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Abstract Heterarchy: Time/State-Scale Re-entrant Form

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Abstract

A heterarchy is a dynamical hierarchical system inheriting logical inconsistencies between levels. Because of these inconsistencies, it is very difficult to formalize a heterarchy as a dynamical system. Here, the essence of a heterarchy is proposed as a pair of the property of self-reference and the property of a frame problem interacting with each other. The coupling of them embodies a one-ity inheriting logical inconsistency. The property of self-reference and a frame problem are defined in terms of logical operations, and are replaced by two kinds of dynamical system, temporal dynamics and state-scale dynamics derived from the same "liar statement". A modified tent map serving as the temporal dynamics is twisted and coupled with a tent map serving as the state-scale dynamics, and this results in a discontinuous self-similar map as a dynamical system. This reveals that the state-scale and temporal dynamics attribute to the system, and shows both robust and emergent behaviors.

Key Words: *Heterarchy, Chaotic liar, Self-reference, Frame problem, Self-similar map*

1. Introduction

What new ideas have been arising in the field of complex systems? Recently, the difference between robustness and stability has been addressed with the aim of accessing a new notion of complexity that is beyond the purview of stability theory. Jen

(2003) claims, “The concept of stability is regarded as an old one that derives from the study of the stability of the solar system. Although both stability and robustness are concepts appropriate to measuring feature persistence, only robustness is relevant for the interplay between dynamics and organization, the role of future, the anticipation of insults, along with other questions even more difficult to formulate relating to creativity, intentionality, and identity”. The key notion is that of a heterarchy (McCulloch, 1945; Keaher and von Goldammer, 1988; Stark, 1999), which is an interconnected, overlapping, hierarchical network constituted by individual components simultaneously belonging to and acting in multiple networks, with the overall dynamics of the system both emerging and governing interactions of these components. In human society, individuals act simultaneously as members of familial, political, and economic groups among others, and this is an example of a heterarchy. Biological signaling processes (Marder and Calabrese, 1996), evolutionary systems (Shapiro, 2002; Voigt et al., 2004) and computation in engineering (Cantwell-Smith, 2002; Gunji and Kamiura, 2004) also yield examples of a heterarchy.

In heterarchical systems robustness may exist on the level of individuals, on an intermediate level, or on the level of the whole system. Conversely, robustness at one level confers a degree of robustness on any of the other levels. Through the interplay between dynamics and organization, emergent levels and/or components of networks can be created. Such interplays cannot be described in advance, and it is possible to define creativity only when one cannot describe all components playing an essential role in maintaining a system in advance. Imagine that an observer is convinced that a hierarchical system is perfectly described by finite numbers of explicit levels. The system, however, has a hidden level that is not described in advance, and the hidden level affects the system such as to reorganize it and create a new explicit component. In this instance, it is possible for an observer to recognize creativity. This is why an essential property of heterarchy is latency of the environment.

There is an essential difficulty in evaluating the property of a heterarchy, because of the difficulty in measuring feature persistence. When a system persists as a unity, the essential cause of the persistence perpetually changes. In other words, even if an observer can find the essential cause of the persistence at one moment, another cause will appear at the next moment. This entails that an observer must keep on describing levels and components constituting the heterarchy. Beyond undertaking such an enumeration, one has to secure an alternative approach to understanding a heterarchy.

One of the possible ways to understand a heterarchy is to construct an abstract model featuring latency of the environment. Formalization of latency of the environment, the outside of the formal description, or the reservations of the formal description is required. On the one hand, an observer has to recognize his own limitations with respect to descriptions, and on the other hand, he has to refer to the outside of the description. Although such perspectives have been previously proposed under the name of internal measurement (Matsuno, 1989; Gunji, 1994, 1995) and/or endophysics (Rössler, 1988, 1994) some of such models (Gunji et al., 1997, 2001) were not actually needed in the field of complex systems till the notion of a heterarchy needed to be explicitly addressed.

How one can construct an abstract model for describing a heterarchy? This can be illustrated by considering computation in the brain. When computation is executed in a particular environment, computation by the brain can be considered as being analogous to computation by a computer. For example, one can see a pen as the object, "pen", when neurons in Wernicke and/or Brocca's area relevant for linguistic comprehension are firing in the brain. If neural activities of the area are regarded as constituting the computation, it is possible to see that neural activities in any other areas of the brain provide a particular environment in which the computation is executed. Such environmental neural activities are also computation. One can see two levels of computation, an explicit piece of computation in the Wernicke's area and the overall environmental computation. It is important to see that the environmental computation cannot be limited to a finite region, and that the environment is destined to be indefinite. That is why computation by the brain yields a typical example of a heterarchy. If one attempts to describe the entirety of a system consisting of both explicit and environmental computations, one tends to describe a phenomenal aspect (e.g., Tye, 1995). In order to progress beyond such phenomenological descriptions, one has to formalize the indefinite and latent environment by using weakly described self-reference (Gunji et al., 1995, 2004).

Here, we propose a model for describing the essence of a heterarchy in the following way. Firstly, we address what a heterarchy is, and reveal that the essence of a heterarchy is robust behavior against logical collapse. Next, robust behavior against logical collapse is formalized by self-reference connected by a frame problem. Although both self-reference and the frame problem reveal a kind of infinite regression, there is a difference with respect to logical status. A dynamical system based on self-reference has been developed by Grim et al. (1994). We develop their ideas, and the self-reference with a latent open environment (or with the frame problem) is expressed as the twisted coupling of two dynamics inheriting an infinite regression. We finally show how such a system reveals robust behavior and discuss the relationship between a heterarchy and robustness.

2. Heterarchy

A heterarchy is a dynamical hierarchical system in which an action at one level simultaneously reveals reactions at other levels. The significance of a heterarchy is manifested with respect to the difference between stability and robustness (Jen, 2003). First, we start by describing a heterarchy of human beings. The allegory helps us to understand the essence of a heterarchy.

A man is not only a member of his family, but also a member of the company in which he is employed. His actions, therefore, affect both his family and company, simultaneously. Imagine if he goes to work on his day off. Although such an action would benefit his company, it would be detrimental to his family. Listening to this, you might think that it satisfies the definition of a heterarchy, i.e., simultaneous interaction among different levels. You should, however, notice that such simultaneous interaction results just from a hidden specific operation such that detrimental (or beneficial) to the man's family is mapped to beneficial (or detrimental) to the company. One cannot recognize "simultaneousness" in his actions until one comprehends both the independence of the two levels (family and company) and their simultaneous interaction. Because of the independence of the two levels, one must take all possible operations between two levels into consideration. Moreover, one has to focus on the process of choosing one operation. Now, let us define a set of values for the family and the company as S = {0(detrimental), 1(beneficial)}. We call all possible operations from the family to the company "Interpretations" -0, -1, -2, and -3. These interpretations are defined as follows.

Interpretation-0: $0\to0$; $1\to0$,   Interpretation-1: $0\to1$; $1\to1$,
Interpretation-2: $0\to0$; $1\to1$,   Interpretation-3: $0\to1$; $1\to0$.

An observer has to describe the man's action of going to work on his day off as a simultaneous process of choosing one interpretation. What is a simultaneous process? A chosen interpretation has a value of S. The situation is, actually, described by the following.

On his day off, a man decides to go to work and puts on his shoes in the hallway of his home, where his son and wife, who were expecting to go to the zoo with him, are disappointed. The husband hesitates as to whether to go work, and considers that going to work would be detrimental to his family but beneficial to the company. This consideration (i.e., choosing Interpretation-3) proceeds within a finite time in the hallway. Therefore, such a process itself can have the value of S, in a family. Meanwhile, the man's wife begins to feel that her husband feels bad about leaving them to go to work, and thinks that her own attitude is making him feel too guilty. She thinks that she should send her husband off to work with a smile. Finally, she decides to hide her disappointment and wishes him farewell with a smile.

The husband hesitates in the hallway. This means that choosing an interpretation makes sense even in a family, and this triggers the emergence of a new value, "smile though disappointed" within S. As a result, the value in a family changes from {0, 1} to {0, 1, 2(smile though disappointed)} since choosing an interpretation that proceeds within a finite time can be a new kind of interpretation.

Here, we generalize such a process as the following. A heterarchy is defined by simultaneous interaction among a plurality of levels. This is replaced by the *simultaneous choice* between intra-level dynamics and inter-level dynamics. In the example of going to work, the intra-level dynamics is just a choice of a value of S (i.e., a value of a particular level) and the inter-level dynamics is a choice of an interpretation from Interpretations-0~3. The simultaneous choice is defined by two properties; (1) a map property, and (2) a simultaneous making value. The map property is defined by the following; for all elements of S, there exists an interpretation. The second property is defined by the following; each possible chosen interpretation must have a value in a level (e.g., family). The map property looks natural, however, it requires all possible correspondences between elements of S and all interpretations. The simultaneous making value is defined so as to expand such a stance.

Imagine that a map is defined by the following; 0→Interpretation-3, and 1→Interpretation-1. Simultaneous choice requires that each interpretation is assigned a value of S when the choice is made. For the choice, one can recognize that Interpretation-1 has a value of 1 and Interpretation-3 has a value of 0, as soon as either interpretation is chosen. However, the property of the simultaneous making value requires a making value for all interpretations. Although Interpretations-0 and -2 are not chosen, they also have to have values of S. Assume that Interpretation-0 has a value of 0. If so, the map property collapses because a value of 0 is mapped to both Interpretations-0 and -3. As a result, the map property and the simultaneous making value have a trade-off relationship therebetween. If each level is defined by a set, S, a set of the inter-level operations is defined by Hom(S, S), that is, a set of functions from

S to S. Simultaneous choice is defined by; $f: S \to \mathrm{Hom}(S, S)$ is a subjective map. The map, f, has to cover all elements of $\mathrm{Hom}(S, S)$ (i.e., for all elements y of the co-domain of f, there is an element x in the domain of f such that $y=f(x)$). Such a requirement is bound to fail, in principle. The map cannot cover all elements of $\mathrm{Hom}(S, S)$ (Gunji, 1992).

However, even though simultaneous choice collapses, the heterarchy proceeds as a real system. In this situation, one has to focus on the notion of a heterarchy as a real system persisting against the collapse of the observer's logical framework. Remember the above example of a man going to work on his day off. A proceeding motion against the collapse happens in that example. The appearance of an emergent state "smile though disappointed" can be explained by an event occurring that proceeds against the collapse. The situation of which choice of interpretation also makes sense in a family is expressed as an assumption of a subjective map from S to $\mathrm{Hom}(S, S)$ (i.e., the map requires simultaneous choice). If one attempts to make a system satisfy simultaneous choice in spite of the collapse of the assumption, one must find a new source that is mapped to possible elements of $\mathrm{Hom}(S, S)$ out of S. In order to avoid one-to-many mapping, a new source of an arrow is constructed out of {beneficial, detrimental}. This is nothing but a new family state, such as "smile though disappointed". The collapse of the assumption termed simultaneous choice makes re-organization of the system possible. This is the essence of a heterarchy; maintenance of feature persistence against logical collapse.

3. Formal notion of indefinite environments

3-1. Self-reference and frame problem

The essence of a heterarchy is proceeding motion against logical collapse. How is it possible to describe such a phenomenon? Our answer is to couple self-reference with a frame problem.

In the above section, it was shown that a heterarchy inherits logical collapse in the form of a subjective map of $f: S \to \mathrm{Hom}(S, S)$. According to Lawvere (1969), this is called a self-referential property, and is the essence of the diagonal argument. Any maps $g: S \to S$ are expressed as $f(x)(x) = g'(x, x)$ for any x in S, where $g'(x, x) = (g(x), g(x))$. Assume that $f: S \to \mathrm{Hom}(S, S)$ is subjective, and there exists y in S such that $f(y)(x) = h\, f(x)(x)$ where $h: S \to S$ is an arbitrary map. In substituting y for x, one obtains that $f(y)(y) = h\, f(y)(y)$. Because it is a fixed point with respect to an *arbitrary map*, h, it shows a contradiction. The assumption of subjectivity allows the ambiguity of indicating both an element of S and Hom(S, S). This is the essence of self-reference.

On the other hand, the frame problem is argued in the field of artificial intelligence (Dreyfuus, 1972, McCarthy and Hayes, 1969). If one attempts to implement decision-making in a machine, one must describe the situation and/or context in a formal way to enable a decision to be made. Such an attempt always fails because it is impossible to distinguish what is necessary and adequate to describe the situation from among all of the constituents of the whole world. The frame problem refers to the notion of an indefinite boundary of a situation that can be generalized to be an indefinite boundary of an entity, matter, and a system.

Recently, in the context of artificial life, it is said that the frame problem has already been resolved. Researchers who are committed to the notion of situational subjects think that the frame problem results from encoding-ism by which an intelligent agent has to connect real entities with their representations in a formal world (Brooks, 1991). To demonstrate the invalidation of the frame problem, they have proposed a multi-agent system in which each agent has no intelligence but has a particular motivation to move (Sterenlny, 1997). For example, an ant-robot is implemented to pick up and carry an object if it currently is carrying no object and put it down if the ant-robot encounters another object. They are also implemented to walk randomly in an arena. In spite of having no particular intelligence, the presence of a large number of ant-robots results in behavior that appears to be intelligent. Objects are gathered and become piled up in certain places. Some researchers think that this can be interpreted as being analogous to the way in which consciousness arises from the global behavior of many neurons, which are similar to simple machines in many ways (Brooks, 1991).

The frame problem can never be resolved using the notion of situational subjects. From that perspective, an observer is separated from an agent, and an observer makes a decision against the frame problem. An observer prepares the environment in which an agent can work. Imagine a situation in which half of a stone is buried in the ground. An ant-robot is unable move it and becomes permanently stuck. There is no intelligent behavior as a result. Therefore, the frame problem is just resolved by a superficial solution in which one ignores that the observer enters the robots' world and makes a decision instead of an ant-robot. The observer enters the robots' world, and the frame problem cannot be resolved in that such an internal observer is not formally described.

Logical self-reference and a frame problem are still the major problems preventing decision-making from being implemented in a formal way. There are few investigations being undertaken into the relationship between self-reference and the frame problem. Although these two problems are similar to each other, and both of them refer to the notion of wholeness and/or of an indefinite boundary of a context, their expressions of the indefinite wholeness are different from each other. In self-reference, the notion of indefinite wholeness is expressed as the ambiguity of indication. In the statement, "This is false", the term, this, indicates both "This" (a part of the statement) and "This is false" (whole statement). Although the ambiguity of indication is used to express the indefinite indication, it is assumed that indicating wholeness is possible. In contrast, the notion of wholeness is negatively expressed in the frame problem. Once a particular context is formally confirmed, other necessary conditions are always re-found. As a result, the environment surrounding an observer making a decision is found *a posteriori*. In this sense, self-reference is based on the positive expression of an indefinite world, and the frame problem is based on the negative expression of it. They are two sides of the same coin.

We think that both self-reference and the frame problem are problems that do not require resolving. As discussed in the perspective of artificial life, universal biology and interactivism, they are only problems if one attempts to implement decision-making using a formal description. Independent of an external observer's description, an internal observer's decision-making perpetually proceeds as materialistic interaction itself (Bickhard and Terveen, 1996). However, if one gives up describing decision-making or interactions, it leads to the erroneous notion that language can be separated from phenomena. Therefore, we construct an external expression for an internal observer with invalidation of the external perspective.

We address the relationship between self-reference and the frame problem in the following. Usually, if one is faced with a particular statement, one believes that it is trivially possible to determine what the frame is, which surrounds the statement. Imagine the statement, "This is false" written on a blackboard, where there is also some graffiti on the same blackboard, such as "NO". If one thinks that the reference of "This" is "This is false", one finds a liar statement. However, if one mixes the graffiti, "NO" with the above statement, one can read the statement as "This is not false", and one does not find a liar statement. There is a problem as to whether one can determine the wholeness that the term "This" indicates (Fig. 1). Even when one finds self-reference in a statement, the situation is exposed by the frame problem.

The coupling of self-reference and the frame problem allows one to make a decision in spite of either self-reference or the frame problem. Imagine that you say, "I am a liar". Such a saying is very ubiquitous in everyday life, although it contains self-reference. How is it possible to say that? Note that, in everyday life, all statements are used in indefinite situations, and this entails the creation of the frame problem. This gives rise to the idea that the frame problem invalidates self-reference and vice versa. If the term "I" indicates both "I" in the sentence "I am a liar" and the whole sentence "I am a liar", the sentence contains self-reference. The frame problem also exists, however, for the sentence, and then the wholeness or the whole sentence cannot be indicated. In this sense, the term "I" has no ambiguous meaning due to the frame problem. The premise of self-reference is invalidated by the frame problem. In contrast, the premise of the frame problem is invalidated by self-reference. The frame problem can be expressed as the following; once a particular situation has been explicitly described, a flaw can be pointed out in the description thereof. The premise of the frame problem is the presence of a subject who recognizes the situation. Such a subject is invalidated by self-reference, because such a subject has to be defined by subject = a subject who recognizes the situation. The term "subject" indicates both part of and the whole of the expression. That is why the saying "I am a liar" is possible despite the occurrence of self-reference.

We propose a model featuring self-reference coupled with the frame problem as the twisted coupling of two dynamics. There has been some research on the relationship between self-reference and dynamics (Grim et al., 1994; Spncer-Brown, 1969). According to Spencer-Brown (1969), time (i.e., time-shift) is a particular device to resolve a contradiction resulting from self-reference, such as x = not(x). If one recognizes time-shift from the right to the left, and, i.e., x^{t+1} = not(x^t), there is no contradiction. We disagree with this. However, the time-shift is introduced, and there is no resolution in principle. The question regarding the origin of initial state cannot be avoided and cannot be answered. Even if the time-shift is introduced, it needs to be coupled with the frame problem.

How can the frame problem be coupled with time-shift dynamics? Let us call x^{t+1} = not(x^t) an example of temporal dynamics, and imagine that a state is defined as a finite binary sequence as the approximation of a real value. A question regarding the origin of an initial state is replaced by a question regarding how to determine a binary sequence. Recall the frame problem. As soon as a particular frame (or premise) for making a decision is stated, a frame including the former frame is found. Although this is an example of an expression of skepticism, it can also be regarded as a positive way of generating a frame. It is a recursive algorithm for generating frames, each frame being generated from the previous frame, one by one. The situation is the same as approximating a real value using a finite binary sequence. Once a digit is determined, a finite sequence of digits is determined, one digit at a time. Such a rule can be expressed using particular dynamics; given a digit and a boundary value of either 0 or 1, the dynamics generates a finite binary sequence. In contrast with temporal dynamics, we call this dynamics state-scale dynamics. In this sense, it seems as though the problem concerning temporal dynamics, the initial value problem, can be settled by using such an algorithm. Strictly speaking, however, it cannot be solved, because a boundary value is required to generate a finite binary sequence. As well as temporal dynamics requiring state-scale dynamics, the problem of a boundary condition has to be settled using temporal dynamics. We no longer use the term settling or solving, but instead just the term invalidation. The initial value problem concerning temporal dynamics is invalidated by state-scale dynamics, and vice versa (Fig. 2).

Our perspective can be applied to a ubiquitous non-linear dynamical system. Any dynamics can be regarded as an expression for an object that cannot be described without a dynamic (i.e., temporal) property. Dynamics is, therefore, an expression resulting from a contradiction or self-reference. A problem resulting from self-reference also remains as a problem concerning the origin of the initial condition. In computing it using a digital computer, the origin of boundary values of digits also remains. In our perspective, instead of dynamics, a pair of temporal dynamics and state-scale dynamics is defined, and two kinds of problems are invalidated complementarily. That is an abstract expression for the essence of a heterarchy inheriting self-reference invalidated by the coupling with the frame problem.

3-2. Chaotic liar

We construct an abstract model for describing a heterarchy as follows. Given a contradictory logical expression, we define a complementary pair of temporal dynamics and state-scale dynamics derived from the same logical expression. If the former carries the property of self-reference and the latter the property of the frame problem, it is reasonable to consider that state-scale dynamics and temporal dynamics invalidate the problem of the origin of an initial value concerning the temporal dynamics and the problem of a boundary value concerning the state-scale dynamics, respectively. Therefore, we examine the property of self-reference and the frame problem in terms of a contradictory logical expression, and their dynamical expression. For this purpose, we examine the previous research conducted by Grim et al. (1992).

Grim et al. (1992) describe the detailed relationship between dynamics and a liar-statement such as “This is false”. They first introduce Lukasiewictz logic in order to describe an endomorphism in a real number interval [0.0, 1.0]. Functions in classical propositional calculus or Boolean algebra are replaced by the following. Truth values, {True, False} or {0, 1} are replaced by [0.0, 1.0]. The two-valued function, xANDy for $x, y \in \{0, 1\}$ is replaced by min(x, y) for $x, y \in [0.0, 1.0]$, and xORy is replaced by max(x, y), where min(x, y) = x, if $x < y$; y, otherwise, and max(x, y) = x, if $x > y$; $= y$, otherwise. The negation operator, NOTx, is also replaced by not$x := 1 - x$. Clearly, min and max satisfy the definition of infimum and supremum in a lattice (Birkhoff, 1967);

$$\min(x, y) \leqq x, y \qquad (\max(x, y) \geqq x, y) \tag{1}$$

$$z \leqq x, y \Rightarrow z \leqq \min(x, y), \quad (z \geqq x, y \Rightarrow z \geqq \max(x, y)) \tag{2}$$

$$x = \min(x, y) \Leftrightarrow x \leqq y \qquad (x = \max(x, y) \Leftrightarrow x \geqq y). \tag{3}$$

A lattice is a partial ordered set (L, $\leqq$) closed with respect to infimum (the greatest lower bound) and supremum (the least upper bound). A binary relation, $\leqq$, satisfies reflective ($x \leqq x$), anti-symmetric ($x \leqq y$, $y \leqq x \Leftrightarrow x = y$) and transitive ($x \leqq y$, $y \leqq z \Rightarrow x \leqq z$) laws. Given X is a subset of L, the lower bound and upper bound of X are defined by c such that for all x in X $c \leqq x$ and $x \leqq c$, respectively. The infimum and supremum of X are defined by the greatest lower bound and the least upper bound, respectively, and are represented by $\wedge X$ and $\vee X$, respectively. In particular, if X consists of two elements, x and y, infimum and supremum are represented by $x \wedge y$ and $x \vee y$, respectively. In this sense, condition (1) implies that min(x, y) is the lower bound of $\{x, y\}$, and condition (2) implies that if z is the lower bound, min(x, y) is the greatest lower bound. Condition (3) is also verified only by the properties of infimum and supremum. In this sense, in a two-valued function, min and max exactly correspond to infimum and supremum, respectively. In addition, Boolean algebra is defined by a complemented distributive lattice. A complemented lattice is defined as a lattice having a greatest element of 1 and a least element of 0, in which, for any element x, there exists x^c such that $x \wedge x^c = 0$ and $x \vee x^c = 1$. A distributive lattice is a lattice in which, for any elements x, y and z, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The x^c is negation of x, and then the law of excluded middle holds in Boolean algebra.

The negation in Lukasiewictz logic is no longer a Boolean negation, however, because it does not satisfy the excluded middle law; min(x, $1-x$) = 0 and max(x, $1-x$) = 1. This, therefore, leads to the result that the implication function, x IMPy, does not coincide with max($1-x$, y), while xIMPy = NOT(x)ORy in Boolean algebra. In focusing on a liar statement, the most fundamental function is an equivalence relation, denoted by EQ. In Boolean algebra, $x\mathrm{EQ}y$ = (x IMPy) AND (yIMPx) = (NOT(x)ORy) AND (NOT(y)ORx). If one adopts this form as an equivalence relation in Lukasiewictz logic, x eq y : = max (min(1−y, x), min(1−x, y)). This form is controversial because it does not hold that x eq x = 1 for all x in [0.0, 1.0]. That is why Grim et al. introduce the definition;

$$x \text{ eq } y := 1-\mathrm{abs}(y-x), \tag{4}$$

where abs represents a function taking an absolute value.

Grim et al. propose two kinds of liar statement, namely, a classical liar and a chaotic liar. A classical liar is expressed as

$$x = \mathrm{not}x = 1-x. \tag{5}$$

A liar statement that is a classical liar is, for example, "This is false". "This" is represented by x and "is" is represented by the equivalence relation =.

A chaotic liar is defined as, for example, the statement, "x is as true as x is false". This statement implies that a liar knows what he says is self-referential. A chaotic liar is expressed as

$$x = x \text{ eq } (\mathrm{not}x) = x \text{ eq } (1-x) = 1-\mathrm{abs}((1-x)-x) = 1-\mathrm{abs}(1-2x). \tag{6}$$

The final expression implies

$$\begin{aligned} &\text{if } 0 \leqq x < 0.5,\ x = 2x; \\ &\text{if } 0.5 \leqq x < 1.0,\ x = 2(1-x). \end{aligned} \tag{7}$$

If one regards this expression as a map (i.e., one recognizes time-shift, $t \rightarrow t+1$, from the right term to the left term), one finds a tent map such as

$$\begin{aligned} x^{t+1} &= 2x^t \qquad \text{if } 0 \leqq x^t < 0.5; \\ &= 2(1-x^t) \quad \text{otherwise} \end{aligned} \tag{8}$$

As well as a chaotic liar, a classical liar can be replaced by dynamics by introducing time-shift. Compared with a chaotic liar, dynamics derived from a classical liar shows just a simple oscillation. Because a chaotic liar inherits self-understanding of self-reference, the behavior of the dynamics shows chaotic behavior. In other words, a chaotic liar inherits indefiniteness of his understanding of the world by making his own description of the world carry the property of self-reference. A chaotic liar reserves his own description, and that evokes the property of self-reference. Tsuda and Tadaki (1997) also enhance the aspect of the liar's self-understanding regarding self-reference. They use a simultaneous equation for a chaotic liar [20], such as $x^{t+1} = f(x^t, y^t)$ and $y^{t+1} = g(x^t, y^t)$, and examine the significance for which only $x^{t+1} = f(x^t, y^t)$ and $y^{t+1} = g(x^{t+1}, y^t)$ show chaotic behavior. According to them, the latter has an internal time attribute to the system in its own right because x^{t+1} appears as an argument of $g(x^{t+1}, y^t)$. They call such asynchronous updating "internal measurement", in contrast to synchronous updating called "external measurement". This scheme can be regarded as a way of making a point of self-understanding of self-reference.

The essential point here is that a contradiction resulting from a chaotic liar cannot be resolved in principle. As mentioned before, remaining problems are expressed as the origin of the initial value and/or the origin of the boundary value. Although they cannot be resolved, we propose complementary interaction between temporal dynamics and state-scale dynamics. Hereinafter, we define a method for deriving two kinds of dynamics from a chaotic liar, and then propose a description of the complementary interaction between them.

4. Time-State-Scale Re-Entrant Form

4-1. Temporal dynamics and state-scale dynamics derived from chaotic liar

It is easy to see the property of self-reference and a frame problem as a contradiction if one investigates Cantor's diagonal argument (Whitehead and Russell 1925) in the following form. Assume an infinite set, S, can cover its power set. If so, all elements of S counted as s_1, s_2, … can correspond to elements of its power-set, and then elements of the power set are also counted as p_1, p_2, … Because each element of the power set is a subset of S, it can be expressed as a binary sequence such as $p_i = (q_{i1}, q_{i2}, \ldots)$, where $q_{ij} = 1$ if s_j is included in the subset; otherwise $q_{ij} = 0$. One obtains a table with infinite columns and rows by arranging $p_i = (q_{i1}, q_{i2}, \ldots)$, for $i = 1, 2., \ldots$. In the diagonal argument, an infinite diagonal sequence $(q_{11}, q_{22}, \ldots)$ is taken and is modified by $f(q_{ii}) = 1 - q_{ii}$. As a result, one obtains $x = (f(q_{11}), f(q_{22}), \ldots)$. Because of the above assumption, one can add the sequence, x, to the infinite table, and that leads to a contradiction at the crossing point between x and the diagonal sequence.

In the diagonal argument, it is assumed that wholeness can be indicated, and then it is proved that the assumption leads to a contradiction. This proves the notion of indefinite wholeness through a contradiction. We apply this aspect to use the procedure in a diagonal argument as a model for indefinite wholeness, positively. To achieve this aim, we search for a way to avoid a contradiction. As a result, such a way can be regarded as an operational model for indefinite wholeness. In this argument, there are two ways to avoid a contradiction, while one has to give up logical consistency. One way is to introduce the notion of growing wholeness. When one designates the wholeness as W, adding of the modified diagonal sequence is expressed as W OR (NOT W). If one introduces the growing wholeness such as

$$W \text{ OR } (\text{NOT } W) > W, \tag{9}$$

one cannot identify the infinite wholeness with the wholeness with NOT W. That is why one can avoid a contradiction. The other way to avoid a contradiction is to introduce the contracted wholeness such as

$$W \text{ AND } (\text{NOT } W) > (\text{NOT } W). \tag{10}$$

Because (NOT W) is an empty set, one can avoid the crossing point between the diagonal sequence and the modified diagonal sequence set as a row. In other words, the statement (10) gives up the statement such that wholeness consists of separable parts, because it breaks in the wholeness consisting of separable parts. Therefore, a contradiction in the diagonal argument can be invalidated. In this paper, we introduce these two ways as expressed by conditions (9) and (10) to construct dynamics carrying either the property of self-reference or the property of a frame problem.

If the growing wholeness approach is taken, the environment attribute to the system cannot be determined. As soon as the environment is determined, another definition of the environment is found. Therefore, the property of the frame problem is inherited. In contrast, if the contracted wholeness approach is taken, the intersection between a part and the wholeness falls into infinite regression. This is the property of self-reference, and it makes a contradiction at the crossing point between diagonal and modified diagonal sequence invalidated. The contracted wholeness, therefore, inherits the property of self-reference. Finally, it is clear to see that if the function, OR, in the Lukasiewictz logic satisfies condition (9), it inherits the property of a frame problem. If AND satisfies condition (10), it inherits the property of self-reference.

Next, we return to the statement of an equivalence relation in Lukasiewictz logic. The reason why Grim et al. (1994) introduce x eq y : = 1−abs(x−y) is to achieve x eq x = 1 for all x in [0.0, 1.0]. Instead of Grim's definition, we define xEQy = (NOT(x)ORy) AND (NOT(y)ORx) in Boolean algebra. By taking such a form, even in Lukasiewictz logic, one obtains x eq y = min(max(1−x, y), max(1−y, x)). We redefine MIN and MAX to achieve x eq x = 1 for all x in [0.0, 1.0]. First, we redefine MIN instead of min, so that the condition (2) is satisfied such that $z \leqq x, y \Rightarrow z \leqq \mathrm{MIN}(x, y)$ and condition (10) is satisfied such that MIN(W, 1− W)>1− W, and x eq x =1. Finally, we define

$$\mathrm{MIN}(x, y) := \min(x, y)/\max(x, y). \tag{11}$$

It is clear to see that this definition satisfies

$$x\mathrm{eq}x = \mathrm{MIN}(\max(1-x, x), \max(1-x, x))=1. \tag{12}$$

Second, we redefine MAX instead of max, by satisfying condition (1), max(x, y) $\geqq x, y$, and (9), max(W, 1− W)> W, and x eq x =1. Finally, we define

$$\mathrm{MAX}(x, y) := \mathrm{sgn}(x+y). \tag{13}$$

where sgn(z)=z if z<1; =1 otherwise. The definition satisfies the condition that

$$x\mathrm{eq}x = \min(\mathrm{MAX}(1-x, x), \mathrm{MAX}(1-x, x))=\min(1-x+x, 1-x+x)=1. \tag{14}$$

Remember that there are two independent ways to avoid a contradiction in the diagonal argument, and MIN and MAX take over that property. The definition of MIN is based on the contracted wholeness, and the notion of separable parts is abandoned. This does not refer to the frame problem, but rather refers to an infinite regression of partitions. In this sense, a contradiction between a part and the wholeness can be obtained definitely, in the sense of self-reference. In contrast, the definition of MAX based on the growing wholeness refers to the contradiction resulting from the frame problem. Finally, the property of self-reference is implanted in a chaotic liar by the replacement of min with MIN, and the property of the frame problem by the replacement of max with MAX. These two functions are no longer infimum or supremum.

In introducing MIN and MAX, two kinds of dynamics are derived from a chaotic liar such that $x = x$ eq (not$x$) = min(max($1-x$, $1-x$), max(x, x)). Therefore, by replacing min by MIN,

$$
\begin{aligned}
x &= \mathrm{MIN}(\max(1-x,\ 1-x),\ \max(x,\ x)) = \min(x,\ 1-x)/\max(x,\ 1-x) \\
&= x/(1-x) \qquad \text{if } 0 \leqq x < 1/2; \\
&\quad (1-x)/x \qquad \text{otherwise,}
\end{aligned}
\tag{15}
$$

is obtained. Because it inherits the property of self-reference, the temporal shift from left to right is introduced as well as Grim's temporal shift. Equation (15) is re-expressed as

$$
\begin{aligned}
x^{t+1} &= x^t/(1-x^t) \qquad \text{if } 0 \leqq x^t < 1/2; \\
&\quad (1-x^t)/x^t \qquad \text{otherwise.}
\end{aligned}
\tag{16}
$$

This map is called a modified tent map, and $x^t \in [0.0,\ 1.0]$. Because we examine the notion of an approximation of computing, a real value is approximated by a binary sequence, and eq. (16) is applied to a binary sequence, as specified later. The development of a binary sequence computed by a modified tent map is shown in Fig. 3B. A return map is obtained by the transformation from a binary sequence obtained by eq. (16) to a real value.

In contrast, replacement of max by MAX in a chaotic liar leads to the following expression as a tent map:

$$
\begin{aligned}
x &= \min(\mathrm{MAX}(1-x,\ 1-x),\ \mathrm{MAX}(x,\ x)) = \min(2(1-x),\ 2x) \\
&= 2x \qquad \text{if } 0 \leqq x < 1/2; \\
&\quad 2(1-x) \qquad \text{otherwise.}
\end{aligned}
\tag{17}
$$

Because MAX is derived from the notion of growing wholeness that can avoid a logical contradiction, the expression with MAX is re-defined by the state-scale shift such as

$$
\begin{aligned}
x_{s+1} &= 2x_s \qquad \text{if } 0 \leqq x_s < 1/2; \\
&\quad 2(1-x_s) \qquad \text{otherwise.}
\end{aligned}
\tag{18}
$$

In this equation, we use $x_s \in [0.0,\ 1.0]$ as a boundary value in computing the approximated value. To make the suffix, *s,* represent the digit of the state scale, a boundary value is regarded as a binary sequence along the temporal shift, t, of eq. (16). Actually, a real value, x_s, is expressed as a binary sequence, and is transformed by a tent map given by eq. (18). As well as the argument on the frame problem, once a particular digit is determined, a binary sequence of the boundary value is determined by eq. (18). The temporal shift, t, is shown vertically, and the state-scale shift, s, is shown horizontally. The development of a binary sequence as a boundary value is shown in Fig. 3A.

Finally, two kinds of expression derived from a particular expression of a chaotic liar are obtained as eqs. (16) and (18). Although a chaotic liar is a specific logical form, any dynamics can be regarded as formal expressions inheriting a contradiction that has to be solved by introducing a time shift. Then, our method, with which two kinds of dynamics are derived from one logical expression, can be applied to any other formal expressions.

A dynamical system consists of a function and a set of states. Because they are independently separated from each other, the problem, the origin of the initial state remains. If one applies dynamics to a finite binary sequence by using a digital computer, the question of how one can determine a terminal value of a binary sequence also appears. This is the origin of the boundary value. In our perspective, two kinds of dynamics complementarily invalidate the problem. A state is defined by applying temporal dynamics (modified tent map) as a finite binary sequence representing x^t, and the sequence is placed horizontally (Fig. 3B). The origin of the initial state can be computed by state-scale dynamics. Given a particular digit, one can place a binary sequence representing x_s vertically at the digit. The binary sequence placed vertically is repeatedly transformed, and that gives rise to a binary matrix (see Fig. 3A). The top horizontal sequence yields a binary sequence of the initial state for temporal dynamics. Conversely, the boundary vertical sequence is yielded by temporal dynamics. In the next section, we define in detail the relationship between computation performed by temporal dynamics and computation performed by state-scale dynamics.

4-2. Definition of re-entrance

We define the relationship between temporal dynamics and state-scale dynamics derived from a chaotic liar as the relationship between eqs. (16) and (18). If one starts from a particular dynamics, the problem of determining the initial condition always occurs. Generally, the problem of determining the initial condition falls into infinite regression, and it is considered that the problem becomes embedded in an infinite history. In contrast, we create state-scale dynamics (i.e., eq. (18)) to determine the initial condition or the state of the temporal dynamics (i.e., eq. (16)). First, the state of the state-scale dynamics (space state) is designated by x_s^t in [0.0, 1.0] and is transformed along the state-scale axis according to eq. (18). The space state is expressed as a binary sequence as

$$x_s^t = \sum_{i=0}^{N-1} 2^{-(i+1)} a_s^t(i), \tag{19}$$

with $a_s^t(i) \in \{0, 1\}$ and the state-scale transition of x_s^t is determined by

$$x_{s+1}^t = f(x_s^t) \tag{20}$$

where f: [0.0, 1.0] → [0.0, 1.0] is defined by eq. (18), a tent map. The state of temporal dynamics is obtained via the state of y^t in [0.0, 1.0] expressed as

$$y^t = \sum_{s=0}^{N-1} 2^{-(N-s)} a_s^t(0). \tag{21}$$

This state plays a role in the interface between the space state and the temporal state. From the state of y^t, the observable state (temporal state) is obtained as z^t as

$$z^t = g(y^t) \tag{22}$$

where g : [0.0, 1.0] → [0.0, 1.0] is defined by eq. (16), a modified tent map. The observed state, z^t, is also expressed as a binary sequence as

$$z^t = \sum_{k=0}^{N-1} 2^{-(k+1)} b^t(k), \tag{23}$$

with $b^t(k) \in \{0, 1\}$. A binary sequence $b^t(k)$ is provided for the next state of the state-scale dynamics by

$$a_0^{t+1}(k) = b^t(N-1-k), \tag{24}$$

with k = 0, 1, …, $N-1$. Note that there is torsion in a binary sequence from x_0^t to y^t, because the boundary condition provided as x_0^t is regarded as the smallest bit surrounded by an indefinite environment. There is also the procedure of inversing a binary sequence from z^t to x_0^{t+1} (i.e., eq. (24)), because the largest bit in z^t should greatly affect the next state of y^{t+1}. The procedure of the time-state-scale re-entrant form is schematically shown in Fig. 4.

Before investigating the re-entry of the observed value in the form of eq. (24), we estimate behaviors of the system in which a state-scale state x_s^t is given independently separated from a temporal state z^t. The next state of a state-scale state is given by

$$a_0^{t+1}(k) = a_0^t(k+1), \tag{25}$$

and then $a_0^{t+1}(N-1)$ is randomly chosen from {0, 1}. That estimation plays a role in manifesting the significance of the re-entrant interaction between the temporal and state-scale dynamics.

The state-scale transition is executed, given $a_0^t(k)$ with k = 0, 1, ..., $N-1$, however the tent map, f, needs $a_0^t(N)$, $a_0^t(N+1)$, ... , $a_0^t(2N)$, for the calculation of $a_1^t(N-1)$, $a_2^t(N-1)$, ..., $a_N^t(N-1)$. Because of the re-entry of the observed value such as that given by eq. (24), only $a_0^t(k)$ with k = 0, ..., $N-1$ are given. As a result, it is assumed that all $a_0^t(N)$, $a_0^t(N+1)$, ... , $a_0^t(2N)$ are 0. In particular, $a_N^t(0)$, which is the largest bit of y^t, is obtained resulting from $a_0^t(N)$=0.

Figure 5A shows the time transition of y^t in a binary sequence fashion, and Fig. 5B shows plots of y^t against y^{t+1}. Note that the plot of y^{t+1} against y^t is not a map, and is a map giving a one-to-many-type mapping. That map is expressed as

$$\begin{aligned} y^t = {} & 2y^{t+1} && \text{if } 0 \leqq y^{t+1} < 1/2; \\ & 2y^{t+1} - 1/2, && \text{if } 1/2 \leqq y^{t+1} < 3/4; \\ & 2y^{t+1} - 3/2, && \text{otherwise.} \end{aligned} \tag{26}$$

Because of the torsion of a binary sequence from to x_0^t to to y^t, a shift map rather than a tent map is obtained. The form of eq. (26) can be analytically obtained in Appendix.

4-3. Basic property of single re-entrant system

We now return to considering the temporal-state-scale re-entrant system in the form of eq. (24). In this case, a finite binary sequence observed at the t-th step re-enters as the boundary condition at the (t+1)-th step. A binary sequence progressing from larger to smaller digits is substituted for the boundary condition as a sequence progressing from the future (below) to the past (above), as shown in Fig. 4. A return map of y^t is shown in Fig. 6A. Basically, these maps are determined by modifying maps $z^{t+1} = z^t / (z^t+2)$, (A-9), (A-10) and (A-11). Figure 6B shows a return map from z^t to z^{t+1}. Different from the case in which a boundary condition is given at random, the return map shows a self-similar gasket structure.

We now focus on the behavior of this system. As suggested by the structure of the return map of z^t (Fig. 6B), the state of z^t is expected to be stable near 0.0, and to then suddenly jump to a higher value. The state of z^t can yield intermittency. Actually, a real valued expression of z^t shows intermittent pulses (Fig. 6C). When the whole gasket is divided into left and right parts, it is found that the lower margin of the left part crosses a diagonal line. If the marginal curve is expressed as $z^{t+1} = h_{\mathrm{low}}(z^t)$, it is clear that $|h_{\mathrm{low}}'(0)|<1$. Therefore, when focusing only on this aspect in terms of $z^{t+1} = h_{\mathrm{low}}(z^t)$, a periodic point $(z^t, z^{t+1})=(0, 0)$ is a hyperbolic attractor (Deverney, 1986; Jackson, 1991). On the other hand, when the upper margin of the left part of the gasket is expressed as $z^{t+1} = h_{\mathrm{up}}(z^t)$, it is found that $|h_{\mathrm{low}}'(0)|>1$. This means that a periodic point $(z^t, z^{t+1})=(0, 0)$ is a hyperbolic repellor. This is why the state of z^t is attracted into 0 along $z^{t+1} = h_{\mathrm{low}}(z^t)$, however, the state of z^t jumps to a higher value along $z^{t+1} = h_{\mathrm{up}}(z^t)$. When focusing on the right part of the gasket, both upper and lower margins of the right part lead to a periodic point $(z^t, z^{t+1})=(0, 0)$ being a hyperbolic repellor. However, some curves connecting points in the gasket can be a map in which a periodic point $(z^t, z^{t+1})=(0, 0)$ is a hyperbolic attractor. That is why it is possible for the state to reach a point $(z^t, z^{t+1})=(1, 1)$ while it is quite instable.

Both a tent map and a modified tent map show stationary fluctuations. In spite of using a tent map and modified tent map, the time-space re-entrant system dispatches intermittent pulses.

4-4. Interactive system

The time-state-scale re-entrant system is based on the perspective that a real entity is perceived with an indefinite environment by an observer and they are observed as a heterarchy. In this sense, the perception is not closed in its own right, and is open to the world. The property of wholeness always accompanies the interface between an entity and its environment. In the time-state-scale re-entrant system, the feature of an indefinite environment is expressed as state-scale-dynamics in the form of a tent map, accompanied with temporal dynamics being expressed in the form of a modified tent map. This perspective is consistent with the perspective of internal measurement and/or endophysics. An indefinite environment is always constructed through internal measurement.

Plurality of matters in the world is independent of the perspective of an internal observer; it is empirically constructed, and then we can count matters and entities. We call such a world consisting of plural entities the empirical world. Our aim is to explain the empirical world consisting of heterarchies. We define the empirical world using many time-state-scale re-entrant systems interacting with each other, and estimate its behaviors.

The j–th element of the empirical world is a time-state-scale re-entrant system and is represented by a triplet of $(x_s^t(j))$, $y^t(j)$, $z^t(j)$). The first element is defined by

$$x_s^t(j)= \sum_{i=0}^{N-1} 2^{-(i+1)}\, a_s^t(i,\ j), \tag{27}$$

with $a_s^t(i,\ j), \in \{0,\ 1\}$. The initial condition as $a_s^0(i,\ j)$ is randomly chosen. The state-scale transition of $x_s^t(j)$ is determined by

$$x_{s+1}^t(j) = f\,(x_s^t(j)), \tag{28}$$

where f is defined by eq. (18), a tent map. The time state is obtained via the state of $y^t(j)$ in [0.0, 1.0] expressed as

$$y^t(j) = \sum_{s=1}^{N} 2^{-(N+1-s)} a_s^t\,(0,\ j). \tag{29}$$

The observable state of each element is obtained as $z^t(j)$ as

$$z^t(j) = g\,(y^t(j)) \tag{30}$$

where g is defined by eq. (16), a modified tent map. The observed state, $z^t(j)$, is also expressed as a binary sequence as

$$z^t(j) = \sum^{N-1} 2^{-(k+1)}\, b^t(k,\ j), \tag{31}$$

$k = 0$

with $b^t(k, j) \in \{0, 1\}$. This state is provided for the interaction.

The interaction is implemented among the nearest neighbors, $(x_s^t(j)), y^t(j), z^t(j))$, $(x_s^t(j-1)), y^t(j-1), z^t(j-1))$ and $(x_s^t(j+1)), y^t(j+1), z^t(j+1))$. If $z^t(j-1) > z^t(j+1)$ (or $z^t(j-1) < z^t(j+1)$), the binary sequence corresponding to $z^t(j-1)$ (or $z^t(j+1)$) is provided for the boundary condition for the j-th element, and that

$$a_0{}^{t+1}(k, j) = b^t(N-k, j-1), \text{ (or } a_0{}^{t+1}(k, j) = b^t(N-k, j+1)) \tag{32}$$

with $k = 0, 1, \ldots, N-1$. If $z^t(j-1) = z^t(j+1)$, the bit sequence corresponding to $(z^t(j-1) + z^t(j+1))/2$ is provided for $a_0{}^{t+1}(k, j)$.

Figure 7 shows the behaviors of the time-state-scale re-entrant systems constituting the empirical world. The value of N represents the length of each bit sequence. In each columnar pattern, the vertical axis represents time and the horizontal axis represents the line arrangement of re-entrant systems. If the value of $z^t(j)$ exceeds 0.8, then it is represented by a dot, otherwise it is left blank. Independent of the length of a binary sequence, re-entrant systems constituting the empirical world show similar behaviors. Although autonomous perturbation suddenly appears, it rapidly disappears and the whole world recovers robust totality. Robust behavior is maintained for a long time, however it is also truncated by an autonomous fluctuation.

Figure 8 shows the behaviors of the control experiment, where each j-th element is denoted by $z^t(j)$. In each element,

$$z^{t+1}(j) = w(gf(z^{t+1}(j-1)), gf(z^{t+1}(j)), gf(z^{t+1}(j+1))), \tag{33}$$

where $w(a, b, c) = a$, if $a > c$; $= c$ if $a < c$; $= (a+c)/2$ if $a = c$ for a, b, c in [0.0, 1.0]. The maps f and g represents a tent map and a modified tent map, respectively. Therefore, the implemented interaction mimics the interaction among the time-state-scale re-entrant systems although there is no interaction between states and a boundary condition for the finite binary sequences. The behaviors of the control experiments are, however, different from those of the time-state-scale re-entrant systems constituting the empirical world. The turbulent-like behavior is perpetually maintained, and no robust behavior can be observed. Such turbulent-like behavior is expected in the control world because the time series generated by both the tent map and modified tent map generates stationary fluctuations. Through the comparison between the empirical world consisting of the time-state-scale re-entrant systems and the control experiment, one can estimate that the robustness as the wholeness results from the re-entrance or the interaction between the temporal state and state-scale state.

All calculations of the time-state-scale re-entrant system are implemented as the calculation of binary sequences. If a binary sequence is arranged in a sequential line, each digit is used as a state of computation. Such a state can correspond to a state in a Turing machine. In the re-entrant system, the boundary condition or the state-scale state is used in that fashion. On the other hand, if a binary sequence is arranged in a parallel line and used as a binary expression of a real number, the calculation is executed as that of a real number and is different from a calculation performed in a Turing machine. Such a state is used as a temporal state in the re-entrant system. This is why the interaction between temporal- and state-scale states is executed as the interaction between a symbolic manipulation and a real number calculation. This is the key notion for achieving global robustness of the empirical world consisting of re-entrant systems.

Given an N-length binary sequence as an initial boundary condition, an $N\times N$ binary square is calculated in the time-space re-entrant system (Fig. 4). As a result, one binary sequence in an $N\times N$ binary square is provided for the calculation of the temporal state. Therefore, redundant binary sequences placed horizontally co-exist with one binary sequence that is y^t. In other words, some anticipatory states are hidden accompanying a real explicit state. Because these states are generated by the re-entrant boundary condition involving constructed future states, we call these states anticipatory states. In an empirical world consisting of many elements, the interaction among elements mixes hidden anticipatory states. To estimate such a property, the dynamics of anticipatory states are examined.

The anticipatory state denoted as $y^t(p)$ in a single system is defined by

$$y^t(p) = \sum_{s=1}^{N} 2^{-(N+1-s)} a_s^t\,(p). \tag{34}$$

As well as a single system, the state of the j-th element in a many-element world denoted by $y^t(j;\,p)$ is defined by $y^t(j;\,p) = \Sigma\ 2^{-(N+1-s)} a_s^t\,(j;\,p)$. The final anticipatory state corresponding to $y^t(p)$ is denoted by $z^t(p)$, and is defined as

$$z^t(p) = g(y^t(p)). \tag{35}$$

The final anticipatory state of the j-th element in the many-element world is similarly defined and is denoted by $z^t(j;\,p)$. Note that $z^t(0)= z^t$ and p is shown in Fig. 4.

Figure 9 shows the generated return map of $z^t(1)$, $z^t(2)$, $z^t(3)$, $z^t(4)$ and $z^t(5)$. The larger the taken p is (i.e., the deeper the taken anticipatory state is), the more dots occupy the Cartesian space of $(z^t(p), z^{t+1}(p))$. With increasing p, a particular network structure is dominant, while it is also a self-similar pattern. The network is constructed by compaction and extension of a unit-like self-similar gasket found in a return map of $(z^t(0), z^{t+1}(0))$. These complex hidden anticipatory states may affect the robustness and/or autonomous perturbation in the empirical world consisting of many re-entrant systems.

5. Obstacles in Self-reference and Autopoiesis

The coupling of self-reference with the frame problem is apparent when we consider the concept of autopoiesis (Maturana and Varela, 1980; Varela, 1979; Maturana and Varela, 1998, Luisi 2003). Autopoiesis, in the form of self-referential systems, has been proposed as a model for life. In this section, we discuss the significance of the interface between a system and its exterior. If a system is self-referential, the interface couples the self-reference to the frame-problem. But autopoiesis lacks this kind of interface. If a dynamical system lacks such an interface, its behavior is periodic or chaotic. Since chaotic behavior is defined as a periodic orbit with infinite period, it can be expressed as the negation, or the limit, of an orbit with finite period (they are two sides of the same coin). In other words, a system without an interface can refer only to a fixed point.

Luisi (2003) points out that the autopoietic analysis of life is based on cellular life, and that an autopoietic unit is a system that is capable of sustaining itself owing to an inner network of reactions that regenerate all the system's components. Thus an autopoietic system can also refer to a hierarchical system consisting of two levels: the molecular components and a system maintaining its own identity. The interaction between the two levels contains the self-reference: for example, bounded system (generates)$\rightarrow$ internal network producing molecular components (emergently generates) $\rightarrow$ bounded system.

We think a problem remains concerning whether emergence is really contained in this cycle. This problem is related to the problem of interface. If a self-sustaining system contains a gap between the molecular components and the bounded system, there is ambiguity in determining the boundary of the autopoietic unit. If so, an autopoietic unit is continually generated against indeterminacy, and that leads to emergence. In other words, emergence makes sense only if there is ambiguity or undecidability. An autopoietic unit, however, is based on a definite link between its components and its production-boundary, since the aim of an autopoietic unit is defined to be the maintenance of its own identity (thus, it is not poiesis but autopoiesis). While Luisi accepts emergence in autopoiesis, he argues that autopoiesis is not concerned with the origin of life or with the transition from non-living to living. This means that self-maintenance never contains emergence in our sense.

Generally, emergent properties are novel properties that are not present in the components themselves and arise only when a collective structure is formed (e.g., see Luisi, 2003). Although this definition of emergent properties is widely accepted, the definition is based on an observer who can survey both levels, parts and whole, and can compare concepts appropriate to the lower level with those of the upper level. That is how one can demonstrate that a property not present in the components occurs in the collective structure. The concept of autopoiesis originates from defining identity. Thus, one can propose identity as an essential property that is not present in the components of an autopoietic unit without ambiguity. Since it is assumed that all notions in both components and collective structures can be determined, one can connect properties in components to those in collective structures and vice versa. This reveals a definite link between the two levels and there is no ambiguity in the cycle. If one cannot prove whether a collective structure does or does not contain some properties that are not present in the components, the cycle reveals a vagueness in the gap or discrepancy between the two levels, and there can be emergent properties. But autopoiesis never admits such vagueness.

According to Letelier (2006), the metabolic repair system proposed by Rosen (1959, 1985, 1991) is a minimal mathematical model for autopoiesis. Imagine a set of states of a cell, and a set of metabolic maps. Since the cardinality of the latter set is much more than that of the former set, there is no bijection between them when they are finite sets. But if one takes a subset of metabolic maps, there is a one-to-one and onto relation between a set of states and that subset of metabolic maps (Letelier 2006). This reveals a definite cycle such as state (product) $\rightarrow$ map (producer) $\rightarrow$ state. One can say that there is no separation between producer and product, and that leads to a self-reference that can refer to a fixed point. If one takes infinite sets, one can see the one-to-one and onto relation between the infinite state set and the infinite metabolic set. The cycle based on an infinite set yields some rich structures in the set of fixed points. This kind of richness has been seen in programming languages, such as denotational semantics (Scott, 1970).

If one thinks about life by starting from "definite identity", one focuses on self-reference or a cycle between a producer and its product. On this point, we think that a self-reference makes sense if there is an observer who can think about properties of both the producer and its product. Paradoxically, self-reference is based on a definite link between a producer and its product and leads to a fixed point with a period that can be finite or infinite. A liar map for two-element set {0, 1} actually leads to a fixed point, and a chaotic liar leads to an infinite cycle. Like an autopoietic unit lacking the discrepancy between producer and its product, these dynamical systems lack the discrepancy between a product (one variable form) and a producer (a form containing operations such as $\wedge$ and $\vee$). This results from the definite identity that leads to self-reference.

We have to give up from the very start the idea of definite identity when thinking about life. We have to accept incomplete or indefinite identity. If there is vagueness between producer and product, a system can contain emergence, evolution and the interaction with its environment. In other words, uncertainty in linking producer and product is nothing but vagueness in poiesis and perpetual intervention in the arena of interaction. Also in autopoiesis, the environment participates in the self-sustaining cycle, but through a definite boundary. Autopoiesis does not lead to an incomplete or broken boundary. Only indeterminacy between producer and product can lead to the broken boundary.

Perpetual intervention from outside can be replaced with the frame problem. In our context of a re-entrant system, it is implemented as an obstacle in the self-reference. In a chaotic liar, self-reference is revealed without obstacles. A product is expressed as "self" x, and a producer is expressed in the form of *equivalence to* x (i.e., affirmation of self) or *not* x (negation of self). If a producer expresses affirmation, a real value is increased. If it expresses negation, the value is decreased. This results in the iteration of increasing and decreasing real values, which is an essential mechanism of general chaotic dynamics. An obstacle jammed between a product and a producer restricts the expanding and contracting of the real value. It forces the behavior to the edge of chaos, and that leads to intermittency.

Since the obstacle originates from the system's own exterior, the obstacle can also be expressed by using terms in the system. In our scheme, given an expression, if one focuses on the interior, a temporal re-entrant form (i.e., self-reference) appears and, if one focuses on the exterior, a state-scale re-entrant form (i.e., obstacle) appears. They are two sides of the same coin. In short, frequently researchers start only from self-reference or chaotic dynamics, and do not pay attention to the origin of the self-reference or dynamics. If they consider the origin, they take it to be an environment, the world or the outside. This emphasizes the difference between the time and state-scales (i.e., space), and between self-reference and the frame problem. In general we can say that the frame problem plays the role of an obstacle in self-reference, inhibiting the increase and decrease of a real value, and forces a system to the edge of chaos (Bak, 1996).

6. Discussion and Conclusion

A heterarchy is a dynamical hierarchical structure, and is maintained as a robust unity inheriting perpetual re-organization. A heterarchy is ubiquitously found, because any elements are accompanied within a particular context in which elements are regarded as individual elements. Therefore, an object consisting of elements is recognized as one-ity, (i.e., two-oneness), or being both a pair of parts and whole. Because an object is maintained as a unity accompanied by an indefinite environment, a pair of parts and the whole inherit inconsistency between them. Such a pair is similar to the concept a pair of extent and intent. In mathematics or an ideal world, extent is equivalent to intent. For example, an even number (concept) is defined not only by intent, $2n$, but also by extent, $\{0, 2, 4, \ldots\}$. In contrast, in natural science, a concept is replaced by a unity, object, or a phenomenon, and a collection of individual elements (extent, the property of parts) is no longer consistent with a general property of elements (intent, the property of the whole), because an object is surrounded by an indefinite environment. In natural phenomena, the equivalence between intent and extent is just an approximation. A pair of intent and extent, two levels of a system, inherits inconsistency, however, a system is recognized as being a unity. Inconsistency between two levels gives rise to perpetual re-organization of levels, and then such a system is heterarchy. The essence of heterarchy is one-ity inheriting inconsistency.

We propose an abstract system taking after the essence of a heterarchy. Such a system is defined by a computation approximating a real value using a binary sequence. The property, one-ity, inheriting inconsistency is expressed as self-reference coupled with the property of a frame problem. The property of self-reference consists of constituent parts and the whole that are logically different from each other. In spite of the difference, a self-referential structure indicates both parts and the whole, and this gives rise to a contradiction. The property of the frame problem is defined by the impossibility of indicating wholeness, the whole environment, or the context (frame). Once a particular frame has been indicated, a more suitable frame is found, and this gives rise to an infinite regression. Our main idea is based on the coupling between self-reference and a frame problem. A contradiction resulting from dual indication of parts and the whole, can be invalidated by the property of a frame problem. This is a formal expression of one-ity in which inconsistency between two levels can be invalidated.

How can one introduce both self-reference and a frame problem? The two expressions are two perspectives for one contradictory statement. We start from a contradictory statement, a liar statement, and derive two expressions each carrying the property of self-reference or the property of a frame problem. Given a logical statement called a chaotic liar, temporal dynamics, a modified tent map, and state-scale dynamics, a tent map, are reduced. The state-scale state is transformed by the state-scale dynamics and it re-enters the temporal dynamics, and then the temporal state is transformed by the temporal dynamics. As a result, the temporal state within an indefinite environment can progress. This formulation can be generalized for an arbitrary logical statement. Given a logical statement expressed in Lukasiewictz logic, one can replace min by MIN and max by MAX, which is the procedure for finding the form of time (self-reference) and the form of the state scale (the frame problem), respectively. The re-entrant form between time and the state scale can be also defined in the same manner as mentioned in this paper. Finally, one obtains a dual system consisting of the form of time and the form of the state scale.

The time-state-scale re-entrant form expands the relationship between dynamics and its boundary condition in terms of a heterarchy. When a tent map is computed in a machine, a real number is replaced by a finite binary sequence, and the boundary condition next to the smallest digit is given randomly. This means that the outside of a dynamical system interacts with the inside through thermal fluctuations. In contrast, the time-state-scale re-entrant system provides an interface between the inside and outside not as a state, but as an operator. The interface is, therefore, more dynamical than the notion of a boundary condition.

Imagine that chaotic dynamics like a tent map or a modified tent map is applied to a finite binary sequence. If an initial binary sequence consists only of 0, and, for the boundary state, 0 is always given, a binary sequence is maintained as an all-0-sequence. The dynamics, however, inherits marked change. If the boundary state is given as 1, such a small perturbation expands in the system. In the re-entrant system, how to give the boundary state is also determined by the state-scale dynamics that is derived from a chaotic liar. Because the boundary state is controlled by the state-scale dynamics, the binary sequence transformed by the temporal dynamics inherits both robust and emergent behaviors. This results in intermittency such as a pulse-like signal. A system maintains its state as a finite binary sequence and the boundary attributes to the system, and also maintains its own temporal dynamics. As a result, the system is recognized as a unity that has been temporally changed. The duality of the re-entrant system carries such a property. Both robust and emergent behaviors are found in an interactive re-entrant system. In contrast, a coupled map that is defined by a composition of a tent map and a modified tent map can only show fluctuated turbulent behaviors. In the empirical world model consisting of many re-entrant systems, intermittent behaviors are collected and give rise to global clusters. Robust and emergent society appears in this model.

The essence of a heterarchy plays an essential role in explaining the origin of consciousness. Philosophers criticize the machine-based model of a brain by referring to the bald paradox (Tye, 1995). A head of hair consists of numerous hairs, while a bald head has no hairs. Then, does the question of how many hairs would have to be pulled out of a head to make the head bald make sense? Philosophers answer "no" because the question results from a category mistake. A hair is defined as an element and a bald head (or any particular hairstyle) is defined as the whole. They belong to different categories. Of course, here, a hair is a metaphor for a single neuron, and a bald head is one for consciousness. The philosophers' argument sounds adequate, but they forget that a neuron perpetually reproduces its membrane, maintains a metabolic system, and contributes to making the particular environment in which it works. In other words, a neuron is a machine existing in an indefinite environment of execution of computation. This is why a neuron is not a solitary object but an object within a greater world. Taking the perspective of a heterarchy can invalidate the bald paradox, and provide a new approach for talking about the origin of consciousness. The time-state-scale re-entrant system is one of the most promising tools that can be used in formalizing a heterarchy that consists of two levels inheriting inconsistency.

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Appendix. Torsion between temporal and state-scale dynamics

We here show how the transition map eq.(26) is obtained. First, we prove the statement with respect to the transition of y^t; such that

$$a_1{}^t(0) = \begin{cases} a_0{}^{t+1}(0), & \text{if } a_0{}^t(0) = 0; \\ 1 - a_0{}^{t+1}(0), & \text{if } a_0{}^t(0) = 1. \end{cases} \quad \text{(A-1)}$$

Since the transition of $x_0{}^t$, is determined by $a_0{}^{t+1}(k) = a_0{}^t\,(k+1)$, one can list all triplets of ($a_0{}^t$ (0), $a_0{}^t$ (1), $a_0{}^t$ (2)) affecting the statement expressed by eq. (A-1), such as ($a_0{}^t$ (0), $a_0{}^t$ (1), $a_0{}^t$ (2)) = (0,0,0), (0,0,1), …, (1,1,1). It leads to eq. (A-1).

Second, it can be proved that for all s = 2, .., $N-1$, $a_{s+1}{}^t(0) = a_s{}^{t+1}(0)$. In assuming $a_{s+1}{}^t(0) = 1 - a_s{}^{t+1}(0)$, a contradiction occurs by induction. At the t+1-th step, there are two possible cases such as ($a_{s-1}{}^{t+1}(0)$, $a_{s-1}{}^{t+1}(1)$) = (0, $1 - a_{s+1}{}^t(0)$) and ($a_{s-1}{}^{t+1}(0)$, $a_{s-1}{}^{t+1}(1)$) = (1, $a_{s+1}{}^t(0)$). One of them is represented as

$$\begin{array}{ll} a_s{}^{t+1}(0) & a_{s-1}{}^{t+1}(0) \\ & a_{s-1}{}^{t+1}(1) \end{array} \qquad \begin{array}{ll} 1 - a_{s+1}{}^t(0) & 0 \\ & 1 - a_{s+1}{}^t(0) \end{array} \quad \text{(A-2)}$$

In this representation, a tent map is applied to a binary sequence from right to left, where the vertical axis represents the bit sequence of $x_s{}^t$, or i of $a_s{}^t$ (i), and the horizontal axis represents s from right to left. For each case, one can calculate $a_{s+1}{}^t(0)$ due to eq. (A-2).

Next, the sequence at the t-th step is considered. Note that the boundary condition of the computation applied by a tent map at the t-th step and that at the (t+1)-th step are repeated where $a_0{}^{t+1}(k-1)=a_0{}^t(k)$ with k =1, 2, .., $N-1$. That is why the sequences shown in (30) at the (t+1)-th step are also found at the t-th step, where ($a_{s-1}{}^{t+1}(0)$, $a_{s-1}{}^{t+1}(1)$) = ($a_{s-1}{}^t(1)$, $a_{s-1}{}^t(2)$), and $a_s{}^{t+1}(0) = a_s{}^t(1)$. Therefore, the sequences at the t-th step are determined for all s = 2, .., $N-1$, if $a_{s-1}{}^t(0)$ is determined either by 0 or 1. For the case of (30a), there are two possible cases such as ($a_{s-1}{}^t(0)$, $a_{s-1}{}^t(1)$, $a_{s-1}{}^t(2)$) = (0, 0, $1 - a_{s+1}{}^t(0)$) and ($a_{s-1}{}^t(0)$, $a_{s-1}{}^t(1)$, $a_{s-1}{}^t(2)$) = (1, 0, $1 - a_{s+1}{}^t(0)$). As for the first case, $a_{s+1}{}^t(0)$ is obtained by

$$a_{s+1}{}^t(0) \quad a_s{}^t(0) \quad a_{s-1}{}^t(0) \qquad\qquad 1 - a_{s+1}{}^t(0) \quad 0 \quad 0$$

$$a_s^t(1) \quad a_{s-1}^t(1), \qquad\qquad 1-a_{s+1}^t(0) \quad 0 \qquad \text{(A-3)}$$
$$a_{s-1}^t(2), \qquad\qquad 1-a_{s+1}^t(0).$$

In this cases, $a_{s+1}^t(0)=1-a_{s+1}^t(0)$ is obtained by reduction. This is a contradiction. Similarly, for the case of $(a_{s-1}^{t+1}(0), a_{s-1}^{t+1}(1))= (a_{s-1}^t(1), a_{s-1}^t(2))=(1, a_{s+1}^t(0))$, we obtain a contradiction.

Third, we obtain the rule of transition of the largest bit of y^t. Because the smallest bit of x_s^t results in the largest bit of y^t, there are only two possible cases affecting the largest bit of y^t, $a_0^t(N-1)=0$ and 1. Remember that $a_0^t(N)$ is always 0 from the above definition. Consider the case in which $a_0^t(0)=0$, $a_0^t(N-1)=0$, and the occurrence of the number "1" in $S(0)=\{a_{N-2}^t(0), a_{N-3}^t(0), \ldots, a_0^t(0)\}$, denoted by $\#(1, S, t)$, is even. If $a_s^t(0)=1$, for all k, $a_{s+1}^t(k) = 1-a_s^t(k+1)$, and otherwise, $a_{s+1}^t(k) = a_s^t(k+1)$ (i.e., the transition is determined according to the tent map). That is why the occurrence of the number 1 in $S(0)$ controls the value of $a_N^t(0)$, and we obtain

$$a_{N-1}^t(0) = (1-(-1)^{\#(1,\ S,\ t)})/2 + a_0^t(N-1)(-1)^{\#(1,\ S,\ t)} \qquad \text{(A-4)}$$

As well as $a_{N-1}^t(0)$, $a_{N-1}^t(1) = (1-(-1)^{\#(1, S, t)})/2 + a_0^t(N)(-1)^{\#(1, S, t)}$.

Consider results obtained by a tent map at the $(t+1)$-th step. Owing to eq. (A-1); $a_1^t(0) = a_0^{t+1}(0)$, and $a_{s+1}^t(0) = a_s^{t+1}(0)$, if $a_0^t(0) = 0$, $\#(1, S, t+1)$ is not changed and becomes fixed as even. Then, $a_{N-1}^{t+1}(0) = (1-(-1)^{\#(1, S, t+1)})/2 + a_0^{t+1}(N-1)(-1)^{\#(1, S, t+1)}$ $= (1-(-1)^{\#(1, S, t+1)})/2 + a_0^{t+1}(N-1)(-1)^{\#(1, S, t)} = a_0^{t+1}(N-1)$. Depending on whether $a_0^t(N-1)=0$ or 1, binary sequences at the $(t+1)$-th step are obtained as

$$\begin{array}{llllll}
a_N^{t+1}(0)\,a_{N-1}^{t+1}(0)\ \ldots\ a_0^{t+1}(0) & 0\ 0\ \ldots\ a_0^{t+1}(0) & & 1\ 1\ \ldots\ a_0^{t+1}(0) & & \\
\quad a_{N-1}^{t+1}(1)\ldots\ \ \vdots & 0 \qquad \vdots & \text{or} & 0 \qquad \vdots & & \text{(A-5)} \\
\qquad\quad a_0^{t+1}(N-1) & \quad 0 & & \quad 1 & & \\
\qquad\quad a_0^{t+1}(N) & \quad 0 & & \quad 0. & &
\end{array}$$

Analogously, for the case in which $a_0^t(N-1)=0$, $a_0^t(0)=0$, and $\#(1, S, t)$ is odd, binary sequences at the $(t+1)$-th step are obtained as

$$\begin{array}{llllll}
a_N^{t+1}(0)\,a_{N-1}^{t+1}(0)\ \ldots\ a_0^{t+1}(0) & 1\ 1\ \ldots\ a_0^{t+1}(0) & & 0\ 0\ \ldots\ a_0^{t+1}(0) & & \\
\quad a_{N-1}^{t+1}(1)\ldots\ \ \vdots & 0 \qquad \vdots & \text{or} & 0 \qquad \vdots & & \text{(A-6)} \\
\qquad\quad a_0^{t+1}(N-1) & \quad 1 & & \quad 0 & & \\
\qquad\quad a_0^{t+1}(N) & \quad 0 & & \quad 0. & &
\end{array}$$

Owing to the statement expressed by eq. (A-1), $a_1^t(0) = a_0^{t+1}(0)$, if $a_0^t(0) = 0$. Therefore, $a_1^t(0)=0$ does not change $\#(1, S, t)$, however, the statement expressed by $a_{s+1}^t(0) = a_s^{t+1}(0)$, adds 1 to $\#(1, S, t)$. Finally, $\#(1, S, t+1) = \#(1, S, t)+1$, that is even, and then, $a_{N-1}^{t+1}(0) = (1-(-1)^{\#(1,S,t)+1})/2 + a_0^{t+1}(N-1)(-1)^{\#(1,S,t)+1} = a_0^{t+1}(N-1)$.

From these considerations pairs, the transition where $a_0^t(0)=a_0^t(N-1)=0$ is determined as

$a_N^t(0)$	$a_{N-1}^t(0)$	0 0	0 0	0 1	0 1
$a_N^{t+1}(0)$	$a_{N-1}^{t+1}(0)$	0 0	1 1	1 1	0 0.

(A-7)

As well as the case of $a_0^t(0)=a_0^t(N-1)=0$, one can determine the transition of the case of $a_0^t(0)=0$ and $a_0^t(N-1)=1$, and the following results are obtained

$a_N^t(0)$	$a_{N-1}^t(0)$	1 1	1 1	1 0	1 0
$a_N^{t+1}(0)$	$a_{N-1}^{t+1}(0)$	1 0	0 1	0 1	1 0.

(A-8)

Similarly, the case of $(a_0^t(0), a_0^t(N-1))=(1, 0)$ and $(1, 1)$ can be determined, and the same results as the transition of (A-7,8) are obtained.

If $(a_N^{t+1}(0), a_{N-1}^{t+1}(0))=(0, 0)$ in the case of (A-7), it means that $0 \leqq y^{t+1} < 1/2$. According to (A-7), this case leads to $a_N^t(0)=0=a_{N-1}^{t+1}(0)$. If $(a_N^{t+1}(0), a_{N-1}^{t+1}(0))=(0, 1)$ in the case of (A-8), it also means that $0 \leqq y^{t+1} < 1/2$ and $a_N^t(0)=1=a_{N-1}^{t+1}(0)$. In taking $a_{s+1}^t(0) = a_s^{t+1}(0)$, $y^t=2y^{t+1}$ is finally obtained. If $(a_N^{t+1}(0), a_{N-1}^{t+1}(0))=(1, 0)$ in the case of (A-8), it means that $1/2 \leqq y^{t+1} < 3/4$ and $a_N^t(0)=1-a_{N-1}^{t+1}(0)$. Therefore, $y^t=2y^{t+1}-1+1/2=2y^{t+1}-1/2$ is obtained. If $(a_N^{t+1}(0), a_{N-1}^{t+1}(0))=(1, 1)$ in the case of (A-7), it means that $3/4 \leqq y^{t+1} \leqq 1$ and $a_N^t(0)=1-a_{N-1}^{t+1}(0)$. Therefore, $y^t=2y^{t+1}-1-1/2=2y^{t+1}-3/2$ is obtained. Finally, we obtain eq. (26).

Because of the transformation such that $z^t = g(y^t)$, the return map of z^t is determined as the following. A transition of y^t is determined by eq. (26), and we first consider $y^t = 2y^{t+1}$. Because $z^t = y^t / (1-y^t)$, and $z^{t+1} = y^{t+1} / (1-y^{t+1})$, and the interval of z^{t+1} is included in [0.0, 1.0], $z^{t+1} = z^t / (z^t+2)$ is obtained. In focusing on $y^t = 2y^{t+1}$, and $z^t = (1-y^t)/y^t$ and $z^{t+1} = y^{t+1} / (1-y^{t+1})$, a map allowing the interval of z^{t+1} to be included in [0.0, 1.0] is obtained as

$$z^{t+1} = z^t / (2z^t+1) \tag{A-9}$$

Similarly, we can determine all possible transition maps from z^t to z^{t+1} as

$$z^{t+1} = (3z^t + 1)/(z^t + 3) \tag{A-10}$$

$$z^{t+1} = (1 - z^t)/\,(5z^t + 3) \tag{A-11}$$

for $y^t = 2y^{t+1} - 1/2$ and $y^t = 2y^{t+1} - 3/2$. The transition of binary expressions of z^t and a return map from z^t to z^{t+1} is shown in Fig. A-1.

Figure Legends

Fig. 1. Frame problem concerning self-reference. As long as one cannot determine the wholeness of a statement, one never encounters self-reference or a contradiction.

Fig. 2. Relationship between logical self-reference referring to time and frame problem referring to state-scale. If one applies a non-linear map to a finite binary sequence using a digital computer, one is faced with two problems, the origin of the initial state resulting from self-reference and the origin of the boundary digit resulting from a frame problem. The property of self-reference can invalidate the frame problem and vice versa, and this can be utilized in a dynamical system. If the temporal dynamics proceeding vertically and the state-scale dynamics proceeding horizontally are constructed from a formal expression of a phenomenon, each of the dynamics can invalidate the other's problem and vice versa.

Fig. 3. A. Development of time-state according to a modified tent map, and a return map for a modified tent map. B. Development of space-state according to a tent map, and a return map for a tent map.

Fig. 4. Schematic diagram of procedure of the time-space re-entrant form.

Fig. 5. A. The transition of a binary sequence denoted by y^t, that is, the interfacial state between space-state and time-state. B. A return map corresponding to the transition, expressed as eq. (26). Note that the transition from y^t to y^{t+1} is a one-to-many-type mapping, while the transition from y^{t+1} to y^t is a well-defined map.

Fig. 6. A. A return map of y^t, where the time-state re-enters the boundary condition of the space-state in the form of eq. (24). B. A return map of z^t, with the time-space re-entrant form. C. Time development of z^t, where the vertical axis denotes values of z^t and the horizontal axis represents time.

Fig. 7. Simulation results for an empirical universe consisting of interacting time-space re-entrant systems. See text for details.

Fig. 8. Simulation results for the empirical universe consisting of interacting composite tent and modified tent maps. This is a control experiment for the interactive

time-space re-entrant systems. See text for details.

Fig. 9. A return map of anticipatory states in the time-space re-entrant systems, where maps of $z^t(1)$, $z^t(2)$, $z^t(3)$, $z^t(4)$ and $z^t(5)$ are shown from the top to the bottom.

Fig. A-1. A. The transition of a binary sequence denoted by z^t, that is, the time-state finally obtained. B. A return map corresponding to the transition, expressed as $z^{t+1} = z^t / (z^t + 2)$, $z^{t+1} = z^t / (2z^t + 1)$, $z^{t+1} = (3z^t + 1)/(z^t + 3)$ and $z^{t+1} = (1 - z^t)/ (5z^t + 3)$.

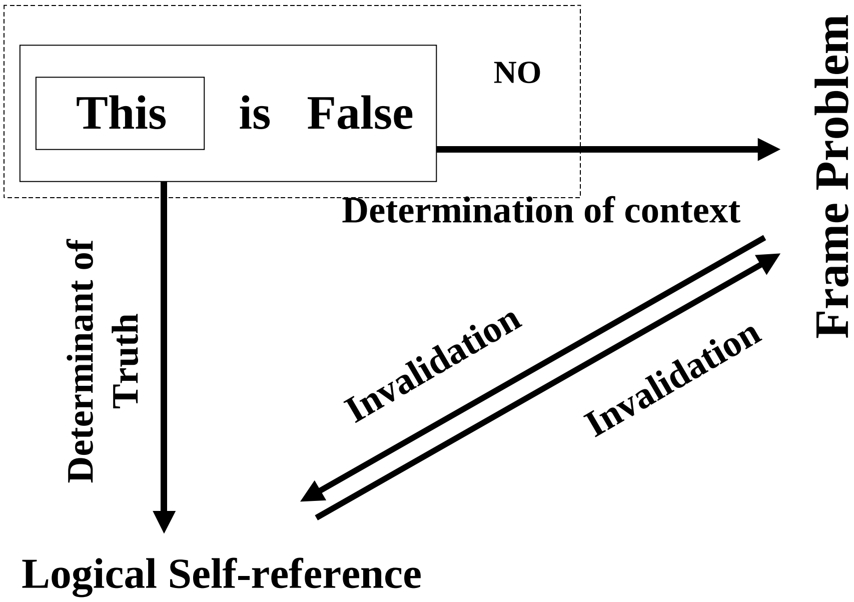

Fig. 1. Gunji, Y.-P. et al.

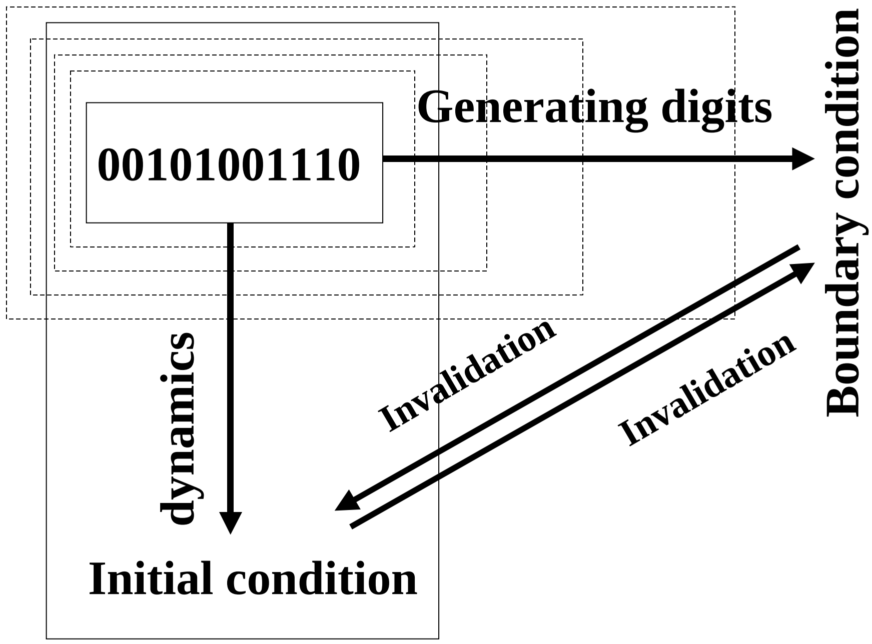

Fig. 2. Gunji, Y.-P. et al.

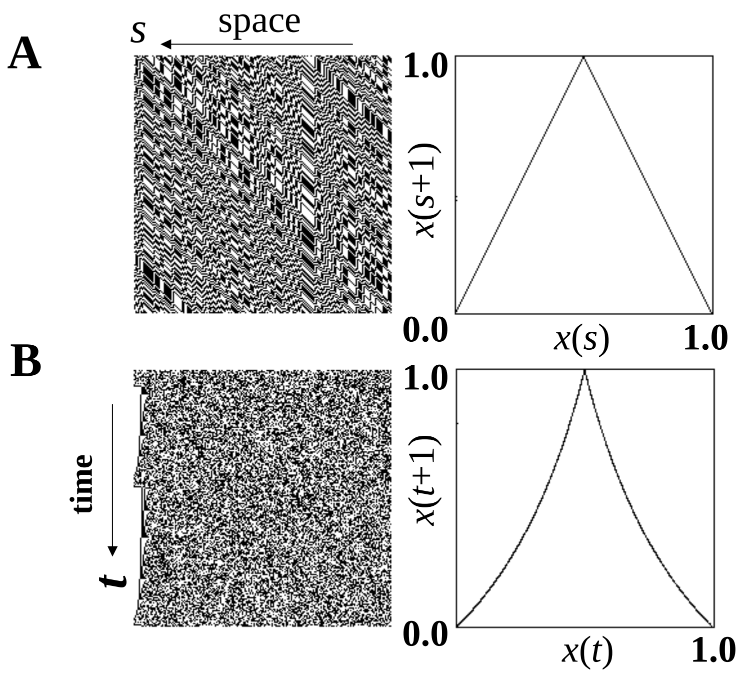

Fig. 3. Gunji, Y.-P. et al.

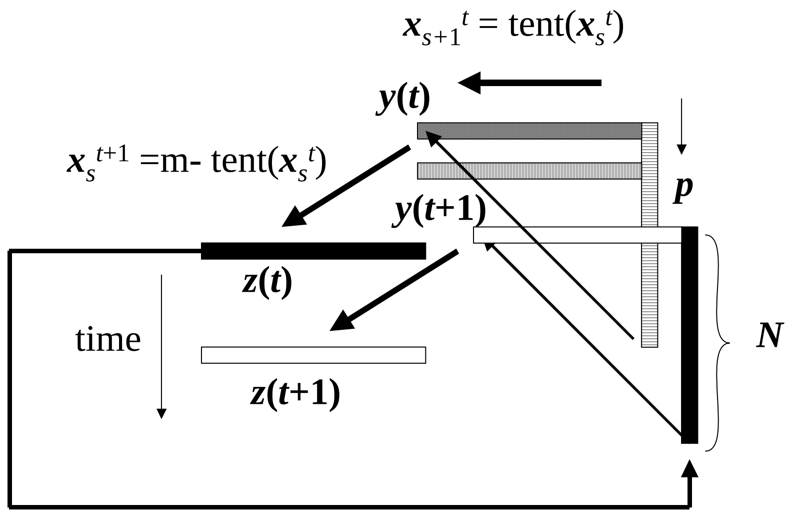

Fig. 4. Gunji, Y.-P. et al.

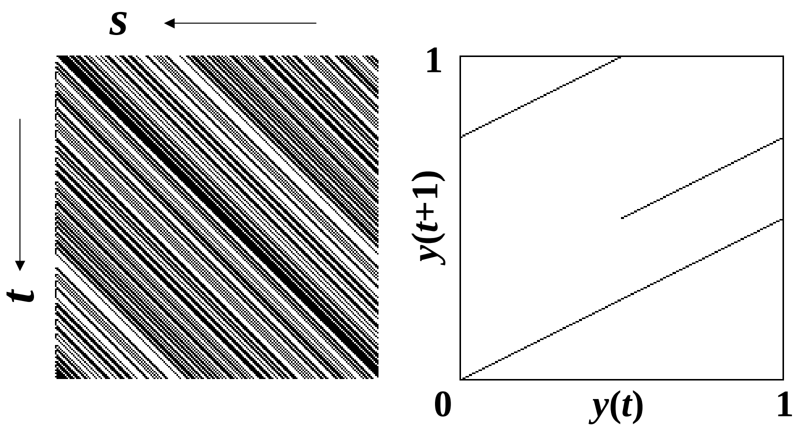

Fig.5. Gunji, Y.-P. et al.

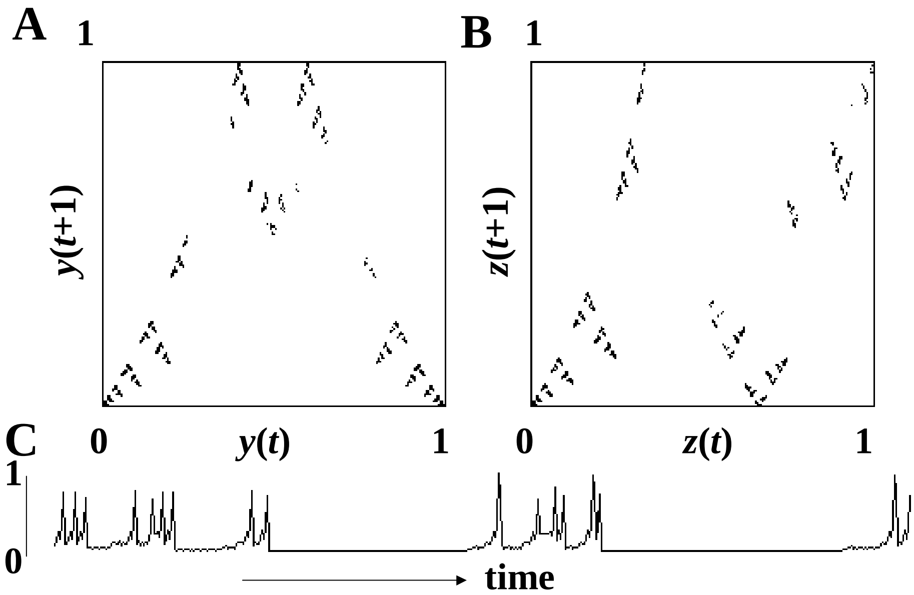

Fig. 6. Gunji et al.

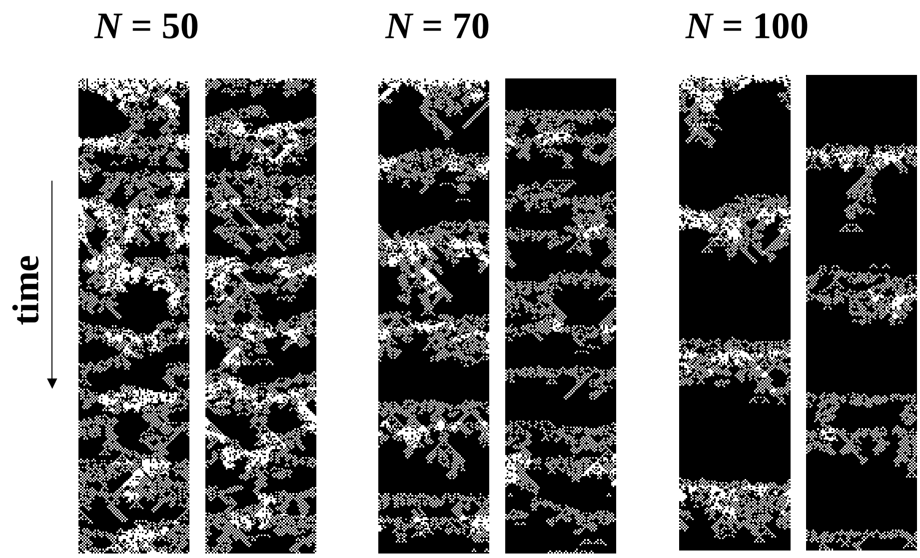

Fig. 7. Gunji, Y. -P. et al.

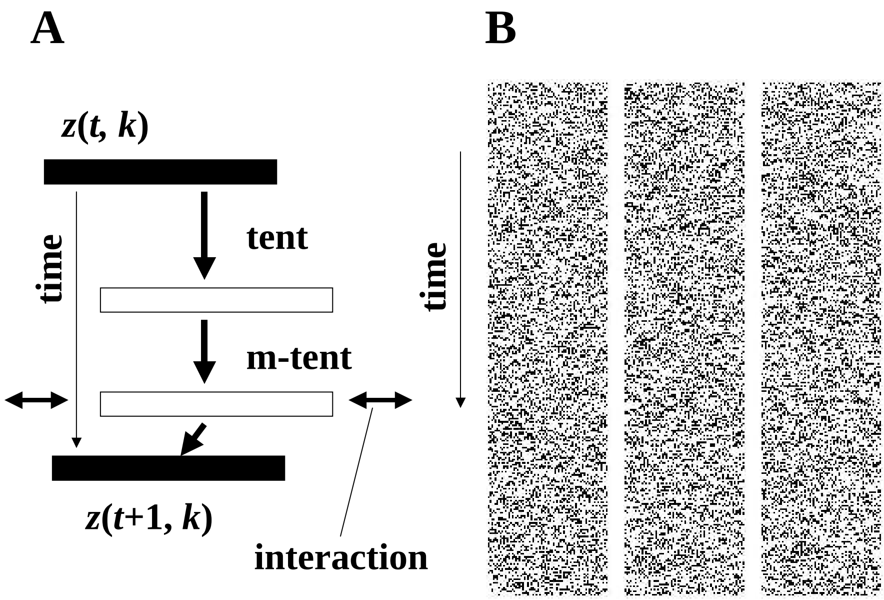

Fig. 8. Gunji, Y.-P. et al.

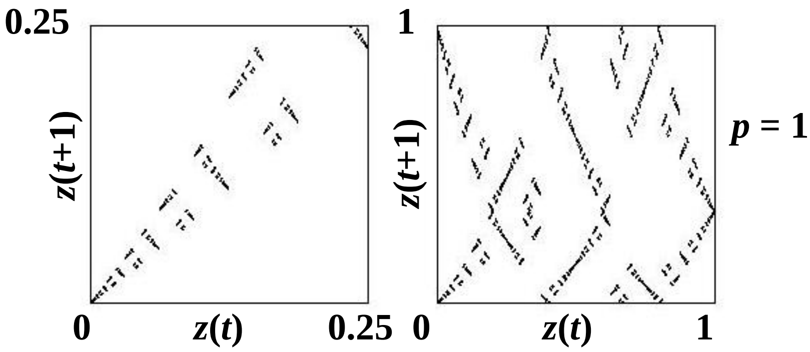

Fig.. 9(part I).　Gunji et al.

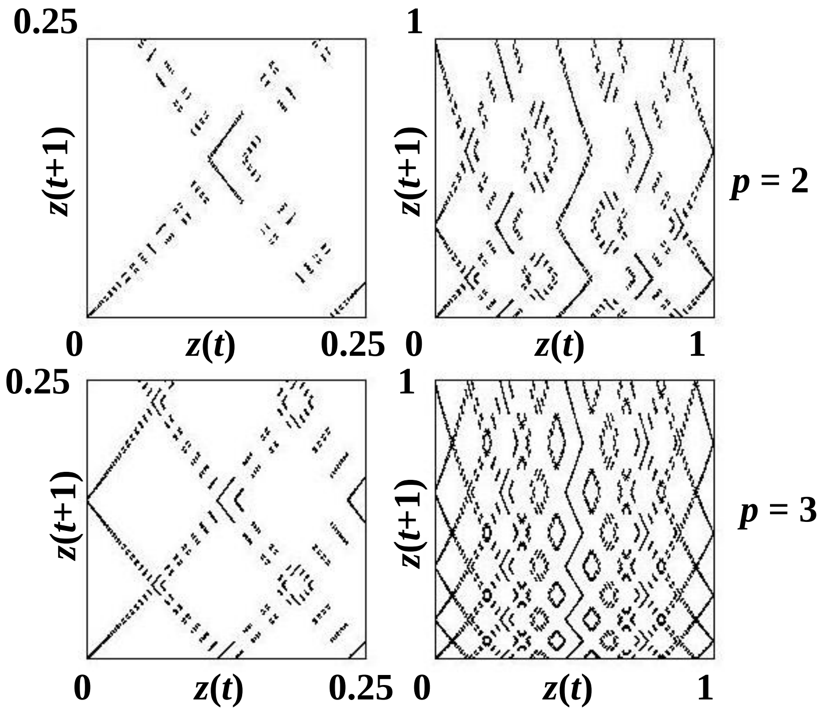

Fig 9 (Part II). Gunji, Y.-P. et al.

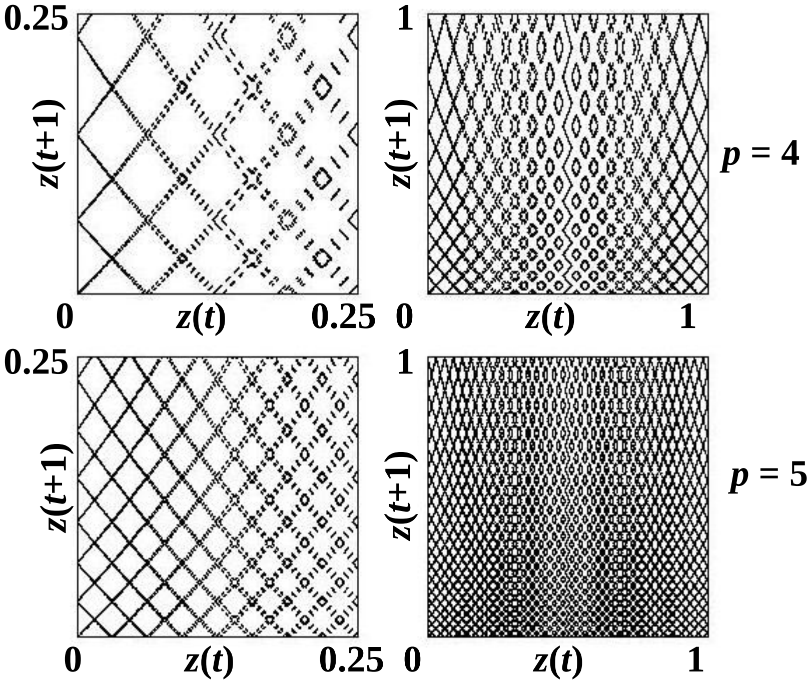

Fig. 9 (Part III). Gunji, Y.-P. et al.

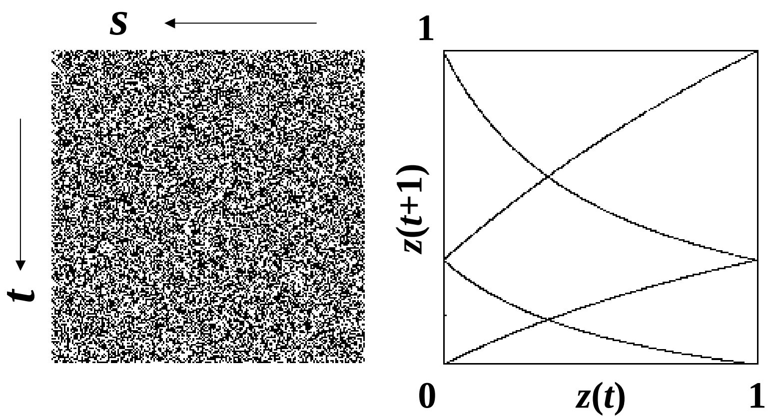

Fig. A-1. Gunji, Y.-P. et al.