# Linear-time algorithm for computing the Bernstein-Bézier coefficients of B-spline basis functions 

Filip Chudy*, Paweł Woźny<br>Institute of Computer Science, University of Wroctaw, ul. Joliot-Curie 15, 50-383 Wrocław, Poland


#### Abstract

A new differential-recurrence relation for the B-spline functions of the same degree is proved. From this relation, a recursive method of computing the coefficients of B-spline functions of degree $m$ in the Bernstein-Bézier form is derived. Its complexity is proportional to the number of coefficients in the case of coincident boundary knots. This means that, asymptotically, the algorithm is optimal. In other cases, the complexity is increased by at most $O\left(m^{3}\right)$. When the Bernstein-Bézier coefficients of B-spline basis functions are known, it is possible to compute any B-spline function in linear time with respect to its degree by performing the geometric algorithm proposed recently by the authors. This algorithm scales well when evaluating the B-spline curve at multiple points, e.g., in order to render it, since one only needs to find the coefficients for each knot span once. When evaluating many B-spline curves at multiple points (as is the case when rendering tensor product B-spline surfaces), such approach has lower computational complexity than using the de Boor-Cox algorithm. The numerical tests show that the new method is efficient. The problem of finding the coefficients of the B-spline functions in the power basis can be solved similarly.


Keywords: B-spline functions, Bernstein-Bézier form, recurrence relations, Bézier curves, B-spline curves, B-spline surfaces, de Boor-Cox algorithm.

## 1. Introduction

The family of Bernstein (basis) polynomials was used by S. N. Bernstein in 1912 in his constructive proof of the Weierstrass approximation theorem. For more details, see [13, §10.3]. Half a century later, they came into prominence when they became a basis for a particular family of parametric curves - Bézier curves.

For a fixed $n \in \mathbb{N}$ and $0 \leq i \leq n, B_{i}^{n}$ is the $i$ th Bernstein (basis) polynomial of degree $n$ given by the formula

$$
\begin{equation*}
B_{i}^{n}(t):=\binom{n}{i} t^{i}(1-t)^{n-i} . \tag{1.1}
\end{equation*}
$$

Remark 1.1. In the sequel, the convention is applied that $B_{i}^{n} \equiv 0$ if $i<0$ or $i>n$.

[^0]The Bernstein polynomial $B_{i}^{n}(0 \leq i \leq n)$ is non-negative and has exactly one maximum value in the interval $[0,1]$ (except for the case $n=0$ ). It is also important that all Bernstein polynomials of the same degree give the partition of unity, i.e., $\sum_{i=0}^{n} B_{i}^{n}(t) \equiv 1$. These properties make the Bernstein basis a very powerful tool in approximation theory, numerical analysis or in CAGD.

For any $t \in \mathbb{R}$, Bernstein polynomials satisfy the recurrence relations connecting the polynomials of two consecutive degrees:

$$
\begin{align*}
B_{k}^{n}(t) & =t B_{k-1}^{n-1}(t)+(1-t) B_{k}^{n-1}(t),  \tag{1.2}\\
B_{k}^{n}(t) & =\frac{n-k+1}{n+1} B_{k}^{n+1}(t)+\frac{k+1}{n+1} B_{k+1}^{n+1}(t) \quad(0 \leq k \leq n) . \tag{1.3}
\end{align*}
$$

It is also well-known that

$$
\begin{equation*}
\left(B_{k}^{n}(t)\right)^{\prime}=n\left(B_{k-1}^{n-1}(t)-B_{k}^{n-1}(t)\right) \quad(0 \leq k \leq n) . \tag{1.4}
\end{equation*}
$$

See [20, §5.1, Properties 2, 11, 13].
Using Eqs. (1.1), (1.3), and (1.4), one can easily obtain the following identities:

$$
\begin{align*}
t\left(B_{k}^{n}(t)\right)^{\prime} & =k B_{k}^{n}(t)-(k+1) B_{k+1}^{n}(t)  \tag{1.5}\\
\left(B_{k}^{n}(t)\right)^{\prime} & =(n-k+1) B_{k-1}^{n}(t)+(2 k-n) B_{k}^{n}(t)-(k+1) B_{k+1}^{n}(t) \tag{1.6}
\end{align*}
$$

where $k=0,1, \ldots, n$.
Remark 1.2. Points in $\mathbb{E}^{d}$, as well as parametric objects which return a point in $\mathbb{E}^{d}$, are denoted using the upper-case letters in the sans-serif font, e.g., $S, P, \ldots$ (as opposed to, e.g., Bernstein polynomials $B_{i}^{n}$, which return real values).

Polynomial Bézier curves are a particular family of parametric curves which is defined as a convex combination of control points. The points are weighted using Bernstein polynomials (1.1). A Bézier curve $\mathrm{P}_{n}:[0,1] \rightarrow \mathbb{E}^{d}$ of degree $n$ with control points $\mathrm{W}_{0}, \mathrm{~W}_{1}, \ldots, \mathrm{~W}_{n} \in \mathbb{E}^{d}$ is defined by the formula

$$
\begin{equation*}
\mathrm{P}_{n}(t):=\sum_{k=0}^{n} B_{k}^{n}(t) \mathbf{W}_{k} \quad(0 \leq t \leq 1) \tag{1.7}
\end{equation*}
$$

Due to the properties of Bernstein polynomials, the curve $\mathrm{P}_{n}$ is in the convex hull of the control points which means that $\mathrm{P}_{n}([0,1]) \subseteq \operatorname{conv}\left\{\mathrm{W}_{0}, \mathrm{~W}_{1}, \ldots, \mathrm{~W}_{n}\right\}$.

One can evaluate a point on a Bézier curve using the famous de Casteljau algorithm which is based on the relation (1.2) and has $O\left(d n^{2}\right)$ complexity.

A new method for computing a point on a polynomial or rational Bézier curve in optimal $O(d n)$ time has been recently proposed by the authors in [29]. The new algorithm combines the qualities of previously known methods for solving this problem, i.e., the linear complexity of the Horner's scheme and the geometric interpretation, the convex hull property, and operating only on convex combinations which are the advantages of the de Casteljau algorithm. The new method can be used not only for polynomial and rational Bézier curves but also for other rational parametric objects, e.g., Bézier surfaces. For more details, see [9, Chapter 2].

For more information about the history of Bézier curves and their properties, see, e.g., $[3,4,5,7,9,15,16,17]$, as well as $[19, \S 1]$ and $[20, \S 4]$.

Despite their elegance and some desirable properties, Bernstein polynomials have a significant drawback. For any $n, i \in \mathbb{N}$ such that $0 \leq i \leq n$, the value of a Bernstein polynomial $B_{i}^{n}(t)$ is non-zero for all $t \in(0,1)$. In practice, when operating on a Bézier curve (1.7), any change in one control point's position changes the curve over its whole length.

To address this issue, B-spline functions can be used. They are constructed in a way which eliminates this drawback. When used as a basis for parametric curves (known as B-spline curves), any change to the control point only has a local effect on a curve.

In order to introduce a B-spline function, the generalized divided differences will be used.
Definition 1.3 ([12, §4.2.1]). The generalized divided difference of a univariate function $f$ at the knots $x_{i}, x_{i+1}, \ldots, x_{k}$ (which may be coincident), denoted by $\left[x_{i}, x_{i+1}, \ldots, x_{k}\right] f$, is defined in the following recursive way:

$$
\left[x_{i}, x_{i+1}, \ldots, x_{i+\ell}\right] f:= \begin{cases}\frac{\left[x_{i+1}, \ldots, x_{i+\ell}\right] f-\left[x_{i}, \ldots, x_{i+\ell-1}\right] f}{x_{i+\ell}-x_{i}} & \left(x_{i} \neq x_{i+\ell}\right) \\ \frac{f^{(\ell)}\left(x_{i}\right)}{\ell!} & \left(x_{i}=\ldots=x_{i+\ell}\right)\end{cases}
$$

In particular, $\left[x_{i}\right] f=\frac{f^{(0)}\left(x_{i}\right)}{0!}=f\left(x_{i}\right)$.
Definition $1.4([25, \S 5.11])$. The B-spline (basis) function $N_{m i}$ of degree $m \in \mathbb{N}$ with knots $t_{i} \leq t_{i+1} \leq \ldots \leq t_{i+m+1}$ is defined as

$$
N_{m i}(u):=\left(t_{i+m+1}-t_{i}\right)\left[t_{i}, t_{i+1}, \ldots, t_{i+m+1}\right](t-u)_{+}^{m}
$$

where the generalized divided difference acts on the variable $t$, and

$$
(x-c)_{+}^{m}:= \begin{cases}(x-c)^{m} & (x \geq c) \\ 0 & (x<c)\end{cases}
$$

is the truncated power function.
The B-spline function $N_{m i}$ with knots $t_{i} \leq t_{i+1} \leq \cdots \leq t_{m+i+1}$ has support $\left[t_{i}, t_{m+i+1}\right]$, i.e., $N_{m i}(u)$ can be non-zero only for $u \in\left[t_{i}, t_{m+i+1}\right]$ (see, e.g., [24, Property 2.2]).

Let $m, n \in \mathbb{N}$. The knots

$$
\underbrace{t_{-m} \leq \ldots \leq t_{-1} \leq t_{0}}_{\text {boundary knots }} \leq \underbrace{t_{1} \leq \ldots \leq t_{n-1}}_{\text {inner knots }} \leq \underbrace{t_{n} \leq t_{n+1} \leq \ldots \leq t_{n+m}}_{\text {boundary knots }}
$$

where $t_{0}<t_{n}$, serve as a support for a B-spline basis of degree $m$ over $\left[t_{0}, t_{n}\right]$. The B-spline functions $N_{m,-m}, N_{m,-m+1}, \ldots, N_{m, n-1}(c f$. Definition 1.4) form a B-spline basis. Splines are commonly used in a wide variety of applications, e.g., in computer-aided geometric design, approximation theory and numerical analysis. See, e.g., [18, 19, 21, 24, 25].

Remark 1.5. In the sequel, a convention is adopted that for any quantity $Q$, if $t_{k}=t_{m+k+1}$ then $\frac{Q}{t_{m+k+1}-t_{k}}:=0$, as well as that $N_{m i} \equiv 0$ for $i<-m$ or $i \geq n$.

Computing the B-spline functions or their derivatives using Definition 1.4 is costly. Instead, one can use the recurrence and differential-recurrence relations.

The B-spline functions satisfy the following de Boor-Mansfield-Cox recursion formula (see, e.g., [21, Eq. (7.8)], [14, §2], [11, Eq. (6.1)]):

$$
\begin{equation*}
N_{m i}(u)=\left(u-t_{i}\right) \frac{N_{m-1, i}(u)}{t_{m+i}-t_{i}}+\left(t_{m+i+1}-u\right) \frac{N_{m-1, i+1}(u)}{t_{m+i+1}-t_{i+1}} \quad(-m \leq i<n) \tag{1.8}
\end{equation*}
$$

(cf. Remark 1.5). Additionally, for $i=0,1, \ldots, n-1$,

$$
N_{0 i}(u)= \begin{cases}1 & \left(u \in\left[t_{i}, t_{i+1}\right)\right),  \tag{1.9}\\ 0 & \text { otherwise } .\end{cases}
$$

The derivative of a B-spline function can be expressed as

$$
\begin{equation*}
N_{m i}^{\prime}(u)=m\left(\frac{N_{m-1, i}(u)}{t_{m+i}-t_{i}}-\frac{N_{m-1, i+1}(u)}{t_{m+i+1}-t_{i+1}}\right) \quad(-m \leq i<n) \tag{1.10}
\end{equation*}
$$

(cf. [24, Eq. (2.7)] and Remark 1.5).
Theorem 1.6 ([24, Property 2.5]). All derivatives of $N_{m i}$ exist in the interior of a knot span (where it is a polynomial). At a knot $N_{m i}$ is $m-k$ times continuously differentiable, where $k$ is the multiplicity of the knot. Hence, increasing $m$ increases continuity, and increasing knot multiplicity decreases continuity.

The B-spline functions, like the family of Bernstein polynomials of an arbitrary degree, have properties which make them a good choice for a parameterization of a family of curves.

Theorem 1.7 ([24, Properties 2.3, 2.4, 2.6]). $N_{m i}(u) \geq 0$ for all $m, i$, u (non-negativity). For an arbitrary knot span, $\left[t_{j}, t_{j+1}\right), \sum_{i=j-m}^{j} N_{m i}(u)=1$ for all $u \in\left[t_{j}, t_{j+1}\right.$ ) (partition of unity). Except for the case $m=0, N_{m i}$ attains exactly one maximum value.

A $B$-spline curve of degree $m$ over the non-empty interval $[a, b]$ with knots

$$
t_{-m} \leq \ldots \leq t_{0}=a \leq t_{1} \leq \ldots \leq b=t_{n} \leq \ldots \leq t_{n+m}
$$

and control points $\mathrm{W}_{-m}, \mathrm{~W}_{-m+1}, \ldots, \mathrm{~W}_{n-1} \in \mathbb{E}^{d}$ is defined as

$$
\mathrm{S}(t):=\sum_{i=-m}^{n-1} N_{m i}(t) \mathrm{W}_{i} \quad(t \in[a, b]) .
$$

One can check that $\mathrm{S}([a, b]) \subseteq \operatorname{conv}\left\{\mathrm{W}_{-m}, \mathrm{~W}_{-m+1}, \ldots, \mathrm{~W}_{n-1}\right\}$, as well as $\mathrm{S}\left(\left[t_{i}, t_{i+1}\right]\right) \subseteq$ $\operatorname{conv}\left\{\mathrm{W}_{i-m}, \mathrm{~W}_{i-m+1}, \ldots, \mathrm{~W}_{i}\right\}$ for $0 \leq i \leq n-1$ (see, e.g., [24, Property 3.5]).

The recurrence relation (1.8) and Eq. (1.9) can be used to evaluate a point on a B-spline curve. This approach, applied to explicitly compute the values of B-spline functions, has been proposed by de Boor in [14, p. 55-57]. The algorithm given in [14, p. 57-59] (see also, e.g, [19, Eq. (8.3)]), which directly computes a point on a B-spline curve is known as the de Boor-Cox algorithm and has $O\left(d m^{2}\right)$ computational complexity.

A popular choice for the boundary knots is to make them coincident with $t_{0}$ and $t_{n}$, i.e.,

$$
\begin{equation*}
t_{-m}=t_{-m+1}=\ldots=t_{-1}=t_{0}=a, \quad b=t_{n}=t_{n+1}=\ldots=t_{n+m} . \tag{1.11}
\end{equation*}
$$

In this case, $\mathrm{S}(a)=\mathrm{W}_{-m}$ and $\mathrm{S}(b)=\mathrm{W}_{n-1}$.
Bézier curves are a particular subtype of B-spline curves for $n=1, t_{-m}=t_{-m+1}=\ldots=$ $t_{0}=0$ and $t_{1}=t_{2}=\ldots=t_{m+1}=1$,

$$
\mathrm{S}(t)=\sum_{i=-m}^{0} N_{m i}(t) \mathrm{W}_{i}=\sum_{i=0}^{m} B_{i}^{m}(t) \mathrm{W}_{i-m} \quad(t \in[0,1]) .
$$

See, e.g., 24, Property 3.1].
The paper is organized as follows. Section 22 comprises the problem statement for finding the adjusted Bernstein-Bézier coefficients of B-spline functions over each knot span. In Section 3 we prove the new differential-recurrence relation between the B-spline functions of the same degree. It will be the foundation for new recurrence relations which can be used to formulate an algorithm which computes the required coefficients. In Section 4 , the algorithm for finding the adjusted Bernstein-Bézier coefficients of all B-spline functions over each knot span in the case $t_{-m}=t_{0}, t_{n}=t_{n+m}$ and all inner knots $t_{1}, t_{2}, \ldots, t_{n-1}$ having multiplicity 1 is given. The computational complexity of the method is $O\left(n m^{2}\right)$. This means that, asymptotically, the algorithm is optimal. Section 5 expands upon using the new algorithm to compute multiple points on multiple B-spline curves ( $\$ 5.1)$. The numerical tests which are presented there show that the new method is efficient. In $\$ 5.2$, the results of applying the new algorithm to evaluating tensor product B-spline surfaces are given. The assumptions about knot multiplicity which were made in Section 4 are then relaxed in Section 6 to cover all cases (cf. Remark 2.4). Section 7 gives a simplified version of the recurrence relations for the case of uniform knots.

## 2. The Problem: Bernstein-Bézier and power coefficients of B-spline functions

In this paper, the following problem is considered.
Let the adjusted Bernstein-Bézier basis form of the B-spline function $N_{m i}$ over a single non-empty knot span $\left[t_{j}, t_{j+1}\right) \subset\left[t_{0}, t_{n}\right](j=i, i+1, \ldots, i+m)$ be

$$
\begin{equation*}
N_{m i}(u)=\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}\left(\frac{u-t_{j}}{t_{j+1}-t_{j}}\right) \quad\left(t_{j} \leq u<t_{j+1}\right), \tag{2.1}
\end{equation*}
$$

with $b_{k}^{(i, j)} \equiv b_{k, m}^{(i, j)}$.
Problem 2.1. Find the adjusted Bernstein-Bézier basis coefficients $b_{k}^{(i, j)}(0 \leq k \leq m)$ (cf. (2.1)) of all functions $N_{m i}$ over all non-trivial knot spans $\left[t_{j}, t_{j+1}\right) \subset\left[t_{0}, t_{n}\right]$, i.e., for $j=0,1, \ldots, n-1$ and $i=j-m, j-m+1, \ldots, j$.

Notice that if the coefficents $b_{k}^{(i, j)}$ are already known, it is possible to use the geometric algorithm proposed in [29] to compute any B-spline function in linear time with respect to its degree. By using the computed values of B-spline functions, one can efficiently evaluate a point on a B-spline curve. Additionally, the knowledge of the coefficients of a B-spline function can allow to perform more operations, such as differentiation or integration, analytically, e.g., when solving differential or integral equations using numerical methods.

The adjusted Bernstein-Bézier coefficients of B-spline functions satisfy some easy to prove properties.

Theorem 2.2. For $u \in\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-1)$, the coefficients $b_{k, m}^{(i, j)}(k=0,1, \ldots, m)$ of the adjusted Bernstein-Bézier representation of the B-spline function $N_{m i}$ (cf. Eq. (2.1)) are non-negative.

Theorem 2.3. For $u \in\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-1)$, the following relation holds:

$$
\sum_{i=j-m}^{j} b_{k, m}^{(i, j)}=1 \quad(k=0,1, \ldots, m)
$$

where $b_{k, m}^{(i, j)}$ are the adjusted Bernstein-Bézier coefficients of $N_{m i}$ (cf. (2.1)).
A task similar to Problem 2.1 for the adjusted power basis form of the B-spline functions has been considered in [9].

Let the adjusted power basis form of the B-spline function $N_{m i}$ over a single non-empty knot span $\left[t_{j}, t_{j+1}\right) \subset\left[t_{0}, t_{n}\right](j=i, i+1, \ldots, i+m)$ be

$$
\begin{equation*}
N_{m i}(u)=\sum_{k=0}^{m} a_{k}^{(i, j)}\left(u-t_{j}\right)^{k} \quad\left(t_{j} \leq u<t_{j+1}\right), \tag{2.2}
\end{equation*}
$$

with $a_{k}^{(i, j)} \equiv a_{k, m}^{(i, j)}$.
Explicit expressions for the adjusted power basis coefficients of $N_{m i}$ have been given in [23], and the result can be adapted for the adjusted Bernstein-Bézier form. The serious drawback of this approach, however, is high complexity, which greatly limits the use of this result in computational practice.

Another algorithm for finding the adjusted power basis coefficients of a spline

$$
\begin{equation*}
s(t):=\sum_{i=-m}^{n-1} c_{i} N_{m i}(t) . \tag{2.3}
\end{equation*}
$$

over a knot span $\left[t_{j}, t_{j+1}\right)$ in $O\left(m^{2}\right)$ time can be found in [18, §1.3.2]. By setting

$$
c_{k}:= \begin{cases}1 & (k=i), \\ 0 & \text { otherwise }\end{cases}
$$

(see (2.3)), one can find the coefficients of $N_{m i}$ over $\left[t_{j}, t_{j+1}\right)$ in $O\left(m^{2}\right)$ time. In total, to find the adjusted power basis coefficients over $\left[t_{j}, t_{j+1}\right)$ for all B-spline functions $N_{m i}$ such that $j-m \leq i \leq j$, one has to do $O\left(m^{3}\right)$ operations. Let us assume that there are $n_{e}$ non-empty knot spans $\left[t_{j}, t_{j+1}\right)$ such that $j=0,1, \ldots, n-1$. To find the coefficients of all B-spline functions over all non-empty knot spans $\left[t_{j}, t_{j+1}\right)$ for $j=0,1, \ldots, n-1$, one would need to perform $O\left(n_{e} m^{3}\right)$ operations.

With a similar approach, one can find the Bernstein-Bézier coefficients of $N_{m i}$ over the knot span $\left[t_{j}, t_{j+1}\right)$. One can check that

$$
b_{k}^{(i, j)}=\frac{(m-k)!}{m!} N_{m i}^{(k)}\left(t_{j}\right)-\sum_{\ell=0}^{k-1}(-1)^{k-\ell}\binom{k}{\ell} b_{\ell}^{(i, j)} \quad(k=0,1, \ldots, m)
$$

(cf. [19, Eq. (5.25)] and [22, Theorem 4.1]). Just as in the case of the adjusted power basis, the Bernstein-Bézier coefficients $b_{k}^{(i, j)}$ of $N_{m i}$ over $\left[t_{j}, t_{j+1}\right)$ can be found in $O\left(m^{2}\right)$ time. In total,
to find these coefficients for all B-spline functions over all non-empty knot spans $\left[t_{j}, t_{j+1}\right)$ for $j=0,1, \ldots, n-1$, it is required to perform $O\left(n_{e} m^{3}\right)$ operations.

The approach given in [26] and [8] serves to convert a B-spline curve segment into a Bézier curve. It can be adapted to give an algorithm with $O\left(m^{3}\right)$ complexity for finding the adjusted Bernstein-Bézier coefficients $b_{k}^{(i, j)}$ of a single basis function $N_{m i}$. Doing so for each B-spline function in each non-empty knot span takes $O\left(n_{e} m^{4}\right)$ operations.

The Bernstein-Bézier coefficients in each knot span of a linear combination of B-spline functions can also be found by using the knot insertion method (see [6, 10]) in the following way. After inserting the knots so that each has multiplicity $m+1$, the basis functions become Bernstein polynomials and the coefficients can be easily read. If all inner knots have multiplicity 1 , the complexity of such procedure is $O\left(n m^{2}\right)$.

This does, however, return only the coefficients of one combination of B-spline basis functions. In order to find the coefficients of all B-spline basis functions

$$
N_{m,-m}, N_{m,-m+1}, \ldots, N_{m, n-1},
$$

one needs to do this procedure for each basis function (i.e., with the coefficient 1 for the selected basis function and 0 otherwise). This can be done in $O\left(n m^{3}\right)$ total time.

A generalization of B -splines which is currently gaining prominence are multi-degree Bsplines. In [1, 2, 27], methods of finding the coefficients of multi-degree B-splines are given, which can be adapted for standard (uniform-degree) B-splines. In particular, the method given in [2], which is based on reverse knot insertion, is shown to be numerically stable.

If there are sufficiently many computationally simple recurrence relations for the BernsteinBézier coefficients of the B-spline functions over multiple knot spans, one can instead use them to efficiently find each of the coefficients. Over the course of this paper, they will be derived from a new differential-recurrence relation for the B-spline functions.

Remark 2.4. In the sequel, we assume that no inner knot $t_{1}, t_{2}, \ldots, t_{n-1}$ has multiplicity greater than $m$. This guarantees the $B$-spline functions' continuity in $\left(t_{0}, t_{n}\right)$.

The assumption regarding the multiplicity of the inner knots is very common and intuitive, as it guarantees the continuity of a B-spline curve. It was used, e.g., in [18, 23] and [28, §3].

Let us suppose that there are $n_{e}$ non-empty knot spans $\left[t_{j}, t_{j+1}\right)$ such that $0 \leq j \leq n-1$. One of the main goals of the paper is to give a recursive way of computing all $O\left(n_{e} m^{2}\right)$ coefficients $b_{k}^{(i, j)}$ (cf. (2.1)) of the B-spline functions in $O\left(n_{e} m^{2}\right)$ time, assuming that all boundary knots are coincident.

The possible applications of this result can be as follows. Once the adjusted BernsteinBézier coefficients $b_{k}^{(i, j)}$ are known, each point on a B-spline curve S ,

$$
\begin{equation*}
\mathrm{S}(u):=\sum_{i=-m}^{n-1} N_{m i}(u) \mathrm{W}_{i} \quad\left(t_{0} \leq u \leq t_{n} ; \mathrm{W}_{i} \in \mathbb{E}^{d}\right), \tag{2.4}
\end{equation*}
$$

can be computed in $O\left(m^{2}+m d\right)$ time using the geometric algorithm proposed recently by the authors in [29]. If there are $N$ such points on $M$ curves (each with the same knots) - a situation closely related to rendering a tensor product B-spline surface - the total complexity is $O\left(n_{e} m^{2}+N m^{2}+M N m d\right)$, compared to $O\left(M N m^{2} d\right)$ when using the de Boor-Cox algorithm. Performed experiments confirm that the new method is faster than the de Boor-Cox algorithm
even for low $M \approx 2,3$. Using a similar approach, one can also compute the value of any $N_{m i}$ in $O(m)$ time.

Additionally, if the Bernstein-Bézier coefficients of the B-spline basis functions are known, one can use them to convert a B-spline curve over one non-empty knot span to a Bézier curve:

$$
\mathrm{S}(u)=\sum_{k=0}^{m}\left(\sum_{i=j-m}^{j} b_{k}^{(i, j)} \mathrm{W}_{i}\right) B_{k}^{m}\left(\frac{u-t_{j}}{t_{j+1}-t_{j}}\right) \quad\left(t_{j} \leq u<t_{j+1}\right) .
$$

## 3. New differential-recurrence relation for $B$-spline functions

Using the recurrence relation (1.8) which connects B-spline functions of consecutive degrees, one can find a recurrence relation which is satisfied by their coefficients in the chosen basis. It is, however, not optimal as the recurrence scheme is analogous to the one used in the de Boor-Cox algorithm.

A new differential-recurrence relation for the B-spline functions of the same degree $m$ will be derived. We show that by using this result, it is possible to find all the Bernstein-Bézier coefficients faster (see Section (4).

Using equations (1.8) and (1.10), one can derive new differential-recurrence relations for the B-spline functions of the same degree. For example, this result can be used to efficiently compute the coefficients of the $N_{m i}$ functions (which are polynomial in each of the knot spans) in an adjusted Bernstein-Bézier or power basis.

Theorem 3.1. Let $t_{-m}=t_{-m+1}=\ldots=t_{0}<t_{1}<\ldots<t_{n-1}<t_{n}=t_{n+1}=\ldots=t_{n+m}$ (cf. (1.11)). The following relations hold:

$$
\begin{align*}
& m N_{m,-m}(u)+\left(t_{1}-u\right) N_{m,-m}^{\prime}(u)=0,  \tag{3.1}\\
& N_{m i}(u)+\frac{t_{i}-u}{m} N_{m i}^{\prime}(u)=\frac{t_{m+i+1}-t_{i}}{t_{m+i+2}-t_{i+1}}\left(N_{m, i+1}(u)+\frac{t_{m+i+2}-u}{m} N_{m, i+1}^{\prime}(u)\right)  \tag{3.2}\\
& \quad(i=-m,-m+1, \ldots, n-2), \tag{3.3}
\end{align*}
$$

Proof. Equations (3.1) and (3.3) follow easily from equations (1.8) and (1.10), respectively. The relation (3.2) follows directly from taking the expression for $N_{m+1, i}$ from Eq. (1.8) and differentiating it, then equating it with the expression for $N_{m+1, i}^{\prime}$ given in Eq. (1.10).

Theorem 3.1 can be used to find a recurrence relation satisfied by the adjusted BernsteinBézier coefficients of B-spline functions of the same degree, as will be shown in the next section.

## 4. Recurrence relations for B-spline functions' coefficients in adjusted BernsteinBézier basis

Assume that that the knot sequence is clamped, i.e.,

$$
t_{-m}=t_{-m+1}=\ldots=t_{0}<t_{1}<\ldots<t_{n}=t_{n+1}=\ldots=t_{n+m}
$$

(cf. (1.11)).

For each knot span $\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-1)$, one needs to find the coefficients of $N_{m i}$ ( $i=j-m, j-m+1, \ldots, j$ ) in the following adjusted Bernstein-Bézier basis form:

$$
N_{m i}(u)=\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}(t) \quad\left(t_{j} \leq u<t_{j+1}\right),
$$

where $b_{k}^{(i, j)} \equiv b_{k, m}^{(i, j)}$ and

$$
\begin{equation*}
t \equiv t^{(j)}(u):=\frac{u-t_{j}}{t_{j+1}-t_{j}} \tag{4.1}
\end{equation*}
$$

(cf. Eq. (2.1) and Problem [2.1). Additionally, then, $u=\left(t_{j+1}-t_{j}\right) t+t_{j}$.
Certainly, $N_{m i}(u) \equiv 0$ if $u<t_{i}$ or $u>t_{m+i+1}$, which means that for a given knot span $\left[t_{j}, t_{j+1}\right)$, one only needs to find the coefficients of $N_{m, j-m}, N_{m, j-m+1}, \ldots, N_{m j}$, as all coefficients of other B-spline functions over this knot span are identical to zero. Thus, in each of $n$ knot spans, there are $m+1$ non-zero B -spline functions, each with $m+1$ coefficients.

Solving Problem [2.1] requires computing $n(m+1)^{2}$ coefficients $b_{k}^{(i, j)}$. In this section, it will be shown how to do it in $O\left(\mathrm{~nm}^{2}\right)$ time - proportionally to the number of coefficients. Theorem 3.1 serves as a foundation of the presented approach. More precisely, the theorem will be used to construct recurrence relations for the coefficients $b_{k}^{(i, j)}$ which allow solving Problem 2.1 efficiently.

The results for particular cases will be presented in stages. In 4 4.1 explicit expressions for the coefficients of $N_{m j}$ and $N_{m, j-m}$ over $\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-1)$ will be found. This will, in particular, cover the only non-trivial knot span for $N_{m, n-1}$. In $\$ 4.2$, Eq. (3.2) will be applied to find the coefficients of $N_{m i}$ for $j=n-1, n-2, \ldots, 0$ and $i=j-1, j-2, \ldots, j-m+1$.
Remark 4.1. In the sequel, it will be assumed that $u \in\left[t_{j}, t_{j+1}\right)$. Thus, $t \in[0,1)$ (cf. (4.1)).

### 4.1. Stage 1

For $j=0,1, \ldots, n-1$, one can use Eq. (1.8) for $i=j$, along with the fact that $N_{\ell, j+1} \equiv 0$ over $\left[t_{j}, t_{j+1}\right)(\ell=m-1, m-2, \ldots, 0)$, to find that

$$
N_{m j}(u)=\frac{\left(u-t_{j}\right)^{m}}{\prod_{k=1}^{m}\left(t_{j+k}-t_{j}\right)} N_{0 j}(u)=\frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+k}-t_{j}\right)} B_{m}^{m}(t) \quad\left(u \in\left[t_{j}, t_{j+1}\right)\right)
$$

It means that

$$
\left\{\begin{array}{l}
b_{k}^{(j, j)}=0  \tag{4.2}\\
b_{m}^{(j, j)}=\frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+k}-t_{j}\right)}
\end{array} \quad(k=0,1, \ldots, m-1)\right.
$$

where $0 \leq j \leq n-1$.
Using the same approach for $N_{m, j-m}$ over $\left[t_{j}, t_{j+1}\right)$ gives

$$
N_{m, j-m}(u)=\frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+1}-t_{j+1-k}\right)} B_{0}^{m}(t) \quad\left(u \in\left[t_{j}, t_{j+1}\right)\right)
$$

The coefficients $b_{k}^{(j-m, j)}(k=0,1, \ldots, m)$ are thus given by the following formula:

$$
\left\{\begin{array}{l}
b_{0}^{(j-m, j)}=\frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+1}-t_{j+1-k}\right)},  \tag{4.3}\\
b_{k}^{(j-m, j)}=0
\end{array} \quad(k=1,2, \ldots, m)\right.
$$

where $0 \leq j \leq n-1$. The adjusted Bernstein-Bézier coefficients of $N_{m j}$ and $N_{m, j-m}$ over the knot span $\left[t_{j}, t_{j+1}\right)$ (cf. Eq. (2.11) have been found for $j=0,1, \ldots, n-1$.

In the sequel, the following observation will be of use.
Remark 4.2. Note that

$$
N_{m, n-1}\left(t_{n}\right)=\frac{\left(t_{n}-t_{n-1}\right)^{m-1}}{\prod_{k=1}^{m-1}\left(t_{n+k}-t_{n-1}\right)} B_{m}^{m}(1)=1
$$

since $t_{n}=t_{n+1}=\ldots=t_{n+m}$. The B-spline functions have the partition of unity property and are non-negative (cf. Theorem 1.7), it is thus clear that

$$
N_{m i}\left(t_{n}\right)=0 \quad(i=-m,-m+1, \ldots, n-2) .
$$

Similarly,

$$
N_{m,-m}\left(t_{0}\right)=\frac{\left(t_{1}-t_{0}\right)^{m}}{\prod_{k=1}^{m}\left(t_{1}-t_{1-k}\right)} B_{0}^{m}(0)=1
$$

since $t_{-m}=t_{-m+1}=\ldots=t_{0}$. It follows that $N_{m i}\left(t_{0}\right)=0$ for $i=-m+1,-m+2, \ldots, 0$.

### 4.2. Stage 2

To compute the coefficients of all functions $N_{m i}$ over knot spans $\left[t_{j}, t_{j+1}\right)$ such that $j=$ $n-1, n-2, \ldots, 0$ and $i=j-1, j-2, \ldots, j-m+1$, Eq. (3.2) will be used. The following identity will be useful when operating on Eq. (3.2):

$$
\begin{equation*}
\left(N_{m i}(u)\right)^{\prime}=\frac{d N_{m i}(u)}{d u}=\sum_{k=0}^{m} b_{k}^{(i, j)} \frac{d B_{k}^{m}(t)}{d t} \cdot \frac{d t}{d u}=\left(t_{j+1}-t_{j}\right)^{-1} \sum_{k=0}^{m} b_{k}^{(i, j)}\left(B_{k}^{m}(t)\right)^{\prime} \tag{4.4}
\end{equation*}
$$

with $u \in\left[t_{j}, t_{j+1}\right)$ (cf. (4.1)).
Let

$$
\begin{equation*}
v_{i} \equiv v_{m i}:=\frac{t_{m+i+1}-t_{i}}{t_{m+i+2}-t_{i+1}} \tag{4.5}
\end{equation*}
$$

Substituting the adjusted Bernstein-Bézier forms of $N_{m i}$ and $N_{m, i+1}$ in the knot span [ $t_{j}, t_{j+1}$ ) and applying Eq. (4.4) into Eq. (3.2) gives

$$
\begin{aligned}
\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}(t)+( & \left.\frac{t_{i}-t_{j}}{m\left(t_{j+1}-t_{j}\right)}-\frac{t}{m}\right) \sum_{k=0}^{m} b_{k}^{(i, j)}\left(B_{k}^{m}(t)\right)^{\prime}= \\
& =v_{i}\left(\sum_{k=0}^{m} b_{k}^{(i+1, j)} B_{k}^{m}(t)+\left(\frac{t_{m+i+2}-t_{j}}{m\left(t_{j+1}-t_{j}\right)}-\frac{t}{m}\right) \sum_{k=0}^{m} b_{k}^{(i+1, j)}\left(B_{k}^{m}(t)\right)^{\prime}\right) .
\end{aligned}
$$

After using identities (1.6) and (1.5) and doing some algebra, one gets

$$
\begin{aligned}
& \sum_{k=0}^{m}\left(l_{k i} b_{k-1}^{(i, j)}+d_{k i} b_{k}^{(i, j)}+u_{k i} b_{k+1}^{(i, j)}\right) B_{k}^{m}(t)= \\
& =v_{i} \sum_{k=0}^{m}\left(l_{k, m+i+2} b_{k-1}^{(i+1, j)}+d_{k, m+i+2} b_{k}^{(i+1, j)}+u_{k, m+i+2} b_{k+1}^{(i+1, j)}\right) B_{k}^{m}(t)
\end{aligned}
$$

where

$$
l_{k r}:=k\left(t_{j+1}-t_{r}\right), \quad d_{k r}:=(m-k)\left(t_{j+1}-t_{r}\right)+k\left(t_{r}-t_{j}\right), \quad u_{k r}:=(m-k)\left(t_{r}-t_{j}\right) .
$$

Matching the coefficients of Bernstein polynomials on both sides gives a system of $m+1$ equations of the form:

Theorem 4.3. For $j=0,1, \ldots, n-1$ and $i=j-1, j-2, \ldots, j-m+1$, assuming that the coefficients $b_{k}^{(i+1, j)}(0 \leq k \leq m)$ are known, the values $b_{0}^{(i, j)}, b_{1}^{(i, j)}, \ldots, b_{m}^{(i, j)}$ satisfy a first-order non-homogeneous recurrence relation

$$
\begin{equation*}
\left(t_{j+1}-t_{i}\right) b_{k}^{(i, j)}+\left(t_{i}-t_{j}\right) b_{k+1}^{(i, j)}=A(m, i, j, k) \quad(k=0,1, \ldots, m-1), \tag{4.7}
\end{equation*}
$$

where

$$
A(m, i, j, k):=v_{i}\left(\left(t_{j+1}-t_{m+i+2}\right) b_{k}^{(i+1, j)}+\left(t_{m+i+2}-t_{j}\right) b_{k+1}^{(i+1, j)}\right)
$$

(cf. (4.5)).
Proof. Base case ( $k=0$ and $k=m$ ): the relation holds and is presented in the first and the last equations of the system (4.6).

Induction step $(k \rightarrow k+1)$ : the $(k+2)$ th equation in the system (4.6) is

$$
\begin{aligned}
l_{k+1, i} b_{k}^{(i, j)}+d_{k+1, i} b_{k+1}^{(i, j)} & +u_{k+1, i} b_{k+2}^{(i, j)}= \\
& =v_{i}\left(l_{k+1, m+i+2} b_{k}^{(i+1, j)}+d_{k+1, m+i+2} b_{k+1}^{(i+1, j)}+u_{k+1, m+i+2} b_{k+2}^{(i+1, j)}\right)
\end{aligned}
$$

Subtracting sidewise the induction assumption scaled by $\frac{l_{k+1, i}}{\left(t_{j+1}-t_{i}\right)}=k+1$ gives, after some algebra,

$$
\left(t_{j+1}-t_{i}\right) b_{k+1}^{(i, j)}+\left(t_{i}-t_{j}\right) b_{k+2}^{(i, j)}=v_{i}\left(\left(t_{j+1}-t_{m+i+2}\right) b_{k+1}^{(i+1, j)}+\left(t_{m+i+2}-t_{j}\right) b_{k+2}^{(i+1, j)}\right)
$$

which concludes the proof.
From Theorem 4.3, it follows that there are $m$ independent equations in the system (4.6), as one of them is redundant. One thus needs an initial value to find the values of all $b_{0}^{(i, j)}, b_{1}^{(i, j)}, \ldots, b_{m}^{(i, j)}$ using the recurrence relation (4.7).

If $j=n-1$, Remark 4.2 can be used to find that

$$
N_{m i}\left(t_{n}\right)=b_{m}^{(i, n-1)}=0 \quad(i=n-2, n-3, \ldots, n-m)
$$

In this case, the recurrence relation given in Theorem 4.3 simplifies to

$$
\left(t_{n}-t_{i}\right) b_{k}^{(i, n-1)}=\left(t_{n-1}-t_{i}\right) b_{k+1}^{(i, n-1)}+v_{i}\left(t_{n}-t_{n-1}\right) b_{k+1}^{(i+1, n-1)} .
$$

It means that, for $i=n-2, n-3, \ldots, n-m$, the following relation holds:

$$
\left\{\begin{array}{l}
b_{m}^{(i, n-1)}=0  \tag{4.8}\\
b_{k}^{(i, n-1)}=\frac{t_{n-1}-t_{i}}{t_{n}-t_{i}} b_{k+1}^{(i, n-1)}+\frac{t_{n}-t_{n-1}}{t_{n}-t_{i+1}} b_{k+1}^{(i+1, n-1)} \quad(k=m-1, m-2, \ldots, 0)
\end{array}\right.
$$

For $i=n-2, n-3, \ldots, n-m$, assuming that the coefficients $b_{k}^{(i+1, n-1)}$ are known ( $k=$ $1,2, \ldots, m)$, Eq. (4.8) has an explicit solution

$$
\left\{\begin{array}{l}
b_{m}^{(i, n-1)}=0  \tag{4.9}\\
b_{k}^{(i, n-1)}=\frac{t_{n}-t_{n-1}}{t_{n}-t_{i+1}} \sum_{\ell=0}^{m-k-1}\left(\frac{t_{n-1}-t_{i}}{t_{n}-t_{i}}\right)^{\ell} b_{k+1+\ell}^{(i+1, n-1)} \quad(k=0,1, \ldots, m-1)
\end{array}\right.
$$

To find the initial value, if $j<n-1$ and $i=j-1, j-2, \ldots, j-m+1$, the right continuity condition will be used, i.e.,

$$
N_{m i}\left(t_{j+1}^{-}\right)=N_{m i}\left(t_{j+1}^{+}\right) .
$$

More precisely,

$$
N_{m i}\left(t_{j+1}^{-}\right)=\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}(1)=b_{m}^{(i, j)}
$$

and

$$
N_{m i}\left(t_{j+1}^{+}\right)=\sum_{k=0}^{m} b_{k}^{(i, j+1)} B_{k}^{m}(0)=b_{0}^{(i, j+1)},
$$

which gives the relation

$$
\begin{equation*}
b_{m}^{(i, j)}=b_{0}^{(i, j+1)} \tag{4.10}
\end{equation*}
$$

This completes the recurrence scheme for $j=n-2, n-3, \ldots, 0$ and $i=j-1, j-2, \ldots, j-$ $m+1$ :

$$
\left\{\begin{array}{l}
b_{m}^{(i, j)}=b_{0}^{(i, j+1)},  \tag{4.11}\\
b_{k}^{(i, j)}=\frac{t_{j}-t_{i}}{t_{j+1}-t_{i}} b_{k+1}^{(i, j)}+\frac{v_{i}}{t_{j+1}-t_{i}}\left(\left(t_{j+1}-t_{m+i+2}\right) b_{k}^{(i+1, j)}+\left(t_{m+i+2}-t_{j}\right) b_{k+1}^{(i+1, j)}\right) \\
(k=m-1, m-2, \ldots, 0) .
\end{array}\right.
$$

From Eq. (4.11) follows an explicit formula for the coefficients $b_{k}^{(i, j)}(0 \leq k \leq m)$, assuming that the coefficients $b_{0}^{(i, j+1)}$ and $b_{k}^{(i+1, j)}(k=0,1, \ldots, m)$ are known:

$$
\begin{equation*}
b_{k}^{(i, j)}=\left(\frac{t_{j}-t_{i}}{t_{j+1}-t_{i}}\right)^{m-k} b_{0}^{(i, j+1)}+\sum_{\ell=0}^{m-k-1}\left(\frac{t_{j}-t_{i}}{t_{j+1}-t_{i}}\right)^{\ell} \frac{v_{i}}{t_{j+1}-t_{i}} q_{k+\ell}, \tag{4.12}
\end{equation*}
$$

where

$$
q_{\ell}:=\left(t_{j+1}-t_{m+i+2}\right) b_{\ell}^{(i+1, j)}+\left(t_{m+i+2}-t_{j}\right) b_{\ell+1}^{(i+1, j)},
$$

and $0 \leq j \leq n-2, j-m+1 \leq i \leq j-1$.
The coefficients of $N_{m i}$ have been found for $j=0,1, \ldots, n-1$ and $i=j-1, j-2, \ldots, j-$ $m+1$.

### 4.3. Recurrence scheme and implementation

The results presented in $\$ 4.1$ and $\$ 4.2$ can be combined to prove the following theorem.
Theorem 4.4. Let us assume that

$$
t_{-m}=t_{-m+1}=\ldots=t_{0}<t_{1}<\ldots<t_{n-1}<t_{n}=t_{n+1}=\ldots=t_{n+m}
$$

(cf. (1.11)). The $n(m+1)^{2}$ adjusted Bernstein-Bézier coefficients $b_{k}^{(i, j)}$ of the B-spline functions $N_{m i}$ over each knot span $\left[t_{j}, t_{j+1}\right)(c f$. (2.1) ), for $j=0,1, \ldots, n-1, i=j-m, j-m+$ $1, \ldots, j$ and $k=0,1, \ldots, m$, can be computed in the computational complexity $O\left(n m^{2}\right)$ in the following way:

1. For $j=0,1, \ldots, n-1$ and $k=0,1, \ldots, m$, the coefficients $b_{k}^{(j, j)}$ and $b_{k}^{(j-m, j)}$ are given explicitly in equations (4.2) and (4.3), respectively.
2. For $j=n-1, i=n-2, n-3, \ldots, n-m$ and $k=m, m-1, \ldots, 0$, the coefficients $b_{k}^{(i, n-1)}(k=0,1, \ldots, m)$ are computed by the recurrence relation (4.8) (for their explicit forms, see (4.9) ).
3. For $j=n-2, n-3, \ldots, 0, i=j-1, j-2, \ldots, j-m+1$ and $k=m, m-1, \ldots, 0$, the coefficients $b_{k}^{(i, j)}$ are computed by the recurrence relation (4.11) (for their explicit forms, see (4.12)).

Example 4.5. Let us set $m:=3, n:=5$. Let the knots be

| $t_{-3}$ | $t_{-2}$ | $t_{-1}$ | $t_{0}$ | $t_{1}$ | $t_{2}$ | $t_{3}$ | $t_{4}$ | $t_{5}$ | $t_{6}$ | $t_{7}$ | $t_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 3 | 5 | 6 | 9 | 10 | 10 | 10 | 10 |.

Figure 4.1 illustrates the approach to computing all necessary adjusted Bernstein-Bézier coefficients of B-spline functions, given in Theorem 4.4. Arrows denote recursive dependence. Diagonally striped squares are computed using Eq. (4.2). Horizontally striped squares are computed using Eq. (4.3). White squares are computed using either the recurrence (4.8) (for $u \in\left[t_{4}, t_{5}\right)$ ) or (4.11) (for $\left.u<t_{4}\right)$.

### 4.3.1. Implementation

Algorithm 4.1 implements the approach proposed in Theorem 4.4. This algorithm returns a sparse array $B \equiv B[0 . . n-1,-m . . n-1,0 . . m]$, where

$$
B[j, i, k]=b_{k}^{(i, j)} \quad(0 \leq j<n,-m \leq i<n, 0 \leq k \leq m)
$$

(cf. (2.1)).
For each of the $n$ knot spans, one has to compute the coefficients of $m+1$ functions $\left(n(m+1)^{2}\right.$ coefficients in total). Computing all coefficients of one B-spline function in a given knot span requires $O(m)$ operations. In total, then, the complexity of Algorithm 4.1 is $O\left(n m^{2}\right)$ - giving the optimal $O(1)$ time per coefficient.


Figure 4.1: An illustration of Example 4.5

## 5. Applications

### 5.1. Fast computation of multiple points on multiple $B$-spline curves

Let $u \in\left[t_{j}, t_{j+1}\right)$ and $t:=\frac{u-t_{j}}{t_{j+1}-t_{j}}$. By solving Problem 2.1, the Bernstein-Bézier coefficients of the B-spline functions are found. A point on a B-spline curve (2.4) can thus be expressed as

$$
\mathbf{S}(u)=\sum_{i=j-m}^{j}\left(\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}(t)\right) \mathbf{W}_{i} .
$$

The inner sums

$$
\begin{equation*}
p_{i}(u):=\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}(t) \equiv N_{m i}(u) \quad(i=j-m, j-m+1, \ldots, j) \tag{5.1}
\end{equation*}
$$

can be treated as polynomial Bézier curves with control points $b_{k}^{(i, j)} \in \mathbb{E}^{1}$ and thus can be computed using the geometric algorithm given in [29] in total time $O\left(m^{2}\right)$ - more precisely, $O(m)$ per each of $m+1$ sums. It also means that - when all the coefficients $b_{k}^{(i, j)}$ are already known - any B-spline function may be computed in linear time with respect to its degree.

Example 5.1. A comparison of the new method of evaluating $B$-spline functions and using recurrence relation (1.8) has been done. The results have been obtained on a computer with Intel Core $i 5-6300 \mathrm{U}$ CPU at 2.40 GHz processor and 4 GB RAM, using GNU C Compiler 11.2 .0 (single precision).

For each $n \in\{10,15,20,25,30,35,40,45,50\}$, and $m=3,4, \ldots, 15$, a sequence of knots has been generated 100 times. The knot span lengths $t_{j+1}-t_{j} \in[1 / 50,1](j=0,1, \ldots, n-$

```
Algorithm 4.1 Computing the coefficients of the adjusted Bernstein-Bézier form of the B-
spline functions
procedure \(\operatorname{BSPLINEBBF}\left(n, m,\left[t_{-m}, t_{-m+1}, \ldots, t_{n+m}\right]\right)\)
    \(B \leftarrow\) SparseArray [0..n-1, \(-\mathrm{m} . . \mathrm{n}-1,0 \ldots \mathrm{~m}](f i l l=0)\)
    for \(j \leftarrow 0, n-1\) do
        \(B[j, j, m] \leftarrow \frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+k}-t_{j}\right)}\)
        \(B[j, j-m, 0] \leftarrow \frac{\left(t_{j+1}-t_{j}\right)^{m-1}}{\prod_{k=2}^{m}\left(t_{j+1}-t_{j+1-k}\right)}\)
    end for
    for \(i \leftarrow n-2, n-m\) do
        for \(k \leftarrow m-1,0\) do
            \(B[n-1, i, k] \leftarrow \frac{t_{n-1}-t_{i}}{t_{n}-t_{i}} \cdot B[n-1, i, k+1]+\frac{t_{n}-t_{n-1}}{t_{n}-t_{i+1}} \cdot B[n-1, i+1, k+1]\)
        end for
    end for
    for \(j \leftarrow n-2,0\) do
        for \(i \leftarrow j-1, j-m+1\) do
            \(v \leftarrow \frac{t_{m+i+1}-t_{i}}{t_{m+i+2}-t_{i+1}}\)
                \(B[j, i, m] \leftarrow B[j+1, i, 0]\)
                for \(k=m-1,0\) do
                \(B[j, i, k] \leftarrow \frac{t_{j}-t_{i}}{t_{j+1}-t_{i}} \cdot B[j, i, k+1]+\frac{v}{t_{j+1}-t_{i}} \cdot\left(\left(t_{j+1}-t_{m+i+2}\right) \cdot B[j, i+\right.\)
\(\left.1, k]+\left(t_{m+i+2}-t_{j}\right) \cdot B[j, i+1, k+1]\right)\)
        end for
        end for
    end for
    return \(B\)
end procedure
```

1; $t_{0}=0$ ) have been generated using the rand() C function. The boundary knots are coincident. Then, $50 \cdot n+1$ points such that $t_{j \ell}:=t_{j}+\ell / 50 \times\left(t_{j+1}-t_{j}\right)$ for $j=0,1, \ldots, n-1$ and $\ell=0,1, \ldots, 49$, with the remaining point being $t_{n 0} \equiv t_{n}$, are generated.

At each point $t_{j \ell} \in\left[t_{j}, t_{j+1}\right)$, all $m+1 B$-spline functions $N_{m i}(i=j-m, j-m+1, \ldots, j)$ which do not vanish at $t_{j \ell}$ are evaluated using both algorithms. Due to the size of the table, the resulting running times are available athttps://www. ii. uni. wroc.pl/ ~pwo/programs/BSpline-BF-Examp

The new method consistently performs faster than evaluating $B$-spline functions using recurrence relation (1.8). The new method reduced the running time for any dataset by 29-46\%, while the total running time was reduced by $45 \%$. The source code in $C$ which was used to perform the tests is available at/https://www. ii. uni. wroc.pl/ ${ }^{\text {pwo/programs/BSpline-BF.c. }}$.

Note that the sums $p_{i}$ (cf. (5.1)) do not depend on the control points. Afterwards, computing a convex combination of $m+1$ points from $\mathbb{E}^{d}$, i.e.,

$$
\mathrm{S}(u)=\sum_{i=j-m}^{j} p_{i}(u) \mathrm{W}_{i} \quad\left(u \in\left[t_{j}, t_{j+1}\right)\right),
$$

requires $O(m d)$ arithmetic operations. Observe that these values may also be computed using the geometric method proposed in [29, Algorithm 1.1]. In total, then, assuming that the Bernstein-Bézier coefficients of the B-spline functions over each knot span $\left[t_{j}, t_{j+1}\right)(j=$ $0,1, \ldots, n-1)$ are known, $O(m(m+d))$ arithmetic operations are required to compute a point $\mathrm{S}(u)\left(u \in\left[t_{j}, t_{j+1}\right)\right)$ on a B-spline curve.

When it is required to compute the values of S for many parameters $u_{0}, u_{1}, \ldots, u_{N}$, one would have to perform $O\left(n m^{2}\right)$ arithmetic operations to find the Bernstein-Bézier coefficients of the B-spline functions over each knot span and then do $O(m(m+d))$ operations for each of $N+1$ points that are to be computed. In total, the computational complexity of this approach is $O\left(n m^{2}+N m(m+d)\right)$.

Due to the fact that the sums $p_{i}$ (cf. (5.1)) do not depend on the control points, they can be used for computing a point on multiple B-spline curves, all of degree $m$, with the same knots.

Problem 5.2. For $M B$-spline curves $S_{0}, S_{1}, \ldots, S_{M-1}$ with the knots

$$
t_{-m}=t_{-m+1}=\ldots=t_{0}<t_{1}<\ldots<t_{n}=t_{n+1}=\ldots=t_{n+m} .
$$

and the control points of $S_{k}$ being

$$
W_{k,-m}, W_{k,-m+1}, \ldots, W_{k, n-1} \in \mathbb{E}^{d} \quad(k=0,1, \ldots, M-1),
$$

compute the value of each of the $B$-spline curves $S_{k}$ at points $u_{0}, u_{1}, \ldots, u_{N-1}$ such that $t_{0} \leq u_{k} \leq t_{n}$ for all $k=0,1, \ldots, N-1$. More precisely, for $k=0,1, \ldots, M-1$ and $\ell=0,1, \ldots, N-1$, compute all the points $S_{k}\left(u_{\ell}\right)$.

One can efficiently solve Problem 5.2 in the following way. Using Algorithm 4.1 allows to compute all the adjusted Bernstein-Bézier coefficients of B-spline functions (cf. Problem 2.1) in $O\left(n m^{2}\right)$ time. Now, one needs to compute the values

$$
p_{i}\left(u_{\ell}\right) \quad\left(\ell=0,1, \ldots, N-1, u_{\ell} \in\left[t_{j}, t_{j+1}\right), i=j-m, j-m+1, \ldots, j\right)
$$

(cf. (5.1)), which takes $O\left(\mathrm{Nm}^{2}\right)$ time. Using these values, computing

$$
\mathrm{S}_{k}\left(u_{\ell}\right)=\sum_{i=j-m}^{j} p_{i}\left(u_{\ell}\right) \mathrm{W}_{k i} \quad\left(\ell=0,1, \ldots, N-1, k=0,1, \ldots, M-1, u_{\ell} \in\left[t_{j}, t_{j+1}\right)\right)
$$

takes $O(M N m d)$ time (see [29]). In total, then, the complexity of this approach is $O\left(n m^{2}+\right.$ $\left.N m^{2}+N M m d\right)$, compared to the complexity of using the de Boor-Cox algorithm to solve Problem 5.2, i.e., $O\left(N M m^{2} d\right)$.

Using the recurrence relation (1.8) (see 14, p. 55-57]) to evaluate the B-spline basis functions in $O\left(\mathrm{Nm}^{2}\right)$ and then compute the linear combinations of control points in $O(\mathrm{NMmd})$ time gives a total complexity of $O\left(N m^{2}+N M m d\right)$. While this may appear similar to the complexity of the new method, a closer examination of the number of floating-point operations shows that the new method has $O(\mathrm{Nm})$ divisions, compared to $O\left(\mathrm{Nm}^{2}\right)$ in the approach based on the relation (1.8). In practice, this saves $O\left(N m^{2}\right)$ divisions (see Table 5.2).

A comparison of running times is given in Example 5.3. The new algorithm is compared to executing the de Boor-Cox algorithm and to an alternative way of computing the B-spline functions based on the recurrence relation (1.8) (see [14, p. 55-57]) and then evaluating the point in the same way as in the new method.

Example 5.3. Table 5.1 shows the comparison between the running times of the de Boor-Cox algorithm, an algorithm which computes the values of $B$-spline function using the recurrence relation (1.8) and then computes the points, and the new method described above and using Algorithm 4.1.

The results have been obtained on a computer with Intel Core i5-6300U CPU at 2.40GHz processor and $4 G B$ RAM, using GNU C Compiler 11.2.0 (single precision).

The following numerical experiments have been conducted. For fixed $n=20$ and $d=2$, for each $M \in\{1,5,10,20,50,100\}$ and $m \in\{3,5,7,9,11\}$, a sequence of knots and control points has been generated 100 times. The control points $W_{k i} \in[-1,1]^{d}(i=-m,-m+1, \ldots, n-$ $1, k=0,1, \ldots, M-1)$ and the knot span lengths $t_{j+1}-t_{j} \in[1 / 50,1]\left(j=0,1, \ldots, n-1 ; t_{0}=\right.$ 0) have been generated using the rand() C function. The boundary knots are coincident. Each algorithm is then tested using the same knots and control points. Each curve is evaluated at 1001 points which are $t_{j}+\ell / 50 \times\left(t_{j+1}-t_{j}\right)$ for $j=0,1, \ldots, n-1$ and $\ell=0,1, \ldots, 49$, with the remaining point being $t_{n}$. Table 5.1 shows the total running time of all $100 \times 1001 \times M$ curve evaluations for each method.

On average, the method eval splines had 7.52 , while new method had 7.24 common digits with the result of the numerically stable de Boor-Cox algorithm in single precision (8 digits) computations.

Example 5.4. An experiment similar to Example 5.3, with a wider choice of parameters, has been performed. The results have been obtained on the same computer, software, and precision. More precisely, for each $d \in\{1,2,3\}, n \in\{10,15,20,25,30,35,40,45,50\}, M \in$ $\{1,2,3,4,5,10,15,20,25,30,50,100\}$ and $m=3,4, \ldots, 15$, a sequence of knots and control points has been generated 100 times. The control points $W_{k i} \in[-1,1]^{d}(i=-m,-m+$ $1, \ldots, n-1, k=0,1, \ldots, M-1)$ and the knot span lengths $t_{j+1}-t_{j} \in[1 / 50,1](j=0,1, \ldots, n-$ $1 ; t_{0}=0$ ) have been generated using the rand() C function. The boundary knots are coincident. Each algorithm is then tested using the same knots and control points. Each curve is evaluated at $50 \cdot n+1$ points which are $t_{j}+\ell / 50 \times\left(t_{j+1}-t_{j}\right)$ for $j=0,1, \ldots, n-1$ and $\ell=$ $0,1, \ldots, 49$, with the remaining point being $t_{n}$. Due to the size of the table, the resulting running times are available at https: // www. ii. uni. wroc.pl/ ~pwo/programs/BSpline-BF-Example-5-4.xls

The results show that the new method is significantly faster than the de Boor-Cox algorithm except for the case $M=1$. While the acceleration with respect to the approach which utilizes Eq. (1.8) is smaller, it is also consistent, getting lower running time in all the test cases.

Some statistics regarding the experiments are given in Table 5.3.

### 5.2. Evaluating a tensor product $B$-spline surface

Let $n_{1}, n_{2}, m_{1}, m_{2} \in \mathbb{N}$. Let

$$
T:=\left(t_{-m_{1}}, t_{-m_{1}+1}, \ldots, t_{n_{1}+m_{1}}\right), \quad V:=\left(v_{-m_{2}}, v_{-m_{2}+1}, \ldots, v_{n_{2}+m_{2}}\right)
$$

be the knot sequences. For $i=-m_{1},-m_{1}+1, \ldots, n_{1}-1$ and $\ell=-m_{2},-m_{2}+1, \ldots, n_{2}-1$, let $N_{m_{1}, i}(u ; T)$ be a B-spline function of degree $m_{1}$ with the knot sequence $T$, and $N_{m_{2}, \ell}(w ; V)$ be a B-spline function of degree $m_{2}$ with the knot sequence $V$. The tensor product $B$-spline surface S with control points $\mathrm{W}_{i \ell} \in \mathbb{E}^{d}$ is given by the following formula:

$$
\begin{equation*}
\mathrm{S}(u, w):=\sum_{i=-m_{1}}^{n_{1}-1} \sum_{\ell=-m_{2}}^{n_{2}-1} \mathrm{~W}_{i \ell} N_{m_{1}, i}(u ; T) N_{m_{2}, \ell}(w ; V), \tag{5.2}
\end{equation*}
$$

| $M$ | $m$ | de Boor-Cox | eval splines | new method |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | $\mathbf{0 . 0 1 7}$ | 0.027 | 0.019 |
| 1 | 5 | 0.033 | 0.046 | $\mathbf{0 . 0 3 0}$ |
| 1 | 7 | 0.056 | 0.073 | $\mathbf{0 . 0 4 6}$ |
| 1 | 9 | 0.090 | 0.110 | $\mathbf{0 . 0 6 7}$ |
| 1 | 11 | 0.129 | 0.151 | $\mathbf{0 . 0 8 9}$ |
| 5 | 3 | 0.076 | 0.045 | $\mathbf{0 . 0 3 8}$ |
| 5 | 5 | 0.161 | 0.077 | $\mathbf{0 . 0 6 1}$ |
| 5 | 7 | 0.281 | 0.116 | $\mathbf{0 . 0 8 6}$ |
| 5 | 9 | 0.445 | 0.161 | $\mathbf{0 . 1 1 5}$ |
| 5 | 11 | 0.643 | 0.212 | $\mathbf{0 . 1 4 7}$ |
| 10 | 3 | 0.151 | 0.078 | $\mathbf{0 . 0 6 5}$ |
| 10 | 5 | 0.323 | 0.116 | $\mathbf{0 . 0 9 7}$ |
| 10 | 7 | 0.562 | 0.167 | $\mathbf{0 . 1 3 5}$ |
| 10 | 9 | 0.890 | 0.224 | $\mathbf{0 . 1 7 4}$ |
| 10 | 11 | 1.285 | 0.287 | $\mathbf{0 . 2 1 8}$ |
| 20 | 3 | 0.302 | 0.142 | $\mathbf{0 . 1 1 5}$ |
| 20 | 5 | 0.645 | 0.194 | $\mathbf{0 . 1 7 1}$ |
| 20 | 7 | 1.126 | 0.276 | $\mathbf{0 . 2 3 1}$ |
| 20 | 9 | 1.775 | 0.350 | $\mathbf{0 . 2 9 3}$ |
| 20 | 11 | 2.568 | 0.438 | $\mathbf{0 . 3 5 8}$ |
| 50 | 3 | 0.754 | 0.333 | $\mathbf{0 . 2 6 7}$ |
| 50 | 5 | 1.612 | 0.428 | $\mathbf{0 . 3 9 1}$ |
| 50 | 7 | 2.810 | 0.579 | $\mathbf{0 . 5 1 7}$ |
| 50 | 9 | 4.450 | 0.729 | $\mathbf{0 . 6 4 8}$ |
| 50 | 11 | 6.423 | 0.889 | $\mathbf{0 . 7 8 1}$ |
| 100 | 3 | 1.510 | 0.655 | $\mathbf{0 . 5 2 4}$ |
| 100 | 5 | 3.225 | 0.822 | $\mathbf{0 . 7 6 0}$ |
| 100 | 7 | 5.622 | 1.099 | $\mathbf{1 . 0 0 1}$ |
| 100 | 9 | 8.900 | 1.369 | $\mathbf{1 . 2 4 7}$ |
| 100 | 11 | 12.840 | 1.652 | $\mathbf{1 . 4 9 4}$ |
|  |  |  |  |  |

Table 5.1: Running times comparison (in seconds) for Example 5.3 The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF.c
with $(u, w) \in D \equiv\left[t_{0}, t_{n_{1}}\right] \times\left[v_{0}, v_{n_{2}}\right]$.
Let $(u, w) \in\left[t_{j_{1}}, t_{j_{1}+1}\right) \times\left[v_{j_{2}}, v_{j_{2}+1}\right)$ for $0 \leq j_{1}<n_{1}$ and $0 \leq j_{2}<n_{2}$. Then, Eq. (5.2) simplifies to

$$
\begin{equation*}
\mathrm{S}(u, w)=\sum_{i=j_{1}-m_{1}}^{j_{1}} \sum_{\ell=j_{2}-m_{2}}^{j_{2}-1} \mathrm{~W}_{i \ell} N_{m_{1}, i}(u ; T) N_{m_{2}, \ell}(w ; V) . \tag{5.3}
\end{equation*}
$$

| operation type | new method |
| :--- | :---: |
| + | $(m-1) m(2 n-1)+N(m+2) m+N M(m+1) d$ |
| - | $2(m-1)(4 n-1)+n+N((m+1) m+3)$ |
| $*$ | $2(m-1) m(2 n-1)+2 N(m+2) m+N M(m+1) d$ |
| $/$ | $2(m-1)(3 n-1)+N(m+2)$ |
| $(\cdot)^{m-1}$ | $n$ |

Table 5.2: The number of floating-point operations performed by the new method.

| Algorithm | Total running time $[\mathrm{s}]$ | Relative to new method |
| :--- | :---: | :---: |
| de Boor-Cox | 13993.58 | 6.80 |
| eval splines | 2482.96 | 1.21 |
| new method | 2058.95 | - |


| Algorithm | New method win \% | Max time rel. <br> to new method | Min time rel. <br> to new method |
| :--- | :---: | :---: | :---: |
| de Boor-Cox | $97.01 \%$ | 11.686 | 0.664 |
| eval splines | $100.00 \%$ | 1.877 | 1.056 |

Table 5.3: Statistics for Example 5.4 The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF.c

If one needs to evaluate the B-spline surface S at $N_{1} \cdot N_{2}$ points

$$
\left\{\left(u_{i}, w_{j}\right): 0 \leq i<N_{1} \wedge 0 \leq j<N_{2}\right\} \subset D
$$

a similar approach to the one shown for B-spline curves can be used, i.e.,

1. For all non-empty knot spans, find the Bernstein-Bézier coefficients of the B-spline basis functions in both dimensions $-O\left(n_{1} m_{1}^{2}+n_{2} m_{2}^{2}\right)$ operations.
2. For each $u_{i}\left(0 \leq i<N_{1}\right)$, evaluate $N_{m_{1}, \ell}\left(u_{i} ; T\right)$ for such $\ell$ that the corresponding Bspline basis functions do not vanish. For each $w_{i}\left(0 \leq i<N_{2}\right)$, evaluate $N_{m_{2}, \ell}\left(w_{i} ; V\right)$ for such $\ell$ that the corresponding B-spline basis functions do not vanish. The evaluation can be done using the algorithm given in [29]. In total, this requires $O\left(N_{1} m_{1}^{2}+N_{2} m_{2}^{2}\right)$ operations.
3. For each $(u, w) \in\left\{\left(u_{i}, w_{j}\right): 0 \leq i<N_{1} \wedge 0 \leq j<N_{2}\right\}$, evaluate $\mathrm{S}(u, w)-$ $O\left(N_{1} N_{2} m_{1} m_{2} d\right)$ operations.

In total, this procedure requires $O\left(\left(N_{1}+n_{1}\right) m_{1}^{2}+\left(N_{2}+n_{2}\right) m_{2}^{2}+N_{1} N_{2} m_{1} m_{2} d\right)$ operations.
Example 5.5. Table 5.4 shows the comparison between the running times of the evaluation of points ona a tensor product B-spline surface using the de Boor-Cox algorithm and the new method based of Algorithm 4.1.

The results have been obtained on a computer with Intel Core i5-6300U CPU at 2.40GHz processor and $4 G B$ RAM, using GNU C Compiler 11.2.0 (single precision).

The following numerical experiments have been conducted. For $d=3$ and $n_{1} \in\{10,30,50\}$, $n_{2}=n_{1}, m_{1} \in\{3,5,7,9\}, m_{2} \in\left\{m_{1}-2, m_{1}\right\}$, sequences of knots $T, V$ and control points have been generated 10 times. The control points $W_{i \ell} \in[-1,1]^{3}\left(i=-m_{1},-m_{1}+1, \ldots, n_{1}-1, \ell=\right.$ $\left.-m_{2},-m_{2}+1, \ldots, n_{2}-1\right)$ and the knot span lengths $t_{j_{1}+1}-t_{j_{1}} \in[1 / 50,1]\left(j_{1}=0,1, \ldots, n_{1}-\right.$ $\left.1 ; t_{0}=0\right), v_{j_{2}+1}-v_{j_{2}} \in[1 / 50,1]\left(j_{2}=0,1, \ldots, n_{2}-1 ; v_{0}=0\right)$ have been generated using the rand() C function. The boundary knots are coincident. Each algorithm is then tested using the same knots and control points. Each non-empty knot span in $T$ and $V$ has been sampled 50 times, with uniform distances between samples. Table 5.4 shows the total running time of all $10 \cdot\left(50 n_{1}+1\right) \cdot\left(50 n_{2}+1\right)$ surface evaluations for each method.

Numerical tests show that the new method has, on average, between 7.19 (for $n_{1}=n_{2}=50$ ) and 7.27 (for $n_{1}=n_{2}=10$ ) common digits with the result of the numerically stable de BoorCox algorithm in single precision (8 digits) computations.

Table 5.5 shows some statistics for the experiment.

| $n_{1}=n_{2}$ | $m_{1}$ | $m_{2}$ | de Boor-Cox | new method |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 3 | 1 | 1.185 | 0.477 |
| 10 | 3 | 3 | 2.610 | 0.879 |
| 10 | 5 | 3 | 4.224 | 1.344 |
| 10 | 5 | 5 | 7.766 | 1.958 |
| 10 | 7 | 5 | 10.796 | 2.592 |
| 10 | 7 | 7 | 17.385 | 3.657 |
| 10 | 9 | 7 | 22.301 | 4.568 |
| 10 | 9 | 9 | 32.807 | 5.362 |
| 30 | 3 | 1 | 10.528 | 4.288 |
| 30 | 3 | 3 | 23.451 | 8.010 |
| 30 | 5 | 3 | 38.009 | 12.190 |
| 30 | 5 | 5 | 69.644 | 17.699 |
| 30 | 7 | 5 | 96.775 | 23.408 |
| 30 | 7 | 7 | 155.907 | 32.924 |
| 30 | 9 | 7 | 200.379 | 41.209 |
| 30 | 9 | 9 | 294.417 | 48.231 |
| 50 | 3 | 1 | 29.060 | 11.898 |
| 50 | 3 | 3 | 65.101 | 22.325 |
| 50 | 5 | 3 | 105.349 | 33.948 |
| 50 | 5 | 5 | 193.328 | 49.205 |
| 50 | 7 | 5 | 268.871 | 65.005 |
| 50 | 7 | 7 | 432.882 | 91.607 |
| 50 | 9 | 7 | 554.807 | 114.142 |
| 50 | 9 | 9 | 817.361 | 134.059 |
|  |  |  |  |  |

Table 5.4: Running times comparison (in seconds) for Example5.5 The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF-Surf.c

| Algorithm | Total running time $[\mathrm{s}]$ | Ratio | Max ratio | Min ratio |
| :--- | :---: | :---: | :---: | :---: |
| de Boor-Cox | 3454.94 | 4.73 | 6.12 | 2.44 |
| new method | 730.98 | - | - | - |

Table 5.5: Statistics for Example 5.5 The source code in C which was used to perform the tests is available at https://www.ii.uni.wroc.pl/~pwo/programs/BSpline-BF-Surf.c

Additionally, note that if the Bernstein-Bézier coefficients of the B-spline basis functions $N_{m_{1}, i}(u ; T), N_{m_{2}, \ell}(w ; V)$ are known (in the intervals $\left[t_{j_{1}}, t_{j_{1}+1}\right)$ and $\left[v_{j_{2}}, v_{j_{2}+1}\right)$, respectiely), i.e.,

$$
N_{m_{1}, i}(u ; T)=\sum_{k=0}^{m_{1}} b_{k}^{\left(i, j_{1}\right)} B_{k}^{m_{1}}(t) \quad\left(u \in\left[t_{j_{1}}, t_{j_{1}+1}\right)\right)
$$

and

$$
N_{m_{2}, \ell}(w ; V)=\sum_{k=0}^{m_{2}} d_{k}^{\left(\ell, j_{2}\right)} B_{k}^{m_{2}}(v) \quad\left(w \in\left[v_{j_{2}}, v_{j_{2}+1}\right)\right)
$$

where

$$
t:=\frac{u-t_{j_{1}}}{t_{j_{1}+1}-t_{j_{1}}}, \quad v:=\frac{w-v_{j_{2}}}{v_{j_{2}+1}-v_{j_{2}}},
$$

one can convert the tensor product B-spline patch to a tensor product Bézier surface:

$$
\begin{equation*}
\mathrm{S}(u, w)=\sum_{k_{1}=0}^{m_{1}} \sum_{k_{2}=0}^{m_{2}} \mathrm{~V}_{k_{1}, k_{2}}^{\left(j_{1}, j_{2}\right)} B_{k_{1}}^{m_{1}}(t) B_{k_{2}}^{m_{2}}(v) . \tag{5.4}
\end{equation*}
$$

The points

$$
\mathrm{V}_{k_{1}, k_{2}}^{\left(j_{1}, j_{2}\right)}:=\sum_{i=j_{1}-m_{1}}^{j_{1}} \sum_{\ell=j_{2}-m_{2}}^{j_{2}-1} b_{k_{1}}^{\left(i, j_{1}\right)} d_{k_{2}}^{\left(\ell, j_{2}\right)} \mathrm{W}_{i \ell}
$$

are the control points of a Bézier patch for the domain $\left[t_{j_{1}}, t_{j_{1}+1}\right) \times\left[v_{j_{2}}, v_{j_{2}+1}\right)$ (cf. Theorem (2.3).

## 6. Generalizations

The approach presented in Section 4 can be generalized so that the inner knots may have their multiplicity higher than 1 or the boundary knots are of multiplicity lower than $m+1$.

### 6.1. Inner knots of any multiplicity

When an inner knot has multiplicity over 1 , some knot spans $\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-1)$ are empty. It is only necessary to find the B-spline functions' coefficients over the non-empty knot spans. If there are $n_{e}$ such knot spans, one only needs to find $n_{e}(m+1)^{2}$ coefficients, and the algorithm will have $O\left(n_{e} m^{2}\right)$ complexity. To use the continuity condition, the following definition will be useful.

Definition 6.1. The left neighbor of a given $k n o t t_{k}$ is the knot $t_{\ell}$ if $\ell$ is the largest natural number such that $t_{\ell}<t_{k}$, i.e., $\left[t_{\ell}, t_{\ell+1}\right)$ is non-empty and $t_{\ell+1}=t_{k}$.

The right neighbor of a given knot $t_{k}$ is the knot $t_{r}$ if $r$ is the smallest natural number such that $t_{k}<t_{r}$, i.e., $\left[t_{r-1}, t_{r}\right)$ is non-empty and $t_{k}=t_{r-1}$.

Note that in the case considered in Section 4, the right neighbor of $t_{j}(j=0,1, \ldots, n-1)$ is always $t_{j+1}$.

From Remark 2.4, it follows that each B-spline function is continuous in $\left(t_{0}, t_{n}\right)$. The only modification then is in the continuity condition in Eq. (4.11). Let us consider a non-empty knot span $\left[t_{j}, t_{j+1}\right)(j=0,1, \ldots, n-2)$. Let $t_{r}$ be the right neighbor of $t_{j+1}$, i.e., $t_{r-1}=t_{j+1}$. In this case, the continuity property at $t_{j+1}$ is

$$
\sum_{k=0}^{m} b_{k}^{(i, j)} B_{k}^{m}\left(\frac{t_{j+1}-t_{j}}{t_{j+1}-t_{j}}\right)=\sum_{k=0}^{m} b_{k}^{(i, r-1)} B_{k}^{m}\left(\frac{t_{j+1}-t_{r-1}}{t_{r}-t_{r-1}}\right),
$$

which simplifies to $b_{m}^{(i, j)}=b_{0}^{(i, r-1)}$ (cf. Eq. (4.10)). In such case, the recurrence relation (4.11) takes the form

$$
\left\{\begin{array}{l}
b_{m}^{(i, j)}=b_{0}^{(i, r-1)}, \\
b_{k}^{(i, j)}=\frac{t_{j}-t_{i}}{t_{j+1}-t_{i}} b_{k+1}^{(i, j)}+\frac{v_{i}}{t_{j+1}-t_{i}}\left(\left(t_{j+1}-t_{m+i+2}\right) b_{k}^{(i+1, j)}+\left(t_{m+i+2}-t_{j}\right) b_{k+1}^{(i+1, j)}\right) \\
(k=m-1, m-2, \ldots, 0)
\end{array}\right.
$$

(cf. Eq. (4.5)), where $t_{r}$ is the right neighbor of $t_{j+1}$, and $j=n-2, n-3, \ldots, 0, i=$ $j-1, j-2, \ldots, j-m+1$. It is thus enough to substitute line 15 of Algorithm 4.1 with

$$
B[j, i, m] \leftarrow B[r-1, i, 0]
$$

and to skip the iterations of loops over $j$ in lines 3 and 12 if $t_{j}=t_{j+1}$. Example 6.2 presents this approach.
Example 6.2. Let us set $m:=3, n:=5$. Let the knots be

| $t_{-3}$ | $t_{-2}$ | $t_{-1}$ | $t_{0}$ | $t_{1}$ | $t_{2}$ | $t_{3}$ | $t_{4}$ | $t_{5}$ | $t_{6}$ | $t_{7}$ | $t_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 3 | 3 | 5 | 9 | 10 | 10 | 10 | 10 |.

The knot $t_{1}$ is of multiplicity 2. To compute the adjusted Bernstein-Bézier coefficients of the $B$-spline functions over $\left[t_{0}, t_{1}\right)$ a continuity condition with the knot span $\left[t_{2}, t_{3}\right)$ is used, as $t_{1}=t_{2}$. Figure 6.1 illustrates this approach to computing all necessary coefficients, analogous to Example 4.5 .

### 6.2. Boundary knots of multiplicity lower than $m+1$

First, note that in Section 4, only the assumption that $t_{n}=t_{n+m}$ is used, therefore if that condition holds, Theorem 4.4 and Algorithm 4.1 still apply, regardless of the multiplicity of boundary knots $t_{-m}, t_{-m+1}, \ldots, t_{0}$.

If the boundary knot $t_{n}$ has multiplicity lower than $m+1$ (i.e., the knot sequence is unclamped), the problem can be reduced so that it can be solved using Theorem 4.4. Its drawback, however, is higher complexity.


Figure 6.1: An illustration of Example 6.2

The idea is to inflate the multiplicity of $t_{n+m}$ up to $m+1$. More precisely, let $t_{n+m-\ell-1}<$ $t_{n+m-\ell}=t_{n+m}$, i.e., $t_{n+m-\ell}$ has multiplicity $\ell+1$. Let the $m-\ell$ new knots $t_{n+m+1}=t_{n+m+2}=$ $\ldots=t_{n+2 m-\ell}$ be defined so that $t_{n+m}=t_{n+m+1}$. This allows to execute Algorithm 4.1] with the new arguments $n_{1}:=n+m-\ell, m_{1}:=m$ and the inflated knot sequence

$$
\underbrace{t_{-m} \leq \ldots \leq t_{-1} \leq t_{0}}_{\text {boundary knots }} \leq \underbrace{t_{1}<\ldots<t_{n+m-\ell-1}}_{\text {inner knots }}<\underbrace{t_{n+m-\ell}=t_{n+m-\ell+1}=\ldots=t_{n+2 m-\ell}}_{\text {boundary knots }} .
$$

It remains then to return the coefficients of $N_{m i}$ over $\left[t_{j}, t_{j+1}\right)$ for $j=0,1, \ldots, n-1$ and $i=j-m, j-m+1, \ldots, j$. This approach requires the computation of $O\left((n+m-\ell) m^{2}\right)$ coefficients and is presented in Example 6.3.

Example 6.3. Let us set $m:=3, n:=2$. Let the knots be

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
t_{-3} & t_{-2} & t_{-1} & t_{0} & t_{1} & t_{2} & t_{3} & t_{4} & t_{5} \\
\hline-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5
\end{array} .
$$

After adding the knots $t_{6}=t_{7}=t_{8}$ such that $t_{5}=t_{8}$ (thus increasing $n$ by 3 ), the problem takes the form

| $t_{-3}$ | $t_{-2}$ | $t_{-1}$ | $t_{0}$ | $t_{1}$ | $t_{2}$ | $t_{3}$ | $t_{4}$ | $t_{5}$ | $t_{6}$ | $t_{7}$ | $t_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 |.

Figure 6.2 illustrates the application of Algorithm 4.1 (cf. Example 4.5) computing all the adjusted Bernstein-Bézier coefficients of the inflated problem. The coefficients which are relevant to the solution of the primary problem are in the frame drawn in bold.


Figure 6.2: An illustration of Example 6.3

## 7. Special case: uniform knots

In the case of uniform knots, i.e.,

$$
t_{i}:=t_{0}+i \cdot h \quad(h>0, i=-1,-m+1, \ldots, n+n),
$$

one can check that

$$
N_{m i}\left(\left[t_{j}, t_{j+1}\right)\right) \equiv N_{m, i+1}\left(\left[t_{j+1}, t_{j+2}\right)\right) .
$$

This means that, to solve Problem [2.1, it is enough to find the coefficients of B-spline basis functions over one knot span.

Let us set $j \in \mathbb{N}$ such that $0 \leq j \leq n-1$. Equation (4.2) takes the form

$$
\left\{\begin{array}{l}
b_{k}^{(j, j)}=0  \tag{7.1}\\
b_{m}^{(j, j)}=\frac{1}{m!}
\end{array} \quad(k=0,1, \ldots, m-1),\right.
$$

while Eq. (4.3) simplifies to

$$
\left\{\begin{array}{l}
b_{0}^{(j-m, j)}=\frac{1}{m!},  \tag{7.2}\\
b_{k}^{(j-m, j)}=0
\end{array} \quad(k=1,2, \ldots, m) .\right.
$$

In order to find the coefficients of the basis functions $N_{m i}$ for $i=j-m+1, j-m+$ $2, \ldots, j-1$, one can use Theorem 4.3, which gives $m$ equations of the form
$(j+1-i) b_{k}^{(i, j)}+(i-j) b_{k+1}^{(i, j)}=(j-m-i-1) b_{k}^{(i+1, j)}+(m+i+2-j) b_{k+1}^{(i+1, j)} \quad(k=0,1, \ldots, m-1)$.

To complete the recurrence scheme, it is enough to use the continuity condition

$$
N_{m, i+1}\left(t_{j+1}^{-}\right)=N_{m, i+1}\left(t_{j+1}^{+}\right),
$$

which, due to the knot uniformity, can be formulated as

$$
N_{m, i+1}\left(t_{j+1}^{-}\right)=N_{m i}\left(t_{j}^{+}\right),
$$

which, eventually, gives

$$
b_{0}^{(i+1, j)}=b_{m}^{(i, j)} .
$$

The complete recurrence scheme is thus, for $i=j-1, j-2, \ldots, j-m+1$,

$$
\left\{\begin{array}{l}
b_{m}^{(i, j)}=b_{0}^{(i+1, j)},  \tag{7.4}\\
b_{k}^{(i, j)}=\frac{j-i}{j-i+1} b_{k+1}^{(i, j)}+\frac{j-i-m-1}{j-i+1} b_{k}^{(i+1, j)}+\frac{m+i+2-j}{j-i+1} b_{k+1}^{(i+1, j)} \\
(k=m-1, m-2, \ldots, 0) .
\end{array}\right.
$$

Note that both the coefficients of B-spline basis functions given in Eqs. (7.1) and (7.2), as well as the coefficients in the recurrence scheme (7.4) are rational. This means that all the coefficients of the B-spline basis functions are thus rational and can be computed without errors. This approach requires $O\left(m^{2}\right)$ operations, which is optimal.

## 8. Conclusion

We have discovered a new differential-recurrence relation satisfied by B-spline functions of the same degree. The relation is a foundation of a new asymptotically optimal method of finding the Bernstein-Bézier coefficents of all B-spline basis functions in the clamped case over all knot spans. The algorithm can be generalized for different knot multiplicities, including the unclamped case. The new method allows to accelerate the computations of B-spline basis functions, which leads to faster evaluation of B-spline curves and surfaces. Numerical experiments show that the algorithm is stable. Further research is required to find the application of the new differential-recurrence relation in finding the Bernstein-Bézier coefficients of B-spline basis functions over a single knot span.

## References

[1] C. V. Beccari, G. Casciola, Matrix representations for multi-degree B-splines, Journal of Computational and Applied Mathematics 381 (2021) 113007.
[2] C. V. Beccari, G. Casciola, Stable numerical evaluation of multi-degree B-splines, Journal of Computational and Applied Mathematics 400 (2022) 113743.
[3] P. Bézier, Définition numérique des courbes et surfaces I (in French), Automatisme XI (1966) 625-632.
[4] P. Bézier, Définition numérique des courbes et surfaces II (in French), Automatisme XII (1967) 17-21.
[5] P. Bézier, Procédé de définition numérique des courbes et surfaces non mathématiques (in French), Automatisme XIII (1968) 189-196.
[6] W. Boehm, Inserting new knots into B-spline curves, Computer-Aided Design 12 (4) (1980) 199-201.
[7] W. Boehm, A. Müller, On de Casteljau's algorithm, Computer Aided Geometric Design 16 (1999) 587-605.
[8] W. Böhm, Über die Konstruktion von B-Spline-Kurven (in German), Computing 18 (1977) 161-166.
[9] F. Chudy, New algorithms for Bernstein polynomials, their dual bases, and B-spline functions, Ph.D. thesis, University of Wrocław, available on request (2022).
[10] E. Cohen, T. Lyche, R. Riesenfeld, Discrete B-splines and subdivision techniques in computer-aided geometric design and computer graphics, Computer Graphics and Image Processing 14 (2) (1980) 87-111.
[11] M. G. Cox, The numerical evaluation of B-splines, IMA Journal of Applied Mathematics 10 (2) (1972) 134-149.
[12] G. Dahlquist, Å. Björck, Numerical methods in scientific computing. Vol. I, SIAM, 2008.
[13] K. R. Davidson, A. P. Donsig, Real Analysis with Real Applications, Prentice Hall, Inc., Upper Saddle River, 2002.
[14] C. de Boor, On calculating with B-splines, Journal of Approximation Theory 6 (1) (1972) 50-62.
[15] P. de Casteljau, Courbes et surfaces à pôles (in French), Tech. rep., André Citroën Automobile SA (1959).
[16] P. de Casteljau, Outillage méthodes calcul (in French), Tech. rep., André Citroën Automobile SA (1959).
[17] P. de Casteljau, De Casteljau's autobiography: My time at Citroën, Computer Aided Geometric Design 16 (1999) 583-586.
[18] P. Dierckx, Curve and Surface Fitting with Splines, Clarendon Press, 1993.
[19] G. Farin, Curves and surfaces for Computer-Aided Geometric Design. A practical guide, 5th ed., Academic Press, Boston, 2002.
[20] R. T. Farouki, The Bernstein polynomial basis: A centennial retrospective, Computer Aided Geometric Design 29 (2012) 379-419.
[21] R. Goldman, Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling, Morgan Kaufmann, 2002.
[22] S. Lewanowicz, P. Woźny, Bézier representation of the constrained dual Bernstein polynomials, Applied Mathematics and Computation 218 (2011) 4580-4586.
[23] L. Liu, G. Wang, Explicit matrix representation for NURBS curves and surfaces, Computer Aided Geometric Design 19 (6) (2002) 409-419.
[24] L. A. Piegl, W. Tiller, The NURBS Book, Springer, 1996.
[25] H. Prautzsch, W. Boehm, M. Paluszny, Bézier and B-Spline Techniques, Springer, 2002.
[26] P. Sablonniére, Spline and Bézier polygons associated with a polynomial spline curve, Computer-Aided Design 10 (1978) 257-261.
[27] D. Toshniwal, H. Speelers, H. H. Hiemstra, C. Manni, T. J. R. Hughes, Multi-degree B-splines: Algorithmic computation and properties, Computer Aided Geometric Design 76 (2020) 101792.
[28] P. Woźny, Construction of dual B-spline functions, Journal of Computational and Applied Mathematics 260 (2014) 301-311.
[29] P. Woźny, F. Chudy, Linear-time geometric algorithm for evaluating Bézier curves, Computer Aided-Design 118 (2020) 102760.


[^0]:    * Corresponding author.

    Email addresses: Filip.Chudy@cs.uni.wroc.pl (Filip Chudy), Pawel.Wozny@cs.uni.wroc.pl (Paweł Woźny)

