On universal classes of Lyapunov functions for linear switched systems*

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Abstract

In this paper we discuss the notion of universality for classes of candidate common Lyapunov functions of linear switched systems. On the one hand, we prove that a family of absolutely homogeneous functions is universal as soon as it approximates arbitrarily well every convex absolutely homogeneous function for the C^0 topology of the unit sphere. On the other hand, we prove several obstructions for a class to be universal, showing, in particular, that families of piecewise-polynomial continuous functions whose construction involves at most l polynomials of degree at most m (for given positive integers l, m) cannot be universal.

1 Introduction

Common Lyapunov functions constitute the most popular and powerful tool for the stability analysis of switched systems. Roughly speaking, the use of common Lyapunov functions for stability analysis gathers the global behavior

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of the system and allows to bypass the explicit analysis of single trajectories, which may be extremely complex. Yet, looking for a common Lyapunov function may be a nontrivial task as stability cannot always be checked by means of Lyapunov functions in a simple form, for instance within the class of quadratic forms. Given a family of systems (e.g., the family of all linear switched systems), classes of functions large enough to include a Lyapunov function for each globally asymptotically stable system are called *universal* [3] and a result establishing the existence of such a class is a called a *converse Lyapunov theorem*. The literature dealing with converse Lyapunov theorems, starting from the works by Massera and Kurzweil in the 1950s (see e.g. [10, 7, 15, 8, 6]) is quite rich. The results concerning the existence of smooth Lyapunov functions for nonlinear systems with global asymptotic stability properties require the development of rather sophisticated techniques. Concerning robust asymptotic stability with respect to a closed invariant set in presence of perturbation terms, converse Lyapunov theorems have been derived in [8]. In the context of switched systems (even in a nonlinear setting) such results establish the equivalence between the global uniform asymptotic stability and the existence of a smooth Lyapunov function. For switched linear systems the construction of a Lyapunov function is much more direct and natural due, essentially, to the homogeneous nature of the system and the equivalence between asymptotic and exponential stability (see e.g. [5]). Furthermore, in the linear case and even for the more general class of uncertain systems, it is well-known that the families of piecewise quadratic functions, polyhedral functions, and homogeneous polynomials are universal [11, 12, 13, 3]. On the other hand, for every positive integer m, the family of polynomials of degree less or equal than m is not universal even for the simple class of two-dimensional linear switched systems with two modes [9]. Similarly, it is well accepted in the research community (although, to the authors' knowledge, no explicit proof is available) that families of piecewise quadratic and polyhedral functions whose construction involves a uniformly bounded number of quadratic or linear functions cannot be universal. For this reason, all numerical methods investigating the existence of Lyapunov functions within these classes are affected by a certain degree of conservativeness.

The contribution of this paper is twofold. First, we provide a general sufficient condition for a class of functions to be universal (Proposition 3.1), which is a formalization of fundamental ideas already present in [11, 12, 13, 3]. As a corollary, we recover the universal classes of functions obtained in these references. We next derive the main results of this paper, which provide some necessary conditions for the universality of classes of functions. The first one, Theorem 4.2, is an abstract result which applies to families of real analytic functions. The fact that polynomials with a uniform bound m on their degree do not form a universal class [9] follows as a simple consequence of this result. Finally, Theorem 4.5 states that families of piecewise-polynomial continuous functions whose construction involves at most l polynomials of degree at most m (for given positive integers l, m) cannot be universal.

2 Universal classes of common Lyapunov functions

We consider linear switched systems of the form

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0, \quad x \in \mathbb{R}^n, \tag{$\Sigma_{\mathscr{M}}$}$$

where the switching law A is an arbitrary function belonging to the space $\mathcal{S}_{\mathrm{arb}}(\mathscr{M})$ of measurable functions taking values on a bounded subset $\mathscr{M}$ of the set of $n \times n$ matrices, denoted by $M_n(\mathbb{R})$. We use $\Phi_A(t, s)$ to denote the fundamental matrix from s to t for $(\Sigma_{\mathscr{M}})$ associated with the switching law A so that every solution of $(\Sigma_{\mathscr{M}})$ can be written as $x(t) = \Phi_A(t, 0)x(0)$. We are interested in the following uniform stability properties.

Definition 2.1. *The switched system $(\Sigma_{\mathscr{M}})$ is said to be*

- uniformly stable *if there exists $C > 0$ such that, for every switching law A and $t \geq 0$, $\|\Phi_A(t, 0)\| \leq C$;*
- uniformly exponentially stable *if there exist $C, \gamma > 0$ such that, for every switching law A and $t \geq 0$, $\|\Phi_A(t, 0)\| \leq Ce^{-\gamma t}$.*

Stability in the previous senses may be assessed through common Lyapunov functions, defined below.

Definition 2.2. *We say that a continuous function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is a* nonstrict common Lyapunov function *for $(\Sigma_{\mathscr{M}})$ if it is* positive definite, *that is, $V(0) = 0$ and $V(x) > 0$ for every $x \neq 0$, and V is non-increasing along trajectories of $(\Sigma_{\mathscr{M}})$. If, moreover, V is strictly decreasing along nonzero trajectories of $(\Sigma_{\mathscr{M}})$, we say that V is a* common Lyapunov function *for $(\Sigma_{\mathscr{M}})$.*

Remark 2.3. *If V is a (possibly nonstrict) common Lyapunov function and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and such that $\varphi(0) = 0$, then $\varphi \circ V$ is also a (nonstrict) common Lyapunov function. In particular, the positive multiple of a common Lyapunov function is a common Lyapunov function and if there exists a absolutely homogeneous common Lyapunov function, then for every $\alpha > 0$ there exists a absolutely homogeneous common Lyapunov function of degree α.*

We state here the version for linear switched systems of the classical direct Lyapunov theorem.

Theorem 2.4. *A linear switched system $(\Sigma_{\mathscr{M}})$ admitting a nonstrict common Lyapunov function is uniformly stable. If there exists a strict common Lyapunov function for $(\Sigma_{\mathscr{M}})$, then the latter is uniformly exponentially stable.*

In case the strict common Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ in the above theorem is of class $\mathcal{C}^1$ on $\mathbb{R}^n \setminus \{0\}$, then a standard test for checking the strict decrease of V along non-trivial trajectories of $(\Sigma_{\mathscr{M}})$ goes as follows:

$$\nabla V(x)^\top M x < 0, \qquad M \in \mathscr{M}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{1}$$

Definition 2.5. *A set $\mathcal{P}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}$ is a* universal class of Lyapunov functions *if for every bounded set $\mathscr{M} \subset M_n(\mathbb{R})$ such that $(\Sigma_{\mathscr{M}})$ is uniformly exponentially stable there exists a common Lyapunov function for $(\Sigma_{\mathscr{M}})$ in $\mathcal{P}$.*

An equivalent formulation of the universality of a class $\mathcal{P}$ is that *the converse Lyapunov theorem holds true within $\mathcal{P}$*.

As mentioned in the introduction, the construction of a common Lyapunov function for switched linear systems can be easily obtained. For instance, a locally Lipschitz continuous Lyapunov function may be defined as

$$V(x) = \sup_{A \in \mathcal{S}_{\mathrm{arb}}(\mathscr{M})} \int_0^{+\infty} \|\Phi_A(t,0)x\| dt,$$

and may be regularised outside the origin by convolution with a smooth function (see e.g. [5, 9] for more details). Recalling that a function $V : \mathbb{R}^n \to \mathbb{R}$ is absolutely homogeneous of degree α if $V(\lambda x) = |\lambda|^\alpha V(x)$ for every $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, this classical construction leads to the following result.

Proposition 2.6. *Let $\alpha \geq 1$ and $\mathcal{P}$ be the class of convex absolutely homogeneous functions of degree α on $\mathbb{R}^n$ that are positive and smooth on $\mathbb{R}^n \setminus \{0\}$. Then $\mathcal{P}$ is universal. Moreover, for every bounded set $\mathscr{M} \subset M_n(\mathbb{R})$ such that $(\Sigma_{\mathscr{M}})$ is uniformly exponentially stable, there exists $\varepsilon > 0$ and a common Lyapunov function $V \in \mathcal{P}$ for $(\Sigma_{\mathscr{M}})$ such that $\nabla V(x)^\top Mx \leq -\|x\|^\alpha$ for every $x \in \mathbb{R}^n \setminus \{0\}$ and $M \in \mathscr{M}$.*

Note that a globally smooth Lyapunov function may be constructed by classical regularisation techniques developed in a nonlinear setting (see e.g. [7, 8]), at the price of losing homogeneity.

Similar to Proposition 2.6, the following straightforward converse Lyapunov result links the uniform stability of $(\Sigma_{\mathscr{M}})$ with the existence of a nonstrict common Lyapunov function.

Proposition 2.7. *Assume that $\mathscr{M} \subset M_n(\mathbb{R})$ is bounded and $(\Sigma_{\mathscr{M}})$ is uniformly stable. Then the convex and absolutely homogeneous function of degree one*

$$V(x) = \sup_{t \geq 0, A \in \mathcal{S}_{\mathrm{arb}}(\mathscr{M})} \|\Phi_A(t,0)x\|. \tag{2}$$

is a nonstrict common Lyapunov function for $(\Sigma_{\mathscr{M}})$.

Due to Proposition 2.6, the continuous differentiability outside the origin of the common Lyapunov function is not a restrictive assumption when checking the uniform exponential stability of a linear switched system. On the other hand, the uniform stability of $(\Sigma_{\mathscr{M}})$ does not always imply the existence of a $\mathcal{C}^1$ nonstrict common Lyapunov function (see e.g. [4, Example 3]). Furthermore, even in case of uniform exponential stability, it may be useful to provide a criterion to ensure the existence of a common Lyapunov function in a class of non-differentiable functions, such as piecewise linear or piecewise quadratic ones. For these reasons we introduce below a criterion which generalizes Equation (1) and characterizes the family of (possibly nonstrict) common Lyapunov functions in a nonsmooth setting. We refer to [1, Proposition1] for a similar result in the context of differential inclusions.

We need the following preliminary result, which expresses the variation of a convex function V along a trajectory in terms of the subdifferential of V. Recall that the subdifferential $\partial V(x)$ at a point $x \in \mathbb{R}^n$ is defined as

$$\partial V(x) = \{l \in \mathbb{R}^n \mid l^\top (y - x) \leq V(y) - V(x), \quad \forall y \in \mathbb{R}^n\}.$$

The proof of the lemma is provided for completeness.

Lemma 2.8. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\varphi : I \to \mathbb{R}^n$ be an absolutely continuous function, with $I \subseteq \mathbb{R}$ an open interval. Then $V \circ \varphi$ is absolutely continuous and it holds*

$$\frac{d}{dt} V(\varphi(t)) = l^\top \dot{\varphi}(t), \quad \forall l \in \partial V(\varphi(t)), \quad \text{for a.e. } t \in I.$$

Proof. As V is convex, V is Lipschitz and the composition $V \circ \varphi$ is absolutely continuous. Hence for almost every $t \in I$ the derivatives of both φ and $V \circ \varphi$ are well-defined. By definition of subdifferential, for every $t, s \in I$ and $l \in \partial V(\varphi(t))$ we have

$$l^\top (\varphi(s) - \varphi(t)) \leq V(\varphi(s)) - V(\varphi(t)).$$

We deduce that

$$\begin{aligned} \frac{d}{dt} V(\varphi(t)) &= \lim_{s \to t^+} \frac{V(\varphi(s)) - V(\varphi(t))}{s - t} \\ &\geq l^\top \lim_{s \to t^+} \frac{\varphi(s) - \varphi(t)}{s - t} \\ &= l^\top \dot{\varphi}(t) \end{aligned}$$

holds true for almost every $t \in I$ and for every $l \in \partial V(\varphi(t))$. Similarly, taking the limit as $s \to t^-$, we obtain that $\frac{d}{dt} V(\varphi(t)) \leq l^\top \dot{\varphi}(t)$ for almost every $t \in I$ and for every $l \in \partial V(\varphi(t))$. This concludes the proof of the lemma. $\square$

Here follows an adaptation to the nonsmooth setting of the characterization of common Lyapunov function.

Proposition 2.9. *Let $\mathscr{M}$ be a bounded subset of $M_n(\mathbb{R})$ and $V : \mathbb{R}^n \to \mathbb{R}_+$ be a convex positive definite function. Then V is a nonstrict common Lyapunov function for $(\Sigma_{\mathscr{M}})$ if and only if*

$$l^\top M x \leq 0, \qquad \forall x \in \mathbb{R}^n \setminus \{0\}, \forall l \in \partial V(x), \forall M \in \mathscr{M}. \tag{3}$$

Moreover, if the inequality in (3) *is strict then V is a common Lyapunov function for $(\Sigma_{\mathscr{M}})$.*

Proof. The second part of the proposition and the *if* implication in the first part directly follow from Lemma 2.8. We are left to show that if V is a nonstrict common Lyapunov function for $(\Sigma_{\mathscr{M}})$, then the inequality (3) holds true. By contradiction, suppose that there exist $x \in \mathbb{R}^n, l \in \partial V(x)$, and $M \in \mathscr{M}$ such that $l^\top Mx > 0$. By [14, Theorem 25.6] one may find a differentiability point y of V such that the pair $(y, \nabla V(y))$ is arbitrarily close to (x, l). In particular we may assume $\nabla V(y)^\top My > 0$, that is, V is increasing at $t = 0$ along the trajectory $t \mapsto e^{tM}y$, leading to a contradiction. □

3 Positive universality results

Given a switched linear system $(\Sigma_{\mathscr{M}})$, the family $\mathcal{P}$ identified by Proposition 2.6 is unsuitable for numerical investigation of the existence of a Lyapunov function. With this goal in mind, interesting candidate classes $\mathcal{P}$ are those parametric families of functions for which the property of being positive definite and strictly decreasing along all admissible dynamics can be translated into numerically verifiable algebraic relations or inequalities (e.g., linear matrix inequalities). It is well-known that piecewise-quadratic, polynomial, and polyhedral functions represent examples of such families [11, 12, 13, 3].

We next provide a general sufficient condition for a class $\mathcal{P}$ to be universal. Roughly speaking, we exploit the fact, specific to convex functions defined on compact sets, that being close in the uniform norm is equivalent to possessing "close" subdifferentials.

Proposition 3.1. *Let $\mathcal{P}$ be a subset of the family of convex absolutely homogeneous functions of degree one from $\mathbb{R}^n$ to $\mathbb{R}_+$. Assume that for every convex absolutely homogeneous function $V : \mathbb{R}^n \to \mathbb{R}_+$ of degree one and every $\delta > 0$ there exists a function W in $\mathcal{P}$ such that $\|W(x) - V(x)\| \leq \delta$ for every x in the unit sphere S^{n-1} of $\mathbb{R}^n$. Then $\mathcal{P}$ is a universal class of Lyapunov functions.*

Proof. Let $(\Sigma_{\mathscr{M}})$ be uniformly exponentially stable. Let V be the absolutely homogeneous of degree one common Lyapunov function provided by Proposition 2.6. In order to prove the proposition, it is enough to show that any convex absolutely homogeneous function close enough to V on S^{n-1} in uniform norm is itself a Lyapunov function for $(\Sigma_{\mathscr{M}})$.

We proceed by contradiction: we assume that there exists a sequence of convex absolutely homogeneous functions $(W_k)_{k\in\mathbb{N}}$ converging uniformly to V on S^{n-1} as k goes to infinity and such that each W_k is not strictly decreasing along at least one trajectory of the system. In particular the derivative of W_k along such a trajectory is nonnegative on a set of times of positive measure. By Lemma 2.8 and absolute homogeneity of W_k, we deduce that there exist $x_k \in S^{n-1}$ and $M_k \in \mathscr{M}$ such that, for every fixed $l_k \in \partial W_k(x_k)$, one has $l_k^\top M_k x_k \geq 0$. By compactness, we may assume that x_k tends to $\bar{x} \in S^{n-1}$ as k goes to infinity. Then, by [14, Theorem 24.5], l_k converges to $\nabla V(\bar{x})$, so that $\lim_{k\to\infty} \nabla V(\bar{x})^\top \bar{M}_k \bar{x} = \lim_{k\to\infty} l_k^\top M_k x_k \geq 0$. However, it follows by the choice of V and Proposition 2.6 that $\nabla V(\bar{x})^\top \bar{M}_k \bar{x} \leq -1$, yielding a contradiction. □

Remark 3.2. *By the absolute homogeneity property, the statement of Proposition 3.1 could be equivalently reformulated by fixing* $\delta = 1$.

As an application of the previous result, two classical examples of universal classes of Lyapunov functions (cf. [3, 11, 12, 13]) are recalled in the following corollary.

Corollary 3.3. *The family of polyhedral functions* $\{\max_{k=1,\dots,N} |l_k^\top x| \mid l_k \in \mathbb{R}^n,\ N \in \mathbb{N}\}$ *and that of homogeneous sums of squares* $\{\sum_{k=1}^N (l_k^\top x)^{2d} \mid l_k \in \mathbb{R}^n,\ d, N \in \mathbb{N}\}$ *are universal classes of Lyapunov functions.*

Proof. Let V be a convex absolutely homogeneous function of degree one. Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in S^{n-1}, and $l_i \in \partial V(x_i)$. We consider the increasing sequence of absolutely homogeneous functions of degree one defined by

$$W_i(x) = \max_{j=1,\dots,i} |l_j^\top x|.$$

Observe that each W_i is convex and $W_i(x) \leq V(x)$ for every $x \in \mathbb{R}^n$. Indeed

$$\begin{aligned}
|l_j^\top x| &= \max\{l_j^\top x, l_j^\top(-x)\} \\
&= \max\{l_j^\top(x - x_j), l_j^\top(-x - x_j)\} + l_j^\top x_j \\
&\leq V(x) - V(x_j) + l_j^\top x_j \\
&= V(x),
\end{aligned}$$

for every positive integer j, by definition of subgradient and since $V(x) = V(-x)$ and $V(x_j) = l_j^\top x_j$. We deduce that $W_i(x_k) = V(x_k)$ for every $k \in \mathbb{N}$ and $i \geq k$, hence $\lim_{i\to\infty} W_i(x_k) = V(x_k)$ and we can apply [14, Theorem 10.8] to conclude that the sequence of functions W_i converges to V

uniformly on S^{n-1}. By applying Proposition 3.1 we get that the family of polyhedral functions is a universal class of Lyapunov functions.

Let us now consider the absolutely homogeneous functions of degree one

$$Z_i(x) = \left(\sum_{j=1}^{i} |l_j^\top x|^{2i} \right)^{\frac{1}{2i}}.$$

The function Z_i is convex since it is the composition of the $2i$-norm on $\mathbb{R}^i$, i.e., $\|y\|_{2i} = \left(\sum_{j=1}^{i} y_j^{2i} \right)^{\frac{1}{2i}}$ for $y \in \mathbb{R}^i$, with the linear function from $\mathbb{R}^n$ to $\mathbb{R}^i$ mapping x to $(l_1^\top x, \dots, l_i^\top x)^\top$. Moreover it is immediate to see that $W_i(x) \leq Z_i(x) \leq i^{\frac{1}{2i}} W_i(x)$, and in particular Z_i tends to W_i uniformly on S^{n-1} as d goes to infinity. By applying again Proposition 3.1, it follows that the family of homogeneous sums of squares is a universal class of Lyapunov functions. □

Remark 3.4. *According to Remark 2.3, the first part of Corollary 3.3 remains valid if one replaces the piecewise linear functions* $\max_{k=1,\dots,N} |l_k^\top x|$ *with the functions* $\left(\max_{k=1,\dots,N} |l_k^\top x|\right)^q = \max_{k=1,\dots,N} |l_k^\top x|^q$, *for any given* $q > 1$. *In particular, for* $q = 2$, *we have that the family of piecewise quadratic functions is a universal class of Lyapunov functions.*

Remark 3.5. *The proof of Proposition 3.1 relies on the fact that, whenever a linear switched system is uniformly exponentially stable, there exists a common Lyapunov function which is convex and homogeneous. In the classical construction, convexity and homogeneity are direct consequences of the convexity and homogeneity of the map* $x_0 \mapsto \|\Phi_A(t,0)x_0\|$ *for given* $t \geq 0$ *and* $A \in \mathcal{S}_{\mathrm{arb}}(\mathscr{M})$. *Proposition 3.1, and hence Corollary 3.3, can then be extended to any class of nonlinear switched systems*

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \sigma(t) \in \Sigma,$$

with $\Sigma \subset \mathbb{R}^m$ *a bounded set of parameters, satisfying the following conditions:*

- f_σ *is 1-homogeneous for every* $\sigma \in \Sigma$, *that is* $f_\sigma(\lambda x) = \lambda f_\sigma(x)$ *for every* $x \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$,
- *for every* $R > 0$, $\{f_\sigma|_{B(0,R)} \mid \sigma \in \Sigma\}$ *is a compact subset of* $\mathcal{C}(B(0,R), \mathbb{R}^n)$,
- *the class of admissible switching laws is* $L^\infty(\mathbb{R}_+, \Sigma)$,

- *denoting by $x(t, x_0, \sigma(\cdot))$ the solution at time t of the system starting at x_0 and corresponding to the switching law $\sigma \in L^\infty(\mathbb{R}_+, \Sigma)$, the function $x_0 \mapsto \|x(t, x_0, \sigma(\cdot))\|$ is convex.*

4 Negative universality results

Next, we provide restrictions on the classes of functions which may be candidate to be universal. For this purpose, we introduce the following technical result.

Lemma 4.1. *Let $\mathscr{M}_1, \mathscr{M}_2$ be bounded subsets of $M_n(\mathbb{R})$ and assume that $(\Sigma_{\mathscr{M}_1})$ is uniformly stable. For $\nu > 0$, denote by $\mathscr{M}_2^\nu$ the set of matrices of the form $M - \nu \mathrm{Id}_n$ for $M \in \mathscr{M}_2$, where Id_n is the $n \times n$ identity matrix. Set $\mathscr{M} = \mathscr{M}_1 \cup \mathscr{M}_2^\nu$. Then, the switched system $(\Sigma_{\mathscr{M}})$ is uniformly stable for $\nu > 0$ large enough.*

Proof. Proposition 2.7 guarantees the existence of a convex nonstrict Lyapunov function V for $(\Sigma_{\mathscr{M}_1})$, absolutely homogeneous of degree one. Intuitively speaking, the lemma follows from the fact that, for λ large enough, the vectors Mx, with $M \in \mathscr{M}_2^\lambda$ and $x \in \mathbb{R}^n$, point towards the interior of the sublevel set $V^{-1}([0, V(x)])$. Let us formalize this idea. Since $l^\top x = V(x)$ whenever $l \in \partial V(x)$, by boundedness of $\mathscr{M}_2$ and of $\cup_{x \in V^{-1}(1)} \partial V(x)$ [14, Theorem 24.7], for $\lambda > 0$ large enough one has $l^\top (M - \lambda \mathrm{Id}_n) x = l^\top M x - \lambda l^\top x = l^\top M x - \lambda < 0$ for every $M \in \mathscr{M}_2$, $x \in V^{-1}(1)$, and $l \in \partial V(x)$. The result is then an immediate consequence of Proposition 2.9. ☐

Theorem 4.2. *Let $n \geq 2$ and $\mathcal{P}$ be a compact subset of the space of analytic functions from $\mathbb{R}^n$ to $\mathbb{R}$ endowed with the topology of uniform convergence on bounded sets. Assume that $\mathcal{P}$ does not contain the zero function. Then $\mathcal{P}$ cannot be a universal class of Lyapunov functions.*

Proof. We start by showing the theorem in the case $n = 2$. We proceed by contradiction, assuming that every uniformly exponentially stable switched system in the case where $\mathscr{M}$ consists of two matrices in $M_2(\mathbb{R})$ admits a Lyapunov function in $\mathcal{P}$. We consider a switched system corresponding to $\mathscr{M}^0 = \{M_1^0, M_2^0\} \subset M_2(\mathbb{R})$, where M_1^0, M_2^0 are Hurwitz, the corresponding trajectories rotate clockwise around the origin, the system is uniformly stable, but not attractive, and starting from every initial nonzero condition there exists a unique periodic trajectory, with four switches per period. The existence of such a system is obtained in [2, Theorem 1], where it corresponds to the case **S4**, $\mathcal{R} = 1$. In particular, there exist $t_1, t_2 > 0$ such that $e^{t_1 M_1^0} e^{t_2 M_2^0}$ has an eigenvalue equal to -1, corresponding to an eigenvector x_0. Set $T = t_1 + t_2$ and consider the switched systems associated with $\mathscr{M}^\varepsilon = \{M_1^\varepsilon, M_2^\varepsilon\} \subset M_2(\mathbb{R})$, where $M_i^\varepsilon = M_i^0 - \varepsilon \mathrm{Id}_2$ for $i = 1, 2$. For $\varepsilon \geq 0$ we consider the T-periodic switching sequence $A^\varepsilon(\cdot)$ which takes values M_1^ε for $t \in [0, t_1)$ and M_2^ε for $t \in [t_1, T)$.

Since $\lambda(\mathscr{M}^\varepsilon) = -\varepsilon$, system $(\Sigma_{\mathscr{M}^\varepsilon})$ is uniformly exponentially stable for $\varepsilon > 0$. Hence, by assumption, it admits a Lyapunov function $V_\varepsilon(\cdot)$ in $\mathcal{P}$. Since the latter is compact, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to zero such that $(V_{\varepsilon_k})_{k\in\mathbb{N}}$ converges to some $\bar{V} \in \mathcal{P}$. Moreover, for every $t \geq 0$,

$$\begin{aligned} \bar{V}(\Phi_{A^0}(t,0)x_0) &= \lim_{k\to\infty} V_{\varepsilon_k}(\Phi_{A^{\varepsilon_k}}(t,0)x_0) \\ &\leq \lim_{k\to\infty} V_{\varepsilon_k}(x_0) = \bar{V}(x_0). \end{aligned}$$

Since $\bar{V}(\Phi_{A^0}(2T,0)x_0) = \bar{V}(x_0)$ we deduce that $\bar{V}$ is constant along the trajectory $\Phi_{A^0}(\cdot,0)x_0$. The function $t \mapsto \bar{V}(e^{tM_1^0}x_0)$ is analytic, being the composition of analytic functions, and it is constantly equal to $\bar{V}(x_0)$ for $t \in [0, t_1]$. By analyticity, $t \mapsto \bar{V}(e^{tM_1^0}x_0)$ is constant for all $t > 0$ and therefore it must be identically equal to 0 since $\bar{V}(\lim_{t\to\infty} e^{tM_1^0}x_0) = \bar{V}(0) = 0$. Since every nonzero point of $\mathbb{R}^2$ may be written as $\mu e^{tM_1^0}x_0$ for some positive μ and t, we deduce that $\bar{V}$ must be identically zero, contradicting the assumptions on $\mathcal{P}$.

We are left to prove the result for $n > 2$. For this purpose we consider $\mathscr{M}_1 = \{\bar{M}_1^0, \bar{M}_2^0\}$ with

$$\bar{M}_i^0 = \begin{pmatrix} M_i^0 & 0 \\ 0 & -\mathrm{Id}_{n-2} \end{pmatrix},$$

where the matrices M_i^0 are defined as above. Let $\lambda > 0$ and $\mathscr{M}_2^\lambda$ be given by Lemma 4.1 with $\mathscr{M}_2 = \{M \in \mathrm{so}(n) \mid \|M\| \leq 1\}$, where $\mathrm{so}(n)$ denotes the space of skew-symmetric $n \times n$ matrices. Define $\bar{\mathscr{M}}^0 = \mathscr{M}_1 \cup \mathscr{M}_2^\lambda$ and, for $\varepsilon > 0$, consider the switched system corresponding to $\bar{\mathscr{M}}^\varepsilon = \bar{\mathscr{M}}^0 - \varepsilon \mathrm{Id}_n$. It is clear that $(\Sigma_{\bar{\mathscr{M}}^\varepsilon})$ is uniformly exponentially stable for every $\varepsilon > 0$.

Letting $\Pi_{1,2}$ be the (x_1, x_2) plane, i.e., $\Pi_{1,2} = \{x \in \mathbb{R}^n \mid x_3 = \cdots = x_n = 0\}$, we notice that, starting from every $\bar{x} \in \Pi_{1,2}$, there exists a periodic trajectory of $(\Sigma_{\bar{\mathscr{M}}^0})$ lying on $\Pi_{1,2}$. The restrictions of functions in $\mathcal{P}$ to $\Pi_{1,2}$ form a compact set of analytic functions on this plane. As in the case $n = 2$, we prove by contradiction that $\mathcal{P}$ is not universal. Assume that there exists a sequence $(V_{\varepsilon_k}(\cdot))_{k\in\mathbb{N}}$ of Lyapunov functions in $\mathcal{P}$ for $(\Sigma_{\bar{\mathscr{M}}^{\varepsilon_k}})$ converging to $\bar{V} \in \mathcal{P}$. We can show as before that $\bar{V}$ is equal to 0 on $\Pi_{1,2}$. Because of the choice of $\mathscr{M}_2$ and by construction of $\bar{\mathscr{M}}^0$, every 1-dimensional linear subspace of $\mathbb{R}^n$ may be reached in finite time from $\Pi_{1,2}$ via a trajectory of $(\Sigma_{\bar{\mathscr{M}}^0})$. Since $\bar{V}$ is non-increasing along such a trajectory, we deduce that $\bar{V} \equiv 0$ on $\mathbb{R}^n$, obtaining a contradiction. □

Remark 4.3. *The assumption that the zero function is not in $\mathcal{P}$ cannot be removed from the hypotheses of Theorem 4.2. Indeed, consider the subset of polynomial functions made of the zero polynomial and, for every $N \geq 0$, the polynomials of degree N whose maximum of the absolute value of the coefficients is upper bounded by a positive constant c_N, where the latter is chosen in such a way that the supremum on the ball of radius N of the polynomial is less than or equal to $1/(N+1)$. Since the class $\mathcal{P}$ contains a multiple of any polynomial, it is universal by Corollary 3.3. It is also compact since, given a sequence in $\mathcal{P}$, it has a subsequence with either degree going to infinity or constant degree. In the former case, the subsequence converges to zero for the topology of uniform convergence on bounded sets, while in the latter one the coefficients are uniformly bounded and hence the sequence admits a further converging subsequence.*

As a consequence of the previous result we obtain a partial counterpart to Corollary 3.3 for homogeneous polynomial functions. Namely, we recover that, if we impose a uniform bound on the degree, such functions do not form a universal class of Lyapunov functions, as already established in [9].

Corollary 4.4. *For every $n \geq 2$ and every positive integer m the set of polynomial functions of degree at most m from $\mathbb{R}^n$ to $\mathbb{R}$ is not a universal class of Lyapunov functions.*

Proof. Without loss of generality we normalize the polynomial functions assuming (for instance) that the maximum of the modulus of their coefficients is equal to 1. In this case the coefficients of the polynomials belong to a compact subset of an Euclidean space and, since polynomials depend continuously (in the topology of uniform convergence on bounded sets) on such coefficients, we are reduced to a compact family of nonzero polynomials. The corollary then follows as an immediate consequence of the previous theorem. □

The conclusion of Corollary 4.4 can be proved to hold true for functions involving maxima and minima within a finite family of polynomials such as the class of polyhedral functions V of the form

$$\begin{aligned} V(x) &= \max\{|l_1^\top x|, \dots, |l_N^\top x|\} \\ &= \max\{l_1^\top x, \dots, l_N^\top x, -l_1^\top x, \dots, -l_N^\top x\}, \end{aligned}$$

with $l_1, \dots, l_N \in \mathbb{R}^n$, where N is fixed. This partial counterpart to Corollary 3.3 is a consequence of the following more general result.

Theorem 4.5. *Let $n \geq 2$ and $\mathcal{P}_d^n$ be the family of polynomial functions in $\mathbb{R}^n$ of degree at most d and l be a positive integer. Consider the family*

$$\begin{aligned} \mathcal{P}_{d,l}^n = \{V \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}) \mid \exists V_1, \dots, V_l \in \mathcal{P}_d^n \\ s.t.\ V(x) \in \{V_1(x), \dots, V_l(x)\},\ \forall x \in \mathbb{R}^n\}. \end{aligned}$$

Then, $\mathcal{P}_{d,l}^n$ is not universal.

Proof. We first claim that if $\mathcal{P}_{d,l}^n$ is universal, the same is true for $\mathcal{P}_{d,l}^2$. Indeed, for every $\mathscr{M} \subset M_2(\mathbb{R})$ such that $(\Sigma_{\mathscr{M}})$ is uniformly exponentially stable, consider $\hat{\mathscr{M}} \subset M_n(\mathbb{R})$ given by

$$\hat{\mathscr{M}} = \left\{ \begin{pmatrix} M & 0 \\ 0 & -\mathrm{Id}_{n-2} \end{pmatrix} \mid M \in \mathscr{M} \right\}.$$

If $\hat{V} \in \mathcal{P}_{d,l}^n$ is a common Lyapunov function for $(\Sigma_{\hat{\mathscr{M}}})$, then $V : \mathbb{R}^2 \ni x \mapsto \hat{V}(x, 0)$ is a common Lyapunov function for $(\Sigma_{\mathscr{M}})$ and $V \in \mathcal{P}_{d,l}^2$.

We are left to prove that $\mathcal{P}_{d,l}^2$ is not universal. Consider the switched systems $(\Sigma_{\mathscr{M}^\varepsilon})$ introduced in the proof of Theorem 4.2, which are uniformly exponentially stable for $\varepsilon > 0$, and only uniformly stable for $\varepsilon = 0$. Assume by contradiction that $\mathcal{P}_{d,l}^2$ is universal and, in particular, that for every $\varepsilon > 0$ there exists a Lyapunov function $V^\varepsilon \in \mathcal{P}_{d,l}^2$ for $(\Sigma_{\mathscr{M}^\varepsilon})$. By definition of $\mathcal{P}_{d,l}^2$, for every $\varepsilon > 0$ there exist l polynomials $P_1^\varepsilon, \dots, P_l^\varepsilon$ of degree at most d such that $V^\varepsilon(x) \in \{P_1^\varepsilon(x), \dots, P_l^\varepsilon(x)\}$. Given $j, k \in \{1, \dots, l\}$, we investigate the set of zeroes of the polynomial Q_{jk}^ε defined as the homogeneous polynomial corresponding to the terms of maximal degree of $P_j^\varepsilon - P_k^\varepsilon$. For this purpose, recall that every homogeneous polynomial Q of positive degree m may be factorized as $Q = \prod_{k=1}^m (\alpha_k x_1 + \beta_k x_2)$, where $\alpha_k, \beta_k \in \mathbb{C}$ for $k = 1, \dots, m$, so that its zeroes correspond to the intersection of the unit circle S^1 with at most m lines through the origin. Hence, it follows that either $Q^\varepsilon_{jk} \equiv 0$ (i.e., $P^\varepsilon_j \equiv P^\varepsilon_k$) or Q^ε_{jk} vanishes at most $2d$ times on the unit circle. Moreover, for every $\varepsilon > 0$, the integer $N = 2d\binom{l-1}{2} + 1$ is a strict upper bound for the total number of zeroes of Q^ε_{jk} for $j, k \in \{1, \dots, l\}$. Partitioning the circle into N arcs $\mathcal{C}_1, \dots, \mathcal{C}_N$ of equal length, for every $\varepsilon > 0$ there exists an arc $\mathcal{C}_{n_\varepsilon}$ which contains no zero of the nontrivial polynomials Q^ε_{jk} in its interior. Denote by $\mathcal{A}_{n_\varepsilon}$ the closed middle third of $\mathcal{C}_{n_\varepsilon}$.

We next claim that for every $\varepsilon > 0$ there exists $\nu_\varepsilon > 0$ large enough such that the restriction of V^ε to the dilated arc $\nu_\varepsilon \mathcal{A}_{n_\varepsilon}$ coincides with the restriction to the same arc of one of the polynomials $P^\varepsilon_1, \dots, P^\varepsilon_l$. By definition of the function V^ε and taking into account its continuity, it is enough to prove that, for every $\varepsilon > 0$ there exists $\nu_\varepsilon > 0$ large enough such that, in the interior of the arc $\nu_\varepsilon \mathcal{A}_{n_\varepsilon}$, one has for every $j, k \in \{1, \dots, l\}$ that either $P^\varepsilon_j \equiv P^\varepsilon_k$ or $P^\varepsilon_j - P^\varepsilon_k$ is never vanishing. To see that, it is enough to prove that if Q^ε_{jk} does not vanish on $\mathcal{A}_{n_\varepsilon}$ then $P^\varepsilon_j - P^\varepsilon_k$ does not vanish on $\nu_\varepsilon \mathcal{A}_{n_\varepsilon}$ for $\nu_\varepsilon > 0$ large enough independent of j, k. In that case, one has, for $\nu > 0$ large enough, $x \in S^1$ and $j, k \in \{1, \dots, l\}$,

$$(P^\varepsilon_j - P^\varepsilon_k)(\nu x) = \nu^{d'} \left(Q^\varepsilon_{jk}(x) + o(1) \right),$$

where d' is the positive degree of Q^ε_{jk} and $o(1)$ is a function of x and ν tending to 0 as ν tends to infinity uniformly with respect to $x \in S^1$ and $j, k \in \{1, \dots, l\}$. This concludes the proof of the claim.

Since the arcs $\mathcal{A}_1, \dots, \mathcal{A}_N$ do not depend on ε, there exist one of them, denoted by $\mathcal{A}$, and sequences $(\varepsilon_m)_{m \in \mathbb{N}}, (\nu_m)_{m \in \mathbb{N}}$ in $(0, +\infty)$, and $(V_m)_{m \in \mathbb{N}}$ in $\mathcal{P}^2_d$ such that $\lim_{m \to \infty} \varepsilon_m = 0$ and $V^{\varepsilon_m} = V_m$ on $\nu \mathcal{A}$ for every $m \in \mathbb{N}$ and every $\nu \geq \nu_m$. Let $\hat{V}_m \in \mathcal{P}^2_d$ be the homogeneous term of maximal degree of V_m. Notice that

$$\hat{V}_m(x) = \lim_{\nu \to +\infty} \nu^{-d_m} V_m(\nu x), \qquad \forall x \in \mathbb{R}^2,$$

where d_m denotes the degree of V_m. Up to normalizing V^{ε_m}, we may assume that the maximum of the moduli of the coefficients of the polynomial $\hat{V}_j$ is equal to 1. Thus, up to extracting a subsequence, $\hat{V}_m$ converges uniformly on compact sets to some nonzero $\bar{V} \in \mathcal{P}^2_d$.

Similarly to the proof of Theorem 4.2, we can construct a periodic trajectory $t \mapsto \Phi_{A^0}(t, 0)\bar{x}$ starting at $\bar{x}$ in the interior of the arc $\mathcal{A}$, with $A^0(\cdot)$ piecewise constant taking values in $\mathscr{M}^0$. Consider the switching laws

$A^\varepsilon(\cdot) = A^0(\cdot) - \varepsilon \mathrm{Id}_2$ taking values in $\mathscr{M}^\varepsilon$. For every $t \geq 0$ such that $\Phi_{A^0}(t,0)\bar{x} \in \mathcal{A}$ and for every $m \in \mathbb{N}$, we have

$$\begin{aligned}\hat{V}_m(\Phi_{A^{\varepsilon_m}}(t,0)\bar{x}) &= \lim_{\nu\to+\infty} \nu^{-d_m} V_m(\nu\Phi_{A^{\varepsilon_m}}(t,0)\bar{x})\\ &= \lim_{\nu\to+\infty} \nu^{-d_m} V^{\varepsilon_m}(\Phi_{A^{\varepsilon_m}}(t,0)\nu\bar{x})\\ &\leq \lim_{\nu\to+\infty} \nu^{-d_m} V^{\varepsilon_m}(\nu\bar{x})\\ &= \lim_{\nu\to+\infty} \nu^{-d_m} V_m(\nu\bar{x}) = \hat{V}_m(\bar{x}),\end{aligned}$$

and therefore

$$\begin{aligned}\bar{V}(\Phi_{A^0}(t,0)\bar{x}) &= \lim_{m\to\infty} \hat{V}_m(\Phi_{A^{\varepsilon_m}}(t,0)\bar{x})\\ &\leq \lim_{m\to\infty} \hat{V}_m(\bar{x}) = \bar{V}(\bar{x}).\end{aligned}$$

We then deduce that $t \mapsto \bar{V}(\Phi_{A^0}(t,0)\bar{x}_0)$ is constant on $\{t \geq 0 \mid \Phi_{A^0}(t,0)\bar{x}_0 \in \mathcal{A}\}$. Moreover $\Phi_{A^0}(t,0) = e^{tM_i^0}$ for some $i = 1,2$, for t small enough, and by repeating the argument in the proof of Theorem 4.2 we obtain that $\bar{V} \equiv 0$, yielding a contradiction. $\square$

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