a} < t_{\hat{k}+1}$. Then $s \leq \bar{a} - \delta_{\bar{a}} < s + \Delta$, which implies that $\bar{a} \geq s + h_1$. Let $n = [(h_2 - h_1)/\Delta^*] + 2$. Then

$$t_{\hat{k}+1+n} \geq t_{\hat{k}+1} + n\Delta^* > s + h_1 + n\Delta^*$$

and, whenever $t \geq t_{\hat{k}+1+n}$,

$$t - \delta_t > s + h_1 + n\Delta^* - h_2 \geq s + \Delta.$$

This shows that $M_{s,\Delta} \subset [t_{\hat{k}}, t_{\hat{k}+1+n})$, whence

$$M_{s,\Delta} = \cup_{k=\hat{k}}^{\hat{k}+n} M_{s,\Delta} \cap [t_k, t_{k+1}).$$

But it is easy to see that $\mu(M_{s,\Delta} \cap [t_k, t_{k+1})) \leq \Delta$. Thus $\mu(M_{s,\Delta}) \leq n\Delta$. As this holds for arbitrary $s \geq -h$ and $\Delta \in (0, \Delta^*)$, we have

$$\bar{h} = \limsup_{\Delta\to 0+}\Big(\sup_{s\geq -h}\frac{\mu(M_{s,\Delta})}{\Delta}\Big) \leq n < \infty.$$

That is, (2.2) holds with $\bar{h} \leq [(h_2 - h_1)/\Delta^*] + 2$. We have therefore shown that the function δ_t defined by (2.3) satisfies Assumption 2.2, but it is not differentiable so cannot satisfy Assumption 2.1.

Case 2. Consider the function δ_t from $\mathbb{R}_+$ to $[h_1, \infty)$ which obeys the Lipschitz condition

$$|\delta_t - \delta_s| \leq \Theta(t - s), \ \forall\, 0 \leq s < t < \infty, \tag{2.4}$$

for some constants $h_1 \geq 0$ and $\Theta \in (0, 1)$. We claim that this function satisfies Assumption 2.2 with $h = \delta_0$ and $\bar{h} \leq 1/(1 - \Theta)$. In fact, it follows from (2.4) that $\delta_t - \delta_0 \leq \Theta t$ for all $t \geq 0$. Hence

$$-h = \inf_{0\leq t<\infty}(t - \delta_t) \geq \inf_{0\leq t<\infty}(t - \Theta t - \delta_0) = -\delta_0,$$

namely $h \leq \delta_0$. On the other hand, $-h \leq -\delta_0$, i.e., $h \geq \delta_0$. We must therefore have $h = \delta_0$. We next let $s \geq 0$ and $\Delta \in (0, 1)$ be arbitrary. Still let $\bar{a} = \inf\{t \in M_{s,\Delta}\}$. Obviously, $\bar{a} \in M_{s,\Delta}$, namely $s \leq \bar{a} - \delta_{\bar{a}} < s + \Delta$. If $t \geq \bar{a} + \Delta/(1 - \Theta)$, then

$$\begin{aligned} t - \delta_t - s &\geq t - \delta_t - (\bar{a} - \delta_a) \geq t - \bar{a} - |\delta_t - \delta_{\bar{a}}| \\ &\geq (1 - \Theta)(t - \bar{a}) \geq \Delta. \end{aligned}$$

and hence $t - \delta_t \geq s + \Delta$, i.e., $t \notin M_{s,\Delta}$. In other words, we have shown $M_{s,\Delta} \subset [\bar{a}, \bar{a} + \Delta/(1 - \Theta))$, whence $\mu(M_{s,\Delta})/\Delta \leq 1/(1 - \Theta)$. As this holds for arbitrary $s \geq -h$ and $\Delta \in (0, 1)$, we see from the definition $\bar{h}$ (i.e., (2.2)) that $\bar{h} \leq 1/(1 - \Theta)$. On the other hand, there are many functions which satisfy (2.4) but are not differentiable so they cannot satisfy Assumption 2.1.

It is time to impose some conditions on the coefficients f and g.

Assumption 2.4 *The coefficients f and g are Borel measurable functions and, for each positive constant a, there is a positive constant K_a such that*

$$\begin{aligned} |f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)|^2 &\vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)|^2 \\ &\leq K_a(|x - \bar{x}|^2 + |y - \bar{y}|^2) \end{aligned}$$

for those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq a$ and all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$. Moreover, $\sup_{(i,t)\in\mathbb{S}\times\mathbb{R}_+}(|f(0, 0, i, t)| \vee |g(0, 0, i, t)|) < \infty$.

To solve the SDDE (1.1), we also need the initial data

$$\begin{cases} \{x(t) : -h \leq t \leq 0\} \\ \qquad = \{\xi(t) : -h \leq t \leq 0\} \in C([-h, 0]; \mathbb{R}^d), \\ r(0) = r_0 \in S. \end{cases} \tag{2.5}$$

But, without additional conditions to Assumptions 2.2 and 2.4, the solution of the hybrid SDDE (1.1) with the initial data (2.5) may explode to infinity at a finite time. To state the additional conditions, we need a few more notations. Let $C(\mathbb{R}^d \times [-h, \infty); \mathbb{R}_+)$ denote the family of all continuous functions from $\mathbb{R}^d \times [-h, \infty)$ to $\mathbb{R}_+$. Denote by $C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ the family of all continuous non-negative functions $V(x, i, t)$ defined on $\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$ such that for each $i \in \mathbb{S}$, they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, we define the function

$LV : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} LV(x,y,i,t) &= V_t(x,i,t) + V_x(x,i,t)f(x,y,i,t) \\ &+ \frac{1}{2}\text{trace}[g^T(x,y,i,t)V_{xx}(x,i,t)g(x,y,i,t)] \\ &+ \sum_{j=1}^N \gamma_{ij}V(x,j,t), \end{aligned}$$

where $V_t = \partial V/\partial t$, $V_x = (\partial V/\partial x_1, \cdots, \partial V/\partial x_d)$, $V_{xx} = \big(\partial^2 V/\partial x_i \partial x_j\big)_{d\times d}$. Let us emphasize that LV is defined on $\mathbb{R}^d \times R^d \times \mathbb{S} \times \mathbb{R}_+$ while V on $\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$. With these notations we can state our another assumption.

Assumption 2.5 *There are three functions $V \in C^{2,1}(\mathbb{R}^d \times S \times \mathbb{R}_+; \mathbb{R}_+)$, $U_1, U_2 \in C(\mathbb{R}^d \times [-h,\infty); \mathbb{R}_+)$, and three positive constants c_j $(1 \le j \le 3)$, such that*

$$\lim_{|x|\to\infty}\Big(\inf_{t\ge 0} U_1(x,t)\Big) = \infty, \tag{2.6}$$

$$U_1(x,t) \le V(x,i,t) \tag{2.7}$$

for $(x,i,t) \in R^n \times S \times \mathbb{R}_+$, and

$$\begin{aligned} LV(x,y,i,t) \le\, & c_1[1 + U_1(x,t) + U_1(y,t-\delta_t)] \\ & - c_2U_2(x,t) + c_3U_2(y,t-\delta_t) \end{aligned} \tag{2.8}$$

for $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

It should be pointed out that there are many SDDEs that satisfy Assumption 2.5. For example, in the paragraph below Theorem 2.10 we will show that if the coefficients of an SDDE satisfy (2.23), then the SDDE satisfies Assumption 2.5.

2.2 Global solution

In this sub-section we will establish two new theorems on the existence and uniqueness of the global solution. The reader will see that $h_1 > 0$ or $h_1 = 0$ makes a significant difference and hence we carefully distinguish them.

Theorem 2.6 *Let Assumptions 2.2, 2.4 and 2.5 hold with $h_1 > 0$. Then the SDDE (1.1) with the initial data (2.5) has a unique global solution $x(t)$ on $[-h,\infty)$ and the solution has the properties that, for all $T > 0$,*

$$\sup_{0\le t\le T} \mathbb{E}U_1(x(t),t) < \infty \tag{2.9}$$

and

$$\mathbb{E}\int_0^T U_2(x(t),t)dt < \infty. \tag{2.10}$$

Proof. Assumptions 2.2 and 2.4 guarantee that the hybrid SDDE (1.1) with the initial data (2.5) has a unique maximal local solution, denoted by $x(t)$ on $[-h, e_\infty)$, where e_∞ is the explosion time (see, e.g., [20]). We need to show $e_\infty = \infty$ a.s. For each integer $k \ge \|\xi\|$, define the stopping time

$$\sigma_k = e_\infty \wedge \inf\{t \in [0, e_\infty) : |x(t)| \ge k\},$$

where throughout this paper we set $\inf\emptyset = \infty$. As σ_k is increasing, it has a limit and we set $\sigma_\infty = \lim_{k\to\infty}\sigma_k$. Obviously, $\sigma_\infty \le e_\infty$ a.s. We divide the whole proof into three steps.

Step 1. Restrict $t \in [0,h_1]$. Noting that $-h \le t-\delta_t \le 0$ we see $x(t-\delta_t) = \xi(t-\delta_t)$ which is already known. By the generalized Itô formula (see, e.g., [20]) and Assumption 2.5, we have

$$\begin{aligned} &\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) - V(\xi(0), r_0, 0) \\ &\le \mathbb{E}\int_0^{t\wedge\sigma_k}\Big(c_1[1 + U_1(x(s),s) + U_1(x(s-\delta_s), s-\delta_s)] \\ &- c_2U_2(x(s),s) + c_3U_2(x(s-\delta_s), s-\delta_s)\Big)ds. \end{aligned} \tag{2.11}$$

This implies

$$\begin{aligned} &\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) + c_2\mathbb{E}\int_0^{t\wedge\sigma_k} U_2(x(s),s)ds \\ &\le \beta_1 + c_1\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(x(s),s)ds, \end{aligned} \tag{2.12}$$

where β_1 is a positive number defined by

$$\begin{aligned} \beta_1 = V(\xi(0), r_0, 0) + \int_0^{h_1}\Big(&c_1U_1(\xi(s-\delta_s), s-\delta_s) \\ &+ c_3U_2(\xi(s-\delta_s), s-\delta_s)\Big)ds + c_1h_1. \end{aligned}$$

In particular, it follows from (2.12) that

$$\begin{aligned} &\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) \\ \le&\beta_1 + c_1\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(x(s),s)ds \\ \le&\beta_1 + c_1\int_0^t \mathbb{E}U_1(x(s\wedge\sigma_k), s\wedge\sigma_k)ds. \end{aligned}$$

An application of the well-known Gronwall inequality yields

$$\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) \le \beta_1 e^{c_1h_1} =: \beta_2 \tag{2.13}$$

for all $t \in [0,h_1]$. Define $\rho_k = \inf_{|x|=k, t\ge 0} U_1(x,t)$. By Assumption 2.5, $\rho_k \to \infty$ as $k \to \infty$. Moreover, it follows from (2.13) that

$$\rho_k\mathbb{P}(\sigma_k \le h_1) \le \mathbb{E}U_1(x(h_1\wedge\sigma_k), h_1\wedge\sigma_k) \le \beta_2.$$

Letting $k \to \infty$ we see that $\mathbb{P}(\sigma_\infty \le h_1) = 0$ and hence $\sigma_\infty \ge h_1$ a.s. We can now letting $k \to \infty$ in (2.13) to obtain

$$\sup_{0\le t\le h_1} \mathbb{E}U_1(x(t),t) \le \beta_2. \tag{2.14}$$

Setting $t = h_1$ in (2.12) and then letting $k \to \infty$ we also get

$$c_2\mathbb{E}\int_0^{h_1} U_2(x(s),s)ds \le \beta_1 + c_1\int_0^{h_1} \mathbb{E}U_1(x(s),s)ds.$$

This, together with (2.14), yields

$$\mathbb{E}\int_0^{h_1} U_2(x(s),s)ds \le (\beta_1 + c_1h_1\beta_2)/c_2 := \beta_3. \tag{2.15}$$

Step 2. Restrict $t \in [0, 2h_1]$. Noting that (2.11) holds for $t \in [0, 2h_1]$ as well, we see from (2.11) that

$$\begin{aligned}&\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) + c_2\mathbb{E}\int_0^{t\wedge\sigma_k} U_2(x(s),s)ds \\ &\le \beta_4 + c_1\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(x(s),s)ds,\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}\beta_4 &= V(\xi(0), r_0, 0) + c_1\mathbb{E}\int_0^{2h_1} U_1(x(s-\delta_s), s-\delta_s)ds \\ &+ c_3\mathbb{E}\int_0^{2h_1} U_2(x(s-\delta_s), s-\delta_s)ds + 2c_1h_1.\end{aligned}$$

We need to show $\beta_4 < \infty$. In Step 1, we showed that up to time h_1, $x(t)$ has properties (2.14) and (2.15). We also observe that $-h \le t - \delta_t \le h_1$ whenever $t \in [0, 2h_1]$. In other words, we already have $x(t-\delta_t)$ from Step 1. By Lemma 7.1,

$$\int_0^{2h_1} U_1(x(s-\delta_s), s-\delta_s)ds \le \bar{h}\int_{-h}^{h_1} U_1(x(s),s)ds.$$

Consequently, using (2.14), we have

$$\begin{aligned}&\mathbb{E}\int_0^{2h_1} U_1(x(s-\delta_s), s-\delta_s)ds \\ \le&\bar{h}\int_{-h}^{0} U_1(\xi(s),s)ds + \bar{h}h_1\beta_2 < \infty.\end{aligned} \tag{2.17}$$

Similarly, using (2.15), we can show

$$\begin{aligned}&\mathbb{E}\int_0^{2h_1} U_2(x(s-\delta_s), s-\delta_s)ds \\ \le&\bar{h}\mathbb{E}\int_{-h}^{0} U_2(\xi(s),s)ds + \bar{h}\beta_3 < \infty.\end{aligned} \tag{2.18}$$

We therefore have $\beta_4 < \infty$. We can then show from (2.16) in the similar fashion as in Step 1 that $\sigma_\infty \ge 2h_1$ a.s., $\sup_{0\le t\le 2h_1} \mathbb{E}U_1(x(t),t) < \infty$ and $\mathbb{E}\int_0^{2h_1} U_2(x(s),s)ds < \infty$.

Step 3. Repeating Step 2 for $t \in [0, 3h_1]$ and then $[0, 4h_1]$ etc., we can show that $\sigma_\infty = \infty$ a.s. and assertions (2.9) and (2.10) hold. The proof is therefore complete. □

Theorem 2.6 requires $h_1 > 0$. This is in general a natural condition as δ_t stands for the time delay and in many practical situations we do have $h_1 > 0$. Nevertheless, there are some situations where $h_1 = 0$, for example, hybrid pantograph SDDEs in which the time delay function $\delta_t = \bar{\delta}t$ on $t \ge 0$ for some constant $\bar{\delta} \in (0,1)$ (see, e.g., [2,7,25]). A natural question is: what may happen if $h_1 = 0$? The following theorem shows that Theorem 2.6 still holds but we need require $c_2 > c_3\bar{h}$.

Theorem 2.7 *Let Assumptions 2.2, 2.4 and 2.5 hold with $h_1 = 0$ and $c_2 > c_3\bar{h}$. Then all the assertions of Theorem 2.6 still hold.*

Proof. We still use the same notations as in the proof of Theorem 2.6. Fix $T > 0$ arbitrarily. Observing that (2.11) holds for all $t \in [0, T]$ and $k \ge \|\xi\|$ and applying Lemma 7.1, we obtain easily from (2.11) that

$$\begin{aligned}&\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) + (c_2 - c_3\bar{h})\mathbb{E}\int_0^{t\wedge\sigma_k} U_2(x(s),s)ds \\ &\le \beta_5 + c_1(1+\bar{h})\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(x(s),s)ds,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\beta_5 &= V(\xi(0), r_0, 0) + c_1T \\ &+ \bar{h}\int_{-h}^{0}[c_1U_1(\xi(s),s) + c_3U_2(\xi(s),s)]ds < \infty.\end{aligned}$$

It then follows from (2.19) that

$$\begin{aligned}&\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) \\ \le&\beta_5 + c_1(1+\bar{h})\int_0^t \mathbb{E}U_1(s\wedge\sigma_k), s\wedge\sigma_k)ds.\end{aligned}$$

An application of the Gronwall inequality yields

$$\mathbb{E}U_1(x(t\wedge\sigma_k), t\wedge\sigma_k) \le \beta_5 e^{c_1(1+\bar{h})T} := \beta_6 \tag{2.20}$$

for $t \in [0, T]$. In the same way as we did in the proof of Theorem 2.6, we can hence show that $\sigma_\infty \ge T$ a.s. Since $T > 0$ is arbitrary, we must have $\sigma_\infty = \infty$ a.s. Letting $k \to \infty$ in (2.20) we obtain assertion (2.9). Setting $t = T$ in (2.19) and then letting $k \to \infty$, we get the other assertion (2.10). The proof is therefore complete. □

2.3 Comparisons

Let us compare our new theorems with some known results to see the advances we have made so far before we develop our further results. The first known result to be compared is the main result in [7]. The assumptions imposed in [7] are Assumption 2.1 and the following one.

Assumption 2.8 *There are three functions $V \in C^{2,1}(R^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ and $U_1, U_2 \in C(R^n \times [-h, \infty); \mathbb{R}_+)$, as well as three positive constants c_j ($1 \le j \le 3$) with $c_2 > c_3/(1-\bar{\delta})$, such that (2.6) holds,*

$$U_1(x,t) \le V(x,i,t) \le U_2(x,t), \tag{2.21}$$

for $(x,i,t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, and

$$LV(x,y,i,t) \le c_1 - c_2 U_2(x,t) + c_3 U_2(y, t-\delta_t) \tag{2.22}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

By Lemma 7.2, under Assumption 2.1, the delay function δ_t satisfies Assumption 2.2 with $h_1 = 0$, $h = \delta_0$ and $\bar{h} \le 1/(1-\bar{\delta})$. We hence set $h = \delta_0$ in the initial data (2.5). We cite the following theorem from [7].

Theorem 2.9 *([7, Theorem 4.3]) If Assumptions 2.1, 2.4 and 2.8 hold, then there is a unique global solution $x(t)$ to the hybrid SDDE (1.1) with any initial data (2.5) (in which $h = \delta_0$).*

This theorem is one of the existing results used frequently. In particular, it forms the foundation for several recent papers [5,13,15,29]. Comparing our new Theorem 2.7 with Theorem 2.9, we see the significant advantages:

- Theorem 2.7 does not require the delay function δ_t to be differentiable. Conditions (2.7), (2.8) and $c_2 > c_3\bar{h}$ in Theorem 2.7 are weaker than conditions (2.21), (2.22) and $c_2 > c_3/(1-\bar{\delta})$ in Theorem 2.9, respectively.

In other words, Theorem 2.7 is much more general than Theorem 2.9.

It is also useful to compare our new Theorems 2.6 and 2.7 with each other. As pointed out before, in many practical situations, we have $h_1 > 0$ as δ_t stands for the time delay. In these situations, Theorem 2.6 is applicable without condition $c_2 > c_3\bar{h}$, which is of course a great advantage.

Let us now compare Theorem 2.6 with a very recent result from [3] which is cited below.

Theorem 2.10 *([3, Theorem 2.4]) Let Assumption 2.4 hold. Assume that δ_t is a Borel measurable function from $\mathbb{R}_+$ to $[h_1, h]$ and satisfies (2.2), where $0 < h_1 < h < \infty$. Assume also that there exist positive constants $p, q, \alpha_1, \alpha_2, \alpha_3$ with $p \wedge q > 2$ such that*

$$\begin{aligned} x^T f(x,y,i,t) &+ \frac{q-1}{2}|g(x,y,i,t)|^2 \\ &\le \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p \end{aligned} \tag{2.23}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$. Then there is a unique global solution $x(t)$ to the hybrid SDDE (1.1) with any initial data (2.5).

Comparing Theorem 2.6 with the one above, we see that Theorem 2.6 allows δ_t to be unbounded while the one above needs δ_t to be bounded. Moreover, letting $V(x,i,t) = |x|^q$ we can easily show from (2.23) that there are three positive constants c_1, c_2, c_3 such that

$$LV(x,y,i,t) \le c_1(|x|^q + |y|^q) - c_2|x|^{q+p-2} + c_3|y|^{q+p-2}$$

for $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$. In other words, Assumption 2.5 holds with $U_1(x,t) = |x|^q$ and $U_2(x,t) = |x|^{q+p-2}$. We hence see that Theorem 2.6 is a generalisation of Theorem 2.10.

3 Boundedness

In the previous section, we have not only established the generalized Hasminskii-type theorems on the existence and uniqueness of the solution to the SDDE (1.1) but also obtained the useful estimates (2.9) and (2.10) on the solution in terms of U_1 and U_2. If we know a bit more on U_1 and U_2, for example, $U_1(x,t) = |x|^q$ and $U_2(x,t) = |x|^{q+p-2}$ as in the end of last section, we then see from (2.9) and (2.10) that

$$\sup_{0 \le t \le T} \mathbb{E}|x(t)|^q < \infty \text{ and } \int_0^T \mathbb{E}|x(t)|^{q+p-2} dt < \infty,$$

respectively, which are the familiar moment estimates. In this section, we will establish more useful estimates on the solution in terms of U_1 or U_2. Please also note that we will no longer distinguish $h_1 = 0$ and $h_1 > 0$ as all the results from now on will hold as long as $h_1 \ge 0$. Moreover, unless otherwise specified, we will fix the initial data (2.5) arbitrarily and will not mention it.

Theorem 3.1 *Let Assumptions 2.2, 2.4 and 2.5 hold except condition (2.8) is replaced by the following stronger condition*

$$LV(x,y,i,t) \le c_1 - c_2 U_2(x,t) + c_3 U_2(y, t-\delta_t) \tag{3.1}$$

for $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$ with $c_2 > c_3\bar{h}$. Then the solution of the SDDE (1.1) has the property

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t EU_2(x(s),s)ds \le \frac{c_1}{c_2 - c_3\bar{h}}. \tag{3.2}$$

Proof. By the generalized Itô formula and condition (3.1), it is straightforward to show that

$$
\begin{aligned}
0 \le \mathbb{E}V(\xi(0), r_0, 0) &+ c_1 t - c_2 \mathbb{E}\int_0^t U_2(x(s), s)ds \\
&+ c_3 \mathbb{E}\int_0^t U_2(x(s-\delta_s), s-\delta_s)ds
\end{aligned}
$$

for $t \ge 0$. Applying Lemma 7.1 we obtain

$$
\begin{aligned}
&(c_2 - c_3\bar{h})\int_0^t \mathbb{E}U_2(x(s), s)ds \\
\le &\mathbb{E}V(\xi(0), r_0, 0) + c_1 t + c_3\bar{h}\mathbb{E}\int_{-h}^0 U_2(\xi(s), s)ds. \quad (3.3)
\end{aligned}
$$

Dividing both sides by t and then letting $t \to \infty$ yields the required assertion (3.2). □

The theorem above holds no matter δ_t is bounded or not. In many practical situations, δ_t is bounded. In this case, we can show a better result.

Theorem 3.2 *In addition to the same conditions imposed in Theorem 3.1, we assume that δ_t is bounded from above by a constant $h_2(> h_1)$ (i.e., $\delta_t \le h_2$ for all $t \ge 0$) and, moreover, there is another positive constant c_4 such that*

$$
V(x, i, t) \le c_4(1 + U_2(x, t)) \quad (3.4)
$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$. Then the solution of the SDDE (1.1) has the property that

$$
\limsup_{t\to\infty} \mathbb{E}U_1(x(t), t) \le \frac{1}{\varepsilon_1}(c_1 + \varepsilon_1 c_4), \quad (3.5)
$$

where $\varepsilon_1 > 0$ is the unique root to the following equation

$$
c_2 - \varepsilon_1 c_4 - \bar{h}c_3 e^{\varepsilon_1 h_2} = 0. \quad (3.6)
$$

Proof. Observe that the right hand side of equation (3.6) is a strictly decreasing continuous function of $\varepsilon_1 \ge 0$ which has value $c_2 - \bar{h}c_3 > 0$ when $\varepsilon_1 = 0$ and tends to $-\infty$ as $\varepsilon_1 \to \infty$. Equation (3.6) must therefore have a unique root $\varepsilon_1 > 0$.

By the generalized Itô formula and conditions (3.1) and (3.4), it is not difficult to show that

$$
\begin{aligned}
&e^{\varepsilon_1 t}\mathbb{E}U_1(x(t), t) - V(\xi(0), r_0, 0) \\
\le &\mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(c_1 + \varepsilon_1 c_4 - (c_2 - \varepsilon_1 c_4)U_2(x(s), s) \\
&+ c_3 U_2(x(s-\delta_s), s-\delta_s)\Big)ds. \quad (3.7)
\end{aligned}
$$

But by Lemma 7.1 and the boundedness of δ_t, we can derive

$$
\begin{aligned}
&\int_0^t e^{\varepsilon_1 s}U_2(x(s-\delta_s), s-\delta_s)ds \\
\le &e^{\varepsilon_1 h_2}\int_0^t e^{\varepsilon_1(s-\delta_s)}U_2(x(s-\delta_s), s-\delta_s)ds \\
\le &\bar{h}e^{\varepsilon_1 h_2}\int_{-h}^t e^{\varepsilon_1 s}U_2(x(s), s)ds.
\end{aligned}
$$

Substituting this into (3.7) and making use of (3.6), we obtain

$$
\begin{aligned}
&e^{\varepsilon_1 t}\mathbb{E}U_1(x(t), t) - \mathbb{E}V(\xi(0), r_0, 0) \\
\le &\frac{1}{\varepsilon_1}(c_1 + \varepsilon_1 c_4)e^{\varepsilon_1 t} + c_3\bar{h}e^{\varepsilon_1 h_2}\int_{-h}^0 e^{\varepsilon_1 s}U_2(\xi(s), s)ds.
\end{aligned}
$$

This implies the required assertion (3.5) immediately. □

In applications, it is easier to verify the following assumption than Assumption 2.5 as there is no need to find the functions V, U_1 and U_2.

Assumption 3.3 *There exist non-negative constants $p, q, \alpha_0, \alpha_1, \alpha_2, \alpha_3$ with $p > 2$, $q \ge 2$ and $\alpha_2 > \alpha_3\bar{h}$ such that*

$$
\begin{aligned}
&x^T f(x, y, i, t) + \frac{q-1}{2}|g(x, y, i, t)|^2 \\
\le &\alpha_0 + \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p
\end{aligned} \quad (3.8)
$$

for all $(x, y, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

Letting $V(x, i, t) = |x|^q$ and using Assumption 3.3, we can easily show that

$$
\begin{aligned}
&LV(x, y, i, t) \\
&\le q|x|^{q-2}\Big(\alpha_0 + \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p\Big)
\end{aligned}
$$

for all $(x, y, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$. By the well-known Young inequality, we can further show that

$$
\begin{aligned}
LV(x, y, i, t) \le &q\alpha_0|x|^{q-2} + 2\alpha_1(q-1)|x|^q + 2\alpha_1|y|^q \\
&- q\Big(\alpha_2 - \frac{\alpha_3(q-2)}{p+q-2}\Big)|x|^{p+q-2} + \frac{pq\alpha_3}{p+q-2}|y|^{p+q-2}.
\end{aligned} \quad (3.9)
$$

Given $\alpha_2 > \alpha_3\bar{h}$ and $\bar{h} \ge 1$, we can find $\varepsilon_2 > 0$ sufficiently small for $c_2 > c_3\bar{h}$, where

$$
c_2 := q\Big(\alpha_2 - \frac{\alpha_3(q-2)}{p+q-2}\Big) - \varepsilon_2 \text{ and } c_3 := \frac{pq\alpha_3}{p+q-2} + \varepsilon_2.
$$

It then follows from (3.9) that

$$\begin{aligned} LV(x,y,i,t) &\le q\alpha_0|x|^{q-2} + 2\alpha_1(q-1)|x|^q + 2\alpha_1|y|^q \\ &- (c_2+\varepsilon_2)|x|^{p+q-2} + (c_3-\varepsilon_2)|y|^{p+q-2} \\ &\le c_1 - c_2|x|^{p+q-2} + c_3|y|^{p+q-2}, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} c_1 &:= \sup_{x\in\mathbb{R}^d}\Big(q\alpha_0|x|^{q-2} + 2\alpha_1(q-1)|x|^q - \varepsilon_2|x|^{p+q-2}\Big) \\ &+ \sup_{y\in\mathbb{R}^d}\Big(2\alpha_1|y|^q - \varepsilon_2|y|^{p+q-2}\Big) < \infty. \end{aligned}$$

In other words, we have shown that (3.1) holds with $U_2(x,t) = |x|^{p+q-2}$. Noting that $|x|^q \le 1 + |x|^{p+q-2}$ for all $x \in \mathbb{R}^d$, we also see that (3.4) holds with $c_4 = 1$. Moreover, letting $U_1(x,t) = |x|^q$, we naturally have (2.6) and (2.7). By Theorems 3.1 and 3.2, we obtain the following useful corollary.

Corollary 3.4 *Let Assumptions 2.2, 2.4 and 3.3 hold. Then the solution of the SDDE (1.1) has the property*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}|x(s)|^{p+q-2}ds < \infty. \tag{3.11}$$

If, moreover, δ_t is bounded, then

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^q < \infty. \tag{3.12}$$

4 Stability

In this section we will discuss the stability of the SDDE (1.1). For this purpose we naturally assume that $f(0,0,i,t)=0$ and $g(0,0,i,t)=0$ for all $(i,t)\in\mathbb{S}\times\mathbb{R}_+$. The SDDE has therefore its equilibrium state 0.

Theorem 4.1 *Let Assumptions 2.2, 2.4 and 2.5 hold except condition (2.8) is replaced by the following stronger condition*

$$LV(x,y,i,t) \le -c_2U_2(x,t) + c_3U_2(y,t-\delta_t) \tag{4.1}$$

for $(x,y,i,t)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}\times\mathbb{R}_+$ with $c_2 > c_3\bar{h}$. Moreover, assume that there is a continuous function $U_3:\mathbb{R}^d\to\mathbb{R}_+$ such that $U_3(x)=0$ if and only if $x=0$ while $U_3(x)\le U_2(x,t)$ for all $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$. Then the solution of the SDDE (1.1) has the property

$$\lim_{t\to\infty}x(t) = 0 \quad a.s. \tag{4.2}$$

Proof. The proof is very technical so is divided into four steps.

Step 1. By condition $U_3(x)\le U_2(x,t)$, we can easily derive from (3.3) (bearing in mind $c_1=0$) that

$$\mathbb{E}\int_0^\infty U_3(x(t))dt < \infty. \tag{4.3}$$

This implies

$$\int_0^\infty U_3(x(t))dt < \infty \quad a.s. \tag{4.4}$$

and

$$\liminf_{t\to\infty}U_3(x(t)) = 0 \quad a.s. \tag{4.5}$$

Step 2. We claim

$$\sup_{0\le t<\infty}|x(t)| < \infty \quad a.s. \tag{4.6}$$

If this were not true, then $\varepsilon_3 := \mathbb{P}(\Omega_1) > 0$, where

$$\Omega_1 = \{\omega\in\Omega : \sup_{0\le t<\infty}|x(t,\omega)| = \infty\}.$$

Set $\beta_7 := V(\xi(0),r_0,0) + c_3\bar{h}\int_{-h}^0 U_2(\xi(s),s)ds$ and choose a number $\beta_8 > 2\beta_7/\varepsilon_3$. Define the stopping time $\sigma = \inf\{t\ge 0 : U_1(x(t),t)\ge\beta_8\}$. Recalling (2.6), we see that $0<\sigma<\infty$ for all $\omega\in\Omega_1$. We can then find a sufficiently large number T such that $\mathbb{P}(0<\sigma\le T)\ge 0.5\varepsilon_3$. On the other hand, in a similar way as we did in the proof of Theorem 2.7, we can show

$$\beta_7 \ge \mathbb{E}U_1(x(T\wedge\sigma),T\wedge\sigma) \ge \beta_8\mathbb{P}(0<\sigma\le T) \ge 0.5\beta_8\varepsilon_3,$$

which is in contradiction with $\beta_8 > 2\beta_7/\varepsilon_3$. We therefore must have (4.6).

Step 3. In this step we are going to show

$$\lim_{t\to\infty}U_3(x(t)) = 0 \quad a.s \tag{4.7}$$

If this were false, we can find a number $\varepsilon_4>0$ such that

$$\mathbb{P}(\Omega_2) \ge 3\varepsilon_4, \tag{4.8}$$

where $\Omega_2 = \{\omega\in\Omega : \limsup_{t\to\infty}U_3(x(t,\omega)) > 2\varepsilon_4\}$. Define a sequence of stopping times:

$$\begin{aligned} \nu_1 &= \inf\{t\ge 0 : U_3(x(t))\ge 2\varepsilon_4\}, \\ \nu_{2k} &= \inf\{t\ge\nu_{2k-1} : U_3(x(t))\le\varepsilon_4\},\ k=1,2,\cdots, \\ \nu_{2k+1} &= \inf\{t\ge\nu_{2k} : U_3(x(t))\ge 2\varepsilon_4\},\ k=1,2,\cdots. \end{aligned}$$

By (4.5) and the definition of Ω_2, we see that $\nu_k(\omega)<\infty$ for all $k\ge 1$ whenever $\omega\in\Omega_2$. Moreover, by (2.5) and (4.6), we can choose a positive number $a = a(\varepsilon_4)$ sufficiently large for

$$\mathbb{P}(\Omega_3) \ge 1 - \varepsilon_4, \tag{4.9}$$

where $\Omega_3 = \{\omega \in \Omega : \sup_{-h \le t < \infty} |x(t,\omega)| < a\}$. Define one more stopping time $\eta_a = \inf\{t \ge 0 : |x(t)| \ge a\}$. Obviously, $\eta_a(\omega) = \infty$ whenever $\omega \in \Omega_3$, while

$$\mathbb{P}(\Omega_2 \cap \Omega_3) \ge 2\varepsilon_4. \tag{4.10}$$

With these notations, we can derive from (4.3) that

$$\begin{aligned}
\infty > & \mathbb{E}\int_0^\infty U_3(x(t))dt \\
\ge & \sum_{k=1}^\infty \mathbb{E}\Big[I_{\{\nu_{2k-1}<\infty,\ \nu_{2k}<\infty,\ \eta_a=\infty\}} \int_{\nu_{2k-1}}^{\nu_{2k}} U_3(x(t))dt\Big] \\
\ge & \varepsilon_4 \sum_{i=1}^\infty \mathbb{E}\Big[I_{\{\nu_{2k-1}<\infty,\ \eta_a=\infty\}}(\nu_{2k} - \nu_{2k-1})\Big],
\end{aligned} \tag{4.11}$$

where we have noted from (4.5) that $\nu_{2k} < \infty$ whenever $\nu_{2k-1} < \infty$. To make our notations more simple, we set $f_t = f(x(t), x(t-\delta_t), r(t), t)$ and $g_t = g(x(t), x(t-\delta_t), r(t), t)$ for $t \ge 0$. In view of Assumption 2.4 and $f(0,0,i,t) = 0$, $g(0,0,i,t) = 0$, we see

$$|f_t|^2 \vee |g_t|^2 \le 2a^2 K_a := \bar{K} \ \text{ if } t \le \eta_a. \tag{4.12}$$

Also set $A_k = \{\eta_a \wedge \nu_{2k-1} < \infty\}$ for $k \ge 1$. By Hölder's inequality, Doob's martingale inequality and (4.12), we derive that, for any $\theta > 0$,

$$\begin{aligned}
& \mathbb{E}\Big[I_{A_k} \sup_{0\le t\le\theta} |x(\eta_a \wedge (\nu_{2k-1}+t)) - x(\eta_a \wedge \nu_{2k-1})|^2\Big] \\
\le & 2\mathbb{E}\Big[I_{A_k} \sup_{0\le t\le\theta} \Big| \int_{\eta_a\wedge\nu_{2k-1}}^{\eta_a\wedge(\nu_{2k-1}+t)} f_s ds\Big|^2\Big] \\
& +2\mathbb{E}\Big[I_{A_k} \sup_{0\le t\le\theta} \Big| \int_{\eta_a\wedge\nu_{2k-1}}^{\eta_a\wedge(\nu_{2k-1}+t)} g_s dB(s)\Big|^2\Big] \\
\le & 2\theta\mathbb{E}\Big[I_{A_k} \int_{\eta_a\wedge\nu_{2k-1}}^{\eta_a\wedge(\nu_{2k-1}+\theta)} |f_s|^2 ds\Big] \\
& +8\mathbb{E}\Big[I_{A_k} \int_{\eta_a\wedge\nu_{2k-1}}^{\eta_a\wedge(\nu_{2k-1}+\theta)} |g_s|^2 ds\Big] \\
\le & 2\bar{K}(\theta+4)\theta.
\end{aligned} \tag{4.13}$$

Given U_3 is uniformly continuous in the closed ball $\bar{S}_a := \{x \in \mathbb{R}^n : |x| \le a\}$, we can find a positive number $b = b(\varepsilon_4) > 0$ so small that

$$|U_3(x) - U_3(y)| < \varepsilon_4 \text{ if } |x-y| < b,\ x,y \in \bar{S}_h. \tag{4.14}$$

We then choose $\theta = \theta(\varepsilon_4, a, b) > 0$ sufficiently small for $2\bar{K}(\theta+4)\theta < b^2\varepsilon_4$. It then follows from (4.13) that

$$\begin{aligned}
& \mathbb{P}\Big(A_k \\
& \cap \Big\{\sup_{0\le t\le\theta} |x(\eta_a \wedge (\nu_{2k-1}+t)) - x(\eta_a \wedge \nu_{2k-1})| \ge b\Big\}\Big) \\
& \le \frac{2\bar{K}(\theta+4)\theta}{b^2} < \varepsilon_4.
\end{aligned}$$

Consequently

$$\begin{aligned}
& \mathbb{P}\Big(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\nu_{2k-1}+t) - x(\nu_{2k-1})| \ge b\Big\}\Big) \\
= & \mathbb{P}\Big(\{\eta_a \wedge \nu_{2k-1} < \infty,\ \eta_a = \infty\} \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\eta_a \wedge (\nu_{2k-1}+t)) - x(\eta_a \wedge \nu_{2k-1})| \ge b\Big\}\Big) \\
\le & \mathbb{P}\Big(A_k \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\eta_a \wedge (\nu_{2k-1}+t)) - x(\eta_a \wedge \nu_{2k-1})| \ge b\Big\}\Big) \\
\le & \varepsilon_4.
\end{aligned}$$

Using (4.10), we further derive

$$\begin{aligned}
& \mathbb{P}\Big(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\nu_{2k-1}+t) - x(\nu_{2k-1})| < b\Big\}\Big) \\
= & \mathbb{P}(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\}) \\
& - \mathbb{P}\Big(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\nu_{2k-1}+t) - x(\nu_{2k-1})| \ge b\Big\}\Big) \\
\ge & \mathbb{P}(\Omega_2 \cap \Omega_3) - \varepsilon_4 \ge \varepsilon_4.
\end{aligned}$$

Set

$$\bar{\Omega}_k = \Big\{\sup_{0\le t\le\theta} |U_3(x(\nu_{2k-1}+t)) - U_3(x(\nu_{2k-1}))| < \varepsilon_4\Big\}.$$

Making use of (4.14), we moreover have

$$\begin{aligned}
& \mathbb{P}\Big(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \cap \bar{\Omega}_k\Big) \\
\ge & \mathbb{P}\Big(\{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \\
& \qquad \cap \Big\{\sup_{0\le t\le\theta} |x(\nu_{2k-1}+t) - x(\nu_{2k-1})| < b\Big\}\Big) \\
\ge & \varepsilon_4.
\end{aligned} \tag{4.15}$$

Observing

$$\nu_{2k}(\omega) - \nu_{2k-1}(\omega) \ge \theta \text{ if } \omega \in \{\nu_{2k-1} < \infty,\ \eta_a = \infty\} \cap \bar{\Omega}_k,$$

we derive from (4.11) and (4.15) that

$$\begin{aligned}\infty >&\varepsilon_4 \sum_{k=1}^{\infty} \mathbb{E}\Big[I_{\{\nu_{2k-1}<\infty,\ \eta_a=\infty\}}(\nu_{2k}-\nu_{2k-1})\Big]\\ \geq&\varepsilon_4 \sum_{k=1}^{\infty} \mathbb{E}\Big[I_{\{\nu_{2k-1}<\infty,\ \eta_a=\infty\}\cap\bar\Omega_k}(\nu_{2k}-\nu_{2k-1})\Big]\\ \geq&\varepsilon_4\theta \sum_{k=1}^{\infty} \mathbb{P}(\{\nu_{2k-1}<\infty,\ \eta_a=\infty\}\cap\bar\Omega_k)\\ \geq&\varepsilon_4\theta\sum_{k=1}^{\infty}\varepsilon_4=\infty,\end{aligned}$$

which is a contradiction. So (4.7) must hold.

Step 4. Finally, we can show assertion (4.2). If this were false, then there is an $\Omega_4\subset\Omega$ with $\mathbb{P}(\Omega_4)>0$ such that

$$\limsup_{t\to\infty}|x(t,\omega)|>0,\quad \forall\omega\in\Omega_4.$$

This, along with (4.6) and (4.7), implies there is some $\bar\omega\in\Omega_4$ such that

$$\sup_{0\le t<\infty}|x(t,\bar\omega)|<\infty \text{ and } \lim_{t\to\infty}U_3(x(t,\bar\omega))=0, \quad (4.16)$$

while there is a subsequence $\{x(t_k,\bar\omega)\}_{k\ge1}$ of $\{x(t,\bar\omega)\}_{t\ge0}$ such that

$$|x(t_k,\bar\omega)|\ge\varepsilon_5,\quad \forall k\ge1,$$

for some $\varepsilon_5>0$. Since $\{x(t_k,\bar\omega)\}_{k\ge1}$ is bounded, we can find its subsequence $\{x(\bar t_k,\bar\omega)\}_{k\ge1}$ which converges to z. Clearly, $|z|\ge\varepsilon_5$ so $U_3(z)>0$. However, by (4.16),

$$U_3(z)=\lim_{k\to\infty}U_3(x(\bar t_k,\bar\omega))=0.$$

In other words, we have a contradiction. The required assertion (4.2) must therefore hold. The proof is complete. □

Theorem 4.1 reveals that the solution will converge to 0 but does not show the convergence rate. With a slightly different conditions we are going to show the exponential convergence rate.

Theorem 4.2 *Let Assumptions 2.2, 2.4 and 2.5 hold except condition (2.8) is replaced by (4.1) with $c_2>c_3\bar h$. Assume also that δ_t is bounded by a constant $h_2(>h_1)$ (i.e., $\delta_t\le h_2$ for all $t\ge0$) and, moreover,*

$$V(x,i,t)\le U_2(x,t) \quad (4.17)$$

for all $(x,i,t)\in\mathbb{R}^d\times\mathbb{S}\times\mathbb{R}_+$. Then the solution of the SDDE (1.1) has the properties that

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}U_1(x(t),t))\le-\varepsilon_6 \quad (4.18)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(U_1(x(t),t))\le-\varepsilon_6 \quad a.s. \quad (4.19)$$

where $\varepsilon_6>0$ is the unique root to the following equation

$$c_2-\varepsilon_6-\bar h c_3 e^{\varepsilon_6 h_2}=0. \quad (4.20)$$

Proof. Obviously, the positive root to equation (4.20) exists and is unique. The first assertion can be proved in the same way as Theorem 3.2 was proved so the details are omitted. To show the second assertion, it is sufficient to show

$$\sup_{0\le t<\infty}e^{\varepsilon_6 t}U_1(x(t),t)<\infty \quad a.s. \quad (4.21)$$

If this were false, then $\varepsilon_7:=\mathbb{P}(\Omega_5)>0$, where

$$\Omega_5=\{\omega\in\Omega:\sup_{0\le t<\infty}e^{\varepsilon_6 t}U_1(x(t,\omega),t)=\infty\}.$$

Set $\beta_9 := V(\xi(0),r_0,0)+c_3\bar h e^{\varepsilon_6 h_2}\int_{-h}^0 U_2(\xi(s),s)ds$. Choose a number $\beta_{10}>2\beta_9/\varepsilon_7$. Define the stopping time $\rho=\inf\{t\ge0: e^{\varepsilon_6 t}U_1(x(t),t)\ge\beta_{10}\}$. Obviously, $0<\rho<\infty$ for all $\omega\in\Omega_5$. We can then find a sufficiently large number T such that $\mathbb{P}(0<\rho\le T)\ge0.5\varepsilon_7$. On the other hand, by the generalized Itô formula and conditions (4.1), (4.17) etc., it is not difficult to show that

$$\begin{aligned}&\mathbb{E}\big(e^{\varepsilon_6(\rho\wedge T)}U_1(x(\rho\wedge T),\rho\wedge T)\big)-\mathbb{E}V(\xi(0),r_0,0)\\ \le&\mathbb{E}\int_0^{\rho\wedge T}e^{\varepsilon_6 s}\Big(-(c_2-\varepsilon_6)U_2(x(s),s)\\ &+c_3U_2(x(s-\delta_s),s-\delta_s)\Big)ds. \qquad (4.22)\end{aligned}$$

But by Lemma 7.1 and the boundedness of δ_t, we can derive

$$\begin{aligned}&\int_0^{\rho\wedge T}e^{\varepsilon_6 s}U_2(x(s-\delta_s),s-\delta_s)ds\\ \le& e^{\varepsilon_6 h_2}\int_0^{\rho\wedge T}e^{\varepsilon_6(s-\delta_s)}U_2(x(s-\delta_s),s-\delta_s)ds\\ \le&\bar h e^{\varepsilon_6 h_2}\int_{-h}^{\rho\wedge T}e^{\varepsilon_6 s}U_2(x(s),s)ds.\end{aligned}$$

Substituting this into (4.22) and making use of (4.20), we obtain

$$\mathbb{E}\big(e^{\varepsilon_6(\rho\wedge T)}U_1(x(\rho\wedge T),\rho\wedge T)\big)\le\beta_9.$$

This implies $0.5\varepsilon_7\beta_{10}\le\beta_9$, which contradicts the fact $\beta_{10}>2\beta_9/\varepsilon_7$. We therefore must have (4.21) and hence the second assertion (4.19) follows. □

To obtain a useful corollary, we impose a new assumption which can be verified more easily in applications.

Assumption 4.3 *There exist six positive constants $p, q, \alpha_1, \cdots, a_4$ with $q \geq 2$, $\alpha_1 > \alpha_2 \bar{h}$, $\alpha_3 > \alpha_4 \bar{h}$ and such that*

$$\begin{aligned} & x^T f(x,y,i,t) + \frac{q-1}{2}|g(x,y,i,t)|^2 \\ \leq & -\alpha_1|x|^2 + \alpha_2|y|^2 - \alpha_3|x|^p + \alpha_4|y|^p \end{aligned} \tag{4.23}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

This looks similar to Assumption 3.3 but is different. In particular, p is positive here but may not bigger than 2. Letting $V(x,i,t) = |x|^q$, we can show in the same way as (3.9) was shown that

$$\begin{aligned} LV(x,y,i,t) \leq & -\bar{\alpha}_1|x|^q + \bar{\alpha}_2|y|^q \\ & - \bar{\alpha}_3|x|^{p+q-2} + \bar{\alpha}_4|y|^{p+q-2}, \end{aligned} \tag{4.24}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where

$$\begin{aligned} \bar{\alpha}_1 &= q\alpha_1 - (q-2)\alpha_2, \quad \bar{\alpha}_2 = 2\alpha_2 \\ \bar{\alpha}_3 &= q\alpha_3 - \frac{q(q-2)\alpha_4}{p+q-2}, \quad \bar{\alpha}_4 = \frac{pq\alpha_4}{p+q-2}. \end{aligned} \tag{4.25}$$

Given that $\alpha_1 > \alpha_2 \bar{h}$, $\alpha_3 > \alpha_4 \bar{h}$ and $\bar{h} \geq 1$, we see $\bar{\alpha}_1 > \bar{\alpha}_2 \bar{h}$ and $\bar{\alpha}_3 > \bar{\alpha}_4 \bar{h}$. Define $U_1(x,t) = |x|^q$ and $U_2(x,t) = \bar{\alpha}_2|x|^q + \bar{\alpha}_4|x|^{p+q-2}$. Then (4.17) holds while

$$\bar{\alpha}_1|x|^q + \bar{\alpha}_3|x|^{p+q-2} \geq c_2 U_2(x,t)$$

where $c_2 = (\bar{\alpha}_1/\bar{\alpha}_2) \wedge (\bar{\alpha}_3/\bar{\alpha}_4)$. It is easy to see $c_2 > \bar{h}$. It then follows from (4.24) that

$$LV(x,y,i,t) \leq -c_2 U_2(x,t) + U_2(y, t-\delta_t). \tag{4.26}$$

This means that (4.1) holds with $c_3 = 1$. The following useful corollary hence follows from Theorem 4.2.

Corollary 4.4 *Let Assumptions 2.2, 2.4 and 4.3 hold. Assume also that δ_t is bounded by a constant $h_2 (> h_1)$. Then the solution of the SDDE (1.1) has the properties that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^q) \leq -\varepsilon_8 \tag{4.27}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \leq -\varepsilon_8/q \quad a.s. \tag{4.28}$$

where $\varepsilon_8 > 0$ is the unique root to the following equation

$$c_2 - \varepsilon_8 - \bar{h} e^{\varepsilon_8 h_2} = 0, \tag{4.29}$$

in which $c_2 = (\bar{\alpha}_1/\bar{\alpha}_2) \wedge (\bar{\alpha}_3/\bar{\alpha}_4)$ and $\bar{\alpha}_1, \cdots, \bar{\alpha}_4$ are defined by (4.25).

5 Examples

Let us discuss two examples in this section to illustrate our theory.

Example 5.1 Many practical systems may experience abrupt changes in their parameters. In this example, we will illustrate how our new theory established in the previous sections can be applied to study such systems. For illustration, we only consider a 2-dimensional hybrid SDDE

$$\begin{aligned} dx(t) = & f(x(t), x(t-\delta_t), r(t))dt \\ & + g(x(t), x(t-\delta_t), r(t))dB(t) \end{aligned} \tag{5.1}$$

on $t \geq 0$. Here $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with its generator $\Gamma = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$, while δ_t is defined by (2.3) and hence satisfies Assumption 2.2 in view of Case 1 in Section 2. Moreover, $f, g : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S} \to \mathbb{R}^2$ are define by

$$f(x,y,1) = (-a_{11}x_1^3 + a_{12}x_1y_2, a_{21}x_2y_1 - a_{22}x_2^3)^T,$$

$$f(x,y,2) = (-b_{11}x_1^3 + b_{12}x_1y_2, b_{21}x_2y_1 - b_{22}x_2^3)^T,$$

$$g(x,y,1) = (a_{13}x_1\cos(y_2), a_{23}x_2\sin(y_1))^T,$$

$$g(x,y,2) = (b_{13}x_1\sin(y_2), b_{23}x_2\cos(y_1))^T,$$

where all parameters a_{11}, b_{11} etc. are positive numbers. This is a simple version of the hybrid SDDE food chain model (see, e.g., [1,16]). The coefficients f and g satisfy Assumption 2.4 obviously. Let $q \geq 2$ be arbitrary. It is almost straightforward to show

$$\begin{aligned} & x^T f(x,y,i,t) + \frac{q-1}{2}|g(x,y,i,t)|^2 \\ & \leq a_1|x|^2|y| - a_2|x|^4 + a_3|y|^2 \\ & \leq (a_3 + 0.5a_1^2/a_2)|y|^2 - 0.5a_2|x|^4, \end{aligned} \tag{5.2}$$

where $a_1 = a_{12} \vee a_{21} \vee b_{12} \vee b_{21}$, $a_2 = 0.5(a_{11} \wedge a_{22} \wedge b_{11} \wedge b_{22})$ and $a_3 = 0.5(q-1)(a_{13} \vee a_{23} \vee b_{13} \vee b_{23})^2$. In other words, Assumption 3.3 is satisfied. By Corollary 3.4, we can hence conclude that for any given initial data $\{x(t) : -h \leq t \leq 0\} \in C([-h, 0]; \mathbb{R}^2)$ and $r(0) = 1$ or 2 (please note h is defined in Case 1 in Section 2), there is a unique solution to the SDDE (5.1) which has the property that $\limsup_{t\to\infty} \mathbb{E}|x(t)|^q < \infty$ for any $q \geq 2$.

Example 5.2 Many practical systems may experience abrupt changes in their structures. For example, a stochastic population system (see, e.g., [1,16,22,23]) may change from a delay geometric Brownian motion in the dry mode to a delay Lotka-Volterra equation in the rain mode; a stochastic financial system (see, e.g., [4,12,27,28]) may switch from a geometric Brownian motion to a constant elasticity of volatility (CEV) process. In this example, we will illustrate how our new theory can be applied to study such type of hybrid systems. To make it more understandable, we consider the SDDE (1.1) with a two-state Markov chain, namely, $r(t)$ there is a Markov chain on the state space $\mathbb{S} = \{1,2\}$ with its generator $\Gamma = \begin{pmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{pmatrix}$, where both γ_{12} and γ_{21} are positive numbers. We assume the coefficients f and g satisfy Assumption 2.4 while the delay function δ_t satisfies Assumption 2.2 and is bounded. Assume in mode 1, the coefficients satisfy

$$2x^T f(x,y,1,t) + 3|g(x,y,1,t)|^2 \le a_{11}|x|^2 + a_{12}|y|^2 \tag{5.3}$$

while in mode 2,

$$\begin{aligned} &2x^T f(x,y,2,t) + |g(x,y,2,t)|^2 \\ \le& a_{21}|x|^2 + a_{22}|y|^2 - a_{23}|x|^4 + a_{24}|y|^4 \end{aligned} \tag{5.4}$$

for $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where $a_{11}, a_{21} \in \mathbb{R}$, $a_{12}, a_{22}, a_{24} \in \mathbb{R}_+$ and $a_{23} > 0$. We need some conditions on these parameters. First of all, we assume that

$$\begin{aligned} &2a_{11} + a_{12} < \gamma_{12},\ a_{23} < \gamma_{21}, \\ &a_{11}a_{21} - a_{11}\gamma_{21} - a_{21}\gamma_{12} > 0. \end{aligned} \tag{5.5}$$

These guarantee that the 2×2-matrix $A := -\mathrm{diag}(a_{11}, a_{21}) - \Gamma$ is non-singular as its determinant $|A| = a_{11}a_{21} - a_{11}\gamma_{21} - a_{21}\gamma_{12} > 0$. Set

$$(\theta_1, \theta_2)^T = A^{-1}(1,1)^T, \tag{5.6}$$

namely

$$\theta_1 = (\gamma_{12} + \gamma_{21} - a_{21})/|A|,\ \theta_2 = (\gamma_{12} + \gamma_{21} - a_{11})/|A|.$$

They are both positive by condition (5.5). To apply Theorem 4.2, we define

$$V(x,i,t) = \begin{cases} \theta_1|x|^2 + \eta|x|^4 & \text{if } i = 1, \\ \theta_2|x|^2 & \text{if } i = 2, \end{cases} \tag{5.7}$$

for $(x,i,t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where η is a positive free parameter to be determined later. Making use of (5.3), (5.4) and (5.6), we can easily show that

$$\begin{aligned} LV(x,y,1,t) \le& -|x|^2 - \eta(\gamma_{12} - 2a_{11} - a_{12})|x|^4 \\ &+ \theta_1 a_{12}|y|^2 + \eta a_{12}|y|^4 \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} LV(x,y,2,t) \le& -|x|^2 - (a_{23} - \eta\gamma_{21})|x|^4 \\ &+ \theta_2 a_{22}|y|^2 + \theta_2 a_{24}|y|^4 \end{aligned} \tag{5.9}$$

for $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$. Condition (5.5) guarantees that there is a unique $\eta > 0$ such that $\eta(\gamma_{12} - 2a_{11} - a_{12}) = a_{23} - \eta\gamma_{21}$, namely

$$\eta = \frac{a_{23}}{\gamma_{12} + \gamma_{21} - 2a_{11} - a_{12}}. \tag{5.10}$$

It then follows from (5.8) and (5.9) that

$$\begin{aligned} &LV(x,y,i,t) \le -|x|^2 - \eta_1|x|^4 \\ &+ (\theta_1 a_{12} \vee \theta_2 a_{22})|y|^2 + (\eta a_{12} \vee \theta_2 a_{24})|y|^4 \end{aligned} \tag{5.11}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where

$$\eta_1 = \frac{a_{23}(\gamma_{12} - 2a_{11} - a_{12})}{\gamma_{12} + \gamma_{21} - 2a_{11} - a_{12}}. \tag{5.12}$$

We now impose additional conditions on the parameters

$$a_{12} < \frac{1}{\bar{h}\theta_1} \wedge \frac{\eta_1}{\bar{h}\eta},\ a_{22} < \frac{1}{\bar{h}\theta_2},\ a_{24} < \frac{\eta_1}{\bar{h}\theta_1}. \tag{5.13}$$

With these conditions, we see from (5.11) that

$$LV(x,y,i,t) \le -|x|^2 - \eta_1|x|^4 + \eta_2(|y|^2 + \eta_1|y|^4), \tag{5.14}$$

where $\eta_2 := (\theta_1 a_{12} \vee \theta_2 a_{22}) \vee [(\eta a_{12} \vee \theta_2 a_{24})/\eta_1] < 1/\bar{h}$. Define $U_1(x) = (\theta_1 \wedge \theta_2)|x|^2$ and $U_2(x) = [\theta_1 \vee \theta_2 \vee (\eta/\eta_1)](|x|^2 + \eta_1|x|^4)$. It is easy to see that $U_1(x) \le V(x,i,t) \le U_2(x)$ while it follows from (5.2) that

$$LV(x,y,i,t) \le -c_2 U_2(x) + c_3 U_2(y), \tag{5.15}$$

where $c_2 = 1/[\theta_1 \vee \theta_2 \vee (\eta/\eta_1)]$ and $c_3 = \eta_2 c_2$. We have $c_2 > \bar{h}c_3$ as $\eta_2 < 1/\bar{h}$. Applying Theorem 4.2, we can conclude that under the conditions specified above the SDDE (1.1) is exponentially stable both in mean square and with probability 1.

We perform a computer simulation for the scalar SDDE (1.1), where the coefficients

$$\begin{aligned} f(x,y,t,1) &= 0.5x, & g(x,y,t,1) &= 0.2y, \\ f(x,y,t,2) &= -2(x+x^3), & g(x,y,t,2) &= 0.6y^2 \end{aligned}$$

for $(x,y,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, $B(t)$ is a scalar Brownian motion, the generator of $r(t)$ is $\Gamma = \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}$, while the delay function has the form

$$\delta_t = \sum_{k=0}^{\infty} \big[(0.1 + 0.05(t-2k))I_{[2k,2k+1)}(t) + (0.15 - 0.05(t-2k-1))I_{[2k+1,2(k+1))}(t)\big]. \quad (5.16)$$

In view of Case 2 in Section 2, we see that δ_t is not only bounded but also satisfies Assumption 2.2 with $h_1 = 0.1$, $h = 0.1$ and $\bar{h} \le 1/(1-0.05) = 1.052632$. It is easy to see that (5.3) and (5.4) hold with $a_{11} = 1$, $a_{12} = 0.12$, $a_{21} = -4$, $a_{22} = 0$, $a_{23} = 4$ and $a_{24} = 0.36$. These parameters satisfy condition (5.5). We can then compute $\theta_1 = 0.8181818$, $\theta_2 = 0.3636364$, $\eta = 1.449275$, $\eta_1 = 2.550725$ and check condition (5.13) is satisfied. Accordingly, the solution with the initial data $x(t) = 1 + \sin(t)$ for $t \in [-0.1, 0]$ and $r(0) = 1$ will tend to 0 exponentially with probability 1 (and in mean square too). The computer simulation (Figure 5.1) using the truncated Euler–Maruyama method (see, e.g., [14]) supports the result.

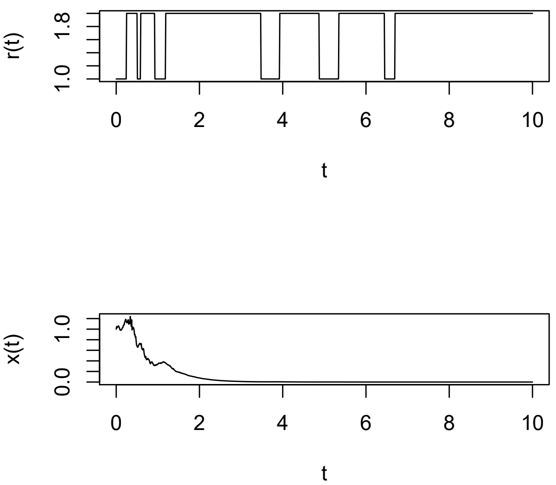


Figure 5.1: Computer simulation of the sample paths of $r(t)$ and $x(t)$ using the truncated Euler–Maruyama method with step size 0.0001.

6 Conclusion

In this paper we have studied a class of highly nonlinear hybrid SDDEs. One of the main advances we have made is that the time delay functions in the SDDEs are no longer required to be differentiable, not to mention their derivatives are less than 1. This is significantly different from the most existing papers, though the very recent paper [3] already tackled this problem initially. We made a comparison between the results in [3] and our new results in Section 2.3 and demonstrated that our new results are much more general. The other main advance we have made is that the coefficients of the hybrid SDDEs are allowed to be highly nonlinear, namely, they do not have to satisfy the classical linear growth condition but satisfy the generalized Hasminskii-type conditions in terms of Lyapunov functions. Comparing with the existing results in Section 2.3, we showed that our new theorems are much more general and can be applied to a much wider class of highly nonlinear SDDEs. We have also established several new theorems on the asymptotic boundedness and stability for the hybrid SDDEs. We have discussed two examples in Section 5 to show that our new theory can be applied to study many practical systems that may experience abrupt changes not only in their parameters but also in structures.

7 Appendix

In this Appendix, we present two useful lemmas which were used before.

Lemma 7.1 *Let Assumption 2.2 hold. Let $\varphi : [-h,\infty) \to \mathbb{R}_+$ be a continuous function. Then for any $T > 0$*

$$\int_0^T \varphi(t-\delta_t)dt \le \bar{h} \int_{-h}^{T-h_1} \varphi(t)dt. \quad (7.1)$$

This lemma was essentially proved in [3]. But our Assumption 2.2 is more general than the corresponding assumption in [3]. We hence give the proof to make our paper more complete.

Proof. Fix $T > 0$ arbitrarily. By Assumption 2.2, for any $\varepsilon > 0$, there is a positive number $\bar{\Delta}$ such that

$$\sup_{s\ge -h} \frac{\mu(M_{s,\Delta})}{\Delta} \le \bar{h} + \varepsilon, \quad \forall \Delta \in (0, \bar{\Delta}). \quad (7.2)$$

Note that $-h \le t - \delta_t \le T - h_1$ for $t \in [0,T]$. Let u be any large integer such that $\Delta := (T - h_1 + h)/u < \bar{\Delta}$. Set $t_v = -h + v\Delta$ for $v = 0, 1, \cdots, u$. By the definition of the Riemann-Lebesgue integral, we have

$$\int_0^T \varphi(t-\delta_t)dt = \lim_{u\to\infty} \sum_{v=0}^{u-1} \mu(M_{t_v,\Delta})\varphi(t_v).$$

But, by (7.2), $\mu(M_{t_v,\Delta}) \le (\bar{h}+\varepsilon)\Delta$. Hence

$$\int_0^T \varphi(t-\delta_t)ds \le \lim_{u\to\infty} \sum_{v=0}^{u-1} (\bar{h}+\varepsilon)\Delta\varphi(t_v) = (\bar{h}+\varepsilon)\int_{-h}^{T-h_1} \varphi(t)dt. \quad (7.3)$$

Letting $\varepsilon \to 0$ yields the required assertion (7.1). □

If we let $\varphi(t) = 1$ for all $t \geq -h$, this lemma shows that $T \leq \bar{h}(T - h_1 + h)$ for any $T > 0$, which implies $\bar{h} \geq \lim_{T\to\infty} T/(T-h_1+h) = 1$. In other words, Assumption 2.2 forces $\bar{h} \geq 1$ inexplicitly. Our next lemma shows that Assumption 2.2 imposed in this paper is weaker than Assumption 2.1 which was used in many existing papers (see, e.g., [2,7,16,25]).

Lemma 7.2 *If δ_t satisfies Assumption 2.1, then it satisfies Assumption 2.2 with $h_1 = 0$, $h = \delta_0$ and $\bar{h} \leq 1/(1 - \bar{\delta})$.*

Proof. If δ_t satisfies Assumption 2.1, then it is easy to deduce that $h_1 = 0$. Let $\eta_t = t - \delta_t$ for $t \geq 0$. Then $d\eta_t/dt = 1 - d\delta_t/dt \geq 1 - \bar{\delta} > 0$. This implies that η_t is a strictly increasing continuous function on $t \in \mathbb{R}_+$ and $\eta_t \to \infty$ as $t \to \infty$. Hence, by the definition of h (i.e., (2.1)),

$$-h = \inf_{0\leq t<\infty}(t - \delta_t) = -\delta_0$$

as required. Moreover, for any $s \geq -h$ and $\Delta > 0$, there is a unique pair of numbers $0 \leq t_1 < t_2 < \infty$ such that $\eta_{t_1} = s$ and $\eta_{t_2} = s + \Delta$ while $\eta_t < s$ for $t < t_1$; $s < \eta_t < s+\Delta$ for $t_1 < t < t_2$ and $\eta_t > s+\Delta$ for $t > t_2$. In other words, we have $M_{s,\Delta} = [t_1, t_2]$. On the other hand,

$$\begin{aligned}\Delta = \eta(t_2) - \eta(t_1) &= \int_{t_1}^{t_2} (d\eta_t/dt)dt \\ &\geq \int_{t_1}^{t_2} (1 - \bar{\delta})dt = (1 - \bar{\delta})(t_2 - t_1).\end{aligned}$$

These imply $\mu(M_{s,\Delta}) = t_2 - t_1 \leq \Delta/(1 - \bar{\delta})$. By the definition of $\bar{h}$ (i.e., (2.2)), we see that

$$\bar{h} = \limsup_{\Delta\to 0+}\Big(\sup_{s\geq -h}\frac{\mu(M_{s,\Delta})}{\Delta}\Big) \leq \frac{1}{1-\bar{\delta}}$$

as required. □

Acknowledgements

We would like to thank the associate editor and reviewers for their useful comments and suggestions. Junhao Hu would like to thank the National Natural Science Foundation of China (61876192), the Fundamental Research Funds for the Central Universities (CZT20020), and Academic Team in Universities (KTZ20051) for their financial support. Wei Mao would like to thank the National Natural Science Foundation of China (11401261), “333 High-level Project” of Jiangsu Province and the Qing Lan Project of Jiangsu Province for their financial support. Xuerong Mao would like to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the Royal Society of Edinburgh (RSE1832), Shanghai Administration of Foreign Experts Affairs (21WZ2503700, the Foreign Expert Program), for their financial support.

References

[1] Bahar, A. and Mao, X., Stochastic delay population dynamics, *J. Int. Appl. Math.* **11(4)** (2004), 377–400.

[2] Baker, C.T.H. and Buckwar, E., Continuous θ-methods for stochastic pantograph equation, *Electron. Trans. Numerical Analysis* **11** (2000), 131–151.

[3] Dong, H. and Mao, X., Advances in stabilization of highly nonlinear hybrid delay systems, *Automatica* **136** (2022), 110086.

[4] Eloe, P., Liu, R., Yatsuki, M., Yin, G. and Zhang, Q., Optimal selling rules in a regime-switching exponential Gaussian diffusion model, *SIAM J. Appl. Math.* **69(3)** (2008), 810–829.

[5] Fei, W., Hu, L., Mao, X. and Shen, M., Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* **82** (2017), 65–70.

[6] Fridman, E., *Introduction to Time-Delay Systems: Analysis and Control*, Birkhauser, 2014.

[7] Hu, L., Mao, X. and Shen, Y., Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Syst. Cont. Lett.* **62** (2013), 178–187.

[8] Huang,L. and Deng, F., Razumikhin-type theorems on stability of neutral stochastic functional differential equations, *IEEE Trans. Automat. Control* **53(7)** (2008), 1718–1723.

[9] Ji, Y. and Chizeck, H.J., Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control* **35** (1990), 777–788.

[10] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, 1986.

[11] Ladde, G.S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, 1980.

[12] Lewis, A.L., *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2000.

[13] Li, X. and Mao, X., Stabilization of highly nonlinear hybrid stochastic differential delay equations by delay feedback control, *Automatica* **112** (2020), 108657, 11 pp.

[14] Li, X., Mao, X. and Yin, G., Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment, and stability, *IMA J. Numer. Anal.* **39(2)** (2019), 847–892.

[15] Lu, Z., Hu, J. and Mao, X., Stabilization by delay feedback control for highly nonlinear hybrid stochastic differential equations, *Discrete Contin. Dyn. Syst. Ser. B.* **24(8)** (2019), 4099–4116.

[16] Mao, X., *Stochastic Differential Equations and Their Applications*, 2nd Edition, Elsevier, 2007.

[17] Mao, X., Stability of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.* **79** (1999), 45–67.

[18] Mao, X., A note on the LaSalle-type theorems for stochastic differential delay equations, *J. Math. Anal. Appl.* **268** (2002), 125–142.

[19] Mao, X., Lam, J., Xu, S. and Gao, H., Razumikhin method and exponential stability of hybrid stochastic delay interval systems, *J. Math. Anal. Appl.* **314** (2006), 45–66.

[20] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[21] Shaikhet, L., Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes* **2(18)** (1996), 180–184.

[22] Wei, F. and Wang, C. Survival analysis of a biomathematical model with fluctuations and migrations between patches, *Appl. Math. Model.* **81** (2020), 113–127.

[23] Wei, F., Jiang, H. and Zhu, Q., Dynamical behaviors of a heroin population model with standard incidence rates between distinct patches, *J. Franklin Inst.-Engin. Appl. Math.* **358(9)** (2021), 4994–5013.

[24] Wu, L., Su, X. and Shi, P., Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems, *Automatica* **48(8)** (2012), 1929–1933.

[25] Xiao, Y., Song, M. and Liu, M., Convergence and stability of the semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equations, *International Journal of Numerical Analysis and Modelling* **8** (2011), 214–225.

[26] Yang, Z., Zhu, E., Xu, Y. and Tan, Y., Razumikhin-type theorems on exponential stability of stochastic functional differential equations with infinite delay, *Acta Appl. Math.* **111(2)** (2010), 219–231.

[27] Yin, G., Liu, R. and Zhang, Q., Recursive algorithms for stock liquidation: a stochastic optimization approach, *SIAM J. Cont. Optim.* **13** (2002), 240–263.

[28] Yin, G. and Zhang, Q., *Continuous-time Markov chains and applications: a singular perturbation approach*, Springer, 2012.

[29] You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q., Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Cont. Optim.* **53(2)** (2015), 905–925.

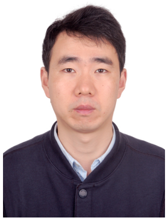
[30] Yue, D. and Han, Q., Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Trans. Automat. Control* **50** (2005), 217–222.

[31] Zhang, J. and Fridman, E., Dynamic event-triggered control of networked stochastic systems with scheduling protocols, *IEEE Trans. Automatic Control*, 2021, DOI: 10.1109/TAC.2021.3061668.

[32] Zhu, E., Wang, Y., Wang, Y. et al., Stability Analysis of Recurrent Neural Networks with Random Delay and Markovian Switching, *J Inequal Appl* **2010**, 191546 (2010).

[33] Zhu, E., Tian, X. and Wang, Y., On pth moment exponential stability of stochastic differential equations with Markovian switching and time-varying delay, *J. Inequal. Appl.* **137** (2015), 1–11.

Junhao Hu received the Ph.D. degree from Control Science and Engineering Department of Huazhong University of Science and Technology, Wuhan, China, in 2007. He was a Post-Doctoral Fellow with the Huazhong University of Science and Technology from 2008 to 2010. He was a Research Associate in the Department of the Mathematics and Statistics, University of Strathclyde, UK from 2011 to 2012. He is now a Professor with the School of Mathematics and Statistics, South-Central University for Nationalities, China. He has published more than 50 international journal papers. His current research interests are in the areas of nonlinear stochastic systems, neural networks.



Wei Mao received the Ph.D. degree from Donghua University, Shanghai, China, in 2020. He was a Research Associate in the Department of Mathematics and Statistics, University of Strathclyde, UK from 2014 to 2015 and then from 2018 to 2019. He is now a Professor with the School of Mathematics and Information Technology, Jiangsu Second Normal University, China. His research interests lie in the theory of stochastic analysis and stochastic differential equations.

Xuerong Mao received the Ph.D. degree from Warwick University, Coventry, U.K., in 1989. He was SERC (Science and Engineering Research Council, U.K.) Post-Doctoral Research Fellow from 1989 to 1992. Moving to Scotland, he joined the University of Strathclyde, Glasgow, U.K., as a Lecturer in 1992, was promoted to Reader in 1995, and was made Professor in 1998 which post he still holds. He was elected as a Fellow of the Royal Society of Edinburgh (FRSE) in 2008. He has authored five books and over 300 research papers. His main research interests lie in the field of stochastic analysis including stochastic stability, stabilization, control, numerical solutions.