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Abstract

In this paper, we present a general approach to studying the problem of finite-time and fixed-time stabilization of a chain of integrators with input delay. To accomplish this, we first reformulate the chain of integrators with input delay as a cascade ODE-PDE system (i.e., a cascade of a linear transport partial differential equation (PDE) with the chain of integrators) where the transport equation models the effect of the delay on the input. Next, we use a nonlinear infinite-dimensional backstepping transformation to convert the cascade system to a suitable target system that is chosen to be finite-time or fixed-time stable. We perform the stability analysis on the target system by means of classical non-asymptotic concepts and tools such as the linear homogeneity and "generalized $\mathscr{KL}$" functions. Then, we use the inverse transformation to transfer back the stability property to the closed-loop system. Finally, we give some characterizations of finite/fixed time predictor-based controllers followed by numerical simulations.

Key words: Finite-time stability, fixed-time stability, linear homogeneity, delay systems, chain of integrators, input delay, cascade ODE-PDE system.

1 Introduction:

Time-delay systems are ubiquitous in control engineering. As time delays may cause performance degradation or instability of the closed-loop system, control design is a central issue; but due to the infinite-dimensional nature of those systems, control, and estimation still continue to be challenging. Most of the existing results on stabilization and estimation for time-delay systems are based on asymptotic or exponential guarantees, though in many applications (e.g., missile guidance, spacecraft docking, trajectory tracking for nonholonomic mobile robots, finite-time deployment and formation control for multi-agent systems,...) the transient process must occur within a given time while also accounting for the effect of the delay. The need to meet time constraints and increase temporal performance has motivated *non-asymptotic stabilization* that can be classified as finite/-fixed and prescribed-time stabilization. Non-asymptotic concepts have been extensively studied within the framework of linear and nonlinear ordinary differential equations (ODEs). The most widely known and used of these notions is finite-time stability (FTS) which means that the studied system is stable and the solutions converge to the equilibrium in a finite time that depends on the initial conditions. If such a time is independent of the initial conditions (in fact, the settling time is uniformly bounded independent of the initial conditions but dependent on some parameters), the type of stability is referred to as fixed-time stability (FxTS) [15]. More recently, the prescribed-time stabilization concept has arisen, which allows the terminal time to be prescribed independently of initial conditions and parameters. It was originally introduced in [18] and has been the basis of several contributions, see e.g., [7,10,19,21,5] for finite-dimensional systems.

However, finite-/fixed-/prescribed-time concepts for time-delay systems still remain sparse and constitute challenging topics. One may refer first to some of the pioneering contributions on non-asymptotic concepts for time-delay systems e.g. [9] (dealing with finite-time stabilization for a class of triangular time-varying systems described by retarded functional differential equations) and [13] (dealing with finite-time stabilization of linear time-delay systems by using Artstein's transformation). In [6] prescribed-time predictor-based controller for LTI systems with input delay is proposed by relying on the PDE-based backstepping approach and making use of time-varying

⋆ This work has been partially supported by ANR Project Finite4SoS (ANR 15-CE23-0007).

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kernels. An extension of the methodology was carried out in [20] (using reduction-based transformation coupled with a backstepping-forwarding transformation) to achieve prescribed-time stabilization of a class of controllable LTI systems with distributed input delay. Furthermore, key recent contributions [24] and [12] deal with FxTS of general LTI systems with input delay. The former builds upon the Artstein–Kwon–Pearson reduction transformation and uses weighted homogeneity to achieve FxTS for general LTI systems with input delay. The latter proposes a novel strategy based on act-and-wait predictor-based control and opens new research avenues on fixed-time control design.

In this paper, we revisit the problem of finite/fixed stabilization of a chain of integrators with input delay and propose a more general approach for the design of finite/fixed-time state-dependent predictor-based controllers. We use a cascade ODE-PDE system (i.e., a cascade of a linear transport partial differential equation (PDE) with the chain of integrators) where the transport equation models the effect of the delay on the input and builds on a nonlinear infinite-dimensional backstepping transformation inspired by [11]. Compared to [6]-which uses a linear transformation and time-varying tools to ensure the prescribed-time stability property for an LTI system with input delay- our approach uses a nonlinear transformation and nonsmooth tools to ensure a different (i.e. finite/fixed) stability property. Both methods bring different challenges and have specific issues. For example, one of the main drawbacks of the time-varying controllers in the framework of prescribed-time stabilization is their implementability. The approach in this paper allows to perform the stability analysis on a suitable target system (chosen to exhibit the desired stability properties, i.e., either finite time or fixed time) while employing classical notions and tools such as Lyapunov-based characterization of finite/fixed-time stability property of ODEs, the "generalized $\mathscr{K}\mathscr{L}$" (in short "$\mathscr{G}\mathscr{K}\mathscr{L}$") functions [8] and linear homogeneity [16]. Hence, we can provide some characterizations of the resulting finite/fixed-time predictor-based controllers.

It is worth mentioning that [13,24] achieve similar results (finite/fixed-time stabilization of LTI systems with input delay) to ours using Artstein's model reduction. Nevertheless, no state estimates of the closed-loop solution are provided. The actuator dynamic is not identified throughout the analysis, either. Moreover, extensions of [13,24] to complex infinite-dimensional systems (including cascaded systems) with constant/time-varying/state-dependent delays, distributed delays are not straightforward. In contrast, our approach does account for the infinite dimensionality of the input, and may allow possible extensions to more complex infinite dimensional systems (e.g., 1D reaction-diffusion PDEs with delayed boundary) or when just cascading finite/fixed-time ISS subsystems.

This paper is organized as follows. In Section 2, we give some preliminary definitions and tools to be used in the rest of the paper. In Section 3, we give the problem statement in which we present the chain of integrators with input delay and its reformulation within an ODE-PDE setting. In Section 4, we give a general approach to stabilize the chain of integrators in a finite time or fixed time. We present the nonlinear backstepping transformation to transform the ODE-PDE setting to a suitable target system and to come up with a finite/fixed-time predictor-based control. Next, in Section 5 we apply our approach to different target systems to attain finite-time stability or fixed-time stability. Then, we give in Section 6 some numerical simulations to illustrate the results. Finally, conclusions and perspectives are given in Section 7.

Notations:

For any real number $a \geq 0$ and for all $x \in \mathbb{R}$, the signed power a of x is defined by $\{x\}^a = \text{sign}(x)|x|^a$. $\mathbb{R}_+$ denotes the set of non negative real numbers. For $j = 1, \cdots, n$, e_j denotes the j^{th} vector of the unit basis of $\mathbb{R}^n$, and $\mathscr{M}_n(\mathbb{R})$ the set of square real matrices of dimension $n \times n$, then the induced matrix norm is denoted as $\|.\|_{\mathscr{M}_n}$.
We denote by $L^2((0,h),\mathbb{R}^n)$ the set of all functions $f : [0,h] \to \mathbb{R}^n$ such that $\int_0^h \|f(x)\|^2 dx < \infty$ (with $\|\cdot\|$ is the euclidean norm of R^n), and for simplicity, we will use the notation $L^2(0,h)$ or L^2 instead of $L^2((0,h),\mathbb{R}^n)$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class-$\mathscr{K}$ function if it is continuous, zero at zero, and strictly increasing. If in addition, α is unbounded with its argument then α is said to be a class-$\mathscr{K}_\infty$.

2 Preliminaries on non-asymptotic concepts

This section recalls some definitions of non-asymptotic concepts (finite/fixed-time stability in the framework of finite-dimensional systems.

Consider the following autonomous system described by:

$$\dot{z} = g(z), \quad z \in \Omega, \tag{1}$$

where $g : \Omega \to \mathbb{R}^n$ is such that $g(0) = 0$, $\Omega \subset \mathbb{R}^n$ is an open connected set containing the origin ($n \in \mathbb{N}\backslash\{0\}$), and such that (1) has the property of existence and uniqueness of solutions in forward time outside the origin. Let $\mathscr{O}$ be a neighborhood of zero.
Definition 1. *The origin of system* (1) *is said to be*

- ***stable*** *if there is $\sigma \in \mathscr{K}$ such that for any $z_0 \in \mathscr{O}$, the solutions are defined and $\|z(t)\| \leq \sigma(\|z_0\|)$ for all $t \geq 0$,*
- ***asymptotically stable*** *if it is stable and $\lim\limits_{t\to+\infty} \|z(t)\| = 0$ for any $z_0 \in \mathscr{O}$,*
- ***finite-time stable (FTS)*** *if it is stable and for any $z_0 \in \mathscr{O}$ there exists $0 \leq T^{z_0} < +\infty$ such that $z(t) = 0$ for all $t \geq T^{z_0}$. The functional $T(z_0) = \inf\{T^{z_0} \geq 0 : z(t) = 0, \forall t \geq T^{z_0}\}$ defines the settling time of the system* (1),

- ***nearly fixed-time stable (nearly FxTS)*** *if it is stable and for any $\rho > 0$ there exists $0 < T_\rho < +\infty$ such that $\|z(t)\| \leq \rho$ for all $t \geq T_\rho$ and all $z_0 \in \mathscr{O}$,*
- ***fixed-time stable (FxTS)*** *if it is FTS and $\sup_{z_0 \in \mathscr{O}} T(z_0) < +\infty$.*

If $\mathscr{O} = \mathbb{R}^n$, then the corresponding properties are called global.

3 Problem statement

We consider the following chain of integrators with input delay:

$$\begin{aligned} \dot{z}_j(t) &= z_{j+1}(t), \quad j = 1, \dots, n-1, \\ \dot{z}_n(t) &= U(t-h), \end{aligned} \tag{2}$$

where $z(t) = (z_1(t), \dots, z_n(t))^\top \in \mathbb{R}^n$ ($n \in \mathbb{N} \backslash \{0\}$) is the instantaneous state of the system, $U(t) \in \mathbb{R}$ is the control input, and $h > 0$ is a known constant delay.

Our goal is to design a nonlinear predictor-based controller for the system (2) to achieve FTS or FxTS. To this end, the methodology developed in this paper relies on representing the actuator delay as a linear transport PDE and builds upon the cascade ODE-PDE setting (i.e., a cascade of linear hyperbolic PDE with an LTI system) of [11].

Remark 1. Note that in [2], the problem of exponential stabilization of the following class of strict-feedback system with delayed integrators and delayed input

$$\begin{aligned} \dot{z}_j(t) &= \sum_{i=1}^{j} a_{ji} z_i(t) + z_{j+1}(t-h_j), \quad j = 1, \dots, n-1, \\ \dot{z}_n(t) &= \sum_{i=1}^{n} a_{ni} z_i(t) + U(t-h_n), \end{aligned} \tag{3}$$

was solved, where the coefficients $(a_{ji})_{1 \leq i \leq j \leq n}$ are real constants, and $(h_j)_{1 \leq j \leq n}$ are positive known delays. This class of systems is more challenging compared to (2), but can be dealt with to achieve finite/fixed-time stabilization, as shown in details in Section 6, using the approach given in this paper. The idea is to use the state transformation (52)-(53) introduced in [2] (with $c_1, \cdots, c_n = 0$) to get rid of the non-delayed terms from (3). Then, we use the following change of variables $\bar{z}_{j+1}(t) = z_{j+1}(t - \sum_{i=1}^{j} h_i)$ for all $j = 1, \dots, n$, which moves the delays to the last equation. As a result, we obtain a similar system to (2) with some additional delayed and non-delayed terms in the last equation. Therefore, studying (3) comes down to studying (2). This motivates applying our approach to the simplest case (2) in order to better communicate the ideas of the approach.

We henceforth represent system (2) as

$$\begin{aligned} \dot{z}_j(t) &= z_{j+1}(t), \quad j = 1, \dots, n-1, \\ \dot{z}_n(t) &= u(t,0), \\ u_t(t,x) &= u_x(t,x), \\ u(t,h) &= U(t). \end{aligned} \tag{4}$$

with $t \geq t_0 \geq 0$, $x \in [0,h]$, and $u(t,\cdot)$ is the transport PDE state whose solution is given by

$$u(t,x) = \begin{cases} u_0(t+x-t_0), & t_0 \leq t+x \leq t_0+h, \\ U(t+x-h), & t+x \geq t_0+h, \end{cases}$$

with u_0 is a bounded function in L^2. We denote by $u_t(t,x)$ (resp. $u_x(t,x)$) the partial derivative of u with respect to the time (resp. space) variable t (resp. x).

The objective of the first part of this paper is to give a general approach to design a controller (predictor-type) for the system (2), to attain FTS and/or FxTS. We employ a nonlinear infinite-dimensional backstepping transformation. The key idea is to transform the original system into a suitable target system that is chosen to exhibit the FTS or FxTS properties.

4 Finite/Fixed-time predictor-based controller via PDE-based backstepping approach

4.1 Nonlinear infinite-dimensional backstepping transformation

Inspired by [3] and [11, Chapter 10], we consider the following nonlinear infinite-dimensional backstepping transformation:

$$\omega(t,x) = u(t,x) - \mathscr{F}(\varphi_1(t,x), \dots, \varphi_n(t,x)), \tag{5}$$

where $\mathscr{F}$ is a suitable nonlinear function to be characterized later on, and $\varphi_1, \dots, \varphi_n$ are the solutions of

$$\begin{aligned} &\varphi_{j,x}(t,x) = \varphi_{j+1}(t,x), \\ &\quad \varphi_j(t,0) = z_j(t), j = 1, \dots, n-1, \end{aligned} \tag{6}$$

$$\begin{aligned} &\varphi_{n,x}(t,x) = u(t,x), \\ &\quad \varphi_n(t,0) = z_n(t). \end{aligned} \tag{7}$$

Notice that $\varphi_i(t,x) = z_i(t+x)$ for all $i = 1, \cdots, n$, all $t \geq t_0$ and all $x \in [0,h]$. Then, by the variation of the constant formula, we obtain:

$$\varphi_i(t,x) = \int_0^x \frac{(x-y)^{n-i}}{(n-i)!} u(t,y) dy + \sum_{j=i}^{n} \frac{x^{j-i}}{(j-i)!} z_j(t). \tag{8}$$

The proof of (8) is as follows:

$$z(t+x) = e^{Ax} z(t) + \int_t^{t+x} e^{A(t+x-s)} B u(s,0) ds,$$

with $z = (z_1, \cdots, z_n)^\top$, $B = e_n$, and $A := \{A_{ij}\} \in \mathbb{R}^{n\times n}$, where $A_{ij} = 1$ if $j = i+1$ and $A_{ij} = 0$ otherwise.

Next, using the change of variables $y = s - t$, we get

$$\begin{aligned} z(t+x) &= e^{Ax}z(t) + \int_0^x e^{A(x-y)}Bu(t+y,0)dy, \\ &= e^{Ax}z(t) + \int_0^x e^{A(x-y)}Bu(t,y)dy.. \end{aligned}$$

Using the expression $e^{Ax} = \sum_{k=0}^{n-1} \frac{x^k}{k!}A^k$, we recover

$$z(t+x) = \sum_{k=0}^{n-1} \frac{x^k}{k!}A^k z(t) + \sum_{k=0}^{n-1} \int_0^x \frac{(x-y)^k}{k!}A^k Bu(t,y)dy.$$

Then, using the fact $A^k z(t) = (z_{k+1}(t), \cdots, z_n(t), 0, \cdots, 0)^\top$ and $A^k B = e_{n-k}$, we obtain

$$\begin{aligned} z_i(t+x) &= \sum_{k=0}^{n-i} \frac{x^k}{k!} z_{k+i}(t) + \int_0^x \frac{(x-y)^{n-i}}{(n-i)!}u(t,y)dy, \\ &= \sum_{j=i}^{n} \frac{x^{j-i}}{(j-i)!} z_j(t) + \int_0^x \frac{(x-y)^{n-i}}{(n-i)!}u(t,y)dy. \end{aligned}$$

Next, using (5) we transform the system (4) into the following nonlinear target system:

$$\begin{aligned} &\dot{z}_j(t) = z_{j+1}(t), \quad j = 1, \ldots, n-1, \\ &\dot{z}_n(t) = \mathscr{F}(z_1(t), \ldots, z_n(t)) + \omega(t,0), \\ &\omega_t(t,x) = \omega_x(t,x), \\ &\omega(t,h) = 0, \end{aligned} \tag{9}$$

with $\omega : [0, +\infty) \times [0,h] \to \mathbb{R}$ is the transport PDE state. The nonlinear function $\mathscr{F}$, to be specified latter on (see Subsection 4.3), is suitably chosen to get FTS/FxTS of the target system when $\omega(t,0)$ becomes zero (this key feature of the transport PDE ω is discussed in Subsection 4.4).

Note that using the fact that $\varphi(t,x) = z(t+x)$ for all $t \geq t_0$ and al $x \in [0,h]$, it is clear that the nonlinear transformation (5) satisfies the PDE part of (9) (i.e. $\omega_t(t,x) = \omega_x(t,x)$).

4.2 *Inverse transformation*

The inverse transformation is given by,

$$u(t,x) = \omega(t,x) + \mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x)), \tag{10}$$

where $\bar{\varphi}_1, \ldots, \bar{\varphi}_n$ are the solutions of:

$$\begin{aligned} &\bar{\varphi}_{j,x}(t,x) = \bar{\varphi}_{j+1}(t,x), \quad j = 1, \ldots, n-1, \\ &\quad \bar{\varphi}_j(t,0) = z_j(t), \ j = 1, \ldots, n-1, \end{aligned} \tag{11}$$

$$\begin{aligned} &\bar{\varphi}_{n,x}(t,x) = \mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x)) + \omega(t,x), \\ &\quad \bar{\varphi}_n(t,0) = z_n(t). \end{aligned} \tag{12}$$

Similarly to the direct transformation, we recover from the inverse transformation: $u_t(t,x) = u_x(t,x)$.

4.3 *On the selection of the Finite/Fixed-time predictor-based controller*

In this section, we give an important assumption on the nonlinear function $\mathscr{F}$ given in (9) to ensure FTS or FxTS properties. Then, we give the expression of our predictor-based controller $U(t)$ using the transformation (5) or (10).

4.3.1 *An assumption on the nonlinear function $\mathscr{F}$*

In order to ensure that the target system (9) is FTS (resp. FxTS), let us assume that $\mathscr{F}$ satisfies the following assumption (for the ODE part of (9)):

Assumption 1. *$\mathscr{F}$ is a continuous nonlinear function, differentiable everywhere except at zero, such that $\mathscr{F}(0, \cdots, 0) = 0$ and the origin of the following system ($\dot{z}_j(t) = z_{j+1}(t)$, $j = 1, \ldots, n-1$, $\dot{z}_n(t) = \mathscr{F}(z_1(t), \ldots, z_n(t))$) is FTS (resp. FxTS) i.e., there exists a class $\mathscr{GKL}$ function β such that the solution of the previous system, satisfies:*

$$\|z(t)\| \leq \beta(\|\bar{z}_0\|, t - \bar{t}_0), \ \forall t \geq \bar{t}_0, \tag{13}$$

where $\bar{z}_0 = (\bar{z}_{1,0}, \ldots, \bar{z}_{n,0})$ is the initial condition at time $\bar{t}_0$. Moreover, there exists an increasing function $T(\cdot)$ such that $\|z(t)\| \to 0$ when $t \to \bar{t}_0 + T(\|z(\bar{t}_0)\|)$, (resp. a positive real constant $T_{\max}$ such that $\|z(t)\| = 0$ when $t \geq \bar{t}_0 + T_{\max}$).

Remark 2. A construction of a Lyapunov function $t \mapsto V(z(t), \omega(t,\cdot))$ such that one has an estimate of this type $\dot{V}(z(t), w(t,\cdot)) \leq -c_1 V^\alpha(z(t), w(t,\cdot)) - c_2 V^\beta(z(t), w(t,\cdot))$, $c_1, c_2 > 0, \alpha \in (0,1), \beta > 1$, could be an alternative yielding the finite/fixed time stability property to the target system and thereby the original one. However, unfortunately, for a cascade nonlinear ODE - transport PDE system (such as (9)), it is still unclear whether one can construct such a Lyapunov function (without even the PDE part). This is one reason why, our approach relies on $\mathscr{GKL}$-class functions β and estimates on the solutions.

4.3.2 *Finite/Fixed-time predictor-based controller*

Under Assumption 1, and from (5) at $x = h$, and using (8), the boundary control is then,

$$U(t) = u(t,h) = \mathscr{F}(\varphi_1(t,h), \ldots, \varphi_n(t,h)), \tag{14}$$

where $\varphi_i(t,h) = \int_0^h \frac{(h-y)^{n-i}}{(n-i)!}u(t,y)dy + \sum_{j=i}^{n} \frac{h^{j-i}}{(j-i)!} z_j(t)$.

Or form (10), at $x = h$,

$$U(t) = u(t,h) = \mathscr{F}(\bar{\varphi}_1(t,h), \ldots, \bar{\varphi}_n(t,h)), \tag{15}$$

where $\bar{\varphi}_1, \ldots, \bar{\varphi}_n$ are resp. the solutions of (11)-(12).

4.4 Stability analysis

In this subsection, we first perform the stability analysis on the target system (9). Then, we use the inverse transformation (10) to establish the boundedness of the state of the original system (4) and its convergence to zero in finite-time (resp. fixed-time) using a suitable norm equivalence.

Proposition 1. *Let $t \geq t_0 + h$, there exists a class $\mathscr{GKL}$ function β_1 such that the solution of system* (11)-(12)*, $\bar{\varphi}(t,x) = (\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))$ satisfies*

$$\|\bar{\varphi}(t,\cdot)\|_{L^2} \leq \beta_1(\|z(t_0+h)\|, t-t_0-h). \quad (16)$$

Moreover, $\|\bar{\varphi}(t,\cdot)\|_{L^2} \to 0$ when $t \to t_0+h+T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$), where $\|\bar{\varphi}(t,\cdot)\|_{L^2}^2 = \sum_{j=1}^{n} \|\bar{\varphi}_j(t,\cdot)\|_{L^2}^2$, $T(\cdot)$ and T_{max} are given in Assumption 1.

Proof. By the method of the characteristics, the solution of the $\omega-$dynamics of the target system (9) for any $t \geq t_0+h$ is zero (i.e. $\omega(t,x) = 0$ for all $x \in [0,h]$ and $t \geq t_0+h$). Then, we can conclude using Assumption 1 (replacing "t" by "x") that the solution of the system (11)-(12) satisfies for $t \geq t_0+h$

$$\|\bar{\varphi}(t,x)\| \leq \beta(\|z(t)\|, x), \ \forall x \in [0,h],$$

where β is a class $\mathscr{GKL}$ function. Moreover, there exists an increasing function $T(\cdot)$ such that $\|\bar{\varphi}(t,x)\| \to 0$ when $x \to T(\|z(t)\|)$ (resp. $x \to T_{max}$). Furthermore, when $t \to t_0 + h + T(\|z(t_0+h)\|)$ (resp. $t \to t_0 + h + T_{max}$) $\|\bar{\varphi}(t,x)\| \to 0$.

Now, using the fact that β is decreasing with respect to the second variable x, we get,

$$\|\bar{\varphi}(t,x)\| \leq \beta(\|z(t)\|, 0), \ \forall x \in [0,h], \ \forall t \geq t_0+h. \quad (17)$$

By squaring and integrating with respect to x from 0 to h, then passing to the square roots, we find,

$$\|\bar{\varphi}(t,\cdot)\|_{L^2} \leq \sqrt{h}\beta(\|z(t)\|, 0), \ \forall t \geq t_0+h.$$

Next, we use inequality (13) from Assumption 1 to obtain,

$$\|\bar{\varphi}(t,\cdot)\|_{L^2} \leq \sqrt{h}\beta(\beta(\|z(t_0+h)\|, t-t_0-h), 0), \ \forall t \geq t_0+h.$$

Then, $\|\bar{\varphi}(t,\cdot)\|_{L^2} \leq \beta_1(\|z(t_0+h)\|, t-t_0-h), \ \forall t \geq t_0+h$, where for any $s,t \in \mathbb{R}_+$, β_1 is a class $\mathscr{GKL}$ function given by, $\beta_1(s,t) = \sqrt{h}\beta(\beta(s,t),0)$. Furthermore, when $t \to t_0 + h + T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$), $\beta(s,t) \to 0$, and by continuity $\beta_1(s,t) \to 0$. ∎

Proposition 2. *There exists a class $\mathscr{GKL}$ function β_2 such that for any $x \in [0,h]$, $\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))$ satisfies for $t \geq t_0+h$,*

$$|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))| \leq \beta_2(\|z(t_0+h)\|, t-t_0-h), \quad (18)$$

and

$$\|\mathscr{F}(\bar{\varphi}_1(t,\cdot), \ldots, \bar{\varphi}_n(t,\cdot))\|_{L^2} \leq \sqrt{h}\beta_2(\|z(t_0+h)\|, t-t_0-h). \quad (19)$$

Moreover, for all $x \in [0,h]$, $|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))| \to 0$ and $\|\mathscr{F}(\bar{\varphi}_1(t,\cdot), \ldots, \bar{\varphi}_n(t,\cdot))\|_{L^2} \to 0$ when $t \to t_0+h+T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$).

Proof. Let $x \in [0,h]$. We can see from Proposition 1 that $\|\bar{\varphi}(t,x)\| \to 0$ when $t \to t_0+h+T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$), and stays zero after. Next, by continuity of $\mathscr{F}$, we also have $\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x)) \to 0$ when $t \to t_0+h+T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$). Then, there exists a class $\mathscr{GKL}$ function β_2 such that $\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))$ satisfies,

$$|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))| \leq \beta_2(\|z(t_0+h)\|, t-t_0-h),$$

for all $x \in [0,h]$ and $t \geq t_0+h$.

Next, by squaring and integrating from 0 to h with respect to x and passing to the square roots, we find,

$$\|\mathscr{F}(\bar{\varphi}_1(t,\cdot), \ldots, \bar{\varphi}_n(t,\cdot))\|_{L^2} \leq \sqrt{h}\beta_2(\|z(t_0+h)\|, t-t_0-h),$$

for all $t \geq t_0+h$. In addition, for all $x \in [0,h]$, we have, $|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))| \to 0$ and $\|\mathscr{F}(\bar{\varphi}_1(t,\cdot), \ldots, \bar{\varphi}_n(t,\cdot))\|_{L^2} \to 0$ when $t \to t_0+h+T(\|z(t_0+h)\|)$ (resp. $t \to t_0+h+T_{max}$). ∎

Proposition 3. *From the transformation* (10)*, the following estimate holds for $t \geq t_0+h$:*

$$\|u(t,\cdot)\|_{L^2} \leq \sqrt{2h}\beta_2(\|z(t_0+h)\|, t-t_0-h), \quad (20)$$

where β_2 is a class $\mathscr{GKL}$ function given in Proposition 2 .

Proof. Using (10), we have

$$|u(t,x)| \leq |\omega(t,x)| + |\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))|, \forall t \geq t_0.$$

Next, squaring the previous inequality and using Young inequality, we get,

$$|u(t,x)|^2 \leq 2|\omega(t,x)|^2 + 2|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))|^2.$$

Now, using $\omega(t,x) = 0, \forall t \geq t_0+h, \forall x \in [0,h]$, we obtain $|u(t,x)|^2 \leq 2|\mathscr{F}(\bar{\varphi}_1(t,x), \ldots, \bar{\varphi}_n(t,x))|^2 \ \forall t \geq t_0+h, \forall x \in [0,h]$. Finally, by integrating from 0 to h with respect to the space variable x and passing to the square roots, we get

$$\|u(t,\cdot)\|_{L^2} \leq \sqrt{2}\|\mathscr{F}(\bar{\varphi}_1(t,\cdot), \ldots, \bar{\varphi}_n(t,\cdot))\|_{L^2}, \ \forall t \geq t_0+h.$$

Using inequality (19) from Proposition 2, we obtain, $\|u(t,\cdot)\|_{L^2} \leq \sqrt{2h}\beta_2(\|z(t_0+h)\|, t-t_0-h)$, $\forall t \geq t_0+h$. ■

Theorem 1. *Let the input initial condition $U_{t_0} : s \in [-h,0] \mapsto U(t_0+s)$ be defined and bounded in $L^2(-h,0)$. Let $h>0$ and $t_0 \geq 0$. Then, the solution of the closed-loop system* (4) *with finite-time (resp. fixed-time) predictor-based controller* (14) *(or* (15)*) is FTS (resp. FxTS) in the following sense: For any initial condition $z_0 \in \mathbb{R}^n$, the quantity $I(t) = \|z(t)\|^2 + \|u(t,\cdot)\|^2_{L^2}$ remains bounded for $t \in [t_0, t_0+h]$, and for all $t \in [t_0+h, t_0+h+T\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty)\right))$ (resp. $t \in [t_0+h, t_0+h+T_{max})$), there exists a class $\mathscr{GKL}$ function β_3 such that,*

$$I(t) \leq \beta_3\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty), t-t_0-h\right), \tag{21}$$

with $\mathscr{B}_h(s_1,s_2) = e^h s_1 + he^{2h}s_2$ for any $s_1, s_2 \geq 0$.

In particular, $I(t) \to 0$ and $|U(t)| \to 0$, as $t \to t_0+h+T\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty)\right)$ (resp. $t \to t_0+h+T_{max}$).

Proof. Let us start by proving the boundedness of $\|z(t)\|$ for all $t \in [t_0, t_0+h]$. By the variation of the constant formula on (2) we recover,

$$\begin{aligned} z(t) &= e^{A(t-t_0)}z_0 + \int_{t_0}^t e^{A(t-y)}BU(y-h)dy, \\ &= e^{A(t-t_0)}\left[z_0 + \int_{t_0}^t e^{A(t_0-y)}BU(y-h)dy\right]. \end{aligned}$$

Using the change of variables $s = y-h-t_0$, we obtain

$$z(t) = e^{A(t-t_0)}\left[z_0 + \int_{-h}^{t-t_0-h} e^{A(-s-h)}BU(t_0+s)ds\right].$$

Using $\|e^{A(t-t_0)}\|_{\mathscr{M}_n} \leq e^h$ and $\|e^{A(-s-h)}\|_{\mathscr{M}_n} \leq e^h$, we get

$$\begin{aligned} \|z(t)\| &\leq e^h\|z_0\| + e^{2h}\int_{-h}^{t-t_0-h}|U_{t_0}(s)|ds, \\ &\leq e^h\|z_0\| + e^{2h}\int_{-h}^{0}|U_{t_0}(s)|ds, \\ &\leq e^h\|z_0\| + he^{2h}\|U_{t_0}\|_\infty, \\ &= \mathscr{B}_h\left(\|z_0\|, \|U_{t_0}\|_\infty\right). \end{aligned} \tag{22}$$

Then, $\|z(\cdot)\|$ is bounded in $[t_0, t_0+h]$.

Next, let us prove inequality (21). Let $t \geq t_0+h$. Using (20) from Proposition 3, we have

$$I(t) \leq \|z(t)\|^2 + 2h\beta_2(\|z(t_0+h)\|, t-t_0-h)^2.$$

By inequality (13) from Assumption 1, we get

$$I(t) \leq \beta(\|z(t_0+h)\|, t-t_0-h) + 2h\beta_2(\|z(t_0+h)\|, t-t_0-h)^2,$$

which leads to $I(t) \leq \beta_3(\|z(t_0+h)\|, t-t_0-h)$, with $\beta_3 = \beta + 2h\beta_2^2$ is a class $\mathscr{GKL}$ function.

Then, using inequality (22), we obtain, $I(t) \leq \beta_3\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty), t-t_0-h\right)$. In particular, we recover that $I(t) \to 0$ when $t \to t_0+h+T\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty)\right)$ (resp. when $t \to t_0+h+T_{max}$) and that $\|z(t)\|$ is bounded for all $t \geq t_0$.

Now, let us prove that $\|u(t,\cdot)\|_{L^2}$ is bounded for all $t \in [t_0, t_0+h]$. Notice that the solution u is given by

$$u(t,x) = \begin{cases} u_0(t+x-t_0), & t \in [t_0, t_0+h-x], \\ U(t+x-h), & t \in [t_0+h-x, t_0+h]. \end{cases}$$

From this last equation, it is easy to deduce the boundedness of $\|u(t,\cdot)\|_{L^2}$ using the transformation (10), the fact that $|U(t+x-h)| \leq |\mathscr{F}(z_1(t+x), \ldots, z_n(t+x))|$ and the boundedness of $||z(t+x)||$ for all $t+x \geq t_0$. As a result, $I(t)$ is bounded for all $t \in [t_0, t_0+h]$.

Finally, let us prove that $|U(t)| \to 0$ as $t \to t_0+h+T\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty)\right)$ (resp. $t \to t_0+h+T_{max}$). From the equation (14), we have,

$$|U(t)| = |\mathscr{F}(\bar{\varphi}_1(t,h), \ldots, \bar{\varphi}_n(t,h))|.$$

and that $|U(t)|$ is bounded for all $t \in [t_0, t_0+h]$ (because $\bar{\varphi}(t,h) = z(t+h)$ bounded for all $t \geq t_0$).

By Proposition 2, we obtain from inequality (18),

$$|U(t)| \leq \beta_2(\|z(t_0+h)\|, t-t_0-h), \forall t \geq t_0+h.$$

Then, using inequality (22), we get

$$|U(t)| \leq \beta_2\left(\mathscr{B}_h(\|z_0\|, \|U_{t_0}\|_\infty), t-t_0-h\right).$$

From where we deduce the desired property. ■

5 Some characterizations of $\mathscr{F}$ for the design of finite-time/fixed-time predictor-based controllers

The previous section provides a general setting in which, as soon as one chooses $\mathscr{F}$ satisfying Assumption 1, one can design a nonlinear predictor-based controller to stabilize the system (4) in finite time or in fixed time. In this section, let us give some characterizations of $\mathscr{F}$. For simplicity let us take $t_0 = 0$.

5.1 *Explicit controllers for double chain of integrators*

For the target system (9) with $n=2$, we propose the following characterization of $\mathscr{F}$ inspired by [4]:

$$\mathscr{F}(z_1(t), z_2(t)) = -k_1\{z_1(t)\}^{\alpha_1} - k_2\{z_2(t)\}^{\alpha_2}, \tag{23}$$

which satisfies Assumption 1 as soon as k_1, k_2 are any positive reals numbers and α_1, α_2 are selected so that weighted homogeneity of negative degree κ is obtained for (9) with $n=2$ and $\omega \equiv 0$: for example by selecting $r > -2\kappa$ and

$$r_1 = r, r_2 = r+\kappa, \alpha_1 = \frac{r+2\kappa}{r}, \alpha_2 = \frac{r+2\kappa}{r+\kappa}.$$

Hence, we can realize the resulting nonlinear predictor-based controller $U(t)$ (14), with $\mathscr{F}$ having the structure in (23), stabilizing the system (4) in finite time.

Let us give now a characterization of $\mathscr{F}$ to get a FxTS counterpart:

Proposition 4. *The $z-$subsystem of the target system* (9) *with $n=2$ is FxTS when $\mathscr{F}$ is selected as follows:*

$$\mathscr{F}(z_1(t), z_2(t)) = -k_{1,0}\{z_1(t)\}^{\alpha_{1,0}} - k_{2,0}\{z_2(t)\}^{\alpha_{2,0}} \\ -k_{1,\infty}\{z_1(t)\}^{\alpha_{1,\infty}} - k_{2,\infty}\{z_2(t)\}^{\alpha_{2,\infty}}, \quad (24)$$

where $k_{1,0}, k_{2,0}, k_{1,\infty}, k_{2,\infty}$ are any positive real numbers and

$\alpha_{1,0} = \frac{r_0+2\kappa_0}{r_0}$, $\alpha_{1,\infty} = \frac{r_\infty+2\kappa_\infty}{r_\infty}$, $\alpha_{2,0} = \frac{r_0+2\kappa_0}{r_0+\kappa_0}$, $\alpha_{2,\infty} = \frac{r_\infty+2\kappa_\infty}{r_\infty+\kappa_\infty}$ *with $\kappa_0 < 0$, $\kappa_\infty > 0$, $r_0 > -2\kappa_0$, $r_\infty > 0$.*

Proof. Consider (9) with $\omega \equiv 0$. Using LaSalle invariance principle with $V(z) = \int_0^{z_1} k_{1,0}\{s\}^{\alpha_{1,0}} + k_{1,\infty}\{s\}^{\alpha_{1,\infty}} ds + \frac{z_2^2}{2}$ combined with [1, Corollary 2.24] ends the proof. ■

5.2 *Implicit Controllers for chain of integrators*

For the $z-$subsystem of the target system (9),we can use the results from [16,22,23] to characterize a new $\mathscr{F}$ from which we can subsequently design a nonlinear predictor-based controller $U(t)$ achieving FTS or nearly FxTS. However, for such a chain of integrators, it appears that $G_{\mathbf{d}}$ has to be of the form $G_{\mathbf{d}} = \mathrm{diag}(r_1, \ldots, r_n)$, $r_i = r+(i-1)\kappa, r > \max(0, -n\kappa), \kappa \in \mathbb{R}$ (κ is the degree of homogeneity) then

$$\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \mathrm{diag}(e^{r_1 s}, \ldots, e^{r_n s}). \quad (25)$$

Note that $AG_{\mathbf{d}} - G_{\mathbf{d}}A = \kappa A$ (the driftless part "Az" is homogeneous) [17] can be rephrased as:

Proposition 5. *[see [17] for details] Let a, b be chosen positive real numbers. For the $z-$subsystem of the target system* (9)*, let*

$$\mathscr{F}(z) = \|z\|_{\mathbf{d}}^{r+n\kappa} k\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z, \quad (26)$$

where $\mathbf{d}$ is the dilation defined by (25) *with $r_i = r+(i-1)\kappa, r > \max(0, -n\kappa)$, $\|z\|_{\mathbf{d}}$ is its associated homogeneous norm and gain $k = yP$ is derived from the solution $X \in \mathbb{R}^{n\times n}$ ($X = P^{-1}$), $y \in \mathbb{R}^{1\times n}$ of the LMIs:*

$$XA^\top + AX + y^\top B^\top + By + aX \le 0, \quad (27)$$
$$X > 0, \quad bX \ge G_{\boldsymbol{d}}X + XG_{\boldsymbol{d}}^\top > 0, \quad (28)$$

where $A = ((0_{(n-1)\times 1}, I_{n-1})^\top, 0_{n\times 1})^\top, B = (0, \ldots, 0, 1)^\top$. Then, the $z-$subsystem of (9) *with $w \equiv 0$ is*

- *globally FTS for $\kappa < 0$ and the settling time is given by*

$$T(z_0) \le \frac{b}{a(-\kappa)}\|z_0\|_{\boldsymbol{d}}^{-\kappa},$$

- *globally uniformly exponentially stable for $\kappa = 0$,*
- *globally nearly fixed-time stable for $\kappa > 0$.*

Similarly, we get:

Proposition 6. *[see [17,22,23] for an equivalent formulation] Select $\kappa_0 < 0, \kappa_\infty > 0$ and $r_0 > -n\kappa_0, r_\infty > 0$. Let us define $r_{i,0} = r_0 + (i-1)\kappa_0, r_{i,\infty} = r_\infty + (i-1)\kappa_\infty$. Set $\boldsymbol{d}_0(s) = e^{G_{\boldsymbol{d}_0}s} = \mathrm{diag}(e^{r_{1,0}s}, \ldots, e^{r_{n,0}s})$ and $\boldsymbol{d}_\infty(s) = e^{G_{\boldsymbol{d}_\infty}s} = \mathrm{diag}(e^{r_{1,\infty}s}, \ldots, e^{r_{n,\infty}s})$. Let $a_0, b_0, a_\infty, b_\infty$ be chosen positive reals. For the $z-$subsystem of the target system* (9)*, let*

$$\mathscr{F}(z) = \begin{cases} \|z\|_{\mathbf{d}_0}^{r_0+n\kappa_0} k_0\mathbf{d}_0(-\ln\|z\|_{\mathbf{d}_0})z & \text{for } \|z\| < 1 \\ \|z\|_{\mathbf{d}_\infty}^{r_\infty+n\kappa_\infty} k_\infty\mathbf{d}_\infty(-\ln\|z\|_{\mathbf{d}_\infty})z & \text{for } \|z\| \ge 1 \end{cases} \quad (29)$$

where the gains k_0 and k_∞ are such that the LMIs

$$X_0A^\top + AX_0 + y_0^\top B^\top + By_0 + a_0X_0 \le 0, \quad (30)$$
$$X_0 > 0, \quad b_0X_0 \ge G_{\boldsymbol{d}_0}X_0 + XG_{\boldsymbol{d}_0}^\top > 0, \quad (31)$$
$$X_\infty A^\top + AX_\infty + y_\infty^\top B^\top + By_\infty + a_\infty X_\infty \le 0, \quad (32)$$
$$X_\infty > 0, \quad b_\infty X_\infty \ge G_{\boldsymbol{d}_\infty}X_\infty + XG_{\boldsymbol{d}_\infty}^\top > 0, \quad (33)$$

have solution X_0 and X_∞ in $\mathbb{R}^{n\times n}$, y_0 and y_∞ in $\mathbb{R}^{1\times n}$ (where $k_0 = y_0P_0, P_0 = X_0^{-1}, k_\infty = y_\infty P_\infty, P_\infty = X_\infty^{-1}$).
Then, the $z-$subsystem of (9) *with $w \equiv 0$ is globally FxTS.*

6 Simulations

In this section, we focus on (3) for $n=2$, i.e.

$$\begin{aligned} \dot{z}_1(t) &= a_{11}z_1(t) + z_2(t-h_1), \\ \dot{z}_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) + U(t-h_2), \end{aligned} \quad (34)$$

where a_{11}, a_{21} and a_{22} are real constants, h_1 and h_2 are positive known delays.

By combining the state transformations (52)-(53) introduced in [2] with the change of variables $\tilde{z}_1(t) = z_1(t)$, $\tilde{z}_2(t) = z_2(t-h_1)$, we recover the following transformations:

$$\begin{aligned} \bar{z}_1(t) &= z_1(t), \\ \bar{z}_2(t) &= z_2(t-h_1) + a_{11}z_1(t). \end{aligned} \quad (35)$$

Thus, (34) is transformed into

$$\begin{aligned} \dot{\bar{z}}_1(t) &= \bar{z}_2(t), \\ \dot{\bar{z}}_2(t) &= -a_{11}a_{22}\bar{z}_1(t) + (a_{11}+a_{22})\bar{z}_2(t) \\ &\quad + a_{21}\bar{z}_1(t-h_1) + U(t-h_1-h_2), \end{aligned} \quad (36)$$

which is rewritten into

$$\begin{aligned}
\dot{\bar{z}}_1(t) &= \bar{z}_2(t),\\
\dot{\bar{z}}_2(t) &= -a_{11}a_{22}\bar{z}_1(t) + (a_{11}+a_{22})\bar{z}_2(t)\\
&\quad + a_{21}\bar{z}_1(t-h_1) + u(t,0),\\
u_t(t,x) &= u_x(t,x),\\
u(t,h_1+h_2) &= U(t),
\end{aligned} \tag{37}$$

with $x \in [0, h_1+h_2]$.

Remark that (36) is similar to (2) and clearly the approach developed in this paper can be adapted to it to obtain the following control

$$\begin{aligned}
U(t) &= \mathscr{F}(\varphi_1(t,h_1+h_2), \varphi_2(t,h_1+h_2)) - a_{21}\varphi_1(t,h_2)\\
&\quad - (a_{11}+a_{22})\varphi_2(t,h_1+h_2) + a_{11}a_{22}\varphi_1(t,h_1+h_2),
\end{aligned} \tag{38}$$

where φ_1 and φ_2 are solutions of

$$\begin{aligned}
\varphi_{1,x}(t,x) &= \varphi_2(t,x),\\
\varphi_{2,x}(t,x) &= -a_{11}a_{22}\varphi_1(t,x) + (a_{11}+a_{22})\varphi_2(t,x)\\
&\quad + a_{21}\varphi_1(t,x-h_1) + u(t,x).
\end{aligned} \tag{39}$$

with $x \in [0, h_1+h_2]$ and $u(t,x)$ the solution of the PDE part of (37).

Let us now give numerical simulations for the closed-loop system (37) with predictor-based controller $U(t)$ in (38). First, using $\mathscr{F}$ given in (23) to attain FTS where we choose the delays $h_1 = 0.75$s, $h_2 = 1$s, and the parameters as follows: $\kappa = -0.2$, $r = 3$, $k_1 = 10$ and $k_2 = 11$. Then, using $\mathscr{F}$ given in (24) to attain FxTS where we take the delays $h_1 = 0.5$s, $h_2 = 0.75$s, and the parameters as follows: $\kappa_0 = -0.5$, $r_0 = 2$, $k_{1,0} = 10$, $k_{2,0} = 11$, $\kappa_\infty = -0.2$, $r_\infty = 3$, $k_{1,\infty} = 11$, $k_{2,\infty} = 10$. Finally, we take the initial time $t_0 = 0$, the coefficients $a_{11} = a_{21} = a_{22} = 1$ and we give the simulations for three different initial conditions: $z_0 = (5,3)^\top$, $10z_0$ and $100z_0$.

Figure 1 shows on the left the evolution of the states z_1 and z_2 of the ODE part of the closed-loop system (37) with predictor-based controller $U(t)$ in (38) (whose time evolution is described in Figure 3 alongside of the time evolution of the norm of (37) for different values of the delays, using the expression of $\mathscr{F}$ in (23) to get FTS. On the right hand we can see the numerical solution $u(t,x)$ of the PDE part of with respect the initial conditions $z(t_0) = (5,3)^\top$ and $u(t_0,x) = 0$, $x \in [0, h_1+h_2]$. Finally, Figure 2 shows in a logarithmic scale the evolution of the norm $\|z(t)\|^2$ of the closed-loop system (34) with predictor-based controller $U(t)$ in (38) on the left using the expression of $\mathscr{F}$ in (23) and on the right using the expression of $\mathscr{F}$ in (24). As it can be observed on the left, the times of convergence depend on the initial conditions (the larger the initial condition, the larger the settling time). On the right-hand side, we can observe that the times of the convergence do not depend on the initial conditions (the settling time is upper bounded by a constant independent of the initial conditions).

7 Conclusion

This paper deals with finite-time and fixed-time stabilization of a chain of integrators with input delay. The chain of integrators is rewritten into an ODE-PDE setting, where the PDE part models the effect of the delay on the input. The predictor-based controller is designed using a nonlinear infinite-dimensional backstepping transformation that links the ODE-PDE setting to the target system. The convergence rate (finite time or fixed time) is ensured by the inverse transformation and using $\mathscr{GKL}$ functions. Some characterizations of $\mathscr{F}$ of the target system are then given followed by numerical simulations to illustrate the results. Future work will extend this result to LTI systems with time-varying input delay or distributed input delay (still under a PDE-based backstepping approach). The problem of the "robustification" of the predictor-based prescribed-time controllers (e.g. [6]) will be also considered by mixing the time-varying tools given in [6] with the results in the present paper, following the same lines of [14]. In addition, the problem of Finite/Fixed-time stabilization of parabolic PDEs with delayed input (either boundary or distributed control) will be studied.

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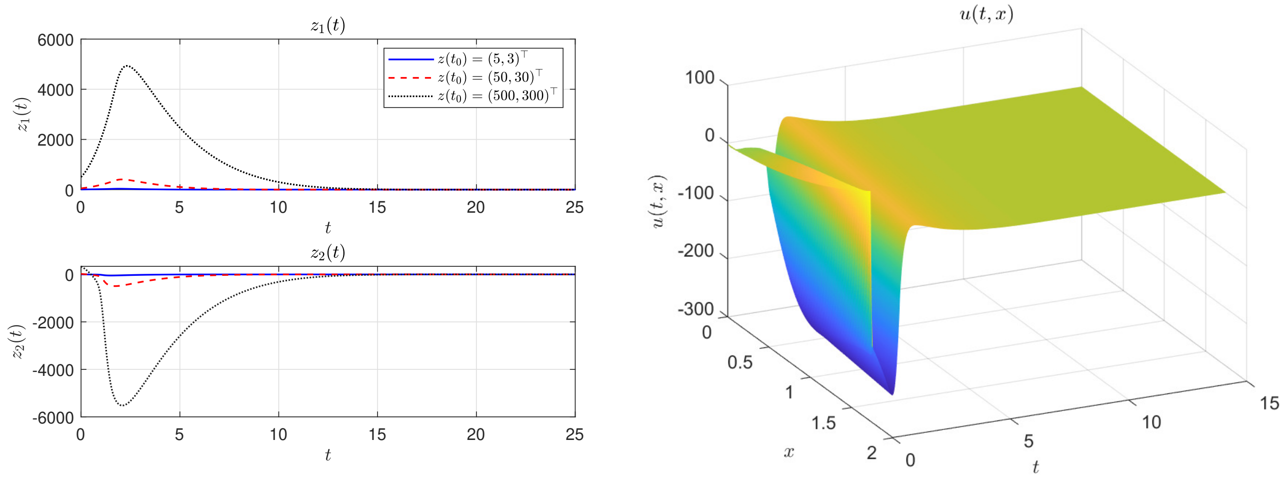

Figure 1. On the left: the evolution of the states $z_1(t)$, $z_2(t)$ of the ODE part of the closed-loop system (37) with predictor-based controller $U(t)$ in (38) and using the expression of $\mathscr{F}$ in (23) to get FTS, in blue solid lines for the initial condition $z(t_0) = (5,3)^\top$, in red dashed lines for $z(t_0) = (50,30)^\top$, and in black dotted lines for $z(t_0) = (500,300)^\top$, with the delays $h_1 = 0.75$s and $h_2 = 1$s. On the right: the evolution of $u(t,x)$ the state of the PDE part of (37) for only the initial condition $z(t_0) = (5,3)^\top$ and $u(t_0,x) = 0$, $x \in [0, h_1 + h_2]$.

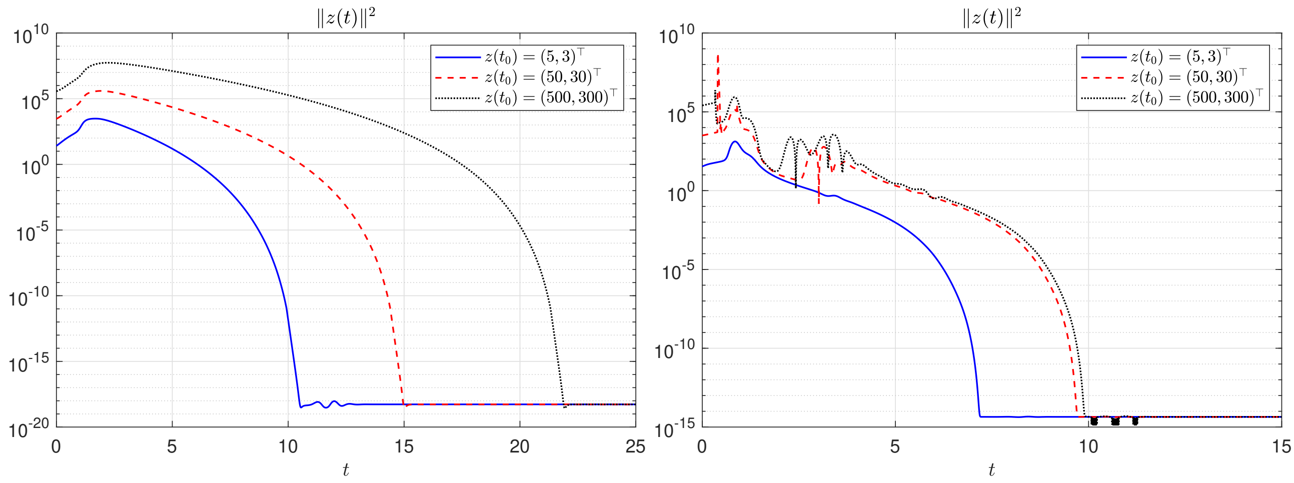

Figure 2. The evolution of the norm $\|z(t)\|^2$ of the closed-loop system (34), with predictor-based controller $U(t)$ in (38) and using on the left the expression of $\mathscr{F}$ in (23) to get FTS with the delays $h_1 = 0.75$s and $h_2 = 1$s, and using on the right the expression of $\mathscr{F}$ in (24) to get FxTS (logarithmic scale) with the delays $h_1 = 0.5$s and $h_2 = 0.75$s, in a blue solid line for the initial condition $z(t_0) = (5,3)^\top$, in a red dashed line for $z(t_0) = (50,30)^\top$, and in a black dotted line $z(t_0) = (500,300)^\top$.

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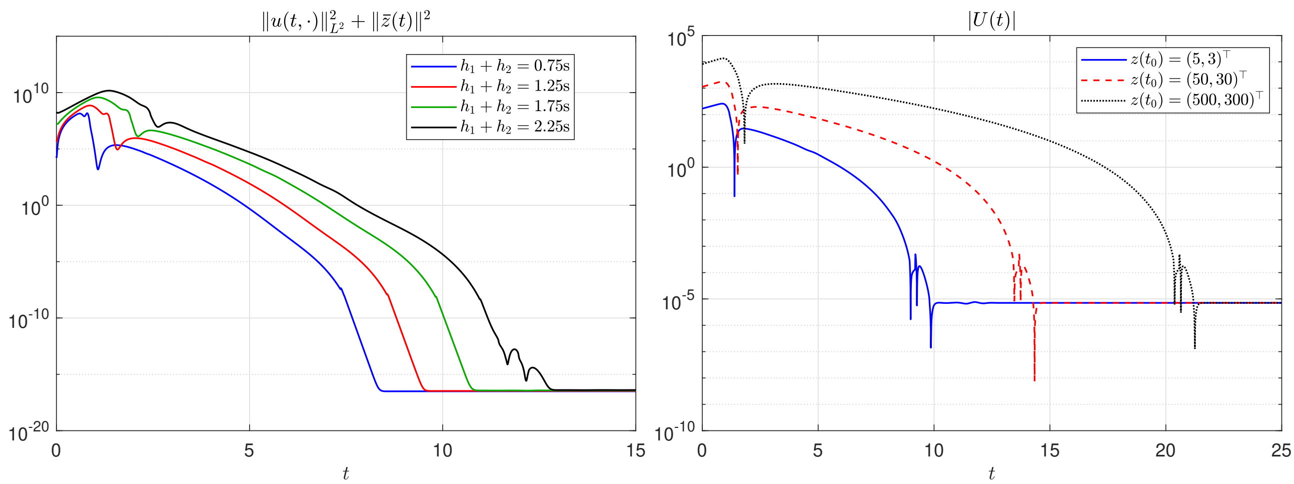

Figure 3. On the left, the evolution of the norm $\|\bar z(t)\|^2 + \|u(t,\cdot)\|^2_{L^2}$ of the closed-loop system (37) with predictor-based controller $U(t)$ in (38) and using the expression of $\mathscr{F}$ in (23) to get FTS (logarithmic scale) for the initial condition $z(t_0) = (5,3)^\top$ with different values for the delays h_1 and h_2. On the right, the evolution of the predictor-based controller $U(t)$ given in (38) using the expression of $\mathscr{F}$ in (23) to get FTS in a blue solid line for the initial condition $z(t_0) = (5,3)^\top$, in a red dashed line for $z(t_0) = (50,30)^\top$, and in a black dotted line for $z(t_0) = (500,300)^\top$, with the delays $h_1 = 0.75$s and $h_2 = 1$s.

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