Dwell-time stability conditions for infinite dimensional impulsive systems

Sergey Dashkovskiy [a] Vitalii Slynko [a]

[a]*Institute of Mathematics, University of Wuerzburg, Emil-Fischer-Str. 40, 97074 Wuerzburg, Germany*
email: {sergey.dashkovskiy,vitalii.slynko}@mathematik.uni-wuerzburg.de

Abstract

We consider nonlinear impulsive systems on Banach spaces subjected to disturbances and look for dwell-time conditions guaranteeing the the ISS property. In contrary to many existing results our conditions cover the case where both continuous and discrete dynamics can be unstable simultaneously. Lyapunov type methods are use for this purpose. The effectiveness of our approach is illustrated on a rather nontrivial example, which is feedback connection of an ODE and a PDE systems.

Key words: Nonlinear systems, impulsive systems, input-to-state stability, Lyapunov methods.

1 Introduction

Hybrid systems accommodate continuous and discontinuous behavior, which allow to model modern practical processes, where a combination of analog and digital effects takes place as well as other processes where, for example, collisions can change the systems state instantaneously. The general theory of hybrid systems includes results on existence, uniqueness, continuous dependence on initial data, stability and robustness of solutions, see [20,19,10]. Impulsive systems are a particular subclass of hybrid ones, see [15,9], where many stability results were developed, in particular by means of the linear approximations and by the direct Lyapunov method as well as by means of the corresponding comparison principle. Such systems can be seen as a combination of continuous and discrete subsystems. Stability conditions for nonlinear impulsive systems using continuously differential Lyapunov functions were provided in [15]. The proofs are based on the comparison principle and require that either of the continuous and discrete subsystems is stable. More general stability conditions were developed in [9] on the base of Lyapunov functions discontinuous in time. These conditions allow also to establish stability even in the case, when both dynamics are unstable. Moreover in this class of Lyapunov functions these conditions are necessary and sufficient [7]. Practical applications of these results lead to difficul-

ties in contructions of suitable Lyapunov functions. This problem was considered in [16,2] for the case of linear finite dimensional impulsive systems with constant coefficients. Even in this relatively simple case these works need to use lyapunov functions, which depend explicitly on time. In general, a construction of Lyapunov function leads to a rather complex boundary value problem for a systems of matrix differential equations [2].

In contrary to the above mentioned works, we will consider nonlinear infinite dimensional impulsive systems. Our aim is to establish stability conditions by means of smooth Lyapunov functions, which do not depend on time at least in case when the right hand sides of the equations do not depend on time. As well we would like to cover the case, where both continuous and discrete dynamics can be unstable. Our class of systems is essentially more general, than in the above mentioned literature and the existing methods cannot be applied directly for our purposes.

Certain results in this direction exist: The work [11] uses second order and the work [5] uses even higher order derivatives of Lyapunov functions, which leads to additional restrictions on the equations constituting the impulsive system. Different related approaches were developed in [17,1], devoted to linear impulsive systems on Banach spaces containing continuous operators only. Also these results cannot be extended directly to the class of systems considered in this paper.

Furthermore, we will consider systems with disturbances

and hence we will work in the ISS framework. This allows for potential applications in view of studying stability of interconnected systems. Several related results in this framework exist, see [4,14,3], for example. The work [4] provides stability conditions of the dwell-time type for systems in Banach spaces, however it was assumed that either discrete or continuous dynamics possesses the ISS property. Interconnections of impulsive systems, such that not all of them are ISS were studied in [14]. The work [3] studies ISS properties of impulsive systems where jump mappings may depend on time and provides dwell-time conditions establishing the ISS property. However only finite dimensional systems were considered in the last two papers.

The main result of our work extends the results of [4] to the general case, where we do not require that either continuous or discrete dynamics is ISS. To this end we use completely different approach. We split the state space into two subsets $X = G^- \cup G^+$ with int $G^- \cap G^+ = \emptyset$ and such that G^+ is forward invariant. We use two auxiliary functions, one is V of the Lyapunov type the other one is W of Chetaev type. The latter one is used to establish the invariance property of G^+.

The Lyapunov function V decays along the trajectories in G^- but can increase in G^+ and allows to estimate the change of solutions between the jumps. It is assumed that the jumps from the state in G^+ are always stabilizing, and there is no such restriction in G^-. Finally, the ISS property is guaranteed by a dwell-time condition. In the particular case, when both continuous and discrete dynamics are not stable this conditions restricts the jumps frequency both from below and above.

We apply our result to a feedback connection of a linear ODE and a nonlinear PDE of the parabolic type. This illustrates how our approach can be applied and demonstrates its powerfulness. Let us note that this example cannot be handled in view of stability by the existing results because of its nonlinearity, possible instability of both dynamics as well as irregular time instants of jumps.

2 Preliminaries

We will use the following classes of continuous functions, frequently called comparison functions
$\mathcal{P} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \gamma(r) = 0 \Leftrightarrow r = 0\}$,
$\mathcal{K} = \{\gamma \in \mathcal{P} \,|\, \gamma$ is strictly increasing$\}$,
$\mathcal{K}_\infty = \{\gamma \in \mathcal{K} \,|\, \gamma$ is unbounded$\}$,
$\mathcal{K}_\infty^2 = \{\gamma : \mathbb{R}_+^2 \to \mathbb{R}_+ \,|\, \gamma(\cdot, s) \in \mathcal{K}_\infty, \gamma(s, \cdot) \in \mathcal{K}_\infty\}$,
$\mathcal{L} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\,$ decreasing with $\lim_{t \to \infty} \gamma(t) = 0\}$,
$\mathcal{KL} = \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \beta(\cdot, t) \in \mathcal{K}, \beta(s, \cdot) \in \mathcal{L}\}$.
For any $\varrho \in \mathcal{K}$ and any $a, b \in \mathbb{R}_+$ it follows by the monotonicity that

$$\varrho(a + b) \le \varrho(2a) + \varrho(2b). \tag{1}$$

$C[0, l]$ denotes the space of functions continuous on $[0, l]$ with values in $\mathbb{R}$ and the norm $\|f\|_{C[0,l]} = \max_{x \in [0,l]} |f(x)|$, $C^k[0, l]$ stands for the space of k-times continuously differentiable functions normed by $\|f\|_{C^k[0,l]} = \max_{p=0,\ldots,k} \max_{x \in [0,l]} |f^{(p)}(x)|$. $H^0[0, l] = L^2[0, l]$ is the Hilbert space of measurable and square integrable functions with the scalar product $(f, g)_{L^2[0,l]} = \int_0^l f(z) g(z) \, dz$. $\mathfrak{L}(L^2[0, l])$ denotes the Banach algebra of linear bounded operators on $L^2[0, l]$.

The Hilbert space $H^k[0, l] \subset L^2[0, l]$ is a subset of $L^2[0, l]$ of functions f such that $f^{(p)} \in L^2[0, l]$ for , $p = 0, \ldots, k$ and the scalar product defined by

$$(f, g)_{H^k[0,l]} = \sum_{p=0}^k \int_0^l f^{(p)}(z) g^{(p)}(z) \, dz. \tag{2}$$

Recall that $H^k[0, l]$ is the completion of $C^k[0, l]$ with respect to the norm $\|f\|_{H^k[0,l]} = \sqrt{(f, f)_{H^k[0,l]}}$. $C_0^\infty[0, l]$ is the space of infinitely smooth on $[0, l]$ functions vanishing in the vicinity of $x = 0$ and $x = l$. $C_0^\infty([0, T], C_0^\infty[0, l])$ is the set of mappings $f : [0, T] \to C_0^\infty[0, l]$ vanishing in the vicinity of $t = 0$ and $t = T$. Completion of $C_0^\infty[0, l]$ with respect to the norm (2) is denoted by $H_0^k[0, l]$. In the space $H_0^1[0, l]$, due to the inequality of Friedrichs, the norms $\|\cdot\|_{H^1[0,l]}$ and $\|\cdot\|_{H_0^1[0,l]}$ defined by $\|f\|_{H_0^1[0,l]} = \|f_z\|_{L^2[0,l]}^2$, are equivalent. $L^\infty(\mathbb{R}_+)$ is the space of measurable and essentially bounded functions $f : \mathbb{R}_+ \to \mathbb{R}$,

For $M \subset \mathbb{R}$ and a Banach space X by $L^\infty(M, X)$ we denote the space of mappings $f : M \to X$ normed by $\|f\|_{L^\infty} = \sup_{m \in M} \|f(m)\|_X$, where in the particular case $M = \mathbb{Z}_+$ we write $L^\infty(\mathbb{Z}_+, X) = l^\infty(X)$. $B_r(x)$ denotes the open ball centered at $x \in X$ of radius $r > 0$. For Banach spaces X, Y and mappings $f : \mathbb{R} \to X$, $g : \mathbb{R} \to Y$ the Cartesian product $f \times g : \mathbb{R} \to X \times Y$ is defined by $(f \times g)(t) = (f(t), g(t))$. For $\alpha \in (0, 1]$ the space of locally Hölder continuous functions $= f : \mathbb{R}_+ \to X$ is denoted by $H_\alpha(\mathbb{R}_+, X)$ and let $H_{loc}(\mathbb{R}_+, X) := \cup_{\alpha \in (0,1]} H_\alpha(\mathbb{R}_+, X)$.

For a linear bounded operator A defined on a Bacnach space by $\sigma(A)$ we denote its spectrum and by $r_\sigma(A)$ its spectral radius.

$\mathbb{R}^{n \times m}$ denotes the linear space of matrices of the size $n \times m$, where in case $m = n$, the space $\mathbb{R}^{n \times n}$ is a Banach algebra. For $A \in \mathbb{R}^{n \times n}$ we denote $\mathbb{R}^n$: $\|A\| = \sup_{\|x\|=1} \|Ax\| = \lambda_{\max}^{1/2}(A^{\mathrm{T}} A)$. $\mathbb{S}^n$ denotes the set of symmetric matreces of the size n. For $P, Q \in \mathbb{S}^n$ we write $P \succ Q$ if the matrix $P - Q$ is positive definite. For

$A \in \mathbb{S}^n$ by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ we denote the smallest and the largest eigenvalues of A respectively. The inequality $P \leq Q$ should be understood element-wise.

The following well-known inequalities will be used

$$xy \leq \frac{x^{p_1}}{p_1} + \frac{y^{p_2}}{p_2}, \quad x \geq 0, \quad y \geq 0, \quad p_1 \in (0, \infty),$$
$$\frac{1}{p_1} + \frac{1}{p_2} = 1,$$

(3)

and for any $f \in L^{p_1}[0, l]$ and $g \in L^{p_2}[0, l]$ it holds that

$$\left| \int_0^l f(t)g(t)\, dt \right| \leq \left(\int_0^l |f(t)|^{p_1}\, dt \right)^{1/p_1} \left(\int_0^l |g(t)|^{p_2}\, dt \right)^{1/p_2},$$
$$p_1 \in (1, \infty), \quad \frac{1}{p_1} + \frac{1}{p_2} = 1,$$

(4)

known as Young's and Hölder's inequalities, in the particular case $p_1 = p_2 = 2$ they are also known as Cauchy inequalities. For any $f \in H_0^1(0, l)$ also holds

$$\|f\|_{H_0^1(0,l)}^2 \geq \frac{\pi^2}{l^2} \|f\|_{L^2[0,l]}^2,$$

(5)

and if additionally $f \in H_0^1(0, l) \cap H^2(0, l)$, then

$$\|\partial_{zz} f\|_{L^2[0,l]}^2 \geq \frac{\pi^2}{l^2} \|\partial_z f\|_{L^2[0,l]}^2,$$

(6)

3   Stability notions

Let us introduce dynamical systems that we will consider adapting definitions from [18,12,13].

Definition 1 *Let X be the state space with the norm $\|\cdot\|_X$ and $\mathcal{U}_1 \subset \{f : \mathbb{R} \to U_1\}$ be the space of input signals normed by $\|\cdot\|_{\mathcal{U}_1}$ with values in a nonempty subset U_1 of some linear normed space and invariant under the time shifts, that is if $d_1 \in \mathcal{U}_1$ and $\tau \in \mathbb{R}$, then $\mathcal{S}_\tau d_1 \in \mathcal{U}_1$, where $\mathcal{S}_s : \mathcal{U}_1 \to \mathcal{U}_1$, $s \in \mathbb{R}$ is the linear operator defined by $\mathcal{S}_s u(t) = u(t + s)$ and satisfying $\|\mathcal{S}_s\| \leq 1$. It is also assume that for all $u_1, u_2 \in \mathcal{U}$ and any $t \geq t_0$ we have that $u(\tau) := u_1(\tau)$ for $\tau \in [t_0, t]$ and $u(\tau) := u_2(\tau)$ for $\tau > t_0$ it holds that $u \in \mathcal{U}$.*

The triple $\Sigma_c = (X, \mathcal{U}_1, \phi_c)$ is called dynamical system with inputs if the mapping $\phi_c : (t, t_0, x, d_1) \mapsto \phi_c(t, t_0, x, d_1)$ defined for all $(t, t_0, x, d_1) \in [t_0, t_0 + \epsilon_{t_0,x,d_1}) \times \mathbb{R} \times X \times \mathcal{U}_1$ for some positive ϵ_{t_0,x,d_1} and satisfies the following axioms

$(\Sigma_c 1)$ for $t_0 \in \mathbb{R}$, $x \in X$, $d_1 \in \mathcal{U}_1$, $t \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$ the value of $\phi_c(t, t_0, x, d_1)$ is well defined and the mapping

$t \mapsto \phi_c(t, t_0, x, d_1)$ is continuous on $(t_0, t_0 + \epsilon_{t_0,x,d_1})$ with $\lim_{t \to t_0+} \phi_c(t, t_0, x, d_1) = x$;

$(\Sigma_c 2)$ $\phi_c(t, t, x, d_1) = x$ for any $(x, d_1) \in X \times \mathcal{U}_1$, $t \in \mathbb{R}$

$(\Sigma_c 3)$ for any $t_0 \in \mathbb{R}$, $(t, x, d_1) \in [t_0, t_0 + \epsilon_{t_0,x,d_1}) \times X \times \mathcal{U}_1$ and $\tilde{d}_1 \in \mathcal{U}_1$ with $d_1(s) = \tilde{d}_1(s)$ for $s \in [t_0, t]$ it holds that $\phi_c(t, t_0, x, d_1) = \phi_c(t, t_0, x, \tilde{d}_1)$;

$(\Sigma_c 4)$ for any $(x, d_1) \in X \times \mathcal{U}_1$ and $t \geq \tau \geq t_0$ with $\tau \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$, $t \in [\tau, \tau + \epsilon_{\tau,\phi(\tau,t_0,x,d_1),d_1}) \cap [t_0, t_0 + \epsilon_{t_0,x,d_1})$ it holds that

$$\phi_c(t, t_0, x, d_1) = \phi_c(t, \tau, \phi(\tau, t_0, x, d_1), d_1),$$

$(\Sigma_c 5)$ for any $(x, d_1) \in X \times \mathcal{U}_1$ and $t \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$, it holds that

$$\epsilon_{t_0+\tau,x,d_1} = \epsilon_{t_0,x,\mathcal{S}_\tau d_1},$$
$$\phi_c(t + \tau, t_0 + \tau, x, d_1) = \phi_c(t, t_0, x, \mathcal{S}_\tau d_1).$$

Note that $(\Sigma_c 5)$ implies that for all $t \in [\tau, \tau + \epsilon_{\tau,x_0,d_1})$, $\tau \leq t$

$$\phi_c(t, \tau, x, d_1) = \phi_c(t - \tau, 0, x, \mathcal{S}_\tau d_1). \qquad (7)$$

Systems with impulsive actions are defined as follows

Definition 2 *Let $\mathcal{E} = \{\tau_k\}_{k=0}^\infty, \tau_k \in \mathbb{R}$ be a strictly increasing time sequence of impulsive actions with $\lim_{k \to \infty} \tau_k = \infty$. Let $\mathcal{U}_2 \subset \{f : \mathbb{Z}_+ \to \mathcal{U}_2\}$ be the space of input signals normed by $\|\cdot\|_{\mathcal{U}_2}$ and taking values in a nonempty subset U_2 of some linear normed space. Let $g : X \times U_2 \to X$ be a mapping defining impulsive actions and the mapping ϕ be defined for all $(t, t_0, x, d_1, d_2) \in \mathbb{R} \times \mathbb{R} \times X \times \mathcal{U}_1 \times \mathcal{U}_2$, $t \geq t_0$.*

The following data $\Sigma = (X, \Sigma_c, \mathcal{U}_2, g, \phi, \mathcal{E})$ defines a (forward complete) impulsive system if

(Σ_1) for all $(k, x, d_1) \in \mathbb{Z}_+ \times X \times \mathcal{U}_1$ the system Σ_c satisfies

$$\tau_{p(t_0)} - t_0 < \epsilon_{t_0,x,d_1}, \quad T_k := \tau_{k+1} - \tau_k < \epsilon_{\tau_k,x,d_1}$$

where we denote $p(t_0) := \min\{k \in \mathbb{Z}_+ : \tau_k \in \mathcal{E}_{t_0}\}$ with $\mathcal{E}_{t_0} = [t_0, \infty) \cap \mathcal{E}$; and

(Σ_2) the mapping ϕ satisfies

$$\phi(t, t_0, x, d_1, d_2) = \phi_c(t, t_0, x, d_1), \quad \text{for all} \quad t \in [t_0, \tau_{p(t_0)}],$$
$$\phi(t, t_0, x, d_1, d_2) = \phi_c(t, \tau_k, g(\phi(\tau_k, t_0, x, d_1, d_2), d_2(k)), d_1)$$
$$\text{for all} \quad t \in (\tau_k, \tau_{k+1}], \quad k \in \mathbb{Z}_+, k \geq p(t_0).$$

We will denote for short

$$\phi(\tau_k^+, t_0, x, d_1, d_2) = g(\phi(\tau_k, t_0, x, d_1, d_2), d_2(k)),$$
$$k \geq p(t_0), \ \tau_k \geq t_0$$

3

The conditions $(\Sigma_c 1)$ and (Σ_2) imply

$$\lim_{t \to \tau_k+} \phi(t, t_0, x, d_1, d_2) = \phi(\tau_k^+, t_0, x, d_1, d_2),$$
$$\lim_{t \to \tau_k-} \phi(t, t_0, x, d_1, d_2) = \phi(\tau_k, t_0, x, d_1, d_2);$$

and $(\Sigma_c 4)$, $(\Sigma_c 5)$, (Σ_2) imply that for $t \geq \tau \geq t_0$, $(x, d_1, d_2) \in X \times \mathcal{U}_1 \times \mathcal{U}_2$ the following holds

$$\phi(t, t_0, x, d_1, d_2) = \phi(t, \tau, \phi(\tau, t_0, d_1, d_2), d_1, d_2). \quad (8)$$

The system Σ_c describes the continuous dynamics of the impulsive system Σ. One can also consider its discrete dynamics separately as a system Σ_d defined next

Definition 3 *A discrete dynamical system with input $\Sigma_d = (X, g, \phi_d, \mathcal{U}_2)$ is defined by a normed state space $(X, \|\cdot\|_X)$; a space of input signals $\mathcal{U}_2 \subset \{f : \mathbb{Z}_+ \to U_2\}$ with norm $\|\cdot\|_{\mathcal{U}_2}$ and values in a nonempty subset U_2 of a linear normed space; a mapping $g : X \times U_2 \to X$; and a mapping $\phi_d : (k, l, x, d_2) \mapsto \phi_d(k, l, x, d_2)$, for $(k, l, x, d_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times X \times \mathcal{U}_2$, $k \geq l$ such that*

$(\Sigma_d 1)$ $d(k, k, x, d_2) = x$ and
$\phi_d(k + 1, l, x, d_2) = g(\phi_d(k, l, x, d_2), d_2(k))$ for all $k \geq l$.

We assume that Σ satisfies the following

Assumption 1 *There exist $\xi, \xi_\tau \in \mathcal{K}_\infty$, $\tau \in \mathbb{R}_+$ and $\eta_\tau, \eta \in \mathcal{K}_\infty$ such that*

$$\|\phi_c(t, 0, x, d_1)\| \leq \xi_\tau(\|x\|) + \eta_\tau(\|d_1\|_{\mathcal{U}_1}), \quad t \in [0, \tau], \quad (9)$$

where $(x, d_1) \in X \times \mathcal{U}_1$ and

$$\|g(x, d_2)\| \leq \xi(\|x\|) + \eta(\|d_2\|_{\mathcal{U}_2}), \quad (10)$$

where $(x, d_2) \in X \times \mathcal{U}_2$.

Now we define the main stability property of this paper.

Definition 4 *For a fixed time sequence $\mathcal{E}$ of impulsive actions the system Σ is called input-to-state stable (ISS) if there exist functions $\beta_{t_0} \in \mathcal{KL}$, $\gamma_{t_0} \in \mathcal{K}_\infty$, such that for all $x \in X$ and all $(d_1, d_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ it holds that*

$$\|\phi(t, t_0, x, d_1, d_2)\|_X \leq \beta_{t_0}(\|x\|_X, t) + \gamma_{t_0}(d), \quad t \geq t_0 \quad (11)$$

where $d := \max\{\|d_1\|_{\mathcal{U}_1}, \|d_2\|_{\mathcal{U}_2}\}$.

Definition 5 *The Lie derivative of a function $V : X \to \mathbb{R}$ is defined by*

$$\dot{V}(x, \xi) = \lim_{t \to 0+} \frac{1}{t}(V(\phi_c(t, 0, x, \xi)) - V(x)), \quad (x, \xi) \in X \times U_1$$

In many practical particular cases there are simpler expressions for calculation of the Lie derivative possible, see Remark 2.15 in [13].

The aim of our work is to establish conditions guaranteeing the ISS property for nonlinear impulsive systems. Next we define the class of functions that we will use as Lyapunov functions for studying the ISS property.

Definition 6 *A continuous function $V : X \to \mathbb{R}_+$ is called ISS-Lyapunov function if for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ it holds that*

$$\alpha_1(\|x\|_X) \leq V(x) \leq \alpha_2(\|x\|_X), \quad x \in X, \quad (12)$$

and there exists a function $W \in C(X, \mathbb{R})$, $W(0) = 0$ such that the Lie derivatives $\dot{V}(x, \xi)$ and $\dot{W}(x, \xi)$ exist for all $(x, \xi) \in X \times U_1$, the sets

$$G^+ = \{x \in X : W(x) \geq 0\},$$
$$G^- = \{x \in X : W(x) \leq 0\}, \quad (13)$$

are not empty and for some $\chi \in \mathcal{K}_\infty$ and $\varphi_i \in \mathcal{P}$, $\psi_i \in \mathcal{P}$, $i = 1, 2$ so that

$$x \in G^-, \quad \|x\|_X \geq \max\{\chi(\|\xi\|_{U_1}), \chi(\|\mu\|_{U_2})\}$$
$$\Rightarrow \begin{cases} \dot{V}(x, \xi) \leq -\varphi_1(V(x)), \\ V(g(x, \mu)) \leq \psi_1(V(x)) \end{cases} \quad (14)$$

$$x \in G^+, \quad \|x\|_X \geq \max\{\chi(\|\xi\|_{U_1}), \chi(\|\mu\|_{U_2})\}$$
$$\Rightarrow \begin{cases} \dot{V}(x, \xi) \leq \varphi_2(V(x)), \\ V(g(x, \mu)) \leq \psi_2(V(x)) \end{cases} \quad (15)$$

and

$$W(x) = 0, \ \xi \neq 0, \ \|x\| \geq \chi(\|\xi\|) \Rightarrow \dot{W}(x, \xi) > 0. \quad (16)$$

This definition defers from the known ones, as for example in [4], due to the auxiliary function W which is of Chetaev type.

4 Main result

Our main results establish conditions guaranteeing the ISS property of Σ.

Theorem 1 *Let the impulsive system Σ satisfy the Assumption 1 and possesses an ISS-Lyapunov function V, satisfying (12)–(15) such that for some constants θ_1 and θ_2 ($\theta_1 \leq \theta_2$) and $\delta > 0$ for all $a > 0$ holds*

$$\int_a^{\psi_1(a)} \frac{ds}{\varphi_1(s)} \leq \theta_1 - \delta, \quad (17)$$

$$\int\limits_{\psi_2(a)}^{a} \frac{ds}{\varphi_2(s)} \geq \theta_2 + \delta. \qquad (18)$$

Then for any $\mathcal{E}$ such that the dwell-time $T_k = \tau_{k+1} - \tau_k$, $k \in \mathbb{Z}_+$ satisfies $\theta_1 \leq T_k \leq \theta_2$ the system Σ is ISS.

The proof of this theorem is split into several steps. Without loss of generality we assume $t_0 \leq \tau_0$, $p(t_0) = 0$.

Proposition 1 *Let Σ satisfy the Assumtion 1 and for some $\beta_{\tau_0^+} \in \mathcal{KL}$, $\gamma_{\tau_0^+} \in \mathcal{K}_\infty$ its solutions satisfy for all $t > \tau_0$ the inequality*

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \\ \leq \beta_{\tau_0^+}(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X, t) + \gamma_{\tau_0^+}(d), \qquad (19)$$

Then Σ is ISS.

Proof. Let us fix any initial state ϕ_0 and disturbance d_1, d_2. From Σ_2, (7) and (9) it follows that for all $t \in [t_0, \tau_0]$ the next estimate holds

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X = \|\phi_c(t - t_0, 0, \phi_0, \mathcal{S}_{t_0}d_1)\|_X \\ \leq \xi_{\tau_0 - t_0}(\|\phi_0\|_X) + \eta_{\tau_0 - t_0}(d). \qquad (20)$$

Also from (10) and (1) we have the next estimate $\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X \leq \xi(\|\phi(\tau_0, t_0, \phi_0, d_1, d_2)\|_X) + \eta(d) \leq \xi(\xi_{\tau_0 - t_0}(\|\phi_0\|_X) + \eta_{\tau_0 - t_0}(d)) + \eta(d) \leq \widehat{\xi}_{\tau_0 - t_0}(\|\phi_0\|_X) + \widehat{\eta}_{\tau_0 - t_0}(d)$, for some fixed $\widehat{\xi}_{\tau_0 - t_0} \in \mathcal{K}_\infty$, $\widehat{\eta}_{\tau_0 - t_0} \in \mathcal{K}_\infty$.

Finally, from (19) and (1) we have

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\| \leq \beta_{\tau_0^+}(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X, t) \\ + \gamma_{\tau_0^+}(d) \leq \beta_{\tau_0^+}(\widehat{\xi}_{\tau_0 - t_0}(\|\phi_0\|_X) + \widehat{\eta}_{\tau_0 - t_0}(d), t) + \gamma_{\tau_0^+}(d) \\ \leq \widehat{\beta}_{\tau_0^+}(\|\phi_0\|_X, t) + \widetilde{\beta}_{\tau_0^+}(\widehat{\eta}_{\tau_0 - t_0}(d), \tau_0^+) + \gamma_{\tau_0^+}(d) \\ \leq \widehat{\beta}_{\tau_0^+}(\|\phi_0\|_X, t) + \widehat{\gamma}_{\tau_0^+}(d), \quad t > \tau_0 \qquad (21)$$

for some $\widehat{\beta}_{\tau_0^+}$, $\widetilde{\beta}_{\tau_0^+} \in \mathcal{KL}$, $\widehat{\gamma}_{\tau_0^+} \in \mathcal{K}_\infty$. Let us define for $s \geq 0$ and $t \geq t_0$

$$b_{t_0}(s, t) := \begin{cases} \xi_{\tau_0 - t_0}(s), & t \in [t_0, \tau_0], \\ \xi_{\tau_0 - t_0}(s)e^{-t + \tau_0}, & t > \tau_0; \end{cases}$$
$$\beta_{t_0}(s, t) := \max\{b_{t_0}(s, t), \widehat{\beta}_{\tau_0^+}(s, t)\};$$
$$\gamma_{t_0}(s) := \max\{\widehat{\gamma}_{\tau_0^+}(s), \eta_{\tau_0 - t_0}(s)\}.$$

From these definitions follows $\beta_{t_0} \in \mathcal{KL}$, $\gamma_{t_0} \in \mathcal{K}_\infty$ and (20) with (21) imply the ISS property for the impulsive system Σ, which proves the proposition.

Lemma 1 *Let V be an ISS-Lyapunov function of Σ satisfying (12)–(15) with φ_1, φ_2 satisfying (17)-(18). Fix*

any $\phi_0 \in X$, $(d_1, d_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ and $r > \chi(d)$. If $\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \geq r$ for all $t \in (\tau_p, \tau_m]$ with some $m > p$, then

$$F(v(\tau_l^+), v(\tau_p^+)) \geq \delta(l - p), \quad p \leq l \leq m \qquad (22)$$

where $v(t) := V(\phi(t, t_0, \phi_0, d_1, d_2))$ and for any $s > 0$, $q > 0$

$$F(s, q) := \int\limits_{s}^{q} \frac{ds}{\widehat{\varphi}(s)}, \quad \widehat{\varphi}(s) := \min\{\varphi_1(s), \varphi_2(s), s\},$$

so that $F(s, q) \to \infty$ for $s \to 0+$ for any fixed $q > 0$.

Proof. For $l = p$ the inequality (22) is trivially satisfied. For $l = p + 1$ we consider the solution $\phi(t, t_0, \phi_0, d_1, d_2)$ of Σ for $t \in (\tau_p, \tau_{p+1}]$ and either of two possible cases 1) $\phi_0 \in G^+$ and 2) $\phi_0 \in \text{int}(G^-)$, recalling that $G^+ \cup G^- = X$ and $G^+ \cap \text{int}(G^-) = \emptyset$ by definition.

1) For $\phi_0 \in G^+$ we will show that $\phi(t, t_0, \phi_0, d_1, d_2) \in G^+$ for $t \in (\tau_p, \tau_{p+1}]$. Assume, this is not the case, that is there exists

$$\widetilde{t} = \sup\{t \in (\tau_p, \tau_{p+1}] : \phi(t, t_0, \phi_0, d_1, d_2) \in G^+\} \in (\tau_p, \tau_{p+1})$$

such that $\phi(t, t_0, \phi_0, d_1, d_2) \in G^+$, $t \in (\tau_p, \widetilde{t}]$ and $\phi(\widetilde{t}, t_0, \phi_0, d_1, d_2) \in \partial G^+$. Let us denote for short $w(t) := W(\phi(t, t_0, \phi_0, d_1, d_2))$, where $t \mapsto W(\phi(t, t_0, \phi_0, d_1, d_2))$ is absolutely continuous. Hence $w(\widetilde{t}) = 0$. By assumptions of the lemma we have $\|\phi(\widetilde{t}, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$, hence due to (16) and (4.8) from [8] follows $\dot{w}(t) > 0$. This means that for some $\epsilon > 0$ for $t \in (\widetilde{t}, \widetilde{t} + \epsilon)$ it holds that $w(\widetilde{t}) \geq 0$, that is $\phi(\widetilde{t}, t_0, \phi_0, d_1, d_2) \in G^+$ contradicting the choice of $\widetilde{t}$.

This means that $\phi(t, t_0, \phi_0, d_1, d_2) \in G^+$ and $\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$ for $t \in (\tau_p, \tau_{p+1}]$, hence (15) imples

$$\dot{v}(t) \leq \varphi_2(v(t)), \quad t \in (\tau_p, \tau_{p+1}].$$

We calculate

$$\int\limits_{v(\tau_p^+)}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} = \int\limits_{\tau_p}^{\tau_{p+1}} \frac{dv(s)}{\varphi_2(v(s))} \leq \int\limits_{\tau_p}^{\tau_{p+1}} ds = \tau_{p+1} - \tau_p = T_p \leq \theta_2.$$

Setting $a = v(\tau_{p+1})$ in (18) we obtain

$$\int\limits_{\psi_2(v(\tau_{p+1}))}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} \geq \theta_2 + \delta \geq \int\limits_{v(\tau_p^+)}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} + \delta.$$

5

which implies

$$\int\limits_{\psi_2(v(\tau_{p+1}))}^{v(\tau_p^+)} \frac{ds}{\varphi_2(s)} \geq \delta.$$

Due to $\|\phi(\tau_{p+1}, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$ from (18) follows $v(\tau_{p+1}^+) \leq \psi_2(v(\tau_{p+1}))$ which implies

$$\int\limits_{v(\tau_{p+1}^+)}^{v(\tau_p^+)} \frac{ds}{\widehat{\varphi}(s)} \geq \int\limits_{v(\tau_{p+1}^+)}^{v(\tau_p^+)} \frac{ds}{\varphi_2(s)} \geq \delta$$

or equivalently

$$F(v(\tau_{p+1}^+), v(\tau_p^+)) \geq \delta. \qquad (23)$$

2) Now let $\phi_0 \in \text{int } G^-$, then either

(i) $\phi(t, t_0, \phi_0, d_1, d_2) \in \text{int } G^-$ for $t \in (\tau_p, \tau_{p+1}]$ or

(ii) $\phi(\widetilde{t}_1, t_0, \phi_0, d_1, d_2) \in G^+$ for some $\widetilde{t}_1 \in (\tau_p, \tau_{p+1}]$.

In case (i) from $\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$ and (14) it follows that

$$\dot{v}(t) \leq -\varphi_1(v(t)), \quad t \in (\tau_p, \tau_{p+1}].$$

which means

$$\int\limits_{v(\tau_p^+)}^{v(\tau_{p+1})} \frac{ds}{\varphi_1(s)} = \int\limits_{\tau_p}^{\tau_{p+1}} \frac{dv(s)}{\varphi_1(v(s))} \leq -(\tau_{p+1} - \tau_p),$$

and hence

$$\theta_1 \leq T_p = \tau_{p+1} - \tau_p \leq \int\limits_{v(\tau_{p+1})}^{v(\tau_p^+)} \frac{ds}{\varphi_1(s)}.$$

Setting $a = v(\tau_{p+1})$ in (17) we obtain

$$\theta_1 \geq \int\limits_{v(\tau_{p+1})}^{\psi_1(v(\tau_{p+1}))} \frac{ds}{\varphi_1(s)} + \delta.$$

That is

$$\int\limits_{v(\tau_{p+1})}^{v(\tau_p^+)} \frac{ds}{\varphi_1(s)} \geq \int\limits_{v(\tau_{p+1})}^{\psi_1(v(\tau_{p+1}))} \frac{ds}{\varphi_1(s)} + \delta,$$

and

$$\int\limits_{\psi_1(v(\tau_{p+1}))}^{v(\tau_p^+)} \frac{ds}{\varphi_1(s)} \geq \delta.$$

From $\phi(\tau_{p+1}, t_0, \phi_0, d_1, d_2) \in G^-$ and $\|\phi(\tau_{p+1}, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$, follows $v(\tau_{p+1}^+) \leq \psi_1(v(\tau_{p+1}))$, and hence

$$\int\limits_{v(\tau_{p+1}^+)}^{v(\tau_p^+)} \frac{ds}{\widehat{\varphi}(s)} \geq \int\limits_{v(\tau_{p+1}^+)}^{v(\tau_p^+)} \frac{ds}{\varphi_1(s)} \geq \int\limits_{\psi_1(v(\tau_{p+1}))}^{v(\tau_p^+)} \frac{ds}{\varphi_1(s)} \geq \delta$$

or in other words

$$F(v(\tau_{p+1}^+), v(\tau_p^+)) \geq \delta. \qquad (24)$$

In case (ii) we define

$$\widehat{t} = \inf\{t \in (\tau_p, \tau_{p+1}] : \phi(t, \tau_p^+, \phi_0, d_1, d_2) \in \text{int } G^+\},$$

so that $\phi(\widehat{t}, t_0, \phi_0, d_1, d_2) \in \partial G^+ \subset G^+$ and $\widehat{t} > \tau_p$. From the properties of W it follows that $\phi(t, t_0, \phi_0, d_1, d_2) \in \text{int } G^-$ for $t \in (\tau_p, \widehat{t})$ and $\phi(t, t_0, \phi_0, d_1, d_2) \in G^+$ for $t \in [\widehat{t}, \tau_{p+1}]$ (similarly to the case 1) above).

From (14) follows

$$v(\widehat{t}) \leq v(\tau_p^+). \qquad (25)$$

Since $\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \geq r > \chi(d)$ and $\phi(t, t_0, \phi_0, d_1, d_2) \in G^+$ for $t \in [\widehat{t}, \tau_{p+1}]$, then from (15) follows

$$\dot{v}(t) \leq \varphi_2(v(t)), \quad t \in [\widehat{t}, \tau_{p+1}].$$

This allows to calculate

$$\int\limits_{v(\widehat{t})}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} = \int\limits_{\widehat{t}}^{\tau_{p+1}} \frac{dv(s)}{\varphi_2(v(s))} \leq \tau_{p+1} - \widehat{t} \leq \tau_{p+1} - \tau_p = T_p \leq \theta_2,$$

so that (25) implies

$$\int\limits_{v(\tau_p^+)}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} \leq \int\limits_{v(\widehat{t})}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} \leq \theta_2.$$

Now we set $a = v(\tau_p)$ into (18) and obtain

$$\int\limits_{v(\tau_p^+)}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} \leq \theta_2 \leq \int\limits_{\psi_2(v(\tau_{p+1}))}^{v(\tau_{p+1})} \frac{ds}{\varphi_2(s)} - \delta$$

6

or

$$\int\limits_{v(\tau_p^+)}^{\psi_2(v(\tau_{p+1}))} \frac{ds}{\varphi_2(s)} \le -\delta.$$

Having $\|\phi(\tau_{p+1}, t_0, \phi_0, d_1, d_2)\|_X \ge r > \chi(d)$, $\phi(\tau_{p+1}, t_0, \phi_0, d_1, d_2) \in G^+$ and (18) we conclude that $v(\tau_{p+1}^+) \le \psi_2(v(\tau_{p+1}))$. Hence

$$\int\limits_{v(\tau_p^+)}^{v(\tau_{p+1}^+)} \frac{ds}{\widehat{\varphi}(s)} \le \int\limits_{v(\tau_p^+)}^{v(\tau_{p+1}^+)} \frac{ds}{\varphi_2(s)} \le \int\limits_{v(\tau_p^+)}^{\psi_2(v(\tau_{p+1}))} \frac{ds}{\varphi_2(s)} \le -\delta.$$

In other words

$$F(v(\tau_{p+1}^+), v(\tau_p^+)) \ge \delta. \tag{26}$$

From (23)—(26) we conclude that (22) is true for $l = p + 1$. Considering the solution between the next two consequent jumps we obtain

$$F(v(\tau_{p+1}^+), v(\tau_l^+)) \ge \delta, \quad l = p, p+1, \ldots, m-1. \tag{27}$$

From the definition of F for any $s \ge z \ge q > 0$ we have $F(s, z) + F(z, q) = F(s, q)$. Since the interval (τ_p, τ_l) is split into $l - p$ subintervals by the time instants of the impulsive actions we finally obtain from (27) that (22) is proved.

Remark 1 Let $F^{-1}(s, \cdot)$ be the inverse function to $F(\cdot, s)$, $s \in \mathbb{R}_+$. From $F(\tau, s) \to \infty$ for $\tau \to 0+$ follows $F^{-1}(s, \tau) \to 0$ for $\tau \to +\infty$. Also note that $F^{-1}(s, \cdot)$ is strictly decreasing whereas $F^{-1}(\cdot, s)$ is strictly increasing for $s > 0$.

Lemma 2 Under the conditions of Theorem 1 let r be such that $r > \chi(d)$, then there exists $\tau_{k_0} \ge \tau_0$ such that $\phi(\tau_{k_0}^+, t_0, \phi_0, d_1, d_2) \in B_r(0)$.

Proof. Assume by contradiction that for all $k \in \mathbb{Z}_+$ we have $\|\phi(\tau_k, t_0, \phi_0, d_1, d_2)\|_X \ge r$. Lemma 1 implies that the sequence $\{v(\tau_k^+)\}_{k=0}^{\infty}$ is decreasing. Being bounded from below it has a limit $v^* \ge 0$. From (22) we have

$$F(v(\tau_m^+), v(\tau_0^+)) \ge \delta m.$$

If $v^* \ne 0$ this inequality leads to a contradiction letting $m \to \infty$. Hence, $v^* = 0$. Due to (12) we have

$$0 \le \alpha_1(r) \le \alpha_1(\|\phi(\tau_m^+, t_0, \phi_0, d_1, d_2)\|_X) \le$$
$$V(\phi(\tau_m^+, t_0, \phi_0, d_1, d_2)) = v(\tau_m^+)$$

and taking the limit for $m \to \infty$ we arrive to $\alpha_1(r) = 0$ which implies $r = 0$ contradicting $r > \chi(d)$. This finishes the proof of the lemma.

Lemma 3 Under the conditions of Theorem 1 the solution ϕ of Σ satisfy $\phi(\tau_{k_0}^+, t_0, \phi_0, d_1, d_2) \in B_r(0)$ for some $\tau_{k_0} \in \mathcal{E}$ and $r > \chi(d)$, then there exists $R \in \mathcal{K}_\infty^2$ such that

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\| \le R(r, d), \quad t > \tau_{k_0}.$$

Proof. We define for $s \ge 0$, $q \ge 0$

$$R(s, q) := \max\{R_1(s, q), R_4(s, q), R_6(s, q), s\},$$

where we use the following combinations of functions from (9)-(10)

$$R_1(s, q) = \xi_{\theta_2}(s) + \eta_{\theta_2}(q), \quad R_2(s, q) = \xi(s) + \eta(q),$$
$$R_3(s, q) = \max\{\xi(R_1(s, q)) + \eta(q), R_2(s, q)\},$$
$$R_4(s, q) = \xi_{\theta_2}(R_3(s, q)) + \eta_{\theta_2}(q),$$
$$R_5(s, q) = (\alpha_1^{-1} \circ \alpha_2)(R_3(s, q)),$$
$$R_6(s, q) = \xi_{\theta_2}(R_5(s, q)) + \eta_{\theta_2}(q).$$

Let $\widehat{t}_1 > \tau_{k_0}$ be such that $\|\phi(\widehat{t}_1, t_0, \phi_0, d_1, d_2)\|_X \le r$ and $\|\phi(\widehat{t}_1^+, t_0, \phi_0, d_1, d_2)\|_X \ge r$ (if such $\widehat{t}_1$ does not exist, then the result is proved). Define

$$\widehat{t}_2 := \sup\{t > \widehat{t}_1 : \|\phi(s, t_0, \phi_0, d_1, d_2)\|_X \ge r$$
$$\text{for} \quad s \in [\widehat{t}_1, t]\} \in [\widehat{t}_1, \infty],$$

so that

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \ge r, \quad t \in [\widehat{t}_1, \widehat{t}_2].$$

It is enough to show that $\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \le R(r, d)$ for $t \in [\widehat{t}_1, \widehat{t}_2]$.

If $\mathcal{E}_{[\widehat{t}_1, \widehat{t}_2]} := [\widehat{t}_1, \widehat{t}_2] \cap \mathcal{E} = \emptyset$, then by the properties (8), Σ_2, (7) for all $t \in [\widehat{t}_1, \widehat{t}_2]$ we conclude

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X$$
$$= \|\phi(t, \widehat{t}_1, \phi(\widehat{t}_1, t_0, \phi_0, d_1, d_2), d_1, d_2)\|_X$$
$$= \|\phi_c(t, \widehat{t}_1, \phi(\widehat{t}_1, t_0, \phi_0, d_1, d_2), d_1)\|_X$$
$$= \|\phi_c(t - \widehat{t}_1, 0, \phi(\widehat{t}_1, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\widehat{t}_1} d_1)\|_X$$
$$\le \xi_{\theta_2}(r) + \eta_{\theta_2}(d) = R_1(r, d).$$

If otherwise $\mathcal{E}_{[\widehat{t}_1, \widehat{t}_2]} \ne \emptyset$ we denote its minimal element by $\tau_p \ge \widehat{t}_1$ and consider two possible cases (i) $\widehat{t}_1 = \tau_p$ and (ii) $\tau_p > \widehat{t}_1$ separately.

In case of (i) from (10) follows

$$\|\phi(\widehat{t}_1^+, t_0, \phi_0, d_1, d_2)\|_X \le \xi(\|\phi(\widehat{t}_1, t_0, \phi_0, d_1, d_2)\|_X)$$
$$+ \eta(\|d_2\|_{\mathcal{U}_2}) \le \xi(r) + \eta(d) = R_2(r, d).$$

In case of (ii) by means of (9) and (10) we obtain

$$\|\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)\|_X \leq \xi(\|\phi(\tau_p, t_0, \phi_0, d_1, d_2)\|_X) + \eta(\|d_2\|_{\mathcal{U}_2})$$

and with help of (8), Σ_2, (7) we get

$$\begin{aligned}
&\|\phi(\tau_p, t_0, \phi_0, d_1, d_2)\|_X \\
&= \|\phi(\tau_p, \hat{t}_1, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), d_1, d_2)\|_X \\
&= \|\phi_c(\tau_p, \hat{t}_1, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), d_1)\|_X \\
&= \|\phi_c(\tau_p - \hat{t}_1, 0, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\hat{t}_1} d_1)\|_X \\
&\leq \xi_{\theta_2}(\|\phi(\hat{t}_1, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(\|d_1\|_{\mathcal{U}_2}) \\
&\leq \xi_{\theta_2}(r) + \eta_{\theta_2}(d) = R_1(r, d).
\end{aligned}$$

Hence,

$$\|\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)\|_X \leq \xi(R_1(r, d)) + \eta(d).$$

In bothe cases (i) and (ii) we see that $\|\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)\|_X \leq R_3(r, d)$ holds.

If $\sharp\mathcal{E}_{[\hat{t}_1, \hat{t}_2]} = 1$, then by means of Σ_2, (7), (9) and (10) we obtain for $t \in (\tau_p, \hat{t}_2]$

$$\begin{aligned}
&\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \\
&= \|\phi_c(t - \tau_p, 0, \phi(\tau_p^+, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\tau_p} d_1)\|_X \\
&\leq \xi_{\theta_2}(\|\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(\|d_1\|_{\mathcal{U}_1}) \\
&\leq \xi_{\theta_2}(R_3(r, d)) + \eta_{\theta_2}(d) = R_4(r, d).
\end{aligned}$$

If $\sharp\mathcal{E}_{[\hat{t}_1, \hat{t}_2]} \geq 2$, then by Lemma 1 and observing that $F(s, q) > 0 \iff s < q$ we obtain for $l \geq p$ such that $\tau_l \in \mathcal{E}_{[\hat{t}_1, \hat{t}_2]}$

$$\begin{aligned}
\alpha_1(\|\phi(\tau_l^+, t_0, \phi_0, d_1, d_2)\|_X) &\leq V(\phi(\tau_l^+, t_0, \phi_0, d_1, d_2)) \\
= v(\tau_l^+) \leq v(\tau_p^+) &= V(\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)) \\
\leq \alpha_2(\|\phi(\tau_p^+, t_0, \phi_0, d_1, d_2)\|_X) &\leq \alpha_2(R_3(r, d)).
\end{aligned}$$

This implies $\|\phi(\tau_l^+, t_0, \phi_0, d_1, d_2)\|_X \leq (\alpha_1^{-1} \circ \alpha_2)(R_3(r, d)) = R_5(r, d)$. Hence for all $t \in (\tau_l, \tau_{l+1}]$, from (9) and properties Σ_2, (7), (8) we obtain

$$\begin{aligned}
&\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \\
&= \|\phi_c(t, \tau_l, \phi(\tau_l^+, t_0, \phi_0, d_1, d_2), d_1)\|_X \\
&= \|\phi_c(t - \tau_l, 0, \phi(\tau_l^+, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\tau_l} d_1)\|_X \\
&\leq \xi_{\theta_2}(\|\phi(\tau_l^+, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(\|d_1\|_{\mathcal{U}_1}) \\
&\leq \xi_{\theta_2}(R_5(r, d)) + \eta_{\theta_2}(d) = R_6(r, d).
\end{aligned}$$

This implies the estimate

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \leq R(r, d) \quad \text{for all} \quad t \in [\tau_p, \tau_{m+1}].$$

If $t \in [\hat{t}_1, \tau_p]$, then from Σ_2, (7), (8) and (9) we get

$$\begin{aligned}
&\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \\
&= \|\phi(t, \hat{t}_1, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), d_1, d_2)\|_X \\
&= \|\phi_c(t, \hat{t}_1, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), d_1)\|_X \\
&= \|\phi_c(t - \hat{t}_1, 0, \phi(\hat{t}_1, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\hat{t}_1} d_1)\|_X \\
&\leq \xi_{\theta_2}(\|\phi(\hat{t}_1, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(d) \\
&= \xi_{\theta_2}(r) + \eta_{\theta_2}(d) = R_1(r, d) \leq R(r, d).
\end{aligned}$$

Since $[\hat{t}_1, \tau_{m+1}] \supseteq [\hat{t}_1, \hat{t}_2]$ holds, the lemma is proved.

Proof of Theorem 1 Take $r = (1 + \varepsilon)\chi(d)$ for some $\varepsilon > 0$ and denote by k_0 the smallest integer for which $\phi(\tau_{k_0}^+, t_0, \phi_0, d_1, d_2) \in B_r(0)$ holds, that is $\|\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)\|_X \geq r$ for all $0 \leq k \leq k_0 - 1$. From Lemma 1 follows $F(v(\tau_k^+), v(\tau_0^+)) \geq \delta k$, hence (see Remark 1) we have

$$v(\tau_k^+) \leq F^{-1}(v(\tau_0^+), k\delta).$$

From (9) and properties Σ_2, (8) and (7) it follows that for $t \in (\tau_k, \tau_{k+1}]$, $k = 0, \ldots, k_0 - 1$ the next inequality holds

$$\begin{aligned}
\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X &= \|\phi_c(t, \tau_k, \phi(\tau_k^+, t_0, \phi_0, d_1, d_2), d_1, d_2)\|_X \\
&= \|\phi_c(t - \tau_k, 0, \phi(\tau_k^+, t_0, \phi_0, d_1, d_2), \mathcal{S}_{\tau_k} d_1)\|_X \\
&\leq \xi_{\theta_2}(\|\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(d).
\end{aligned}$$

Now (12) implies

$$\begin{aligned}
\alpha_1(\|\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)\|_X) &\leq V(\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)) \\
= v(\tau_k^+) &\leq F^{-1}(v(\tau_0^+), \delta k), \\
v(\tau_0^+) = V(\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)) &\leq \alpha_2(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X).
\end{aligned}$$

Defining $\beta_k(s) := F^{-1}(\alpha_2(s), \delta k)$, $s > 0$ and $\beta_k(0) := 0$ we can write

$$\begin{aligned}
&\alpha_1(\|\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)\|_X) \\
&\leq F^{-1}(\alpha_2(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X), \delta k) \\
&= \beta_k(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X).
\end{aligned}$$

Which means

$$\|\phi(\tau_k^+, t_0, \phi_0, d_1, d_2)\|_X \leq (\alpha_1^{-1} \circ \beta_k)(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X).$$

By the properties of F^{-1} (see Remark 1) it follows that $\beta_k \in \mathcal{K}$, $\beta_{k+1}(s) < \beta_k(s)$ and $\lim_{k\to\infty} \beta_k(s) = 0$ for $s > 0$.

For $t \in (\tau_k, \tau_{k+1}]$ the following estimate holds

$$\begin{aligned}
&\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \\
&\leq (\xi_{\theta_2} \circ \alpha_1^{-1} \circ \beta_k)(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X) + \eta_{\theta_2}(d).
\end{aligned}$$

Defining $\widehat{\beta}_k(s) := (\xi_{\theta_2} \circ \alpha_1^{-1} \circ \beta_k)(s)$ for $k \in \mathbb{Z}_+$ and $s \geq 0$ we see that $\widehat{\beta}_k \in \mathcal{K}$, $\widehat{\beta}_{k+1}(s) < \widehat{\beta}_k(s)$ and $\lim_{k \to \infty} \widehat{\beta}_k(s) = 0$ for any $s > 0$. Further, we define

$$\beta_{\tau_0^+}(s,t) = \widehat{\beta}_k(s) + \frac{t - \tau_k}{\tau_{k+1} - \tau_k}(\widehat{\beta}_{k+1}(s) - \widehat{\beta}_k(s)),$$
$$t \in (\tau_k, \tau_{k+1}], \ s \geq 0$$

so that $\beta_{\tau_0^+} \in \mathcal{KL}$ and for $t \in (\tau_0, \tau_{k_0}]$ the following estimate holds

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \leq \beta_{\tau_0^+}(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X, t)$$
$$+ \eta_{\theta_2}(d).$$
$$(28)$$

From Lemma 3 follows that for any $t > \tau_{k_0}$ we have

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \leq \sigma(d), \qquad (29)$$

where we denote $\sigma(s) := R((1 + \varepsilon)s, s)$, $\varepsilon > 0$. Since $\sigma \in \mathcal{K}$ then collecting the estimates (28) and (29) we obtain

$$\|\phi(t, t_0, \phi_0, d_1, d_2)\|_X \leq \beta_{\tau_0^+}(\|\phi(\tau_0^+, t_0, \phi_0, d_1, d_2)\|_X, t)$$
$$+ \gamma_{\tau_0^+}(d), \quad t > \tau_0,$$

where $\gamma_{\tau_0^+}(s) := \sigma(s) + \eta_{\theta_2}(s)$. The statement of the theorem follows then from the Proposition 1.

Remark 2 *In the definition of an ISS-Lyapunov function V we have assumed that the sets G^+ and G^- are not empty. Observe however that in case $G^+ = \emptyset$, $G^- = X$ Theorem 1 reduces to Theorem 1 from [4] and if $G^- = \emptyset$, $G^+ = X$ then Theorem 1 reduces to Theorem 3 from [4].*

5 Nonlinear interconnection of an ODE and a PDE

Let $\widehat{X} = L^2[0, l] \times \mathbb{R}$, $X = H_0^1(0, l) \times \mathbb{R}$, $U_1 = U_2 = H_0^1(0, l) \times \mathbb{R}$ and the spaces of input signals be

$$\mathcal{U}_1 = L^\infty(U_1) \cap (H_{\mathrm{loc}}(\mathbb{R}_+, H_0^1(0, l)) \times H_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})),$$
$$\mathcal{U}_2 = l^\infty(\mathbb{Z}_+, U_2).$$

Consider the following nonlinear impulsive system

$$\partial_t x(z, t) = a^2 \partial_{zz} x(z, t) + \Phi(x(z, t))$$
$$+ B(z)y(t) + d_{11}(z, t), \quad t \neq \tau_k,$$

$$\dot{y}(t) = c^2 y(t) + \int_0^l D(z)x(z, t)\, dz + d_{12}(t), \quad t \neq \tau_k,$$

$$x(z, t^+) = \alpha(z)x(z, t) + \beta(z)y(t) + d_{21}(z, k), \quad t = \tau_k,$$

$$y(t^+) = \int_0^l \gamma(z)x(z, t)\, dz + \delta y(t) + d_{22}(k), \quad t = \tau_k,$$
$$(30)$$

with initial and boundary conditions

$$x(z, 0) = x_0(z) \in H_0^1(0, l), \quad z \in [0, l],$$
$$y(0) = y_0 \in \mathbb{R},$$
$$x(0, t) = x(l, t) = 0, \quad t \in \mathbb{R}_+,$$
$$x_0(0) = x_0(l) = 0.$$
$$(31)$$

Here a, c, l are given positive constants, $\alpha \in C^2[0, l]$, $\beta \in H_0^1(0, l)$, $\gamma \in L^2[0, l]$, $D \in L^2[0, l]$, $B \in H_0^1(0, l)$ are given functions and $d_1(t) = (d_{11}(\cdot, t), d_{12}(t)) \in U_1$, $d_2(k) = (d_{21}(\cdot, k), d_{22}(k)) \in U_2$ are unknown disturbances.

Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies:

(i) $\Phi \in C^1(\mathbb{R})$ with locally Lipshitz Φ' that is for any $s_0 \in \mathbb{R}$ and $\varrho_0 > 0$ there exists $L = L(s_0, \varrho_0) > 0$ such that for all $s \in \mathbb{R}$ with $|s - s_0| \leq \varrho_0$ it holds that $|\Phi'(s) - \Phi'(s_0)| \leq L|s - s_0|$.

(ii) It holds that $\Phi'(s) \leq 0$, $s\Phi(s) \leq 0$ for all $s \in \mathbb{R}$.

The problem (30)–(31) can be written in the following form

$$\frac{d}{dt}\begin{pmatrix} x(\cdot, t) \\ y(t) \end{pmatrix} + \mathcal{A}\begin{pmatrix} x(\cdot, t) \\ y(t) \end{pmatrix} = f(t, x, y, d_1), \quad t \neq \tau_k$$

$$\begin{pmatrix} x(\cdot, t^+) \\ y(t^+) \end{pmatrix} = \mathcal{B}\begin{pmatrix} x(\cdot, t) \\ y(t) \end{pmatrix} + d_2(k), \quad t = \tau_k,$$
$$(32)$$

where $\mathcal{A}$ is the linear operator on $\widehat{X}$ defined by

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a^2 \partial_{zz} x(z) \\ -c^2 y \end{pmatrix}$$

with domain $\mathcal{D}(\mathcal{A}) = (H_0^1(0, l) \cap H^2(0, l)) \times \mathbb{R}$, $\mathcal{B}$ is the linear operator defined by

$$\mathcal{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(z)x(z) + \beta(z)y \\ \int_0^l \gamma(z)x(z)\, dz + \delta y \end{pmatrix}$$

and $d_2(k) = (d_{21}(\cdot, k), d_{22}(k))$.

The operator $\mathcal{A}$ being a direct product of two sectorial operators is sectorial and hence it generates on $\widehat{X}$ an analytic semi-group [6]. The mapping $f : \mathbb{R} \times H_0^1(0, l) \times \mathbb{R} \to L^2[0, l] \times \mathbb{R}$ in (32) is defined by

$$f(t, x, y, d_1) = \begin{pmatrix} \Phi(x(z)) + B(z)y + d_{11}(z, t) \\ \int\limits_0^l D(z)x(z)\, dz + d_{12}(t) \end{pmatrix}$$

We consider classical solutions of (30)—(31) defined as in Definition 3.3.1 of [6].

Remark 3 *For any initial state $(x_0, y_0) \in H_0^1(0, l) \times \mathbb{R}$ and input $(d_1, d_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ there exists a unique solution to the propblem (30)–(31). This follows from the fact the the corresponding problem without impulsive actions*

$$\frac{d}{dt}\begin{pmatrix} x(\cdot, t) \\ y(t) \end{pmatrix} + \mathcal{A}\begin{pmatrix} x(\cdot, t) \\ y(t) \end{pmatrix} = f(t, x, y), \qquad (33)$$

for each $(x_0, y_0) \in H_0^1(0, l) \times \mathbb{R}$ possesses a unique solution defined for $[t_0, t_0 + \epsilon_{t_0, x_0, y_0, d_1}]$, $\epsilon_{t_0, x_0, y_0, d_1} > \theta_2$, and that the mapping $g(x, y, \mu) = \mathcal{B}(x, y) + \mu$ keeps the space X invariant for any $\mu \in U_2$.

The well-posedness of (33) can be established by means of Theorems 3.3.3 and 3.3.4 from [6]. In order to check the conditions of Theorem 3.3.3 (setting $\alpha = 0.5$ there) we need to show that the mapping f is locally Hölder wrt t and locally Lipschitz wrt $(x, y) \in H_0^1(0, l) \times \mathbb{R}$, that is for any $(t_0, x_0, y_0) \in \mathbb{R} \times H_0^1(0, l) \times \mathbb{R}$ there exists

$$O_\varrho(t_0, x_0, y_0) := \{(t, x, y) \in \mathbb{R}_+ \times H_0^1(0, l) \times \mathbb{R} \mid$$
$$|t - t_0| < \varrho,\ \|x - x_0\|_{H_0^1(0,l)} < \varrho,\ |y - y_0| < \varrho\}.$$

such that for any point $(t, x, y) \in O_\varrho(t_0, x_0, y_0)$ it holds that

$$\|f(t, x, y) - f(t_0, x_0, y_0)\|_{\widehat{X}} \leq L(|t - t_0|^{\nu_1}$$
$$+ \|x - x_0\|_{H_0^1(0,l)} + |y - y_0|)$$

for some constants $L > 0$, $\nu_1 \in (0, 1]$.

Indeed,

$$\|f(t, x, y) - f(t_0, x_0, y_0)\|_{\widehat{X}}$$
$$\leq \frac{l}{\pi}\|\Phi \circ x - \Phi \circ x_0 + B(y - y_0) + d_{11}(\cdot, t) - d_{11}(\cdot, t_0)\|_{H_0^1(0,l)}$$
$$+ \left| \int\limits_0^l D(z)(x(z) - x_0(z))\, dz + d_{12}(t) - d_{12}(t_0) \right|$$
$$\leq \frac{l}{\pi}(\|\Phi \circ x - \Phi \circ x_0\|_{H_0^1(0,l)} + \|B\|_{H_0^1(0,l)}|y - y_0|)$$
$$+ \|D\|_{L^2[0,l]}\|x - x_0\|_{L^2[0,l]} + |d_{12}(t) - d_{12}(t_0)|$$
$$+ \frac{l}{\pi}\|d_{11}(\cdot, t) - d_{11}(\cdot, t_0)\|_{H_0^1(0,l)}.$$
$$(34)$$

Consider the first summand separately

$$\|\Phi \circ x - \Phi \circ x_0\|_{H_0^1(0,l)} = \|\partial_z(\Phi \circ x - \Phi \circ x_0)\|_{L^2[0,l]}$$
$$= \|(\Phi' \circ x)\, \partial_z x - (\Phi' \circ x_0)\, \partial_z x_0\|_{L^2[0,l]}$$
$$\leq \|(\Phi' \circ x - \Phi' \circ x_0)\partial_z x\|_{L^2[0,l]}$$
$$+ \|(\Phi' \circ x_0)(\partial_z x - \partial_z x_0)\|_{L^2[0,l]}$$

By the Sobolev embedding theorem we have $x \in C[0, l]$ and satisfies $\|x - x_0\|_{C[0,l]} \leq C_1\|x - x_0\|_{H_0^1(0,l)} \leq C_1\varrho$, for some $C_1 > 0$. Hance, using the condition (i), we obtain

$$|\Phi'(x(z)) - \Phi'(x_0(z))| \leq L_1|x(z) - x_0(z)|$$
$$\leq L_1\|x - x_0\|_{C[0,l]} \leq L_1 C_1\|x - x_0\|_{H_0^1(0,l)},$$

where L_1 is a positive constant, which can depend on x_0 and ρ. Hence

$$\|\Phi' \circ x - \Phi' \circ x_0\|_{C[0,l]} \leq C_2\|x - x_0\|_{H_0^1(0,l)} \leq C_2\varrho,$$

where $C_2 = L_1 C_1$, and

$$\|\Phi \circ x - \Phi \circ x_0\|_{H_0^1(0,l)}$$
$$\leq \|\Phi' \circ x - \Phi' \circ x_0\|_{C[0,l]}\|\partial_z x\|_{L^2[0,l]}$$
$$+ \|\Phi' \circ x_0\|_{C[0,l]}\|\partial_z x - \partial_z x_0\|_{L^2[0,l]}$$
$$\leq C_2\|x - x_0\|_{H_0^1(0,l)}\|x\|_{H_0^1(0,l)} + \|\Phi' \circ x_0\|_{C[0,l]}\|x - x_0\|_{H_0^1(0,l)}$$
$$\leq (C_2(\|x_0\|_{H_0^1(0,l)} + \varrho) + \|\Phi' \circ x_0\|_{C[0,l]})\|x - x_0\|_{H_0^1(0,l)},$$

which together with (34) proves that f is locally Lipschitz.

By means of Theorem 3.3.4 Exercise 1 from the Section 3.3 in [6]) we can show that the solution to (33) exists globally. To this end it is sufficient to check that

$$\frac{\|f(t, x(\cdot, t), y(t), d_1)\|_{\widehat{X}}}{1 + \|x(t)\|_{H_0^1(0,l)} + |y(t)|} \qquad (35)$$

is bounded on the domain of existence of the solution $(x(\cdot, t), y(t)) \in X$.

To verify this fact we introduce the following auxiliary function

$$U(x,y) = \|x\|^2_{L^2[0,l]} + y^2, \quad x \in H_0^1[0,l], \quad y \in \mathbb{R}$$

and use the following

Proposition 2 *For any $(x,y,\xi) \in H_0^1(0,l) \times \mathbb{R} \times U_1$ it holds that*

$$\dot{U}(x,y,\xi) \leq \zeta^{\mathrm{T}}\left(\widetilde{A}_0 + \varepsilon\,\mathrm{id}\right)\zeta + \varepsilon^{-1}\|\xi\|^2_{U_1}, \qquad (36)$$

where we denote

$$\zeta = (\|x\|_{L^2[0,l]}, |y|)^{\mathrm{T}},$$
$$\widetilde{A}_0 = \begin{pmatrix} -\frac{2\pi^2 a^2}{l^2} & \|B+D\|_{L^2[0,l]} \\ \|B+D\|_{L^2[0,l]} & 2c^2 \end{pmatrix},$$

The proof can be found in the Appendix.

Corollary 1 *Let $(x(\cdot,t), y(t))$ be a solution to (33) with initial conditions $x(\cdot,t_0) = x_0$, $y(t_0) = y_0$ and defined on $t \in [t_0, t_0 + \epsilon_{t_0,x_0,y_0,d_1})$, then*

$$\sup_{t\in[t_0,t_0+\epsilon_{t_0,x_0,y_0,d_1})} \|(x,y)\|_{\widehat{X}} \leq C(x_0,y_0,d_1).$$

The proof can be found in the Appendix.

From this corollary and the locally Lipschitzness follows that (35) is bounded, hence the solution $(x(\cdot,t), y(t))$ of (33) exists for all $t \geq t_0$. The properties of α, β guarantee that $g(H_0^1(0,l) \times \mathbb{R} \times U_2) \subset H_0^1(0,l) \times \mathbb{R}$, which demonstrates the well-posedness of the problem (30)–(31).

To state the asymptotic stability conditions of the system (30)–(31) we define the following symmetric matrices

$$A_0 = \begin{pmatrix} -\frac{2\pi^2 a^2}{l^2} & \|B\|_{H_0^1(0,l)} + \frac{l}{\pi}\|D\|_{L^2[0,l]} \\ \|B\|_{H_0^1(0,l)} + \frac{l}{\pi}\|D\|_{L^2[0,l]} & 2c^2 \end{pmatrix},$$

$$B_0 = $$
$$\begin{pmatrix} \|\alpha^2\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\alpha\alpha_{zz}\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\gamma\|^2_{L^2[0,l]} & * \\ \|\alpha_z\beta_z + \gamma\delta\|_{L^2[0,l]}\frac{l}{\pi} + \|\alpha\beta_z\|_{L^2[0,l]} & \delta^2 + \|\beta\|^2_{H_0^1(0,l)} \end{pmatrix},$$

$$\sigma = \|\alpha^2\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\alpha\alpha_{zz}\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\gamma\|^2_{L^2[0,l]}$$

$$+2(\|\alpha_z\beta_z + \gamma\delta\|_{L^2[0,l]}\frac{l}{\pi} + \|\alpha\beta_z\|_{L^2[0,l]}) + \delta^2 + \|\beta\|^2_{H_0^1(0,l)},$$

$$\vartheta := \frac{\pi^2 a^2}{l^2} - \|B\|_{H_0^1(0,l)} - \frac{l}{\pi}\|D\|_{L^2[0,l]} - c^2,$$

let $\varrho^{A_0}_{\max} \in \mathbb{R}^2$ and $\varrho^{B_0}_{\max} \in \mathbb{R}^2$ be eigenvectors of A_0 and B_0, respectively, corresponding to the maximal eigenvalues $\lambda_{\max}(A_0)$ and $\lambda_{\max}(B_0)$, respectively.

Proposition 3 *Let the impulsive system (30) satisfy*

$$c^2 + \frac{\pi^2 a^2}{l^2} > \frac{l}{\pi}\|D\|_{L^2[0,l]} + \|B\|_{H_0^1(0,l)}, \vartheta > 0,$$
$$(\varrho^{A_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\{-1,1\}\varrho^{A_0}_{\max} > 0$$

and additionally let one of the two following conditions hold

(a) $(\varrho^{B_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\{-1,1\}\varrho^{B_0}_{\max} \geq 0$, and θ_1, θ_2 satisfy

$$\frac{1}{\vartheta}\ln\frac{\sigma}{2} < \theta_1 \leq \theta_2 < -\frac{1}{\lambda_{\max}(A_0)}\ln\lambda_{\max}(B_0).$$

(b) $(\varrho^{B_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\{-1,1\}\varrho^{B_0}_{\max} < 0$, and θ_1, θ_2 satisfy

$$\frac{1}{\vartheta}\ln\lambda_{\max}(B_0) < \theta_1 \leq \theta_2 < \frac{1}{\lambda_{\max}(A_0)}\ln\frac{2}{\sigma}.$$

then (30) is ISS for all $\mathcal{E}$ satisfying the dwell-time condition $\theta_1 \leq T_k \leq \theta_2$.

To show how the last proposition can be applied we consider the following

5.1 Specific example

Consider (30) with $a = 1$, $l = \pi$, $D(z) = 0.05z$, $B(z) = 0.05z(\pi - z)$, $c = 0.5$, $\alpha(z) = 1$, $\beta(z) = 0$, $\gamma(z) = 0.05$, $\delta = 0.25$, $\Phi(s) = -s^3$. Then we have

$$A_0 = \begin{pmatrix} -2 & 0.3214875 \\ 0.3214875 & 0.5 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1.007853982 & 0.02215567314 \\ 0.02215567314 & 0.0625 \end{pmatrix},$$

$$\lambda_{\max}(A_0) = 0.54067976, \quad \lambda_{\max}(B_0) = 1.0083729,$$
$$\sigma = 1.1146653, \quad \vartheta = 0.42851243,$$
$$\varrho^{A_0}_{\max} = (0.12553504, -0.9920891862)^{\mathrm{T}},$$
$$\varrho^{B_0}_{\max} = (0.99972578, -0.023417096)^{\mathrm{T}}$$

Then all conditions of the Proposition 3 are satisfied and the dwell-time condition reads as

$$0.01945822 < \theta_1 \leq T_k \leq \theta_2 < 1.0812185.$$

Let us note that with this parameters choice we have that both discrete and continuous dynamics of (30) considered separately are not asymptotically stable already for the unperturbed case $d_1 = 0$, $d_2 = 0$. Indeed, $W(x,y)$ can be used as a Chetaev function for the continuous dynamics of (30), and to see that the discrete dynamics is unstable just check that the spectral radius of the jump operator is larger than 1.

6 Conclusions

Our results provide a dwell-time condition that guarantees the ISS property of a nonlinear impulsive system. In contrary to the existing dwell-time conditions in the literature our result can be in particular applied even to the cases where both discrete and continuous dynamics are unstable simultaneously. In contrary to the results of [4] the ISS property is assured by the analysis of specific interaction of the discrete and continuous dynamics instead of a compensation of the unstable discrete (continuous) dynamics by means of the stable continuous (discrete) one. Our future research will be devoted to the development of constructive approach in order to derive the auxiliary Lyapunov V and Chetaev W functions. Another open problem that needs to be investigated is the derivation of conditions under which a combination of the simultaneously stable discrete and continuous dynamics leads to an unstable dynamics of the overall impulsive system.

7 Appendix

7.1 Proof of Proposition 2

Consider the function

$$U(x,y) = \|x\|^2_{L^2[0,l]} + y^2,$$

Its time derivative $\dot{U}$ with respect to the system (30) is

$$\dot{U}(x,y,\xi) = 2\int_0^l x(z)(a^2\partial_{zz}x(z) + \Phi(x(z)) + B(z)y$$

$$+\xi_1(z))\,dz + 2y(c^2y + \int_0^l D(z)x(z)\,dz + \xi_2)$$

Applying integration by parts and the Friedrich's inequality (5) we get

$$\int_0^l x(z)\partial_{zz}x(z)\,dz = -\int_0^l |\partial_z x(z)|^2\,dz \le -\frac{\pi^2}{l^2}\|x\|^2_{L^2[0,l]}.$$

By means of the Cauchy inequality we can write

$$\dot{U}(x,y,\xi) \le -\frac{2\pi^2 a^2}{l^2}\|x\|^2_{L^2[0,l]} + 2c^2y^2$$
$$+2\|B+D\|_{L^2[0,l]}\|x\|_{L^2[0,l]}|y|$$
$$+2|y||\xi_2| + 2\|\xi_1\|_{L^2[0,l]}\|x\|_{L^2[0,l]}$$
$$\le \left(-\frac{2\pi^2 a^2}{l^2} + \varepsilon\right)\|x\|^2_{L^2[0,l]} + (2c^2 + \varepsilon)y^2$$
$$+2\|B+D\|_{L^2[0,l]}\|x\|_{L^2[0,l]}|y| + \varepsilon^{-1}(\|\xi_1\|^2_{L^2[0,l]} + \xi_2^2).$$

for all $(x,y,\xi) \in L^2[0,l] \times \mathbb{R} \times U_1$ and $\varepsilon > 0$.

Recall that $\zeta = (\|x\|_{L^2[0,l]}, |y|)^{\mathrm{T}}$ hence the last estimate can be written as

$$\dot{U}(x,y,\xi) \le \zeta^{\mathrm{T}}(\widetilde{A}_0 + \varepsilon\,\mathrm{id})\zeta + \varepsilon^{-1}\|\xi\|^2_{U_1}, \qquad (37)$$

which finishes the proof.

7.2 Proof of the Corollary 1

Denote $u(t) := U(x(\cdot,t), y(t))$, $t \in [t_0, t_0 + \epsilon_{t_0,x_0,y_0,d_1})$, $d_1 \in \mathcal{U}_1$, then Proposition 2 implies that

$$\dot{u}(t) \le (\lambda_{\max}(\widetilde{A}_0) + \varepsilon)u(t) + \varepsilon^{-1}\|d_1(t)\|^2_{U_1}$$
$$\le (\lambda_{\max}(\widetilde{A}_0) + \varepsilon)u(t) + \varepsilon^{-1}\|d_1\|^2_{\mathcal{U}_1}.$$

By the comparison principle we obtain

$$u(t) \le e^{(\lambda_{\max}(\widetilde{A}_0)+\varepsilon)(t-t_0)}u(t_0)$$

$$+\varepsilon^{-1}\int_{t_0}^t e^{(\lambda_{\max}(\widetilde{A}_0)+\varepsilon)(t-s)}\,ds\|d\|^2_{\mathcal{U}_1}.$$

By the basic inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, $a, b \ge 0$ we derive

$$\|(x(\cdot,t), y(t))\|_{\widehat{X}} \le e^{(\lambda_{\max}(\widetilde{A}_0)+\varepsilon)\epsilon_{t_0,x_0,y_0,d_1}/2}\|(x_0,y_0)\|_{\widehat{X}}$$

$$+\sqrt{\frac{e^{(\lambda_{\max}(\widetilde{A}_0)+\varepsilon)\epsilon_{t_0,x_0,y_0,d_1}}-1}{\varepsilon(\lambda_{\max}(\widetilde{A}_0)+\varepsilon)}}\|d_1\|_{\mathcal{U}_1},$$

for all $t \in [t_0, t_0 + \epsilon_{t_0,x_0,y_0,d_1})$ which finishes the proof.

7.3 Proof of Proposition 3

Let us define

$$V(x,y) = \|x\|^2_{H_0^1(0,l)} + y^2, \quad W(x,y) = y^2 - \|x\|^2_{H_0^1(0,l)}$$

and calculate

$$\dot{V}(x,y,\xi) = -2\int_0^l \partial_{zz}x(z)(a^2\partial_{zz}x(z) + \Phi(x(z))$$

$$+B(z)y + \xi_1(z))\,dz + 2y(c^2y + \int_0^l D(z)x(z)\,dz + \xi_2)$$

Using the integration by parts

$$\int_0^l \partial_{zz}x(z)\Phi(x(z))\,dz = -\int_0^l \Phi'(x(z))\,(\partial_z x(z))^2\,dz,$$

$$\tag{38}$$

$$\int_0^l \partial_{zz}x(z)B(z)\,dz = -\int_0^l \partial_z x(z)\partial_z B(z)\,dz,$$

$$\int_0^l \partial_{zz}x(z)\xi_1(z)\,dz = -\int_0^l \partial_z x(z)\partial_z \xi_1(z)\,dz,$$

as well as the property $\Phi'(s) \leq 0$ for all $s \in \mathbb{R}$ and (6), we obtain

$$\dot{V}(x,y,\xi) \leq -\frac{2\pi^2 a^2}{l^2}\|x\|_{H_0^1(0,l)}^2$$

$$+2\int_0^l \partial_z x(z)\partial_z B(z)\,dz y + 2\int_0^l \partial_z x(z)\partial_z \xi_1(z)\,dz$$

$$+2c^2 y^2 + 2\int_0^l D(z)x(z)\,dz y + 2\xi_2 y$$

By the Friedrich's inequality (5) we get

$$\dot{V}(x,y,\xi) \leq -\frac{2\pi^2 a^2}{l^2}\|x\|_{H_0^1(0,l)}^2$$

$$+2(\|B\|_{H_0^1(0,l)} + \frac{l}{\pi}\|D\|_{L^2[0,l]})\|x\|_{H_0^1(0,l)}|y|$$

$$+2c^2 y^2 + 2|\xi_2||y| + 2\|x\|_{H_0^1(0,l)}\|\xi_1\|_{H_0^1(0,l)} \quad (39)$$

$$\leq \left(-\frac{2\pi^2 a^2}{l^2} + \varepsilon\right)\|x\|_{H_0^1(0,l)}^2 + 2(\|B\|_{H_0^1(0,l)}$$

$$+\frac{l}{\pi}\|D\|_{L^2[0,l]})\|x\|_{H_0^1(0,l)}|y| + (2c^2 + \varepsilon)y^2$$

$$+\varepsilon^{-1}(\|\xi_1\|_{H_0^1(0,l)}^2 + \xi_2^2).$$

and by the Cauchy inequality with (6) and (38) we obtain

$$\dot{W}(x,y,\xi) = 2y(c^2 y + \int_0^l D(z)x(z)\,dz + \xi_2)$$

$$+2\int_0^l \partial_{zz}x(z)(a^2\partial_{zz}x(z) + \Phi(x(z)) + B(z)y + \xi_1(z))\,dz$$

$$\geq 2c^2 y^2 + 2a^2\int_0^l (\partial_{zz}x(z))^2\,dz + 2y\int_0^l D(z)x(z)\,dz$$

$$-2y\int_0^l \partial_z B(z)\partial_z x(z)\,dz + 2y\xi_2 - 2\int_0^l \partial_z x(z)\partial_z \xi_1(z)\,dz$$

Applying (6) we have

$$\dot{W}(x,y,\xi) \geq 2c^2 y^2 + \frac{2\pi^2 a^2}{l^2}\|x\|_{H_0^1(0,l)}^2$$

$$-2(\frac{l}{\pi}\|D\|_{L^2[0,l]} + \|B\|_{H_0^1(0,l)})\|x\|_{H_0^1(0,l)}|y|$$

$$-2|y||\xi_2| - 2\|x\|_{H_0^1(0,l)}\|\xi_1\|_{H_0^1(0,l)}$$

$$\geq (2c^2 - \varepsilon)y^2 + \left(\frac{2\pi^2 a^2}{l^2} - \varepsilon\right)\|x\|_{H_0^1(0,l)}^2$$

$$-2(\frac{l}{\pi}\|D\|_{L^2[0,l]} + \|B\|_{H_0^1(0,l)})\|x\|_{H_0^1(0,l)}|y|$$

$$-\varepsilon^{-1}(\|\xi_1\|_{H_0^1(0,l)}^2 + \xi_2^2),$$

for all $(x,y,\xi) \in X \times U_1$. Further we have

$$V(g(x,y,\mu)) \leq \int_0^l (\alpha_z(z)x(z) + \alpha(z)x_z(z) + \beta_z(z)y$$

$$+\mu_{1z}(z))^2\,dz + \left(\int_0^l \gamma(z)x(z)\,dz + \delta y + \mu_2\right)^2$$

$$\leq \|\alpha^2\|_{C[0,l]}\|x_z\|_{L^2[0,l]}^2 + \|\beta_z\|_{L^2[0,l]}^2 y^2 + \|\mu_{1z}\|_{L^2[0,l]}^2$$

$$+\|\alpha\alpha_{zz}\|_{C[0,l]}\|x\|_{L^2[0,l]}^2 + 2\|\alpha_z\beta_z + \gamma\delta\|_{L^2[0,l]}\|x\|_{L^2[0,l]}|y|$$

$$+2\|\alpha\beta_z\|_{L^2[0,l]}\|x\|_{H_0^1(0,l)}|y| + 2\|\alpha_z\|_{C[0,l]}\|\mu_1\|_{H_0^1(0,l)}\|x\|_{L^2[0,l]}$$

$$+2\|\beta\|_{H_0^1(0,l)}\|\mu_1\|_{H_0^1(0,l)}|y| + 2\|\alpha\|_{C[0,l]}\|\mu_1\|_{H_0^1(0,l)}\|x\|_{H_0^1(0,l)}$$

$$+\|\gamma\|_{L^2[0,l]}^2\|x\|_{L^2[0,l]}^2 + \delta^2 y^2 + \mu_2^2 + 2|\delta||\mu_2||y|$$

$$+2|\mu_2|\|\gamma\|_{L^2[0,l]}\|x\|_{L^2[0,l]}$$

By the Friedrich's inequality (5) we have

$$V(g(x,y,\mu)) \leq \left(\|\alpha^2\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\alpha\alpha_{zz}\|_{C[0,l]} + \frac{l^2}{\pi^2}\|\gamma\|_{L^2[0,l]}^2\right.$$

$$\left.+\varepsilon(\frac{l^2}{\pi^2}\|\alpha_z\|_{C[0,l]} + \|\alpha\|_{C[0,l]} + \|\gamma\|_{L^2[0,l]}\frac{l^2}{\pi^2})\right)\|x\|_{H_0^1(0,l)}^2$$

$$+2\left(\|\alpha_z\beta_z + \gamma\delta\|_{L^2[0,l]}\frac{l}{\pi} + \|\alpha\beta_z\|_{L^2[0,l]}\right)\|x\|_{H_0^1(0,l)}|y|$$

$$+(\|\beta\|_{H_0^1(0,l)}^2 + \delta^2 + \varepsilon(\|\beta\|_{H_0^1(0,l)} + |\delta|))y^2$$

$$+(1 + \varepsilon^{-1}(\|\alpha_z\|_{C[0,l]} + \|\beta\|_{H_0^1(0,l)} + \|\alpha\|_{C[0,l]}))\|\mu_1\|_{H_0^1(0,l)}^2$$

$$+(1 + \varepsilon^{-1}(|\delta| + \|\gamma\|_{L^2[0,l]}))\mu_2^2$$

Let us denote

$$B_1 = \begin{pmatrix} \frac{l^2}{\pi^2}\|\alpha_z\|_{C[0,l]} + \|\alpha\|_{C[0,l]} + \|\gamma\|_{L^2[0,l]}\frac{l^2}{\pi^2} & 0 \\ 0 & \|\beta\|_{H_0^1(0,l)} + |\delta| \end{pmatrix},$$

$$\kappa(\varepsilon) = \max\{1 + \varepsilon^{-1}(\|\alpha_z\|_{C[0,l]} + \|\beta\|_{H_0^1(0,l)} + \|\alpha\|_{C[0,l]}),$$

$$1 + \varepsilon^{-1}(|\delta| + \|\gamma\|_{L^2[0,l]})\},$$

then the last inequality can be written as

$$V(g(x,y,\mu)) \leq \zeta^T(B_0 + \varepsilon B_1)\zeta + \kappa(\varepsilon)\|\mu\|^2. \quad (40)$$

The inequalities (39) and (40) imply the estimates (9) and (10): Denote $w(t) = \|x(\cdot,t)\|^2_{H^1_0(0,l)} + y^2(t)$, $t \geq 0$, then from (9) follows:

$$\dot{w}(t) \leq (\lambda_{\max}(A_0) + \varepsilon)w(t) + \varepsilon^{-1}\|d_1(t)\|^2_{U_1}$$
$$\leq (\lambda_{\max}(A_0) + \varepsilon)w(t) + \varepsilon^{-1}\|d\|^2_{\mathcal{U}_1}.$$

By the comparison principle we have

$$w(t) \leq e^{(\lambda_{\max}(A_0)+\varepsilon)t}w(0)+$$
$$\varepsilon^{-1}\int_0^t e^{(\lambda_{\max}(A_0)+\varepsilon)(t-s)}\,ds\|d_1\|^2_{\mathcal{U}_1}.$$

and with help of $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $a,\,b \geq 0$ we get

$$\|(x(\cdot,t),y(t))\|_X \leq e^{(\lambda_{\max}(A_0)+\varepsilon)\tau/2}\|(x_0,y_0)\|_X +$$
$$\sqrt{\frac{e^{(\lambda_{\max}(A_0)+\varepsilon)\tau}-1}{\varepsilon(\lambda_{\max}(A_0)+\varepsilon)}}\|d_1\|_{\mathcal{U}_1},$$

for all $t \in [0,\tau]$, which proves (9). Similarly (10) follows from (40).

Let us estimate $\dot{V}(x,y,\xi)$, $V(g(x,y,\mu))$ on the sets

$$G^+ = \{(x,y) \in X \;:\; |y| \geq \|x\|_{H^1_0(0,l)}\},$$
$$G^- = \{(x,y) \in X \;:\; |y| \leq \|x\|_{H^1_0(0,l)}\}$$

and $\dot{W}(x,y,\xi)$ on the set, where $W(x,y) = 0$ holds.

By the conditions of Proposition 3 we have $(\varrho^{A_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\,\{-1,1\}\varrho^{A_0}_{\max} > 0$ and

$$-\vartheta = -\frac{\pi^2 a^2}{l^2} + \|B\|_{H^1_0(0,l)} + \frac{l}{\pi}\|D\|_{L^2[0,l]} + c^2 < 0.$$

Hence by (39) for all $(x,y) \in G^+$ we have

$$\dot{V}(x,y,\xi) \leq (\lambda_{\max}(A_0) + \varepsilon)(\|x\|^2_{H^1_0(0,l)} + y^2) + \varepsilon^{-1}\|\xi\|^2_{U_1}$$
$$\leq (\lambda_{\max}(A_0) + 2\varepsilon)(\|x\|^2_{H^1_0(0,l)} + y^2),$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \varepsilon^{-1}\|\xi\|_{U_1}.$

For all $(x,y) \in G^-$ we have

$$\dot{V}(x,y,\xi) \leq (-\vartheta + \frac{\varepsilon}{2})(\|x\|^2_{H^1_0(0,l)} + y^2) + \varepsilon^{-1}\|\xi\|^2_{U_1}$$
$$\leq (-\vartheta + \varepsilon)(\|x\|^2_{H^1_0(0,l)} + y^2),$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \sqrt{2}\varepsilon^{-1}\|\xi\|_{U_1}.$

Now choosing $\varepsilon > 0$ small enough ($\varepsilon < \vartheta$), we can take $\varphi_1(s) = (\vartheta - \varepsilon)s$, $\varphi_2(s) = (\lambda_{\max}(A_0) + 2\varepsilon)s$.

If the condition (a) is satisfied, that is $(\varrho^{B_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\,\{-1,1\}\varrho^{B_0}_{\max} \geq 0$, then for all $(x,y) \in G^+$ we have

$$V(g(x,y,\mu)) \leq (\lambda_{\max}(B_0) + \varepsilon\|B_1\|)(\|x\|^2_{H^1_0(0,l)} + y^2)$$
$$+\kappa(\varepsilon)\|\mu\|^2_{U_2} \leq (\lambda_{\max}(B_0) + \varepsilon(1 + \|B_1\|))(\|x\|^2_{H^1_0(0,l)} + y^2).$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \sqrt{\varepsilon^{-1}\kappa(\varepsilon)}\|\mu\|_{U_2}.$

For all $(x,y) \in G^-$ we have

$$V(g(x,y,\mu)) \leq \left(\frac{\sigma}{2} + \varepsilon(1 + \|B_1\|)\right)V(x,y),$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \sqrt{\varepsilon^{-1}\kappa(\varepsilon)}\|\mu\|_{U_2}.$

In this case $\psi_1(s) = \left(\frac{\sigma}{2} + \varepsilon(1 + \|B_1\|)\right)s$, $\psi_2(s) = (\lambda_{\max}(B_0) + \varepsilon(1 + \|B_1\|))s.$

If the condition (b) is satisfied, that is $(\varrho^{B_0}_{\max})^{\mathrm{T}}\,\mathrm{diag}\,\{-1,1\}\varrho^{B_0}_{\max} < 0$, then for all $(x,y) \in G^+$ we have

$$V(g(x,y,\mu)) \leq \left(\frac{\sigma}{2} + \varepsilon(1 + \|B_1\|)\right)V(x,y),$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \sqrt{\varepsilon^{-1}\kappa(\varepsilon)}\|\mu\|_{U_2}.$

For all $(x,y) \in G^-$ we have

$$V(g(x,y,\mu)) \leq (\lambda_{\max}(B_0) + \varepsilon\|B_1\|)(\|x\|^2_{H^1_0(0,l)} + y^2)$$
$$+\kappa(\varepsilon)\|\mu\|^2_{U_2} \leq (\lambda_{\max}(B_0) + \varepsilon(1 + \|B_1\|))(\|x\|^2_{H^1_0(0,l)} + y^2).$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \sqrt{\varepsilon^{-1}\kappa(\varepsilon)}\|\mu\|_{U_2}.$ In this case $\psi_1(s) = (\lambda_{\max}(B_0) + \varepsilon(1 + \|B_1\|))s$, $\psi_2(s) = \left(\frac{\sigma}{2} + \varepsilon(1 + \|B_1\|)\right)s.$

If

$$c^2 + \frac{\pi^2 a^2}{l^2} > \frac{l}{\pi}\|D\|_{L^2[0,l]} + \|B\|_{H^1_0(0,l)},$$

then $W(x,y) = 0$ implies

$$\dot{W}(x,y,\mu) \geq \left(c^2 + \frac{\pi^2 a^2}{l^2} - (\frac{l}{\pi}\|D\|_{L^2[0,l]}\right.$$
$$\left.+\|B\|_{H^1_0(0,l)}) - 2\varepsilon\right)(\|x\|^2_{H^1_0(0,l)} + y^2) > 0,$$

if $\sqrt{\|x\|^2_{H^1_0(0,l)} + y^2} > \varepsilon^{-1}\|\xi\|_{U_1}$, with small enough ε.

14

Now applying Theorem 1 and choosing positive constant ε and δ small enough, we see that the conditions of Proposition 3 guarantee the ISS property for the system (30), which finishes the proof.

References

[1] V. O. Bivzyuk and V. I. Slynko. Sufficient conditions for the stability of linear differential equations with periodic impulse action. *Mat. Sb.*, 210(11):3–23, 2019.

[2] C. Briat. Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems. *Nonlinear Anal. Hybrid Syst.*, 24:198–226, 2017.

[3] S. Dashkovskiy and P. Feketa. Input-to-state stability of impulsive systems and their networks. *Nonlinear Anal. Hybrid Syst.*, 26:190–200, 2017.

[4] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of nonlinear impulsive systems. *SIAM J. Control Optim.*, 51(3):1962–1987, 2013.

[5] A. I. Dvirnyĭ and V. I. Slynko. On the stability of solutions of nonlinear nonstationary systems of differential equations with impulse action in a critical case. *Nelīnīĭnī Koliv.*, 14(4):445–467, 2011.

[6] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840. Springer, 2006.

[7] A. O. Ignatyev. On the stability of invariant sets of systems with impulse effect. *Nonlinear Anal.*, 69(1):53–72, 2008.

[8] B. Jacob, A. Mironchenko, J. R. Partington, and F. Wirth. Noncoercive Lyapunov functions for input-to-state stability of infinite-dimensional systems. *SIAM J. Control Optim.*, 58(5):2952–2978, 2020.

[9] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov. *Theory of impulsive differential equations*, volume 6 of *Series in Modern Applied Mathematics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[10] D. Liberzon. *Switching in systems and control*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2003.

[11] X. Liu and A. Willms. Stability analysis and applications to large scale impulsive systems: a new approach. *Canad. Appl. Math. Quart.*, 3(4):419–444, 1995.

[12] A. Mironchenko, I. Karafyllis, and M. Krstic. Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances. *SIAM J. Control Optim.*, 57(1):510–532, 2019.

[13] A. Mironchenko and C. Prieur. Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Rev.*, 62(3):529–614, 2020.

[14] A. Mironchenko, G. Yang, and D. Liberzon. Lyapunov small-gain theorems for networks of not necessarily ISS hybrid systems. *Automatica J. IFAC*, 88:10–20, 2018.

[15] A. M. Samoĭlenko and N. A. Perestyuk. *Impulsive differential equations*, volume 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*. World Scientific Publishing Co., Inc., River Edge, NJ, 1995. With a preface by Yu. A. Mitropolskiĭ and a supplement by S. I. Trofimchuk, Translated from the Russian by Y. Chapovsky.

[16] V. I. Slynko. On conditions for the stability of motion of linear impulsive systems with delay. *Prikl. Mekh.*, 41(6):130–138, 2005.

[17] V. I. Slyn'ko, O. Tunç, and V. O. Bivziuk. Application of commutator calculus to the study of linear impulsive systems. *Systems Control Lett.*, 123:160–165, 2019.

[18] E. D. Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2013.

[19] A. R. Teel, R. G. Sanfelice, and R. Goebel. Hybrid control systems. In *Mathematics of complexity and dynamical systems. Vols. 1–3*, pages 704–728. Springer, New York, 2012.

[20] H. Ye, A. N. Michel, and L. Hou. Stability analysis of systems with impulse effects. *IEEE Trans. Automat. Control*, 43(12):1719–1723, 1998.