On inherent limitations in robustness and performance for a class of prescribed-time algorithms

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Abstract

Prescribed-time algorithms based on time-varying gains may have remarkable properties, such as regulation in a user-prescribed finite time that is the same for every nonzero initial condition and that holds even under matched disturbances. However, at the same time, such algorithms are known to lack robustness to measurement noise. This note shows that the lack of robustness of a class of prescribed-time algorithms is of an extreme form. Specifically, we show the existence of arbitrarily small measurement noises causing considerable deviations, divergence, and other detrimental consequences. We also discuss some drawbacks and trade-offs of existing workarounds as motivation for further analysis.

Key words: prescribed-time controllers, prescribed-time observers, prescribed-time differentiators, robustness analysis.

1 Introduction

Design methodologies that arbitrarily prescribe the convergence time bound of dynamical systems, such as closed control loops or observer error dynamics, have recently seen a great deal of attention. Specifically, such prescribed-time algorithms achieve a so-called fixed convergence-time bound (cf. Polyakov, 2012) that is arbitrarily prescribed and independent of the initial condition. A subclass of these methodologies uses time-varying gains (TVG) that tend to infinity as the time approaches the prescribed convergence time. This is the case of Song et al. (2019, 2017); Holloway and Krstic (2019); Orlov et al. (2022); Aldana-López et al. (2021); Gómez-Gutiérrez (2020); Tran and Yucelen (2020) and Orlov (2022). Compared to time-invariant approaches, such as Seeber et al. (2021); Sánchez-Torres et al. (2018), TVG-based approaches have been shown to have some remarkable advantages. On the one hand, controllers for an integrator chain can be designed to achieve exact tracking *at* a fixed, prescribed time that is the same for

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all trajectories (Song et al., 2017, 2019), which maintain a prescribed-time convergence in the presence of bounded disturbances even without knowledge on its bound. Such methods can be extended for the output tracking problem for systems described by partial differential equations (Steeves et al., 2020). On the other hand, the observers by Holloway and Krstic (2019) allow to reconstruct the system's state at a prescribed time instant, which can be extended to maintain the prescribed-time convergence property even under input delay (Espitia et al., 2022). Aldana-López et al. (2021) and Orlov et al. (2022) designed online differentiators with a prescribed upper bound for the convergence time.

Despite the singularity of the TVG at the desired convergence time instant or convergence-time bound, the magnitude of controllers and the error correction functions of differentiators, in the absence of measurement noise, have been shown to remain bounded with the discussed approaches or even to tend to zero. However, for practical purposes, analyzing robustness under measurement noise is of paramount importance. To our best knowledge, a formal analysis of the sensitivity to the noise is missing in the prescribed-time literature based on TVGs.

Hence, in this note, we analyze the sensitivity to measurement noise of a class of prescribed-time controllers and differentiator algorithms characterized by what we call an *absolute deadline*. The defining property of systems with such an absolute deadline is that the convergence-time bound stays the same for every trajectory, rather than shifting along with the *initial time instant* as it would be the case for a time-invariant system with fixed-time convergence. We show that many existing prescribed-time algorithms based on TVGs exhibit an absolute deadline. Furthermore, we prove some significant *inherent* performance limitations and lack of robustness to measurement noise appearing arbitrarily close to such an absolute deadline. In particular, for nonscalar systems, we show that arbitrarily small noise may result in arbitrarily large control or observation errors at the absolute deadline and can also lead to diverging trajectories. We also discuss why some popular workarounds have important drawbacks, to motivate future works to further analyze the sensitivity to measurement noise and the suggested workarounds.

Notation: Boldface lowercase and capital letters denote vectors and matrices, respectively. $\mathbb{R}$ is the set of real numbers. Given a vector $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}$, where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$. Given a scalar $v \in \mathbb{R}$, $|v|$ represents its absolute value. The i-th element of a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted by x_i and $\|\mathbf{x}\|_\infty = \max_i |x_i|$. One-sided limits of a function f at a time instant T from below are written as $\lim_{t\to T^-} f(t)$, $\limsup_{t\to T^-} f(t)$. In the formal proofs, the convention $\operatorname{sign}(0) = 1$ is used.

2 Preliminaries and definitions

In this work, we study systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\eta}, d) \tag{1}$$

defined for $t \in [0, T)$, $T > 0$. Therein, $\boldsymbol{\eta}(t) \in \mathbb{R}^n$ is a time-varying noise, and the disturbance $d(t) \in \mathbb{R}$ and $\mathbf{f} : [0, T) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are assumed to be such that system (1) has a unique Filippov solution $\mathbf{x}(t) \in \mathbb{R}^n$ defined on $t \in [0, T)$ (Filippov, 1988) for $\boldsymbol{\eta}(t) \equiv \mathbf{0}$. Moreover, we assume that both d and $\boldsymbol{\eta}$ are Lebesgue measurable, and that the noise $\boldsymbol{\eta}$ is uniformly bounded as $\|\boldsymbol{\eta}(t)\| \le \bar{\eta}$ for some $\bar{\eta}$ and all $t \in [0, T]$.

Definition 1 (Uniform Lyapunov Stability [1]) *Given $T > 0$, we say that the origin of system* (1) *is* uniformly Lyapunov stable *on* $[0, T)$ *if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $s \in [0, T)$, $\|\mathbf{x}(s)\| \le \delta$ implies $\|\mathbf{x}(t)\| \le \epsilon$ for all $t \in [s, T)$*

In particular, we are interested in studying systems (1) that satisfy the following property:

Definition 2 (Absolute Deadline) *Given $T > 0$, we say that system* (1) *has an* absolute deadline *at $t = T$ if, for any initial time instant $s \in [0, T)$ and $\boldsymbol{\xi} \in \mathbb{R}^n$, every solution $\mathbf{x} : [s, T) \to \mathbb{R}^n$ of* (1) *with $\boldsymbol{\eta} = \mathbf{0}$ and $\mathbf{x}(s) = \boldsymbol{\xi}$ satisfies $\lim_{t\to T^-} \mathbf{x}(t) = \mathbf{0}$.*

Note that although its definition involves trajectories with different initial time instants, the absolute deadline is a mathematical property of system (1) that is unrelated to the actual time instant when a controller or differentiator is switched on, commonly called t_0 in the literature, cf. Holloway and Krstic (2019). The definition of absolute deadline does not require system (1) to be undefined for $t \ge T$. Therefore, this definition and all further results apply also in the case of systems defined on the unbounded time interval $[0, \infty)$.

In this note, we focus on the subclass of prescribed-time algorithms exhibiting an absolute deadline (prescribed-time algorithms ensure convergence before a user-defined time). However, note that not every prescribed-time algorithm exhibits an absolute deadline, such is the case of the time-invariant algorithm in (Seeber et al., 2021), or the time-varying algorithm with uniformly bounded TVG in Aldana-López et al. (2022). Two specific structures of system (1) are considered that are particularly relevant in the context of either control or

[1] Notice that uniform Lyapunov stability is a property that is usually defined on the time interval $[0, \infty)$, see (Khalil, 2002, Definition 4.2). For simplicity in the terminology, we define uniform Lyapunov stability restricted to the interval of interest $[0, T)$.

observation. For control, the form

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i \in \{1, \ldots, n-1\} \\ \dot{x}_n &= v(t, \mathbf{x}+\boldsymbol{\eta}) + d(t)\end{aligned} \tag{2}$$

with $n \geq 2$ and $v : [0, T) \times \mathbb{R}^n \to \mathbb{R}$ is considered. Such a system is obtained as a closed loop when applying a TVG-based control law $v(t, \mathbf{x})$ to a perturbed integrator chain. For observation, a system of the form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \phi_i(t, x_1 + \eta_1), \quad i \in \{1, \ldots, n-1\} \\ \dot{x}_n &= d(t) + \phi_n(t, x_1 + \eta_1)\end{aligned} \tag{3}$$

with $n \geq 2$, $\phi_1, \ldots, \phi_n : [0, T) \times \mathbb{R} \to \mathbb{R}$, and measurement noise η_1 with $|\eta_1(t)| \leq \bar{\eta}$. Such a system models error dynamics when constructing a differentiator, i.e., an observer for the state of a perturbed integrator chain.

For example, consider the system with $n = 2$ and $T = 1$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\tfrac{6}{(1-t)^2} x_1 - \tfrac{4}{1-t} x_2 + d(t) \tag{4}$$

with $d(t) = (1-t)^2$. Given an initial condition $[\xi_1, \xi_2]^T$ at $t = s$, the unique solution to (4) can be written as

$$\begin{aligned}x_1(t) &= \left(\tfrac{3(1-t)^2}{(1-s)^2} - \tfrac{2(1-t)^3}{(1-s)^3}\right)\xi_1 + \left(\tfrac{(1-t)^2}{(1-s)} - \tfrac{(1-t)^3}{(1-s)^2}\right)\xi_2 \\ &\quad + \tfrac{1}{2}(s-t)^2(1-t)^2 \\ x_2(t) &= \left(\tfrac{6(1-t)^2}{(1-s)^3} - \tfrac{6(1-t)}{(1-s)^2}\right)\xi_1 + \left(\tfrac{3(1-t)^2}{(1-s)^2} - \tfrac{2(1-t)}{(1-s)}\right)\xi_2 \\ &\quad - (s-t)^2(1-t) - (s-t)(1-t)^2\end{aligned} \tag{5}$$

for any $s \in [0, 1)$ and $\xi_1, \xi_2 \in \mathbb{R}$, and hence satisfies $\lim_{t\to 1^-} \mathbf{x}(t) = \mathbf{0}$. Thus, system (4) has an absolute deadline at $t = T = 1$. The following proposition shows that every time-varying linear system with a fixed (prescribed) convergence time has an absolute deadline.

Proposition 3 *Consider system* (1) *with* $\mathbf{f}(t, \mathbf{x}, \mathbf{0}, d) = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)d$ *with* $\mathbf{A} : [0, T) \to \mathbb{R}^{n\times n}$ *and* $\mathbf{b} : [0, T) \to \mathbb{R}^n$ *continuous. Suppose that every solution of* (1) *with* $\boldsymbol{\eta} = \mathbf{0}$ *starting at time* $t = 0$ *satisfies* $\lim_{t\to T^-} \mathbf{x}(t) = \mathbf{0}$. *Then,* (1) *has an absolute deadline*[2] *at* T.

It follows from Proposition 3, that some algorithms proposed in the literature induce an absolute deadline, e.g., Song et al. (2017); Holloway and Krstic (2019); Song et al. (2019). In other cases, existence of Filippov solutions with an absolute deadline can be shown through a time-scaling argument, e.g., Pal et al. (2020); Aldana-López et al. (2021); Orlov et al. (2022); Tran and Yucelen (2020). These algorithms have been used for control in Song et al. (2017); Gómez-Gutiérrez (2020); Pal et al. (2020); Song et al. (2019), for a system (1) with $n \geq 2$ of the form (2). Usually, the disturbance $d(t)$ therein is restricted to a class of admissible functions, e.g. measurable signals bounded by a constant $L > 0$ for all $t \geq 0$.

Similarly, systems with an absolute deadline have been used for differentiation by Holloway and Krstic (2019); Aldana-López et al. (2021); Orlov et al. (2022) by studying error systems of the form (3). For example, take the system from Example 1 in Holloway and Krstic (2019):

$$\begin{aligned}\dot{x}_1 &= -\left(\ell_1 + \tfrac{6}{T-t}\right)x_1 + x_2 \\ \dot{x}_2 &= -\left(\ell_2 + \tfrac{3\ell_1}{T-t} + \tfrac{6}{(T-t)^2}\right)x_1\end{aligned} \tag{6}$$

which is of the form (3) with $n = 2, d(t) = 0$ and arbitrary $T, \ell_1, \ell_2 > 0$. It was shown by Holloway and Krstic (2019) that for any $\mathbf{x}(0) = \boldsymbol{\xi}$, (6) satisfies $\lim_{t\to T^-} \mathbf{x}(t) = \mathbf{0}$. Thus, (6) has an absolute deadline at $t = T$ by virtue of Proposition 3. With similar arguments being applicable to the linear time-varying controllers by Song et al. (2017); Holloway and Krstic (2019); Song et al. (2019) discussed above and the mentioned time-scaling argument being applicable to other approaches such as (Aldana-López et al., 2021; Orlov et al., 2022), one can see that a significant number of prescribed-time algorithms exhibit an absolute deadline.

3 Robustness and performance limitations

Despite the benefits of algorithms equipped with an absolute deadline, these systems have inherent limitations in terms of stability, robustness, and practical feasibility.

3.1 Controllers for integrator chains

The following theorem establishes a set of consequences of systems of the form (2) with $n \geq 2$ and an absolute deadline, i.e., perturbed integrator chains under prescribed-time control.

Theorem 4 *Let* $n \geq 2$, $T > 0$ *and suppose that system* (2) *has an absolute deadline at* T. *Then,*

- **i)** *For all* $\bar{\eta} > 0$ *and* $\mathbf{x}_0 \in \mathbb{R}^n$ *there exists a piecewise continuous and bounded noise* $\boldsymbol{\eta} : [0, T) \to \mathbb{R}^n$ *with countably many discontinuities and* $\|\boldsymbol{\eta}(t)\| \leq \bar{\eta}, \forall t \in [0, T]$ *such that the solution* $\mathbf{x}(\cdot)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies* $\limsup_{t\to T^-} \|\mathbf{x}(t)\| = \infty$.
- **ii)** *For all* $\epsilon > 0$, *if there exists a continuous* $\boldsymbol{\xi}_\epsilon : [0, T] \to \mathbb{R}^{n-1}$ *satisfying* $\boldsymbol{\xi}_\epsilon(T) = 0$ *and* $\limsup_{t\to T^-} |v(t, [\boldsymbol{\xi}_\epsilon(t)^T, -2\epsilon]^T)| < \infty$, *then, for all* $\bar{\eta} > 0$ *and all* $\boldsymbol{x}_0 \in \mathbb{R}^n$ *there exists a piecewise continuous noise* $\boldsymbol{\eta} : [0, T) \to \mathbb{R}^n$ *with two discontinuities and* $\|\boldsymbol{\eta}(t)\| \leq \bar{\eta}$ *such that the solution* $\mathbf{x}(\cdot)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies* $\|\lim_{t\to T^-} \mathbf{x}(t)\| \geq \epsilon$.
- **iii)** *For all* $\delta > 0$, $\sup_{\|\mathbf{x}\|_\infty \leq \delta, t\in[0,T)} |v(t, \mathbf{x})| = \infty$.

[2] The same conclusion holds for every system defined by $\mathbf{f}(t, \mathbf{x}, \mathbf{0}, d(t)) =: \mathbf{g}(t, \mathbf{x})$ globally Lipschitz in $\mathbf{x}$, uniformly over t in every compact subinterval of $[0, T)$.

iv) *The origin of* (2) *is not uniformly Lyapunov stable.*

Remark 5 *Theorem 4-* **i**) *and* **ii**) *show that arbitrarily small noises can lead to trajectories with arbitrarily large or even diverging control error at the deadline T. As a consequence, arbitrarily bad tracking performance on the interval* $[0, T)$ *is obtained even if such noise is removed in a vicinity of the deadline T. Theorem 4-***iii**) *and* **iv**) *show the main reasons for this lack of robustness: unboundedness of the controller* $v(t, \mathbf{x})$ *in t, which is consistent with literature on prescribed-time control based on TVGs, and more importantly, lack of uniform Lyapunov stability.*

Remark 6 *Theorem 4-***iv**) *implies lack of uniform Lyapunov stability also in the classical sense of (Khalil, 2002, Definition 4.2), if system* (1) *is defined on the unbounded time interval, i.e., for* $t \in [0, \infty)$.

Remark 7 *Note that the additional condition in Theorem 4-***ii**) *is very mild. Often it is possible to achieve even* $v(t, \boldsymbol{\xi}_\epsilon(t)^T, -2\epsilon]^T) = 0$. *In system* (4), *this is achieved with* $\xi_\epsilon(t) = \epsilon(4/3)(1-t)$, *which satisfies* $\xi_\epsilon(1) = 0$.

3.2 Differentiators

Similarly to the previous section, the following theorem establishes a set of consequences of systems of the form (3) with $n \geq 2$ and an absolute deadline, i.e., differentiation error dynamics with prescribed-time convergence.

Theorem 8 *Let* $n \geq 2$, $T > 0$ *and suppose that system* (3) *has an absolute deadline at T. Then,*

- **i**) *For all* $\bar{\eta} > 0$ *and* $\mathbf{x}_0 \in \mathbb{R}^n$, *there exists a locally Lipschitz continuous noise* $\eta_1 : [0, T) \to [-\bar{\eta}, \bar{\eta}]$ *such that the corresponding solution* $\mathbf{x}(\cdot)$ *with* $\mathbf{x}(0) = \mathbf{x}_0$ *satisfies* $\limsup_{t \to T^-} \|\mathbf{x}(t)\| = \infty$.
- **ii**) *For all* $\bar{\eta} > 0$ *and* $\epsilon > 0$, *there exists a Lipschitz continuous noise* $\eta_1 : [0, T) \to [-\bar{\eta}, \bar{\eta}]$, *such that all corresponding solutions satisfy* $\|\lim_{t \to T^-} \mathbf{x}(t)\| \geq \epsilon$.
- **iii**) *For all* $\delta > 0$, $\sup_{|x_1| \leq \delta, t \in [0,T)} |\boldsymbol{\phi}_i(t, x_1)| = \infty$ *for some* $i \in \{1, \ldots, n\}$.

Remark 9 *Note that unlike Theorem 4, lack of uniform Lyapunov stability is not shown in Theorem 8. Nevertheless, an equivalent set of consequences to those in Theorem 4 is also shown in this case. In fact, Theorem 8-***ii**) *is even stronger than Theorem 4-***ii**), *because an arbitrarily small noise signal yields arbitrarily bad performance* for all initial conditions *in that case.*

Remark 10 *A robust exact differentiator, in the sense of Levant (1998); Seeber and Haimovich (2023), is one which differentiates noise-free signals exactly after a finite (possibly prescribed) time, and whose behavior under bounded noise tends uniformly to the behavior in the absence of noise as the noise bound tends to zero. An important corollary of Theorem 8 is that* a robust exact differentiator with an absolute deadline cannot exist.

4 Discussion

In the prescribed-time literature based on TVG it is often acknowledged that it is problematic, when measurement noise is present, to have TVG that tends to infinity, see e.g., Section 3.2 in Song et al. (2017) and Section 2.A in Holloway and Krstic (2019). To avoid this problem, some workarounds have been suggested in the literature.

A common workaround proposed by Song et al. (2017); Holloway and Krstic (2019) is to switch off the algorithm at a time t_{stop} before the absolute deadline T, thus maintaining the TVG uniformly bounded. With such a workaround, an absolute deadline is no longer present, eliminating the lack of robustness exposed above, but at the same time also the convergence to zero in prescribed time. As a result, the error then does not reach zero at time t_{stop}, and for the case of linear time-varying systems with bounded dynamic matrix on $[0, t_{\text{stop}}]$, the remaining error grows linearly with the initial condition. Moreover, our results show that the system becomes more sensitive to measurement noise as t_{stop} approaches T.

Another workaround proposed by Song et al. (2017) is to switch off the algorithm when the error trajectory enters a desired deadzone on the error. A particular case of this workaround is essentially used in Orlov et al. (2022), with a deadzone of zero width. However, with such an approach, due to the nature of the algorithms in Song et al. (2017, 2019); Holloway and Krstic (2019); Orlov et al. (2022), unperturbed trajectories with a large initial condition will enter such deadzone arbitrarily close to T with arbitrarily large TVG. The presence of additional arbitrarily small noise may furthermore prevent the trajectory from entering the deadzone at all, thus lacking robustness despite the workaround. Indeed, in the case of the zero-width deadzone workaround, used e.g., in Orlov (2022); Orlov et al. (2022); Verdés Kairuz et al. (2022), it can be shown, using a time-scale transformation argument, that the workaround does not eliminate the absolute deadline property.

5 Illustrative Example

Recall system (4) under $d(t) = (1-t)^2$, which was shown above to exhibit an absolute deadline at $t = T = 1$. Consider a noise signal $\boldsymbol{\eta}(t) = [\eta(t), 0]^T$ with $\eta(t) = \bar{\eta}\sqrt{1-t}$ satisfying $\|\boldsymbol{\eta}(t)\| = |\eta(t)| \leq \bar{\eta}, \forall t \in [0, 1)$ such that system (4) becomes:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\tfrac{6}{(1-t)^2}(x_1 + \eta(t)) - \tfrac{4}{1-t}x_2 + d(t) \quad (7)$$

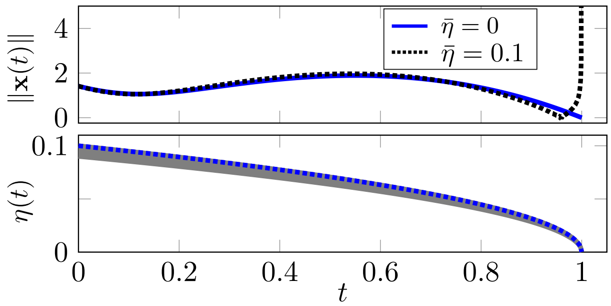

Fig. 1. Above: Solutions to (7) with explicit expression given in (8), with $\xi_1 = \xi_2 = 1$, and $\bar{\eta} = 0$ and $\bar{\eta} = 0.1$, respectively. Below: The dashed blue line represents the noise signal $\eta(t) = \bar{\eta}\sqrt{1-t}$. Any noise signal contained in the gray region $\bar{\eta}\sqrt{1-t}[0.9, 1]$ causes a divergent trajectory.

It can be verified that $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ with

$$\begin{aligned} x_1(t) =&(3(1-t)^2 - 2(1-t)^3)\xi_1 \\ &+ ((1-t)^2 - (1-t)^3)\xi_2 + \frac{1}{2}t^2(1-t)^2 \\ &+ \frac{4\bar{\eta}}{5}\left(5(1-t)^2 - 3(1-t)^3 - 2\sqrt{1-t}\right) \\ x_2(t) =&(6(1-t)^2 - 6(1-t))\xi_1 - t^2(1-t) \\ &+ (3(1-t)^2 - 2(1-t))\xi_2 + t(1-t)^2 \\ &+ \frac{4\bar{\eta}}{5}\left(9(1-t)^2 - 10(1-t)\right) + \frac{4\bar{\eta}}{5\sqrt{1-t}} \end{aligned} \tag{8}$$

is the unique solution of (7) for $t \in [0, 1)$ with $\mathbf{x}(0) = [\xi_1, \xi_2]^T$. In case $\bar{\eta} = 0$, $\lim_{t\to T^-} \mathbf{x}(t) = \mathbf{0}$, verifying that the system has an absolute deadline at $t = T$. As already discussed, $\mathbf{x}(t_{\text{stop}})$ is an unbounded function of the initial condition $\mathbf{x}(0)$ and hence switching off the algorithm at a time $t_{\text{stop}} < T$ is not sufficient for convergence.

In the case $\bar{\eta} > 0$, regardless of how small it is, the last term of $x_2(t)$ in (8) is divergent at the deadline and thus $\limsup_{t\to T^-} \|\mathbf{x}(t)\| = \infty$. The trajectories $\mathbf{x}(t)$ for $\bar{\eta} = 0$ and $\bar{\eta} = 0.1$, respectively, with $\xi_1 = \xi_2 = 1$ are illustrated in the first plot of Fig. 1. Moreover, it can be verified that any noise satisfying $\eta(t) \in (1-t)^\alpha \bar{\eta}[\beta, 1]$ with $\frac{6-3\alpha}{6-2\alpha} < \beta \leq 1, \alpha \in (0, 1)$ would also produce a divergent trajectory. The second plot of Fig. 1, illustrates this region in gray for $\alpha = \frac{1}{2}, \beta = 0.9, \bar{\eta} = 0.1$, together with the noise $\eta(t) = 0.1\sqrt{1-t}$ in dashed line.

6 Conclusion

In this note, we analyzed the behavior under measurement noise of a class of non-scalar fixed-time algorithms characterized by an absolute deadline. This analysis exposes some inherent performance limitations and lack of robustness, mainly when noise appears arbitrarily close to the absolute deadline. We show, for instance, that an arbitrarily small noise signal may result in arbitrarily large errors at the absolute deadline and can also lead to divergence. In line with existing literature, our analysis focuses on controllers for integrator chains and differentiators. In future work, we consider extending the study to more general forms of control algorithms, such as controllers for nonlinear systems and systems in a strict feed-forward form exhibiting an absolute deadline.

A Appendix

A.1 Proof of Proposition 3

Consider an arbitrary $\mathbf{x}(s) = \boldsymbol{\xi}$ with $s \in [0, 1)$. Hence, due to the proposition's assumptions, the right-hand side of (1) is continuous in $t \in [0, s]$ and there exists $L > 0$ such that $\|\mathbf{f}(t, \mathbf{x}_1, \mathbf{0}, d) - \mathbf{f}(t, \mathbf{x}_2, \mathbf{0}, d)\| \leq L\|\mathbf{x}_1 - \mathbf{x}_2\|$ for all $t \in [0, s]$ and all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$. Therefore, the Cauchy–Lipschitz existence theorem ensures that the solution $\mathbf{x}(t)$ can be continued backwards in time towards $\mathbf{x}(0) = \boldsymbol{\xi}'$. Hence, by the proposition assumptions, it follows that $\lim_{t\to T^-} \mathbf{x}(t) = \mathbf{0}$.

A.2 Proof of Theorem 4

First, we show the following auxiliary lemma:

Lemma 1 *If there exists $T > 0$ such that (2) with $n \geq 2$ has an absolute deadline at $t = T$ then, for any pair $\delta, \epsilon > 0$ there exists $s \in (0, T)$ such that if $|x_1(s')| \geq \delta$ for any $s' \in [s, T)$, then $\|\mathbf{x}(t)\| > \epsilon$ for some $t \in (s', T)$.*

PROOF. Given $\delta, \epsilon > 0$ choose $s = T - \delta/\epsilon'$ for any $\epsilon' > \max(\epsilon, \delta)$ such that $s \in [0, T)$. Note that $\lim_{t\to T^-} x_1(t) = 0$ from the absolute deadline property. Assume $|x_1(s')| \geq \delta$ for arbitrary $s' \in [s, T)$. By virtue of being a solution, $x_2(\bullet)$ is absolutely continuous and since $\dot{x}_1 = x_2$, then x_1 is differentiable everywhere in (s, T). Hence, by the mean value theorem, there must exist $t \in (s', T)$ such that $\dot{x}_1(t) = x_2(t) = \frac{-x_1(s')}{T-s'}$. Therefore, $\|\mathbf{x}(t)\| \geq |x_2(t)| = \frac{|x_1(s')|}{T-s'} \geq \frac{|x_1(s')|}{T-s} \geq \frac{\delta}{\delta/\epsilon'} = \epsilon' > \epsilon$.

Item i): Consider strictly increasing sequences $\{t_k\}_{k=0}^\infty$, $\{t'_k\}_{k=0}^\infty$ with $t_0 = t'_0 = 0$ and $\{\epsilon_k\}_{k=0}^\infty$ with $\lim_{k\to\infty} \epsilon_k = \infty$. Given $\delta \in (0, \bar{\eta})$, we construct a noise $\boldsymbol{\eta}(t) = \eta_1(t)\mathbf{b}_1$ with $\eta_1(t) = \delta\text{sign}(x_1(t_k)), \forall t \in [t_k, t_{k+1})$ which is possible due to causality of (2). Note that $\|\boldsymbol{\eta}(t)\| \leq \bar{\eta}, \forall t \in [0, \lim_{k\to\infty} t_k)$. Now, we construct the rest of $\{t_k\}_{k=1}^\infty, \{t'_k\}_{k=1}^\infty$ as follows. Given the pair $\delta, \epsilon'_k > 0$ with $\epsilon'_k = \epsilon_k + \bar{\eta}$, use $s > 0$ as in Lemma 1 to define any $t_k \in (\max(s, t_{k-1}, t'_{k-1}, T - 1/\epsilon_k), T)$ picked such that in the case of $\bar{\eta} = 0$, $|x_1(t_k)| \geq \delta$ implies $\|\mathbf{x}(t'_k)\| > \epsilon_k$ for some $t'_k \in (t_k, T)$ as from Lemma 1. Note that $t_k \geq T - 1/\epsilon_k$ such that both $\lim_{k\to\infty} t_k = \lim_{k\to\infty} t'_k = T$. Given this construction of $\boldsymbol{\eta}(t)$, analyze the interval $[t_k, t_{k+1})$. Let $\mathbf{z}(t) = \mathbf{x}(t) + \boldsymbol{\eta}(t)$. For this interval, we have $\dot{z}_i = z_{i+1}$ for $i = 1, \ldots, n-1$ and $\dot{z}_n = v(t, \mathbf{z}) + d(t)$, which is precisely (3) with $\bar{\eta} = 0$. Moreover, note that $|z_1(t_k)| = |x_1(t_k) + \delta \text{sign}(x_1(t_k))| = |x_1(t_k)| + \delta \geq \delta$ Thus, $\|\mathbf{z}(t'_k)\| > \epsilon'_k$ by Lemma 1 and the definition of t'_k which implies $\epsilon_k + \bar{\eta} = \epsilon'_k < \|\mathbf{z}(t'_k)\| = \|\mathbf{x}(t'_k) + \boldsymbol{\eta}(t'_k)\| \leq \|\mathbf{x}(t'_k)\| + \bar{\eta}$. Thus, $\|\mathbf{x}(t'_k)\| > \epsilon_k$. As a consequence, $\limsup_{t\to T^-} \|\mathbf{x}(t)\| \geq \limsup_{k\to\infty} \|\mathbf{x}(t'_k)\| > \sup_{t\to\infty} \epsilon_k = \infty$ concluding the proof for this item.

Item ii): Let $\bar{\eta}' = \bar{\eta}/\sqrt{n}$, $\boldsymbol{\xi}' = [\boldsymbol{\xi}_\epsilon^T, 0]^T$, $\boldsymbol{q} = [\mathbf{0}^T, -2\epsilon]^T$ and $\Psi_{n+1}(t) = v(t, \boldsymbol{\xi}'(t) + \boldsymbol{q})$. By assumption, $s > \max(T - \bar{\eta}'/(12\epsilon), T - 1/2)$ exists such that $(T-s)|\Psi_{n+1}(t)| \leq \min(\epsilon, \bar{\eta}'/2)$ and $\|\boldsymbol{\xi}_\epsilon(t)\|_\infty \leq \bar{\eta}'/2$ for all $t \in [s, T)$. Define functions $\Psi_i : [s, T) \to \mathbb{R}$ for $i = 1, \ldots, n$ recursively via $\dot{\Psi}_i(t) = \Psi_{i+1}(t)$ with $\Psi_n(s) = -2\epsilon$ and arbitrary $\Psi_i(s) \in [-\bar{\eta}'/4, \bar{\eta}'/4]$ for $i < n$. From the bound on $|\Psi_{n+1}(t)|$, obtain $\Psi_n(t) \in [-3\epsilon, -\epsilon]$ and $|\Psi_n(t) - \Psi_n(s)| \leq \bar{\eta}'/2$. From $t - s \leq \bar{\eta}'/(12\epsilon)$, then $|\Psi_{n-1}(t)| \leq \bar{\eta}'/2$, and using $t - s \leq 1/2$ yields $|\Psi_i(t)| \leq \bar{\eta}'/2$ for all $i = 1, \ldots, n-1$. With $\boldsymbol{\Psi} = [\Psi_1, \ldots, \Psi_n]^T$, then $\|\boldsymbol{\Psi}(t) - \boldsymbol{q}\|_\infty \leq \bar{\eta}'/2$. Now, define the noise as $\boldsymbol{\eta}(t) = \boldsymbol{\xi}'(t) + \boldsymbol{q} - \boldsymbol{\Psi}(t)$, which satisfies $\|\boldsymbol{\eta}(t)\|_\infty \leq \|\boldsymbol{\xi}'(t)\|_\infty + \bar{\eta}'/2 \leq \bar{\eta}'$ and hence $\|\boldsymbol{\eta}(t)\| \leq \bar{\eta}'\sqrt{n} = \bar{\eta}$. Then, $\mathbf{x}(t) = \boldsymbol{\Psi}(t)$ is a solution of (2) on $[s, T]$, because $\dot{x}_i(t) = x_{i+1}(t)$ for $i < n$ and $\dot{x}_n(t) = \Psi_{n+1}(t) = v(t, \boldsymbol{\xi}'(t) + \boldsymbol{q}) = v(t, \mathbf{x}(t) + \boldsymbol{\eta}(t))$ by construction, with $\|\lim_{t\to T^-} \mathbf{x}(t)\| \geq |\lim_{t\to T^-} \Psi_n(t)| \geq \epsilon$. To steer every initial condition $\mathbf{x}(0)$ to an $\mathbf{x}(s)$ of the required form, apply a constant noise $\boldsymbol{\eta}(t) = [\bar{\eta}'/8, 0, \ldots, 0]^T$ initially, leading to $\|\boldsymbol{x}(s_0) - \boldsymbol{\eta}(s_0)\| \leq \min(\bar{\eta}'/16, \epsilon)$ for some $s_0 > \max(T - \bar{\eta}'/(32\epsilon), T - 1/2)$, i.e., $x_1(s_0) \in [\bar{\eta}'/16, 3\bar{\eta}'/16]$, $|x_i(s_0)| \leq \bar{\eta}'/16$ for $i = 2, \ldots, n$, and $|x_n(s_0)| \leq \epsilon$. Removing the noise, i.e., setting $\boldsymbol{\eta}(t) = \mathbf{0}$ starting at $t \geq s_0$, then yields $\mathbf{x}(s)$ of the required form (possibly with reversed sign) at $s = \inf\{\sigma \geq s_0 : |x_n(\sigma)| \geq 2\epsilon\}$.

Item iii): Assume $\sup_{\mathbf{x}\in[-\delta,\delta]^n, t\in[0,T]} |v(t, \mathbf{x})| = \epsilon$ for some $\epsilon \geq 0$. Consider $s = T - \delta/\epsilon'$ with $\epsilon' > \max(\epsilon, \delta)$ and an arbitrary initial condition $\mathbf{x}(s)$ for (2) with $\bar{\eta} = 0$ and $\mathbf{x}(s) \in [-\delta, \delta]^n$. The absolute deadline property $\lim_{t\to T^-} \mathbf{x}(t) = 0$ implies the existence of $s' = \inf\{t \in [s, T) : \mathbf{x}(t') \in [-\delta, \delta]^n, \forall t' \geq t\}$ and, by absolute continuity of $\mathbf{x}(t)$, $x_i(s') = \delta$ for some $i \in \{1, \ldots, n\}$. Assume $i \neq n$; by the mean value theorem there then exists $t \in (s', T)$ with $\dot{x}_i(t) = x_{i+1}(t) = \frac{-x_i(s')}{T-s'}$. Therefore, $|x_{i+1}(t)| = \frac{|x_i(s')|}{T-s'} \geq \frac{|x_i(s')|}{T-s} = \frac{\delta}{\delta/\epsilon'} = \epsilon' > \delta$ which is a contradiction of $\mathbf{x}(t) \in [-\delta, \delta]^n, \forall t \in [s', T)$. Hence, $i = n$. The function x_n is absolutely continuous and thus differentiable almost everywhere. Therefore, it can be shown that there must exist $t_1, t_2 \in (s', T)$ such that $v(t_1, \mathbf{x}(t_1)) \leq \frac{-x_n(s')}{T-s'}$ and $v(t_2, \mathbf{x}(t_2)) \geq \frac{-x_n(s')}{T-s'}$. Therefore, there exists $t \in \{t_1, t_2\}$ so that $|v(t, \mathbf{x}(t))| \geq \frac{|x_n(s')|}{T-s} = \epsilon' > \epsilon$. The previous fact, in addition to $\mathbf{x}(t) \in [-\delta, \delta]^n$, contradicts the initial assumption.

Item iv): We will show that for any $\delta, \epsilon > 0$, there exist s, t with $0 \leq s < t \leq T$ and a trajectory of (2) which satisfies both $\|\mathbf{x}(s)\| \leq \delta$ and $\|\mathbf{x}(t)\| > \epsilon$, which implies that uniform Lyapunov stability does not hold. Given $\delta, \epsilon > 0$, choose $s \in (0, T)$ as in Lemma 1 and a trajectory of (2) passing through $\mathbf{x}(s) = \delta \mathbf{b}_1$ satisfying $\|\mathbf{x}(s)\| = |x_1(s)| = \delta$. Hence, Lemma 1 implies $\|\mathbf{x}(t)\| > \epsilon$ for some $t \in (s, T)$.

A.3 Proof of Theorem 8

Item i): Consider strictly increasing sequences $\{t_k\}_{k=0}^\infty$, $\{\epsilon_k\}_{k=0}^\infty$ with $t_0 = \epsilon_0 = 0$ and $\lim_{k\to\infty} \epsilon_k = \infty$. Now, let $\eta_1(t) = -(\bar{\eta}\text{sign}(\eta_1(t_k)) + \eta_1(t_k))(t - t_k)/(T - t_k) + \eta_1(t_k)$ for all $t \in [t_k, t_{k+1})$ and $\eta_1(0) = \bar{\eta}$, which is locally Lipschitz continuous. It can be verified that since $|\eta_1(0)| \leq \bar{\eta}$, then $|\eta_1(t)| \leq \bar{\eta}, \forall t \in [0, \sup\{t_k\}_{k=0}^\infty)$. Now, note that $|\dot{\eta}_1(t)| > \epsilon_0, \forall t \in [t_0, t_1)$ for arbitrary t_1. Hence, we construct the rest of the t_k recursively as follows. First, assume $|\dot{\eta}_1(t)| = (\bar{\eta} + |\eta_1(t_k)|)/(T - t_k) > \epsilon_k, \forall t \in [t_k, t_{k+1})$. Then, given $\delta > 0$ there exists $t'_{k+1} \in (t_k, T)$ such that for $t \in [t_{k+1}, t_{k+2})$ we have $|\dot{\eta}_1(t)| = (\bar{\eta} + |\eta_1(t_{k+1})|)/(T - t_{k+1}) > \epsilon_k + \delta$ for arbitrary $t_{k+1} \in (t'_{k+1}, T)$ and $t_{k+2} \in (t_{k+1}, T)$. Moreover, on the time interval $[t_k, t_{k+1})$, consider new state variables $z_1 = x_1 + \eta_1$, $z_2 = x_2 + \dot{\eta}_1$, and $z_i = x_i$ for $i > 2$ leading to

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \phi_i(t, z_1), \quad i \in \{1, \ldots, n-1\} \\ \dot{z}_n &= d(t) + \phi_n(t, z_1) \end{aligned} \tag{A.1}$$

Hence, given $\delta > 0$ there exists $t''_{k+1} \in (t_k, T)$ such that $|z_2(t_{k+1})| \leq \delta$, equivalently $|x_2(t_{k+1}) - \dot{\eta}_1(t_{k+1})| \leq \delta, \forall t_{k+1} \in (t''_{k+1}, T)$ by virtue of the absolute deadline property. Hence, choose $t_{k+1} \in (\max(t'_{k+1}, t''_{k+1}, T - 1/\epsilon_k), T)$ implying that $|x_2(t_{k+1})| > \epsilon_k$ and $\sup\{t_k\}_{k=0}^\infty = T$. Similarly to the proof of Theorem 4-**i**), it follows that $\limsup_{t\to T^-} \|\mathbf{x}(t)\| = \infty$.

Item ii): Let $s = T - 2\bar{\eta}/\epsilon$ and define the noise $\eta_1(t) = -\bar{\eta}$ for $t \in [0, s)$ and $\eta_1(t) = -\bar{\eta} + (t - s)\epsilon$ for $t \in [s, T)$. Note that η_1 is Lipschitz continuous and satisfies $|\eta_1(t)| \leq \bar{\eta}$ for all $t \in [0, T]$. On the time interval $(s, T]$, consider new state variables $z_1 = x_1 + \eta_1$, $z_2 = x_2 + \epsilon$, and $z_i = x_i$ for $i > 2$. Since $\dot{\eta}_1(t) = \epsilon$ for $t \in (s, T]$ this leads to dynamics of the form (A.1) on this interval, and consequently $\lim_{t\to T^-} z_2(t) = 0$ by virtue of the absolute deadline property. Hence, $\lim_{t\to T^-} x_2(t) = -\epsilon$ and $\|\lim_{t\to T^-} \mathbf{x}(t)\| \geq |\lim_{t\to T^-} x_2(t)| = \epsilon$.

Item iii): Assume that $\sup_{x_1\in[-\delta,\delta], t\in[0,T]} \|\boldsymbol{\phi}(t, x_1)\| = \epsilon \geq 0$, let $\epsilon' > \max(\delta, \epsilon + \delta)$ and define $s, s' > 0$ and initial condition $\mathbf{x}(s)$ in the same way as in the proof of Theorem 4-**iii**). Hence, there is $i \in \{1, \ldots n\}$ with $x_i(s') = \delta$ and $\mathbf{x}(t) \in [-\delta, \delta]^n, \forall t \in [s', T)$. Assume $i \neq n$ such that by the mean value theorem and the absolute deadline property there exists $t \in [s', T)$ with $|x_{i+1}(t) + \phi_i(t, x_1(t))| = \frac{|x_i(t)|}{T-s} \geq \epsilon' > \epsilon + \delta$. But $|x_{i+1}(t) + \phi_i(t, x_1(t))| \leq |x_{i+1}(t)| + |\phi_i(t, x_1(t))| \leq |\phi_i(t, x_1)| + \delta$. Hence, $\|\boldsymbol{\phi}(t, x_1(t))\| \geq |\phi_i(t, x_1(t))| > \epsilon$ and $x_1(t) \in [-\delta, \delta]$ contradicting the initial assumption. The proof follows in a similar way when $i = n$.

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