A Robust Control Approach to Asymptotic Optimality of the Heavy Ball Method for Optimization of Quadratic Functions

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Abstract

Among first order optimization methods, Polyak's heavy ball method has long been known to guarantee the asymptotic rate of convergence matching Nesterov's lower bound for functions defined in an infinite-dimensional space. In this paper, we use results on the robust gain margin of linear uncertain feedback control systems to show that the heavy ball method is provably worst-case asymptotically optimal when applied to quadratic functions in a finite dimensional space.

Keywords: Optimization methods; the heavy ball method; analysis of systems with uncertainties.

1 Introduction

First order methods for solving optimization problems

$$\min_{x} f(x) \tag{1}$$

iteratively approximate the minimum point x^* of f using linear combinations of the previous iterates and the gradients of f computed at those previous iterates. Such methods find widespread applications in machine learning and its applications to control. A fundamental problem regarding such methods is to characterize their rate of convergence for a given class of objective functions f [3,8,4].

For two times differentiable unimodal functions $f : \mathbf{R}^n \to \mathbf{R}$ whose Hessian at the stationary point $\Delta = \nabla^2 f(x^*)$ satisfies

$$mI \le \Delta \le LI,\tag{2}$$

Polyak's heavy ball method [10,11],

$$x_{t+1} = x_t - \alpha \nabla f(x_t) + \beta (x_t - x_{t-1}), \quad t = 0, 1, \dots, (3)$$

guarantees that

$$r_{\{x_t\}} = \limsup_{t \to \infty} \|x_t - x^*\|^{1/t} \le \rho^*, \quad \rho^* \triangleq \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}},$$
(4)

when it is initiated sufficiently close to x^* . Here $\nabla f(\cdot)$ denotes the gradient of f. The quantity $r_{\{x_t\}}$ is the rootconvergence factor of the sequence $\{x_t\}$ [9]; it characterizes the asymptotic rate of convergence of $\{x_t\}$ to x^* . When the set of functions f is restricted to include only quadratic functions satisfying (2) (we denote this class of functions $\mathscr{D}_{m,L}^n$), the method (3) converges globally [11], and the inequality (4) is tight in the worst-case:

$$\sup_{f \in \mathcal{Q}_{m,L}^n} r_{\{x_t\}} = \rho^*.$$
(5)

Polyak commented [11, p.74], that for large-scale problems with quadratic functions in $\mathscr{Q}_{m,L}^n$, where the dimension of the vector x is greater than the number of iterations T required to reach x^* with a sufficient accuracy, one cannot expect any first order method to converge at a rate faster than a geometric sequence with the heavy ball method's ratio ρ^* .

The same ratio ρ^* appears in the (often misquoted) infinite dimensional Nesterov's lower bound. Nesterov showed [8, Theorem 2.1.13] that among *m*-strongly convex continuously differentiable functions with *L*-Lipschitz gradient, defined in an infinite dimensional Hilbert space, there exists a 'bad' quadratic function *f* for which

$$||x_t - x^*|| \ge (\rho^*)^t ||x_0 - x^*|| \quad \forall t = 0, 1, \dots,$$
 (6)

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for any first order optimization method. As a result, any such method applied to this quadratic function will produce a sequence of iterates whose root-convergence factor is bounded from below:

$$r_{\{x_t\}} = \limsup_{t \to \infty} \|x_t - x^*\|^{1/t} \ge \rho^*.$$
(7)

For functions in the finite dimensional space \mathbb{R}^n , lower bounds similar to (6) hold only over the first T steps, and T is linked to the dimension of the space \mathbb{R}^n [3,4]. These results have led the community to believe that in optimization of quadratic functions $\mathbb{R}^n \to \mathbb{R}$, no other first order method can provide a better root-convergence factor than the heavy ball method. However, this conclusion cannot be drawn formally from the existing finite-dimensional results, since they do not hold as $T \to \infty$ while the dimension n of the search space remains fixed.

In this paper, we consider the set $\mathscr{Q}_{m,L}^n$ of finite-dimensional quadratic functions $f : \mathbf{R}^n \to \mathbf{R}$ with the Hessian Δ satisfying condition (2) and fixed-parameter first-order methods

$$x_{t+1} = x_t + \sum_{j=0}^{k-1} \beta_j (x_{t-j} - x_{t-j-1}) - \sum_{j=0}^l \alpha_j \nabla f(y_{t-j}),$$
$$y_t = \sum_{\nu=0}^{k-l} \gamma_\nu x_{t-\nu}, \quad t = 0, 1, \dots.$$
(8)

The number of past iterates k and the number of gradient evaluations $l \leq k$ used at every step can be arbitrary, but they do not change with time. The coefficients α_j , β_j and γ_{ν} are scalar constants.

Our main result shows that for any converging method (8), the worst-case root-convergence factor within the class $\mathscr{Q}_{m,L}^n$ is bounded from below by the same value ρ^* which appears on the right-hand side of equation (4); see inequality (16) in Theorem 1. This conclusion applies to any optimization algorithm (8) including the fixed-step gradient descent method [8,11], Polyak's heavy-ball method [10,11], the triple momentum method [12], and Nesterov's fixed parameter accelerated method [8]. From this result and equation (5) it immediately follows that the heavy ball method is *worst-case optimal* among methods (8) in the sense that for quadratic functions in $\mathscr{Q}_{m,L}^n$ it guarantees the best worst-case asymptotic convergence rate.

Our approach uses robust control theory. Specifically, our derivation of the worst-case lower bound on the root-convergence factor of the method (8) employs the results on the robust gain margin of feedback control systems [7].

2 Optimal lower bound on the convergence rate of first order methods applied to quadratic functions

In this section, we derive the optimal lower bound on the root-convergence factor of the method (8) applied to quadratic functions f of the class $\mathscr{Q}_{m.L}^n$. First we recall some basic definitions which formalize the notion of the asymptotic rate of convergence of an iterative process.

Definition 1 ([9]) Let $\{x_t\}$ be a sequence that converges to x^* . Then the number

$$r_{\{x_t\}} = \limsup_{t \to \infty} \|x_t - x^*\|^{1/t}$$
(9)

is the root-convergence factor, or R-factor of $\{x_t\}$. If \mathscr{I} is an iterative process with limit point x^* , and $\mathscr{C}(\mathscr{I}, x^*)$ is the set of all sequences generated by \mathscr{I} which converge to x^* , then

$$r_{\mathscr{I}} = \sup\{r_{\{x_t\}} : \{x_t\} \in \mathscr{C}(\mathscr{I}, x^*)\}$$
(10)

is the *R*-factor of \mathscr{I} at x^* .

Iterative processes of the form (8) can be written in the form of a nonlinear dynamic system of Luré type [5,6],

$$X_{t+1} = AX_t + BU_t,$$

$$Y_t = CX_t, \quad U_t = -\phi(Y_t),$$
(11)

whose state, output and nonlinearity are respectively

$$X_t = \left[x_{t-k}^T \dots x_t^T \right]^T, \quad Y_t = \left[(C_l X_t)^T \dots (C_0 X_t)^T \right]^T,$$

$$\phi(Y) = \left[(\nabla f(C_l X))^T \dots (\nabla f(C_0 X))^T \right]^T.$$

The matrices $A \in \mathbf{R}^{(k+1)n \times (k+1)n}$, $B \in \mathbf{R}^{(k+1)n \times (l+1)n}$, $C \in \mathbf{R}^{(l+1)n \times (k+1)n}$ and $C_j \in \mathbf{R}^{n \times (k+1)n}$, $j = 0, \ldots, l$, are defined as

$$A = A_0 \otimes I_n, \quad B = B_0 \otimes I_n, \quad C = \begin{bmatrix} C_l^T \dots C_0^T \end{bmatrix}^T,$$

$$A_0 = \begin{bmatrix} \mathbf{0} & I_k \\ -\beta_{k-1} & \beta_{k-1} - \beta_{k-2} & \beta_{k-2} - \beta_{k-3} \dots 1 + \beta_0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} \mathbf{0} \\ \alpha_l & \alpha_{l-1} \dots & \alpha_0 \end{bmatrix}, \quad C_j = C_j^0 \otimes I_n,$$

$$C_j^0 = \underbrace{\begin{bmatrix} \mathbf{0} \dots & \mathbf{0} \\ -\beta_{k-1} & \beta_{k-1} \dots & \gamma_0 \\ 0 \dots & 0 \end{bmatrix}}_j.$$

In the above equations, the symbol \otimes denotes the Kronecker product, I_n is the $n \times n$ identity matrix, and **0** denotes a zero matrix or vector. The dimensions of the matrices A_0 , B_0 and C_j^0 are $(k + 1) \times (k + 1)$, $(k + 1) \times (l + 1)$, and $1 \times (k + 1)$, respectively.

Assumption 1 It holds that

$$\sum_{j=0}^{l} \alpha_j \neq 0, \quad \sum_{\nu=0}^{k-l} \gamma_{\nu} = 1.$$
 (12)

Assumption 1 holds for many practical methods. For instance, in Nesterov's constant-step accelerated gradient method [8, constant step scheme III, p.94], l = 0, k = 1, $\alpha_0 = \frac{1}{L} \neq 0$, $\beta_0 = \frac{1 - \sqrt{m/L}}{1 + \sqrt{m/L}}$, and $\gamma_0 = (1 + \beta_0)$, $\gamma_1 = -\beta_0$. Thus, $\gamma_0 + \gamma_1 = 1$.

Lemma 1 Suppose f(x) has a unique stationary point x^* . Under Assumption 1, x^* is the unique fixed point of the method \mathscr{I} in equation (8).

Proof: Substituting the $x_t = x_{t-1} = \ldots = x_{t-k} = x^*$ into the left-hand side of equation (8) and using the second identity (12) and the fact that $\nabla f(x^*) = 0$ yields

$$x_{t+1} = x^* - \sum_{j=0}^{l} \alpha_j \nabla f\left(\sum_{\nu=0}^{k-l} \gamma_\nu x^*\right)$$
$$= x^* - \left(\sum_{j=0}^{l} \alpha_j\right) \nabla f\left(\left(\sum_{\nu=0}^{k-l} \gamma_\nu\right) x^*\right) = x^*.$$

This shows that x^* is a fixed point of the mapping (8).

To show uniqueness, suppose that \check{x} is a fixed point of the method \mathscr{I} . Then

$$\left(\sum_{j=0}^{l} \alpha_j\right) \nabla f\left(\left(\sum_{\nu=0}^{k-l} \gamma_\nu\right) \check{x}\right) = 0$$

According to (12) this implies that $\nabla f(\check{x}) = 0$, i.e., \check{x} is a stationary point of f. Since f is assumed to have a unique stationary point, the fixed point of the method \mathscr{I} must be unique.

It follows from Lemma 1 that when f is unimodal, $X^* = [\underbrace{1 \dots 1}_{k+1}]^T \otimes x^*$ is the unique equilibrium of the system (11).

For a quadratic function $f(x) = \frac{1}{2}(x-x^*)^T \Delta(x-x^*) + f_0$, using the change of variable $\bar{X}_t = X_t - X^*$ and the condition $\sum_{\nu=0}^{k-l} \gamma_{\nu} = 1$ from Assumption 1, the system (11) can be written as the linear system

$$\bar{X}_{t+1} = \bar{A}\bar{X}_t$$
 where $\bar{A} \triangleq A_0 \otimes I_n - (B_0 \otimes \Delta)C.$ (13)

The system (13) is stable if and only if the spectral radius of the matrix \overline{A} , denoted $\rho(\overline{A})$, satisfies $\rho(\overline{A}) < 1$. Moreover, $\rho(\overline{A})$ characterizes the degree of stability of the system (13) [1]. That is, when the system (13) is stable, the states X_t approach X^* at least as fast as $\rho(\overline{A})^t$. In terms of the root convergence factor, this observation reads

$$r_{\mathscr{I}} = \rho(\bar{A}) \tag{14}$$

for any $f \in \mathscr{Q}_{m,L}^n$ for which the matrix \overline{A} is stable.



Fig. 1. A linear uncertain control system with a robust stabilizing compensator.

When the function f is not known in advance and it is known only that it belongs to the set $\mathscr{Q}^n_{m,L}$, the system (13) is uncertain. When it is stable for all $f \in \mathscr{Q}^n_{m,L}$, its degree of stability in the face of this uncertainty is characterized by $\sup_{mI \leq \Delta \leq LI} \rho(\bar{A})$, and so the states X_t approach X^* at least as fast as $(\sup_{mI \leq \Delta \leq LI} \rho(\bar{A}))^t$. That is, the worstcase root convergence rate of the method (8) is given by

$$\sup_{f \in \mathcal{Q}^n_{m,L}} r_{\mathscr{I}} = \sup_{mI \le \Delta \le LI} \rho(\bar{A}).$$
(15)

We are now in a position to present the main result of the paper which establishes the lower bound on the quantity in (15). Note that $\sup_{mI \le \Delta \le LI} \rho(\bar{A})$ can be greater than or equal to 1, however the expression on the left-hand side of (15) is well-defined only when the sequences $\{x_t\}$ and $\{X_t\}$ asymptotically converge. This is the standing assumption of this result.

Theorem 1 For any $k \ge 1$ and $0 \le l \le k$ and any collection of parameters $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_l)$, $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{k-1})$ and $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_{k-l})$ which satisfies Assumption 1 and for which the algorithm (8) asymptotically converges for all $f \in \mathcal{Q}_{m,L}^n$, it holds that

$$\sup_{f \in \mathcal{Q}^n_{m,L}} r_{\mathscr{I}} = \sup_{\Delta : mI \le \Delta \le LI} \rho(\bar{A}) \ge \rho^*.$$
(16)

The proof of this theorem employs the result of [7] regarding the robust gain margin of linear feedback control systems. It is presented in Lemma 2 given below. This lemma is a special case of a more general theoretical development in [7]. To make our presentation self-contained, we include a direct proof of Lemma 2 in the Appendix.

Lemma 2 Consider the uncertain linear feedback system shown in Fig. 1 consisting of the plant $P(z) = \frac{1}{z-1}$, an uncertain constant gain λ and a compensator K(z). Let $\rho \in (0, 1)$ be a constant. A proper real rational compensator K(z) that places all poles of this system in the interior of the disk $|z| < \rho$ for all $\lambda \in [m, L]$ exists if and only if

$$\rho > \rho^*, \tag{17}$$

where ρ^* is the constant defined in (4), i.e., $\rho^* \triangleq \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}}$.

Proof of Theorem 1: Since the algorithm (8) is assumed to converge asymptotically for all $f \in \mathcal{Q}_{m,L}^n$, then $\rho(\bar{A}) < 1$ for every symmetric matrix Δ which satisfies (2). We now show that $\sup_{mI \leq \Delta \leq LI} \rho(\bar{A}) < 1$. Indeed, the set $mI \leq \Delta \leq LI$ is compact in the finite dimensional metric space of symmetric matrices equipped with the norm induced by the Euclidean norm in \mathbb{R}^n . This conclusion follows from the fact that this set is a closed bounded set. Furthermore, $\rho(\bar{A})$ depends continuously on Δ ; see (13). Thus, by the Weierstrass extreme value theorem, there exists a symmetric matrix Δ^* within that set which attains the supremum. However, for this matrix Δ^* the conditions of the theorem state that the spectral radius of the corresponding matrix (13) $\bar{A}^* = A_0 \otimes I_n - (B_0 \otimes \Delta^*)C$ is less than 1. Thus we conclude that

$$\sup_{mI \le \Delta \le LI} \rho(\bar{A}) = \max_{mI \le \Delta \le LI} \rho(\bar{A}) = \rho(\bar{A}^*) < 1.$$
(18)

Now consider an arbitrary function $f \in \mathcal{Q}_{m,L}^n$. The corresponding matrix Δ is symmetric, therefore there exists an orthogonal matrix T such that $\Delta = T^T \Lambda T$, where Λ is the diagonal matrix whose diagonal consists of the eigenvalues of Δ . Using the matrix T, let us change coordinates in (13),

$$\widetilde{X}_t = \widetilde{T}\overline{X}_t \triangleq (I_{k+1} \otimes T)\overline{X}_t.$$

In the new coordinates, the system (13) becomes

$$\widetilde{X}_{t+1} = \widetilde{T}\overline{A}\widetilde{T}^T\widetilde{X}_t = \widetilde{A}\widetilde{X}_t, \tag{19}$$

where $\widetilde{A} = \widetilde{T} \overline{A} \widetilde{T}^T$. The state transformation does not affect the spectral radius of a matrix. Therefore, $\rho(\overline{A}) = \rho(\widetilde{A})$. Since the matrix $\widetilde{T} = I_{k+1} \otimes T$ is orthogonal, $\widetilde{A} = A_0 \otimes I - N \otimes \Lambda$. Here N is a $(k+1) \times (k+1)$ matrix of the form

$$N = \left[\frac{\mathbf{0}}{n_k \, n_{k-1} \dots n_0}\right],\tag{20}$$

where **0** is the $k \times (k+1)$ zero matrix. The elements of the last row of N are the coefficients of the product polynomial $(\sum_{j=0}^{l} \alpha_j z^j) (\sum_{\nu=0}^{k-l} \gamma_{\nu} z^{\nu})$,

$$n_j = \sum_{\nu=0}^{j} \alpha_{\nu} \gamma_{j-\nu}, \quad j = 0, \dots, k.$$
 (21)

Here we use the standard convention that if the index extends beyond the length of a vector, the corresponding element of the extended vector is taken to be 0.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Δ , and so $\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_n]$. It is easy to see by permuting the columns and rows of the matrix $A_0 \otimes I - N \otimes \Lambda$ that the eigenvalues

of the matrix \widetilde{A} are the same as those of the block diagonal matrix comprised of the companion matrices

$$g_{i} = A_{0} - \lambda_{i} N$$

$$= \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -\beta_{k-1} - n_{k}\lambda_{i} \beta_{k-1} - \beta_{k-2} - n_{k-1}\lambda_{i} \dots 1 + \beta_{0} - n_{0}\lambda_{i} \end{bmatrix},$$

$$i = 1, \dots, n.$$

Therefore, we arrive at the conclusion that

$$\bar{\rho} \stackrel{\Delta}{=} \sup_{mI \le \Delta \le LI} \rho(\bar{A}) = \max_{i} \sup_{\lambda_i \in [m,L]} \rho(g_i) = \sup_{\lambda_i \in [m,L]} \rho(g_i).$$
(22)

In the last identity, \max_i is dropped since all eigenvalues λ_i lie in the same interval [m, L], and therefore $\bar{\rho} = \sup_{\lambda_i \in [m, L]} \rho(g_i)$ and does not depend on *i*.

Next we observe that $\rho(g_i)$ represents the radius of the smallest disk in the complex plane which contains all roots of the characteristic equation of the matrix g_i ,

$$(z-1)\left(z^{k} - \sum_{j=0}^{k-1} \beta_{j} z^{k-j-1}\right) + \lambda_{i} \sum_{j=0}^{k} n_{j} z^{k-j} = 0.$$
(23)

This equation can be written as $1 + \lambda_i P(z)K(z) = 0$, where P(z) and K(z) are given by

$$P(z) = \frac{1}{z - 1}, \quad K(z) = \frac{N(z)}{D(z)},$$
$$N(z) \triangleq \sum_{j=0}^{k} n_j z^{k-j}, \quad D(z) \triangleq z^k - \sum_{j=0}^{k-1} \beta_j z^{k-j-1}.$$

Thus, we conclude that $\bar{\rho} = \sup_{\lambda_i \in [m,L]} \rho(g_i)$ is the radius of the smallest disk that contains the poles of the SISO feedback control system in Fig. 1 consisting of the uncertain plant $\lambda_i P(z)$ and the compensator K(z). According to (18), this radius $\bar{\rho}$ is in the interval (0, 1). Hence one can select $\rho \in (\bar{\rho}, 1)$ such that all poles of this system lie in the interior of the disk $|z| < \rho$ for any $\lambda_i \in [m, L]$. From Lemma 2, a proper compensator K(z) which ensures such pole placement for the family of plants $\lambda_i P(z)$ exists if and only if $\rho > \rho^*$. Taking infimum with respect to $\rho \in (\bar{\rho}, 1)$ proves (16).

3 Conclusions

This paper uses results on the robust gain margin of linear uncertain systems to establish a theoretical lower bound on the rate of asymptotic convergence of iterative first order optimization methods applied to quadratic functions. One conclusion that follows from our results is that among multistep methods of the form (8), Polyak's heavy ball method [10,11] in fact guarantees the best worst-case rate of asymptotic root convergence for the class of quadratic functions. Also, our result confirms that Nesterov's bound indeed holds asymptotically for functions defined in finite-dimensional space when one considers the worst case over quadratic functions. Consequently it also holds for any class of strongly convex twice differentiable functions that are *m*-convex and have *L*-Lipschitz gradient in the vicinity of x^* , containing $\mathcal{Q}_{m,L}^n$ as a subset.

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Appendix: Proof of Lemma 2

Let N(z), D(z) be the numerator and denominator of the compensator K(z). Introduce the sensitivity function

$$S(z) = \left(1 + \frac{m+L}{2}P(z)K(z)\right)^{-1}.$$
(24)

Note that the pole of the plant P(z), z = 1, lies outside the disk $|z| < \rho$ and is a zero of S(z). Also, $z = \infty$, the unique zero of P(z), is a zero of 1 - S(z) since the compensator K(z) is proper. Thus, we conclude that

$$S(1) = 0, \quad S(\infty) = 1.$$
 (25)

The following proposition adapts Lemma 2.3 from [7] to the problem setting of this paper.

Proposition 1 The closed loop system in Fig. 1 has all its poles in the disk $|z| < \rho$ for all $\lambda \in [m, L]$ if and only if S(z) in (24) is real rational and analytic in the region $\widetilde{\mathscr{H}}_{\rho} \triangleq \{|z| \ge \rho\} \cup \{\infty\}$ and

$$S(z) \notin \mathscr{G} \triangleq \left(-\infty, \frac{2m}{m-L}\right] \cup \left[\frac{2L}{L-m}, +\infty\right)$$
$$\forall z \in \tilde{\mathscr{H}}_{\rho}. \tag{26}$$

The proof of this proposition follows, *mutatis mutandis*, the proof of Lemma 2.3 in [7]. Based on this proposition, we conclude that the poles of the closed loop system under consideration can be placed in the interior of the open disk $|z| < \rho$ for all $\lambda \in [m, L]$ if and only if there exist polynomials D(z), N(z) with real coefficients, such that the degree of N is less than or equal to the degree of D and the function S(z) in (24) is analytic in $\hat{\mathscr{H}}_{\rho}$, maps $\hat{\mathscr{H}}_{\rho}$ into the complement of the set \mathscr{G} , denoted \mathscr{G}^c (i.e., $\mathscr{G}^c = \mathbb{C} \setminus \mathscr{G}$ where \mathbb{C} is the set of complex numbers), and satisfies (25). Following [7], we observe that the existence of such polynomials and the function S(z) is essentially the Nevanlinna-Pick interpolation problem.

The classical Nevanlinna-Pick problem [2] is concerned with the following. Given the points $\zeta_1, \zeta_2, \ldots, \zeta_l$ in the interior of the unit disk $\overline{\mathscr{D}} = \{z : |z| \leq 1\}$ and the array of values b_1, b_2, \ldots, b_l , the problem is to find a rational function s(z) with no poles in $\mathscr{D} = \{z : |z| < 1\}$, for which $\sup_{z \in \mathscr{D}} |s(z)| < 1$ and such that $s(\zeta_i) = b_i, i = 1, \ldots, l$. The equivalence between this classical formulation and the interpolation problem stated above follows from the following diagram

$$\begin{array}{ccc} \tilde{\mathcal{H}}_{\rho} & \stackrel{S}{\longrightarrow} & \mathcal{G}^{c} \\ \downarrow^{\varphi} & & \downarrow^{\theta} \\ \bar{\mathcal{D}} & \stackrel{s}{\longrightarrow} & \mathcal{D} \end{array}$$

In this diagram, $\varphi(z) = \rho z^{-1}$ and $\theta(z)$ is the conformal mapping which maps the set \mathscr{G}^c onto the open disk \mathscr{D} [7]:

$$\theta(z) = \left(1 - \sqrt{\frac{1 - \frac{L - m}{2L}z}{1 - \frac{m - L}{2m}z}}\right) \left(1 + \sqrt{\frac{1 - \frac{L - m}{2L}z}{1 - \frac{m - L}{2m}z}}\right)^{-1}.$$
(27)

According to this diagram, $S(z) = \theta^{-1}(s(\rho z^{-1}))$, $s(z) = \theta(S(\rho z^{-1}))$. The interpolation data for s(z) are obtained from (25), $s(0) = \theta(1) = \rho^*$, $s(\rho) = \theta(0) = 0$. A function s(z) which solves the Nevanlinna-Pick problem with these data exists if and only if

$$\begin{bmatrix} 1 - (\theta(1))^2 & 1\\ 1 & \frac{1}{1 - \rho^2} \end{bmatrix} > 0;$$
(28)

e.g., see [2, Theorem 18.1]. Condition (28) is equivalent to $\rho > \theta(1) = \rho^*$. This concludes the proof.