

# Topology of 2D and 3D Rational Curves

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## Abstract

In this paper we present algorithms for computing the topology of planar and space rational curves defined by a parametrization. The algorithms given here work directly with the parametrization of the curve, and do not require to compute or use the implicit equation of the curve (in the case of planar curves) or of any projection (in the case of space curves). Moreover, these algorithms have been implemented in Maple; the examples considered and the timings obtained show good performance skills.

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## 1 Introduction

The topology of planar algebraic curves, implicitly given, is a well-studied problem (see (1), (10), (11), (12), (13), and the more recent works (8), (17), among others); more recently, the problem for space algebraic curves has also received certain attention (see (2), (7), (9)). In all these works it is assumed that the curve is given by means of implicit equations, and the considered algorithms deal with the curves in this form. However, in this paper we address the problem, apparently not discussed up to now, of computing the topology of a rational curve (i.e. constructing a planar or space graph describing the shape of the curve) starting directly from its parametrization, without computing or making use of the implicit equation of the curve. This question may be of special interest in the field of computer-aided geometric design (CAGD), where many of the curves used are rational and even directly provided in parametric form (e.g. Bezier curves, B-splines, NURBS).

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<sup>1</sup> Supported by the Spanish “Ministerio de Ciencia e Innovación” under the Project MTM2008-04699-C03-01

Perhaps the reason for the absence of previous studies in this direction is the common belief that if the parametric equations of a curve are available, the curve is easy to visualize. This is essentially true, but if the goal is to get a global idea of how the curve is like, then there are still some difficulties. On the one hand, one should previously compute a parameter interval such that the plotting of the curve over the interval shows the main features of the curve; this includes handling the case of possible missing points/branches (which happens if some point of the curve is generated when the parameter of the curve tends to infinity, see for example (4)). On the other hand, the plotting, as pointed out by Gonzalez-Vega and Necula in the introduction to (12), may not always provide a clear idea of the topology of the curve, and hence auxiliary tools for describing the shape of the curve may be of help.

In this paper we address both planar and space rational curves. As in other topology algorithms, we require the input curves to satisfy certain conditions that can be achieved with generality. In the planar case it is required that the curve has neither vertical asymptotes nor vertical components, and that the parametrization is *proper* (see Section 2). Initially, the algorithm works in a similar way to existing algorithms, i.e. first one computes the critical points and the points of the curve lying on the lines  $x = \alpha_i$  containing some critical point, and then one appropriately connects these points. However, the connection phase is carried out not in the usual way, but taking advantage of the fact that a parametrization is available (see Theorem 8 in Section 3). More precisely, the algorithm computes how the *parameter* values are connected; so, two points are joined whenever the algorithm detects that the parameter values giving rise to them need to be connected. In particular, and unlike many classical algorithms, this strategy does not require the curves to be in generic position (as defined in (12)).

The method is specially profitable in the case of space rational curves. Existing implicit algorithms compute the topology of the curve by projecting it onto a plane (the  $xy$ -plane, in our case), and then lifting to space the topology of this projection. In (2), this lifting phase is carried out in general by using a second, auxiliary projection; however, in (7), (9) no auxiliary projection is needed. In any case, the lifting of the singularities of the projection is a delicate operation. In our case, we use a similar strategy for 3D curves. However, here the lifting operation (which is performed without auxiliary projections) presents no difficulties since the space points are identified by the parameter values giving rise to them (previously computed when addressing the projection). In the case of 3D curves, our requirements are: (i) the curve has no asymptotes or components normal to the  $xy$ -plane; (ii) the projection onto the  $xy$ -plane fulfills the requirements of the 2D algorithm.

We have implemented the algorithms in Maple 13; outputs and timings of several examples are given in Section 5. In our implementation we give to the

user the option of computing isolated points of the curve or not. The reason for this is that isolated points correspond to points generated by complex, non-real, values of the parameter, and therefore they may not be of interest for certain users; moreover, the number of isolated points is certified by means of Hermite's method (see (6)) and therefore it may be time-consuming.

The structure of the paper is the following. In Section 2 we provide the necessary background on rational curves; hence, notions like properness, normality, critical and singular points are reviewed here, jointly with related results. In Section 3 we provide the algorithm for the 2D case. In Section 4, the algorithm for the 3D case is given. Finally, in Section 5 we describe some details of the implementation, and we provide the outputs and timings of different examples in 2D and 3D. The parametrizations used in the examples are given in Appendix I and Appendix II.

## 2 Background on Rational Curves

In this section we briefly recall the background on affine rational curves that we need in order to develop our results. So, in the sequel we will consider an affine rational curve  $\mathcal{C}$  defined by a rational parametrization

$$\varphi(t) = (x_1(t), x_2(t), \dots, x_n(t)) = \left( \frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}, \dots, \frac{p_n(t)}{q_n(t)} \right)$$

where  $\gcd(p_1, q_1) = \gcd(p_2, q_2) = \dots = \gcd(p_n, q_n) = 1$  and  $p_i(t), q_i(t) \in \mathbb{Z}[t]$  for all  $i = 1, \dots, n$ . In our case  $n = 2$  or  $n = 3$ ; so, we will usually write  $x, y, z$  instead of  $x_1, x_2, x_3$ . Moreover, since the parametrization is assumed to be real, we have that  $\mathcal{C}$  is a real curve (i.e. that it consists of infinitely many real points), although for theoretical reasons when necessary we will see the curve embedded in  $\mathbb{C}^n$ . Nevertheless, our goal will always be the description of the shape of its real part.

A point  $P_0 \in \mathbb{R}^n$  is *reached* by the parametrization  $\varphi(t)$  if there exists  $t_0 \in \mathbb{C}$  such that  $\varphi(t_0) = P_0$ ; in this case, we will also say that  $t_0$  generates  $P_0$ . Notice that the value of the parameter generating a real point may be either real or complex, and that there may be points generated by several (real or complex) values of the parameter. In this sense, we will say that the parametrization  $\varphi(t)$  is **proper** if almost all points of  $\mathcal{C}$  are reached by just one value of the parameter  $t$ , i.e. if  $\varphi(t)$  is injective for almost all the points of  $\mathcal{C}$ . So, if  $\varphi(t)$  is proper then there are just finitely many points of  $\mathcal{C}$  generated by several different values of the parameter, corresponding to the *self-intersections* of the curve. In order to check whether  $\varphi(t)$  is proper, we will use the following

criterion. Let

$$\begin{aligned}\tilde{G}_1(t, s) &= p_1(t)q_1(s) - p_1(s)q_1(t) \\ \tilde{G}_2(t, s) &= p_2(t)q_2(s) - p_2(s)q_2(t) \\ &\vdots \\ \tilde{G}_n(t, s) &= p_n(t)q_n(s) - p_n(s)q_n(t) \\ \tilde{G}(t, s) &= \gcd(\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n)\end{aligned}$$

Then, the following theorem, directly deducible from Proposition 7 in (15) (see also Theorem 4.30 in (19), for the planar case), holds.

**Theorem 1** *The parametrization  $\varphi(t)$  is proper iff  $\tilde{G}(t, s) = t - s$ .*

On the other hand, we will say that  $\varphi(t)$  is **normal** if every point in  $\mathcal{C}$  is reached by at least one value of the parameter, i.e. if  $\varphi(\mathbb{C}) = \mathcal{C}$ . If  $\varphi(t)$  is not normal, then (see Proposition 4.2 in (4)) there is just one point of  $\mathcal{C}$  non-reached by the parametrization, namely the point

$$P_\infty = \lim_{t \rightarrow \pm\infty} \varphi(t)$$

Notice that  $P_\infty$  exists if and only if  $\deg(p_i) \leq \deg(q_i)$  for all  $i \in \{1, 2, \dots, n\}$ . Furthermore, if  $P_\infty$  exists, it may still be reached by some (real or complex) value of the parameter. If we denote  $P_\infty = (a_1, a_2, \dots, a_n)$ ,  $P_\infty$  is reached iff

$$\deg(\gcd(a_1q_1(t) - p_1(t), a_2q_2(t) - p_2(t), \dots, a_nq_n(t) - p_n(t))) \geq 1$$

Also, observe that if  $P_\infty$  exists then it is obtained as the limit of a sequence of real points of  $\mathcal{C}$ , and therefore it cannot be isolated. Hence, if  $P_\infty$  is reached by some value  $t_a$  then it is a self-intersection of the curve, because it is a crossing of two branches of the curve, one corresponding to  $t \rightarrow \pm\infty$  and the other corresponding to  $t_a$ . On the other hand, if  $P_\infty$  exists but it is not reached, one can reparametrize the curve so that it is reached (see Theorem 7.30 in (19)). However, reparametrizations may complicate the equations of the curve, or bring other difficulties, like impropersness or the introduction of algebraic numbers. Hence, in our case whenever we meet this phenomenon, we will understand that this reparametrization has not been performed.

If every point of  $\mathcal{C}$  is reachable via  $\varphi(t)$  by real values of the parameter one says that  $\varphi(t)$  is  **$\mathbb{R}$ -normal**. We refer to (4), (19) for a thorough study of this phenomenon. If  $\varphi(t)$  is not  **$\mathbb{R}$ -normal**, then there exist real points  $P \in \mathcal{C}$  reachable only by complex values of the parameter. Moreover, the following result (see Proposition 4.2 in (4)) clarifies the nature of these points.

**Proposition 2** *Let  $\varphi(t)$  be a proper parametrization of  $\mathcal{C}$ . Then  $P \neq P_\infty$ ,  $P \in \mathcal{C} \cap \mathbb{R}^n$  is non-reached by any real value of the parameter if and only if it is a real isolated point of  $\mathcal{C}$ .*

## 2.1 Critical Points of Planar Rational Curves

In the rest of the section we assume that  $n = 2$ , i.e. that  $\mathcal{C} \subset \mathbb{R}^2$  is a real rational curve parametrized by  $\varphi(t) = (x(t), y(t))$ . Let  $f \in \mathbb{R}[x, y]$  be its implicit equation; then we have the following classical definitions:

**Definition 3** *A point  $P \in \mathcal{C}$  is called: (a) a critical point if  $f(P) = \frac{\partial f}{\partial y}(P) = 0$ ; (b) a singular point, if it is critical and  $\frac{\partial f}{\partial x}(P) = 0$ ; (c) a ramification point if it is critical, but non-singular; (d) a regular point if it is not critical.*

One may easily see that ramification points correspond to those points satisfying that  $x'(t) = 0$  but  $y'(t) \neq 0$ , and that singular points correspond to either points where  $x'(t) = y'(t) = 0$ , or to self-intersections of the curve. Singularities of a rational parametrization can be computed directly from the parametrization, without converting to implicit form. More precisely, the following result holds (see Theorem 10 and Theorem 11 in (14)). Here, we denote  $G_1 = \tilde{G}_1/\tilde{G}$ ,  $G_2 = \tilde{G}_2/\tilde{G}$ , and we write  $M(t) = \text{Res}_s(G_1, G_2)$ .

**Theorem 4** *Let  $\varphi(t)$  be a parametrization of  $\mathcal{C}$ , and let  $P_0 \in \mathcal{C}$  be an affine singularity of  $\mathcal{C}$ , reachable by some value  $t_0 \in \mathbb{C}$  of the parameter. Then,  $M(t_0) = 0$ .*

**Remark 1** *If  $P_\infty$  is reached by some  $t_0 \in \mathbb{C}$  (in that case it is a self-intersection of the curve, and therefore a singularity, as we observed before), then  $t_0$  must be a root of  $M(t)$  (see Theorem 10 in (14)).*

Whenever  $\varphi(t)$  is proper, one may deduce that  $M(t)$  is not identically 0; therefore, in that situation  $M(t)$  has finitely many roots and from Theorem 4, the  $t$ -values generating reachable singularities are among these roots. Now let us denote the numerator of  $x'(t)$  by  $N(t)$ , and let us write the square-free part of  $M(t) \cdot N(t)$  as  $\tilde{m}(t)$ ; also, let  $\tilde{q}(t) = \text{lcm}(q_1, q_2)$ , and let  $m(t) = \tilde{m}(t)/\text{gcd}(\tilde{m}(t), \tilde{q}(t))$ . Then, the following corollary on the real critical points of  $\mathcal{C}$  can be deduced.

**Corollary 5** *The real critical points of  $\mathcal{C}$  are included in the (finite) set consisting of: (i)  $P_\infty$  (if it exists); (ii) the real points generated by (real or complex) roots of  $m(t)$ .*

### 3 Computation of the Graph Associated with a Planar Curve

Let  $\mathcal{C} \subset \mathbb{R}^2$  be a planar algebraic curve, parametrized by

$$\varphi(t) = (x(t), y(t)) = \left( \frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right), \quad \gcd(p_1(t), q_1(t)) = \gcd(p_2(t), q_2(t)) = 1$$

In this section we address the problem of algorithmically computing a graph  $\mathcal{G}$  homeomorphic to the curve  $\mathcal{C}$ . In order to do so, we will follow the usual strategy widely used in the implicit case (see (8), (12), (13), (17)):

- (1) Compute the critical points of  $\mathcal{C}$  (see Definition 3 in Subsection 2.1). Let  $a_1 < \dots < a_m$  be the  $x$ -coordinates of the critical points of  $\mathcal{C}$ ; also, let  $a_0 = -\infty$ ,  $a_{m+1} = +\infty$ .
- (2) Compute the points of  $\mathcal{C}$  lying on the vertical lines  $x = a_i$ ,  $i = 1, \dots, m$  passing through the critical points; we will refer to these lines as **critical lines**.
- (3) For  $i = 1, \dots, m - 1$ , compute the points of  $\mathcal{C}$  lying on the vertical line  $x = (a_i + a_{i+1})/2$ ; similarly for  $x = a_1 - 1$ ,  $x = a_m + 1$ . We will refer to these lines as “non-critical” lines.
- (4) Connect, by means of segments, the points of  $\mathcal{C}$  lying on each non-critical line, with the points in the critical lines immediately on its right and on its left, respectively.

In our case, we will take advantage of the fact that a parametrization of the curve is available; this will be specially useful in order to carry out step (4). Moreover, in order to apply the method presented in this section, we need that certain hypotheses are fulfilled by  $\mathcal{C}$ . These hypotheses are introduced in Subsection 3.1. Then, in Subsection 3.2 and Subsection 3.3 we show how to compute the vertices and edges, respectively, of the planar graph. Finally, in the last subsection we provide the full algorithm. The reader may find several examples of the output of this algorithm in Section 5.

#### 3.1 Hypotheses

In the rest of the section, we assume that the following hypotheses are fulfilled:

- (i)  $\varphi(t)$  is proper.
- (ii)  $\mathcal{C}$  has no vertical asymptotes; in particular, it is not a vertical line.

The first hypothesis guarantees that  $\mathcal{C}$  is traced just once when following the parametrization  $\varphi(t)$ . In order to check this hypothesis, Theorem 1 can be applied. Moreover, if this hypothesis does not hold, one can always reparametrize

the curve (see Chapter 6.1 in (19)) so that it is fulfilled. In order to check the second hypothesis, one can use the following result, which is easy to prove.

**Lemma 6**  $\mathcal{C}$  has a vertical asymptote iff one of the following conditions occurs: (a)  $q_2(t)$  has some real root which is not a real root of  $q_1(t)$ ; (b)  $\deg(p_2) > \deg(q_2)$  but  $\deg(p_1) \leq \deg(q_1)$ .

If  $\mathcal{C}$  has some vertical asymptote, one proceeds in the following way:

- If  $\mathcal{C}$  has no horizontal asymptotes (which can be checked by appropriately adapting Lemma 6), then by interchanging the axes  $x$  and  $y$  the condition is fulfilled. Notice that this is an affine transformation, which therefore does not change the topology of the curve.
- If  $\mathcal{C}$  has also horizontal asymptotes, then almost all changes of coordinates of the type  $\{x = X + \mu Y, y = Y\}$ , with  $\mu \in \mathbb{Q}$ , set the curve properly (see Proposition 3.2 in (11)). Observe that if  $\varphi(t)$  is proper, the curve  $\mathcal{C}_\mu$  obtained by applying such a transformation is properly parametrized by  $\varphi_\mu(t) = (x(t) - \mu y(t), y(t))$ .

### 3.2 Vertices of the Graph.

The notable points of  $\mathcal{C}$  are the real critical points. Now from Corollary 5, we have that these are among the following points:

- $P_\infty$  (if it exists).
- The points of  $\mathcal{C}$  generated (via  $\varphi(t)$ ) by the real roots of the polynomial  $m(t)$  in Corollary 5.
- The real points of  $\mathcal{C}$  generated (via  $\varphi(t)$ ) by complex roots of  $m(t)$ .

The computation of  $P_\infty$  is described in Section 2. Moreover, once the real roots of  $m(t)$  are computed, the points in (ii) are obtained by evaluating  $x(t), y(t)$  at these roots. Now we consider as vertices of the graph  $\mathcal{G}$  not only these points, but also the points of  $\mathcal{C}$  lying on the vertical lines containing the points in (i) and (ii). In order to compute these points, we recall the definition of the polynomials  $\tilde{G}_1, \tilde{G}_2, \tilde{G}$ , introduced in Section 2, and we consider the polynomials

$$G_1(t, s) = \frac{\tilde{G}_1(t, s)}{\tilde{G}(t, s)}, \quad G_2(t, s) = \frac{\tilde{G}_2(t, s)}{\tilde{G}(t, s)}.$$

Then, given a point  $P_r = (x_r, y_r) = \varphi(t_r)$ ,  $t_r \in \mathbb{R}$ , the real roots of  $G_1(t, t_r)$  provide the  $t$ -values of the points lying in the line  $x = x_r$ ; then, the coordinates of those points can be obtained by evaluating  $x(t), y(t)$  at these  $t$ -values. Observe that we get not only the coordinates, but also the  $t$ -values generating the points, via  $\varphi(t)$ . This is important for the connection phase.

So, let us consider the points in (iii). If a point in (iii) is also generated by a real value of the parameter, then it will have already been computed as a point in (ii). So, if this is not the case, by Proposition 2 it is an isolated point. Now these points might be computed by seeking complex roots of  $m(t)$  giving rise (when evaluating  $x(t), y(t)$ ) to real points of  $\mathcal{C}$ . However, in the sequel we will provide an alternative way for carrying out this computation, that allows to certify the existence or non-existence of this kind of points. For this purpose, we denote a complex value of the parameter  $t = u + iv$ , where  $i^2 = -1$  and  $u, v \in \mathbb{R}$ , and we represent the complex modulus as  $|\cdot|$ . Also, we write

$$\frac{p_1(u + iv) \cdot \overline{q_1(u + iv)}}{|q_1|^2} = \frac{1}{|q_1|^2} \cdot (a(u, v) + ib(u, v))$$

and

$$\frac{p_2(u + iv) \cdot \overline{q_2(u + iv)}}{|q_2|^2} = \frac{1}{|q_2|^2} \cdot (c(u, v) + id(u, v))$$

Then the following result, that can be easily verified, holds. Here, we denote the result of substituting  $t = u + iv$  in  $\tilde{q}(t) = \text{lcm}(q_1, q_2)$ , as  $\tilde{q}(u, v)$ .

**Lemma 7** *Let  $P_0 \in \mathcal{C} \cap \mathbb{R}^2$ . Then,  $P_0$  is generated by a complex value of the parameter  $t_0 = u_0 + iv_0$  if and only if there exists  $w_0 \in \mathbb{R}$  satisfying that  $(u_0, v_0, w_0)$  is a real solution of the system*

$$\left\{ \begin{array}{l} b(u, v) = 0 \\ d(u, v) = 0 \\ v \cdot |\tilde{q}(u, v)|^2 \cdot w - 1 = 0 \end{array} \right. \quad (1)$$

In order to certify the number of real solutions of System (1) we apply Hermite's method (see for example (6)). However, these solutions include the complex values of the parameter generating real points that are also reached by real values of the parameter. In order to identify the existence of those solutions, we compute, also by Hermite's method, the number of real solutions of the system obtained by adding the following equations to System (1):

$$\left\{ \begin{array}{l} x(t) = \frac{a(u, v)}{|q_1(u, v)|^2} \\ y(t) = \frac{b(u, v)}{|q_2(u, v)|^2} \\ v \cdot \tilde{q}(t) \cdot w - 1 = 0 \end{array} \right. \quad (2)$$

So, real isolated points of  $\mathcal{C}$  correspond to solutions of System (1) which are not solutions of System (2).

### 3.3 Edges of the Graph.

In this section, we address the problem of connecting the vertices of  $\mathcal{G}$  (to compute the edges of the graph). For this purpose, the idea is to introduce between two consecutive critical lines an intermediate “non-critical” line, and to connect the points of  $\mathcal{C}$  on each “non-critical” line with the points of  $\mathcal{C}$  on the critical line immediately on its right or on its left. In order to do this, we take advantage of the fact that a parametrization of the curve is available, and we connect the points just by comparing the parameters generating the points in the two vertical lines (one of them critical, and the other one “non-critical”). The idea is made precise in the following theorem. Here, we will consider  $P_\infty$  as “generated” by both  $+\infty$  and  $-\infty$ , besides other real values that may also generate it; as usual,  $-\infty$  (resp.  $+\infty$ ) is considered less (resp. greater) than any other real number compared with it, and  $-\infty < +\infty$ . This result is illustrated by Figure 1.

**Theorem 8** *Let  $x_a, x_b \in \mathbb{R}$  satisfying that: (i)  $x_a < x_b$  (resp.  $x_a > x_b$ ); (ii) there is no critical line  $x = x_c$  such that  $x_a \leq x_c < x_b$  (resp.  $x_a \geq x_c > x_b$ ). Also, let  $P_a$  be a real point of  $\mathcal{C}$  lying on the line  $x = x_a$ , generated by  $t_a \in \mathbb{R}$ , and let  $\mathcal{V}_b = \{t_{b,1}, \dots, t_{b,n_b}\}$  (including  $-\infty, +\infty$ , if  $P_\infty$  belongs to the line  $x = x_b$ ) be the set of real values generating the real points of  $\mathcal{C} \cap \{x = x_b\}$ . The following statements are true:*

- (1) *If  $x'(t_a) > 0$ , then  $P_a$  must be connected with the point  $P_b$  of  $\mathcal{C} \cap \{x = x_b\}$  generated by the least (resp. greatest) element of  $\mathcal{V}_b$  which is greater (resp. less) than  $t_a$ .*
- (2) *If  $x'(t_a) < 0$ , then  $P_a$  must be connected with the point  $P_b$  of  $\mathcal{C} \cap \{x = x_b\}$  generated by the greatest (resp. least) element of  $\mathcal{V}_b$  which is less (resp. greater) than  $t_a$ .*

**Proof.** We prove (1) for the case when  $x_a < x_b$ ; the proofs of (1) for the case  $x_a > x_b$ , and of (2) in both cases, are analogous. Now let  $t_c \in \mathcal{V}_b$  be the least element of  $\mathcal{V}_b$  which is greater than  $t_a$ . Since by hypothesis  $\mathcal{C}$  has no vertical asymptotes, then  $P_a$  must be connected either with exactly one real point of  $\mathcal{C} \cap \{x = x_b\}$  generated by a real value of the parameter, or with  $P_\infty$ . Now, we distinguish two different cases, depending on whether  $P_\infty$  belongs to the line  $x = x_b$ , or not. We begin with the case when  $P_\infty$  does not belong to  $x = x_b$ . So,  $P_a$  is connected with a point of  $\mathcal{C} \cap \{x = x_b\}$  generated by some  $\tilde{t} \in \mathcal{V}_b$ . First of all, observe that  $\tilde{t} > t_a$ . Indeed, by hypothesis  $\mathcal{C}$  has no vertical asymptotes. Then,  $x(t)$  is defined for every  $t$  between  $t_a$  and  $\tilde{t}$ , and since  $x(t)$  is a quotient of polynomials,  $x'(t)$  is also differentiable there. Moreover,  $x'(t)$  cannot vanish between  $t_a$  and  $\tilde{t}$  because by hypothesis there does not exist any critical line between  $x = x_a$  and  $x = x_b$ . Hence, the sign of  $x'(t)$  is constant in the interval lying between  $t_a$  and  $\tilde{t}$ , and since  $x'(t_a) > 0$ , then  $x'(t) > 0$  in that interval;

therefore,  $x(t)$  is increasing there. So, since  $x_a = x(t_a) < x(\tilde{t})$  we deduce that  $t_a < \tilde{t}$ .

Now our aim is to prove that  $\tilde{t} = t_c$ . For this purpose, observe that  $\tilde{t} \geq t_c$  because  $t_c$  is the least element of  $\mathcal{V}_b$  greater than  $t_a$ ; hence, we just have to prove that  $\tilde{t} > t_c$  cannot occur. Assume by contradiction that  $\tilde{t} > t_c$ . Since  $x(t_c) = x_b = x(\tilde{t})$  and  $x(t)$  is differentiable along  $[t_a, \tilde{t}]$ , by Rolle's Theorem  $x'(t)$  must vanish at some point of  $(t_c, \tilde{t})$ . However, this is absurd because  $x'(t)$  is strictly positive in  $[t_a, \tilde{t}]$ , which contains  $(t_c, \tilde{t})$ .

Finally, let us consider the case when  $P_\infty$  belongs to the line  $x = x_b$ . If there exists  $\hat{t} \in \mathcal{V}_b$ ,  $\hat{t} \neq +\infty$ , with  $\hat{t} > t_a$ , then  $P$  must be connected with  $\hat{P} = \varphi(\hat{t})$ , since otherwise by adapting the above argument one reaches a contradiction. On the other hand, if  $t_a$  is greater than every real element of  $\mathcal{V}_b$ , then  $P_a$  cannot be connected with any other point of  $\mathcal{C} \cap \{x = x_b\}$  but  $P_\infty$ ; however, since we consider  $P_\infty$  generated by  $+\infty$ , and  $t_a < \infty$ , the rule also holds in this case.  $\square$

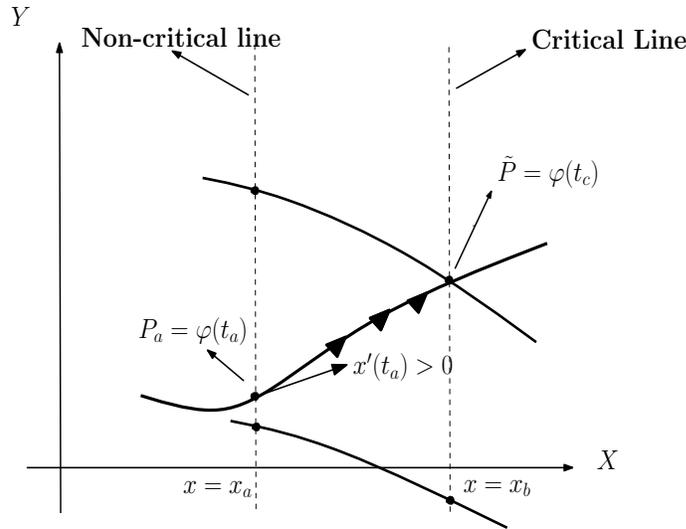


Fig. 1. Connecting Points

### 3.4 Full Algorithm

The following algorithm **Planar-Top** can be derived from the preceding subsections.

Planar-Top Algorithm:

**Input:** a planar curve  $\mathcal{C}$ , parametrized by

$$\varphi(t) = (x(t), y(t)) = \left( \frac{p_1(t)}{q_2(t)}, \frac{p_2(t)}{q_2(t)} \right),$$

fulfilling: (i)  $p_i(t), q_i(t) \in \mathbb{Z}[t]$  for  $i = 1, 2$ ,  $\gcd(p_i, q_i) = 1$  for  $i = 1, 2$ ; (ii)  $\varphi(t)$  is proper; (iii)  $\mathcal{C}$  has no vertical asymptotes.

**Output:** a planar graph  $\mathcal{G}$  homeomorphic to the curve.

(1) (Critical Points) Compute the polynomial  $m(t)$  in Corollary 5, and the real roots of  $m(t)$ . Then, compute:

(1.1) The critical points of  $\mathcal{C}$  (by evaluating  $x(t), y(t)$  at the real roots of  $m(t)$ ). Store these points in a list

$$\mathcal{L}_{crit} = [P_1, \dots, P_r],$$

For each of these points, store its coordinates, and the list of real  $t$ -values generating them.

(1.2) The point  $P_\infty = (x_\infty, y_\infty)$  (if it exists), and the list of  $t$ -values generating it.

(2) (Points of  $\mathcal{C}$  on Critical Lines)

(2.1) For  $\ell$  from 1 to  $r$ , compute the real points of  $\mathcal{C}$  lying on the line  $x = x_\ell$ . Store these points in a list

$$\mathcal{L}_\ell = [P_{\ell,1}, \dots, P_{\ell,j_\ell}]$$

For each of these points, store its coordinates, and the list of real  $t$ -values generating them.

(2.2) Check whether  $P_\infty$  belongs to some of the above lines  $x = x_\ell$ . In the affirmative case, go to (3); otherwise, compute the real points of  $\mathcal{C}$  lying on  $x = x_\infty$ . Store these points in a list

$$\mathcal{L}_\infty = [P_{\infty,1}, \dots, P_{\infty,m}]$$

For each of these points, store its coordinates, and the list of real  $t$ -values generating them.

(3) (Points of  $\mathcal{C}$  on Non-Critical lines)

(3.1) Let  $\mathcal{N} = \{a_1, \dots, a_s\}$ ,  $a_1 < \dots < a_s$ , be the set consisting of the  $x$ -coordinates of the critical points computed in (1.1) and (1.2). Also, let  $\bar{a}_0 = a_1 - 1$ ,  $\bar{a}_s = a_s + 1$ , and for  $i$  from 1 to  $s - 1$  let  $\bar{a}_i = (a_i + a_{i+1})/2$ .

(3.2) For  $j$  from 0 to  $s$ , compute the real points of  $\mathcal{C}$  lying on the line  $x = \bar{a}_j$ ; store these points in a list

$$\mathcal{N}_j = [\bar{P}_{j,1}, \dots, \bar{P}_{j,\alpha_j}]$$

For each of these points, store its coordinates, and the list of real  $t$ -values generating them.

- (4) (Edges)
  - (4.1) For  $i$  from 0 to  $s - 1$ , connect the points of  $\mathcal{C}$  lying on  $x = \bar{a}_i$  and  $x = a_{i+1}$  by applying Theorem 8.
  - (4.2) For  $i$  from 1 to  $s$ , connect the points of  $\mathcal{C}$  lying on  $x = \bar{a}_i$  and  $x = a_i$  by applying Theorem 8.
- (5) (Isolated vertices) Compute the real isolated points of the curve, and add them to the graph.

We will provide several examples of the output of the algorithm in Section 5.

## 4 Computation of the Graph Associated with a Space Curve

In this section we let  $\mathcal{C} \subset \mathbb{R}^3$  be a real curve, parametrized by

$$\varphi(t) = (x(t), y(t), z(t)) = \left( \frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}, \frac{p_3(t)}{q_3(t)} \right)$$

where  $\gcd(p_1, q_1) = \gcd(p_2, q_2) = \gcd(p_3, q_3) = 1$ . In the sequel, we consider the problem of algorithmically computing a graph  $\mathcal{G}$  homeomorphic to  $\mathcal{C}$ . In order to do that, we will follow the strategy used to address the implicit case in (2), (7), (9); more precisely, we need to perform the following steps:

- (1) Project the curve onto a coordinate plane (the  $xy$ -plane, in our case)
- (2) Compute the graph  $\bar{\mathcal{G}}$  associated with the projection (by using the algorithm given in Section 3)
- (3) Lift the graph  $\bar{\mathcal{G}}$  of the projection, to get  $\mathcal{G}$ .

As in (7) and (9), here we will require just one projection in order to perform the lifting phase. Now in the following subsections we first describe the hypotheses that we request on the input curve (essentially, that it is properly parametrized, and that it is “correctly placed” in space); then, we present the ideas and results needed for computing the vertices and edges of the graph, and finally we provide the full algorithm.

### 4.1 Hypotheses

Since  $\mathcal{C}$  is rational, if it is not a line parallel to the  $z$ -axis, then its projection onto the  $xy$  plane, denoted as  $\pi_{xy}(\mathcal{C})$ , is an algebraic rational curve and can be parametrized by

$$\psi(t) = (x(t), y(t)) = \left( \frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right)$$

Thus, in the sequel we assume that the following hypotheses hold:

- (i)  $\mathcal{C}$  has no asymptotes parallel to the  $z$ -axis (in particular, it is not normal to the  $xy$ -plane).
- (ii)  $\psi(t)$  is a proper parametrization of  $\pi_{xy}(\mathcal{C})$ .
- (iii)  $\pi_{xy}(\mathcal{C})$  has no asymptotes parallel to the  $y$ -axis.

In particular, hypotheses (ii) and (iii) imply that the graph of  $\pi_{xy}(\mathcal{C})$  can be computed by using the Planar-Top Algorithm. Now if the parametrization  $\varphi(t)$  of  $\mathcal{C}$  is not proper, then  $\psi(t)$  cannot be a proper parametrization of  $\pi_{xy}(\mathcal{C})$  either; then, in particular (ii) implies that  $\mathcal{C}$  is properly parametrized. However, the converse does not necessarily hold, i.e. it can happen that  $\varphi(t)$  is proper, but  $\psi(t)$  is not. From Section 3, we know how to check hypotheses (ii) and (iii). In order to check hypothesis (i), the next lemma, analogous to Lemma 6, can be applied.

**Lemma 9**  $\mathcal{C}$  has an asymptote parallel to the  $z$ -axis iff one of the following conditions happen: (a)  $q_3(t)$  has some real root, which is not a real root of  $q_1(t) \cdot q_2(t)$ ; (b)  $\deg(p_3) > \deg(q_3)$  but  $\deg(p_2) \leq \deg(q_2)$ ,  $\deg(p_1) \leq \deg(q_1)$ .

Moreover, hypothesis (i) implies the following relationship between the points  $Q_\infty = \lim_{t \rightarrow \pm\infty} \psi(t)$ ,  $P_\infty = \lim_{t \rightarrow \pm\infty} \varphi(t)$ . Recall from Section 2 that they are the only points of  $\pi_{xy}(\mathcal{C})$  and  $\mathcal{C}$ , respectively, that may not be reached by any complex value of the parameter.

**Lemma 10** Assume that hypothesis (i) holds. Then,  $P_\infty$  exists iff  $Q_\infty$  exists, and  $\pi_{xy}(P_\infty) = Q_\infty$ .

**Proof.** If  $P_\infty$  exists, then it is clear that  $Q_\infty$  exists and is the projection of  $P_\infty$ . Conversely, if  $Q_\infty = (x_\infty, y_\infty) \in \mathbb{R}^2$  then  $P_\infty$  exists because  $\mathcal{C}$  has no asymptotes.  $\square$

On the other hand, hypothesis (ii) leads to the following result. Here, the notion of *birationality* arises; essentially, the projection of  $\mathcal{C}$  is said to be *birational* if there are not two different branches of  $\mathcal{C}$  whose projections overlap (see Chapter 5 in (5) for further information on birationality).

**Theorem 11** Assume that  $\mathcal{C}$  is not a line parallel to the  $z$ -axis. Then, if  $\psi(t)$  is proper, the projection of  $\mathcal{C}$  onto the  $xy$ -plane is birational. Conversely, if  $\varphi(t)$  is proper and the projection of  $\mathcal{C}$  onto the  $xy$ -plane is birational, then  $\psi(t)$  is proper.

**Proof.** Let us see ( $\Rightarrow$ ). For this purpose, let  $Q \in \pi_{xy}(\mathcal{C})$ ,  $Q \neq Q_\infty$ , satisfying that there are at least two different points  $P, \tilde{P} \in \mathcal{C}$  projecting onto  $Q$ . Since  $Q \neq Q_\infty$ , by Lemma 10 none of these points is  $P_\infty$ , and hence both are reached by  $\varphi(t)$ . Let  $t_p \neq \tilde{t}_p$  be the  $t$ -values generating  $P, \tilde{P}$ , respectively. Then,

$\psi(t_p) = \psi(\tilde{t}_p)$ , and thus  $Q$  is generated by two different values of the parameter. But since  $\psi(t)$  is proper, this can only happen for finitely many points, and thus the projection is birational. Conversely, given any  $Q \in \pi_{xy}(\mathcal{C})$ ,  $Q \neq Q_\infty$ , not generated by any root of  $q_3(t)$  (notice that we are excluding finitely many points), the  $t$ -values reaching  $Q$  are exactly those ones generating the points of  $\mathcal{C}$  that are projected onto  $Q$ . Since  $\varphi(t)$  is proper, almost all points of  $\mathcal{C}$  are generated by just one value of the parameter. And since the projection is birational, we conclude that almost all points of  $\pi_{xy}(\mathcal{C})$  come from just one point of  $\mathcal{C}$ , and therefore almost all points of  $\pi_{xy}(\mathcal{C})$  are generated by just one value of the parameter. So,  $(\Leftarrow)$  holds.  $\square$

It is well-known that almost all affine transformations of the type  $\{X = x + az, y = y + bz, z\}$  transform  $\mathcal{C}$  so that its  $xy$ -projection is birational. So, if  $\varphi(t)$  is proper, almost all of these transformations set  $\mathcal{C}$  proper. Moreover, if  $\varphi(t)$  is not proper there exist reparametrization algorithms (see (3), (16)). Therefore, in the sequel we will assume that the above hypotheses hold.

#### 4.2 Definition of the Space Graph.

By assuming the hypotheses of the preceding subsection hold, we can compute the graph  $\overline{\mathcal{G}}$  associated with  $\pi_{xy}(\mathcal{C})$  with the Planar-Top Algorithm described in Section 3. Hence, in the following we will assume that this process has already been carried out.

Now we make precise the definition of the graph  $\mathcal{G}$  that we want to compute.

**Definition 12** *Let  $\mathcal{C}$  be a space curve in the above conditions. Then, the graph associated with  $\mathcal{C}$ ,  $\mathcal{G}$ , is the following graph:*

- (i) *Its vertices are the real points of  $\mathcal{C}$  giving rise (by projection) to the vertices of  $\overline{\mathcal{G}}$ .*
- (ii) *Its edges are the result of “lifting” to space the edges of  $\overline{\mathcal{G}}$ , i.e. of computing, for each edge  $\ell$  of  $\overline{\mathcal{G}}$ , an space segment  $\ell'$  corresponding to the branch of  $\mathcal{C}$  giving rise (by projection) to  $\ell$ .*

Hence, we have to lift to space the vertices and edges of  $\overline{\mathcal{G}}$  in order to compute  $\mathcal{G}$ . Let us see that this lifting operation is well-defined.

**Theorem 13** *Every vertex of  $\overline{\mathcal{G}}$ , except perhaps the isolated vertices, lifts to at least one real space point of  $\mathcal{C}$ .*

**Proof.** Every point of  $Q \in \pi_{xy}(\mathcal{C}) \cap \mathbb{R}^2$  fulfills one of the following conditions: (1)  $Q = Q_\infty$ ; (2) there exists  $t_0 \in \mathbb{R}$  satisfying that  $Q = \psi(t_0)$ ; (3)  $Q$  does not fulfill (2), but there exists  $t_0 \in \mathbb{C}$  such that  $Q = \psi(t_0)$ . In the first case,  $Q$  is

lifted to  $P_\infty$  by Lemma 10. If  $Q$  belongs to the second group, then it is lifted to  $P = \varphi(t_0)$  because  $\mathcal{C}$  has no asymptotes parallel to the  $z$ -axis. Finally, if  $Q$  belongs to the third group then it is an isolated point of  $\pi_{xy}(\mathcal{C})$ ; in this case,  $Q$  comes from a real point of  $\mathcal{C}$  iff  $z(t_0) \in \mathbb{R}$ .  $\square$

**Remark 2** *Real isolated points of  $\pi_{xy}(\mathcal{C})$  may come from real isolated points of  $\mathcal{C}$ , or from points of  $\mathcal{C}$  whose  $z$ -coordinate is complex. In any case, thanks to hypothesis (i) they do not come from branches of  $\mathcal{C}$  normal to the  $xy$ -plane.*

Now let us consider the lifting of the edges of the planar graph. The next result guarantees that, under the considered hypotheses, this lifting process can be always carried out. In particular, it implies that there are no real branches of  $\pi_{xy}(\mathcal{C})$  coming from complex components of  $\mathcal{C}$  (which is a phenomenon that in general can happen when working with space algebraic curves; see for example p. 734 in (2)).

**Theorem 14** *Under the considered hypothesis, for every edge  $\ell$  of  $\overline{\mathcal{G}}$  there exists one and just one branch of  $\mathcal{C}$  giving rise to  $\ell$ .*

**Proof.** Let  $\ell$  be an edge of  $\overline{\mathcal{G}}$ . By construction of the graph provided in Section 3, , if  $Q_\infty$  exists, it is always included as a vertex of  $\overline{\mathcal{G}}$ . So there exists a real open interval  $I \subset \mathbb{R}$  such that  $\psi(I)$  generates the real branch of  $\pi_{xy}(\mathcal{C})$ , that we denote by  $\mathcal{L}$ , corresponding to  $\ell$ . On the other hand, for every  $t \in I$  we have that  $z(t)$  must be defined, because otherwise  $\mathcal{C}$  has an asymptote parallel to the  $z$ -axis. Then,  $\varphi(t)$  is defined for every  $t \in I$ , and gives rise to a real connected branch of  $\mathcal{C}$  projecting as  $\mathcal{L}$ . Furthermore, since  $\varphi(t)$  is proper by hypothesis, the projection onto the  $xy$ -plane is birational by Theorem 11. Hence, there are just finitely many points of  $\mathcal{C}$  giving rise, by projection, to the same point of  $\pi_{xy}(\mathcal{C})$ ; but none of these points can give rise to a point of  $\mathcal{L}$ , because such a point would create a singularity of  $\pi_{xy}(\mathcal{C})$  which would split  $\ell$  into two different edges, and  $\ell$  is already an edge of  $\overline{\mathcal{G}}$ . Then, we conclude that  $\mathcal{L}$  lifts to a unique connected real branch of  $\mathcal{C}$ .  $\square$

### 4.3 Computation of the Vertices

From Definition 12, this process is the lifting of the vertices of  $\overline{\mathcal{G}}$ . From the construction of the planar graph, one may see that for each vertex  $Q_i = (x_i, y_i)$  of  $\overline{\mathcal{G}}$  the algorithm stores the real values  $t_{i,1}, \dots, t_{i,r}$  of the parameter generating it. For a fixed  $i$ ,  $z(t_{i,j})$  is well-defined for  $j \in \{1, \dots, r\}$ , since otherwise  $\mathcal{C}$  has an asymptote parallel to the  $z$ -axis. Hence,  $Q_i$  is lifted to the space points

$$P_{i,1} = \varphi(t_{i,1}), \dots, P_{i,r} = \varphi(t_{i,r})$$

Furthermore, if  $Q_\infty$  exists, then it is lifted to  $P_\infty$  and to the space points reached by the real values of the parameter generating  $Q_\infty$ , if any. Proceeding

this way, the only remaining space vertices are the real isolated ones (which, by Proposition 2, are generated by complex values of the parameter). So, in the rest of the subsection we consider this kind of points.

From Theorem 13, the real isolated vertices of  $\pi_{xy}(\mathcal{C})$  do not necessarily come from real isolated points of  $\mathcal{C}$  (since they may be the projection of complex space points). Conversely, a real isolated point of  $\mathcal{C}$  does not necessarily project as an isolated point of  $\pi_{xy}(\mathcal{C})$ , because its projection may coincide with the projection of some other real point of  $\mathcal{C}$  which is not isolated. However, the next result ensures that isolated points of  $\mathcal{C}$  always project as vertices of  $\overline{\mathcal{G}}$ ; therefore, these points are computed when lifting the planar vertices.

**Lemma 15** *Let  $P \in \mathcal{C}$  be a real isolated point. Then,  $\pi_{xy}(P)$  is a vertex of  $\overline{\mathcal{G}}$ .*

**Proof.** If  $\pi_{xy}(P)$  is an isolated point of  $\pi_{xy}(\mathcal{C})$ , then the statement is true. Otherwise, there exists a point  $P' \neq P$  in a real branch of  $\mathcal{C}$  such that  $\pi_{xy}(P) = \pi_{xy}(P')$ . Observe that  $P$  cannot be  $P_\infty$  because it is isolated. Therefore, suppose that it is reached via  $\varphi(t)$  by  $t_p \in \mathbb{C}$ . Now we distinguish the cases  $P' \neq P_\infty$  or  $P' = P_\infty$ , respectively. If  $P' \neq P_\infty$ , then  $P' = \varphi(t_{p'})$  with  $t_{p'} \in \mathbb{R}$ . Thus,  $\pi_{xy}(P)$  is generated via  $\varphi(t)$  by two different values of the parameter, namely  $t_p, t_{p'}$ , and since  $\varphi(t)$  is proper,  $\pi_{xy}(P)$  is a self-intersection of  $\pi_{xy}(\mathcal{C})$ . Hence, it is a singularity of  $\pi_{xy}(\mathcal{C})$ , and the statement follows. Finally, if  $P' = P_\infty$  then  $\pi_{xy}(P') = Q_\infty$  and therefore it is also a vertex of  $\overline{\mathcal{G}}$ .  $\square$

Then, we might recover isolated singularities of  $\mathcal{C}$  by determining the complex values of the parameter that generate (by projection) vertices of  $\overline{\mathcal{G}}$ , and by computing those real points of  $\mathcal{C}$  which are generated by those values. Nevertheless, in the sequel we consider an alternative method, analogous to that in Subsection 3.2. For this purpose, the following lemma is needed. Here, we denote a complex value of the parameter  $t$  as  $t = u + iv$ , where  $i^2 = -1$  and  $u, v \in \mathbb{R}$ . Also, we write

$$\frac{p_1(u + iv) \cdot \overline{q_1(u + iv)}}{|q_1|^2} = \frac{1}{|q_1|^2} \cdot (a(u, v) + ib(u, v))$$

$$\frac{p_2(u + iv) \cdot \overline{q_2(u + iv)}}{|q_2|^2} = \frac{1}{|q_2|^2} \cdot (c(u, v) + id(u, v)),$$

and

$$\frac{p_3(u + iv) \cdot \overline{q_3(u + iv)}}{|q_3|^2} = \frac{1}{|q_3|^2} \cdot (e(u, v) + ih(u, v))$$

Then the following result, analogous to Lemma 7, holds. Here,  $\tilde{q}(u, v)$  denotes the result of substituting  $t = u + iv$  in  $\text{lcm}(q_1, q_2, q_3)$ . As in Lemma 7, by applying the following result one computes a finite set of complex points which contains the complex points generating the isolated singularities of the space curve.

**Lemma 16** *Let  $P \in \mathcal{C} \cap \mathbb{R}^3$ . Then,  $P$  is generated by a complex value of the parameter  $t_0 = u_0 + iv_0$  if and only if there exists  $w_0 \in \mathbb{R}$  satisfying that  $(u_0, v_0, w_0)$  is a real solution of the system*

$$\left\{ \begin{array}{l} b(u, v) = 0 \\ d(u, v) = 0 \\ h(u, v) = 0 \\ v \cdot |\tilde{q}(u, v)|^2 \cdot w - 1 = 0 \end{array} \right. \quad (3)$$

As in the planar case, one can certify the number of real solutions of the system by Hermite's method; also, one can construct another system whose solutions correspond to complex values of the parameter generating points that are also reached by real values of the parameter, and proceed as in the 2D case.

#### 4.4 Computation of the Edges

The method consists of the lifting of the edges of  $\overline{\mathcal{G}}$ . So, let  $\ell$  be an edge of  $\overline{\mathcal{G}}$ ; by Theorem 14,  $\ell$  is lifted to an space edge  $\ell' \in \mathcal{G}$ . In order to compute  $\ell'$ , the crucial observation is that the computation of the edges of  $\overline{\mathcal{G}}$  is in fact done by connecting not points, but values of the parameter  $t$ . Hence, each edge  $\ell$  can be identified with a pair

$$[t_a, \tilde{t}]$$

where  $t_a, \tilde{t}$  belong to  $\mathbb{R} \cup \{-\infty, +\infty\}$ , and where the vertices of  $\overline{\mathcal{G}}$  defining  $\ell$  are  $Q_a = \psi(t_a)$ ,  $\tilde{Q} = \psi(\tilde{t})$  (see also Figure 1); here,  $Q_\infty = \psi(\pm\infty)$ . Hence,  $\ell$  is lifted to the space segment connecting the points  $P_a = \varphi(t_a)$ ,  $\tilde{P} = \varphi(\tilde{t})$ ; also,  $P_\infty = \varphi(\pm\infty)$ . Notice that this idea works perfectly when  $\tilde{Q}$  is the projection of several real points of  $\mathcal{C}$ .

#### 4.5 Full Algorithm

The following algorithm **Space-Top** can be derived from the preceding subsections.

##### Space-Top Algorithm:

**Input:** a space curve  $\mathcal{C}$ , parametrized by

$$\varphi(t) = (x(t), y(t), z(t)) = \left( \frac{p_1(t)}{q_2(t)}, \frac{p_2(t)}{q_2(t)}, \frac{p_3(t)}{q_3(t)} \right),$$

fulfilling: (i)  $p_i(t), q_i(t) \in \mathbb{Z}[t]$  for  $i = 1, 2, 3$ ,  $\gcd(p_i, q_i) = 1$  for  $i = 1, 2, 3$ ; (ii)  $\mathcal{C}$  has no asymptotes parallel to the  $z$ -axis; (iii)  $\psi(t) = \left( \frac{p_1(t)}{q_2(t)}, \frac{p_2(t)}{q_2(t)} \right)$  is proper; (iv)  $\pi_{xy}(\mathcal{C})$  has no asymptotes parallel to the  $y$ -axis.

**Output:** a space graph  $\mathcal{G}$  homeomorphic to  $\mathcal{C}$ .

- (1) (Projection) Compute the graph  $\overline{\mathcal{G}}$  associated with the projection  $\pi_{xy}(\mathcal{C})$  of  $\mathcal{C}$  onto the  $xy$ -plane, parametrized by  $\psi(t)$ , by applying **Planar-Top**.
- (2) (Lifting phase)
  - (2.1) (Vertices) For each vertex of  $\overline{\mathcal{G}}$ : if  $P$  is generated by  $t_1, \dots, t_p$  where  $t_1, \dots, t_p \in \mathbb{R} \cup \{-\infty, +\infty\}$ , then  $P$  lifts to the points  $\varphi(t_1), \dots, \varphi(t_p)$ ;  $Q_\infty$ , if it exists, lifts to  $P_\infty$ , and we write  $P_\infty = \varphi(\pm\infty)$ .
  - (2.2) (Edges) For each edge of  $\overline{\mathcal{G}}$ : if  $\ell$  is identified (according to Subsection 4.4) with  $[t_a, t_b]$ , where  $t_a, t_b \in \mathbb{R} \cup \{-\infty, +\infty\}$ , then it is lifted to the space edge obtained by connecting  $\varphi(t_a), \varphi(t_b)$  by means of a segment.
- (3) (Isolated vertices) Add to  $\mathcal{G}$  the real isolated singularities of  $\mathcal{C}$ .

Several examples of the output of this algorithm are presented in the next section.

## 5 Experimentation and Examples

The algorithm has been implemented in **Maple 13**, and the examples run on an Intel Core 2 Duo processor with speeds revving up to 1.83 GHz. The implementation allows the option of computing isolated points, or not. The reason for introducing this option is that the number of isolated points is certified by means of Hermite's method, and this method may be costly.

On the other hand, the user can decide the number of digits used in the computation. Suppose we denote such a number by  $n$ . Then, when running the algorithm, the computing starts using  $n$  digits. However, if the algorithm detects that the number of points in a vertical line is not the right number, the precision is automatically increased by 5 more digits and the whole process starts again. In our experimentations, we usually set  $n = 10$ , the default value of **Digits** variable in **Maple**, and in the implementation, the number of digits is limited to a maximum of 500, although we have never needed more than 70 digits.

Next, we first present examples of the 2D algorithm. In Table 1, we include, for each curve, the degree of the parametrization (i.e. the maximum exponent of the parameter in the numerators and denominators of the components of the parametrization,  $d_p$ ), the total degree of the implicit equation ( $d_i$ ), the

number of terms of the implicit equation (n.terms), the timings in seconds corresponding to the graph without computing isolated points ( $t_0$ ) or computing them ( $t_1$ ), and the number of digits used in the computations. The parametrizations corresponding to these examples are given in Appendix I.

Example	$d_p$	$d_i$	n.terms	$t_0$	$t_1$	Digits
1	3	6	16	0.359	1.672	10
2	8	8	25	0.891	1.078	10
3	8	8	9	10.250	71.172	40
4	4	4	7	0.109	0.110	10
5	6	6	28	0.203	11.859	10
6	8	8	21	0.171	2.50	10
7	23	23	335	49.797	> 1 h.	10
8	6	12	49	13.625	> 1 h.	10
9	17	17	171	1.656	> 1 h.	10

**Table 1:** 2D Examples.

One may notice that as the degree increases (see Example 7 or Example 9) the computation of the isolated points turns very costly. An alternative for those cases could be to detect isolated points directly by checking the existence of complex values of the parameter corresponding to real singular points; users interested in certifying rigorously the number of isolated singularities, can choose to apply Hermite's method later.

The pictures corresponding to the examples in Table 1 can be found in Figure 2; from left to right we have Examples 1, 2, 3 in the first row, 4, 5, 6 in the second row and 7, 8, 9 in the third one. Examples 2 and 6 are the offsets of the cardioid and of the cubical cusp, respectively; furthermore, Example 4 corresponds to the epitrochoid. Notice that the curves in Examples 2, 3 and 4 are not in generic position.

Finally, we present examples of the 3D algorithm. In Table 2, for each curve we include: the degree of the parametrization ( $d_p$ ), the total degree of the implicit equation of the projection onto the  $xy$ -plane ( $d_i$ ), the number of terms of this projection (n.terms), the timing without computing isolated points ( $t_0$ ), the timing including the computation of isolated points ( $t_1$ ), and the number of digits used. As in the 2D-case, in all cases the computations start with 10 digits, and the algorithm increases the number of digits when it is necessary. The parametrizations corresponding to these examples are given in Appendix

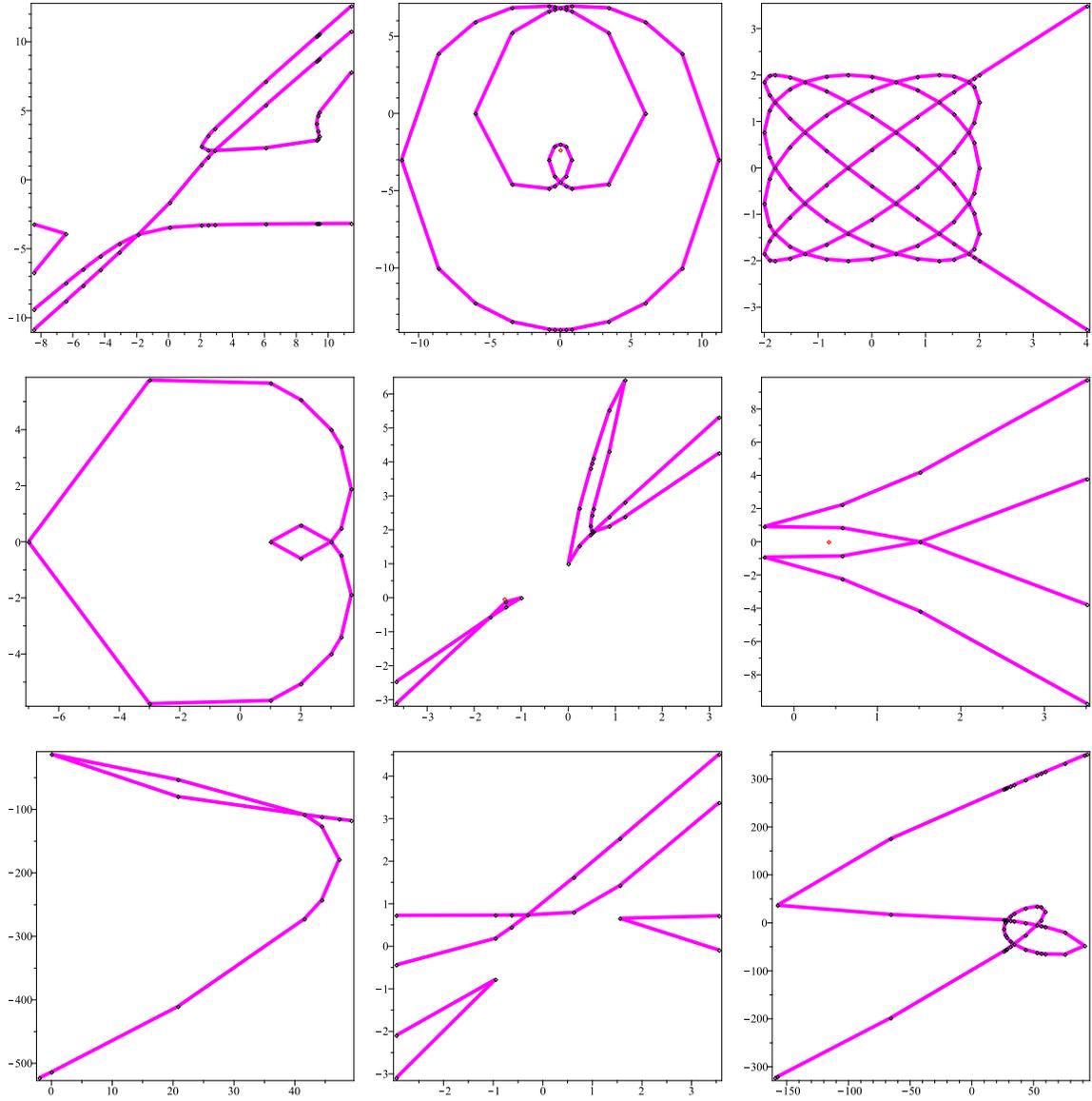


Fig. 2. Examples of the 2D algorithm.

II.

Example	$d_p$	$d_i$	n.terms	$t_0$	$t_1$	Digits
1	8	8	38	5.578	6.188	30
2	10	10	65	3.516	3.297	10
3	21	21	234	4.453	4.515	10
4	4	7	8	0.657	0.625	10
5	6	6	28	0.437	0.266	10
6	8	4	5	0.141	0.109	10
7	4	4	15	0.125	0.500	10
8	12	12	91	1.015	0.875	10
9	16	16	142	74.00	74.094	65

**Table 2:** 3D Examples.

The pictures corresponding to these curves can be found in Figure 3. The diamond in each picture points out the origin of the system of coordinates; moreover, in Example 7 we have not plotted the axes for the isolated point to be better appreciated.

## 6 Acknowledgements

We want to thank Prof. J.R. Sendra for his excellent ideas and the time and energy he devoted to us. We also want to thank Prof. González-Vega for suggesting the problem and discussing it with us.

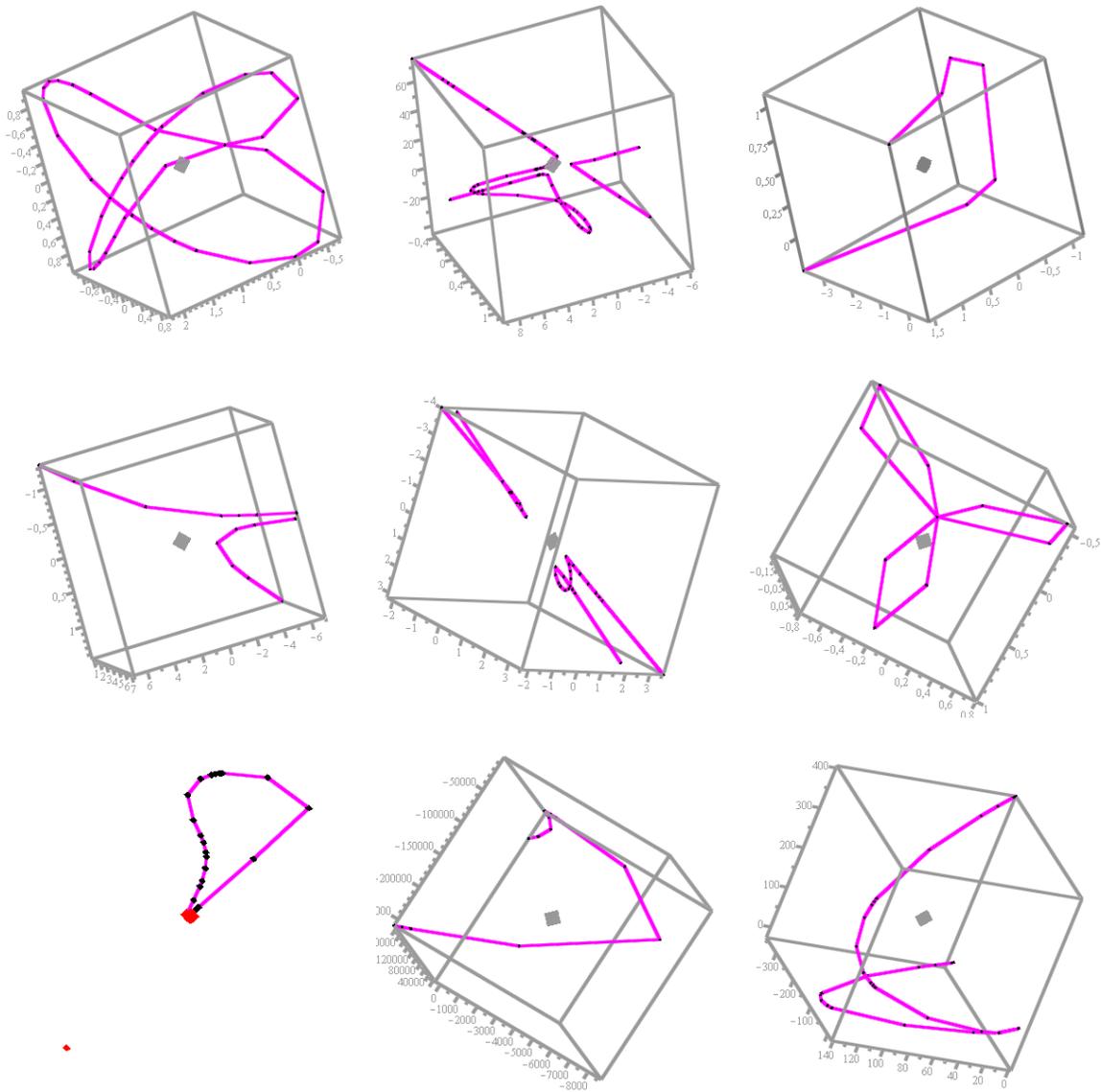


Fig. 3. Examples of the 3D algorithm.

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## Appendix I: Parametrizations of the planar curves used in the experimentation

Example 1:

$$\varphi(t) = \left( \frac{44 + 37t^3 - 23t^2 + 87t}{10 + 29t^3 + 98t^2 - 23t}, \frac{95 - 61t^3 - 8t^2 - 29t}{40 + 11t^3 - 49t^2 - 47t} \right)$$

Example 2:

$$\varphi(t) = \left( \frac{3456t^5 - 31104t^3 + 6t^8 - 756t^6 + 61236t^2 - 39366}{486t^4 + 36t^6 + 2916t^2 + t^8 + 6561}, \frac{-18t(864t^3 - 16t^5 - 1296t + 6t^6 - 126t^4 - 1134t^2 + 4374)}{486t^4 + 36t^6 + 2916t^2 + t^8 + 6561} \right)$$

Example 3:

$$\varphi(t) = (t^8 - 8t^6 + 20t^4 - 16t^2 + 2, t^7 - 7t^5 + 14t^3 - 7t)$$

Example 4:

$$\varphi(t) = \left( \frac{-7t^4 + 288t^2 + 256}{t^4 + 32t^2 + 256}, \frac{-80t^3 + 256t}{t^4 + 32t^2 + 256} \right)$$

Example 5:

$$\varphi(t) = \left( \frac{3t^2 + 3t + 1}{-3t - 1 + t^6 - 2t^4}, \frac{t^2(t^4 - 2t + 2)}{-3t - 1 + t^6 - 2t^4} \right)$$

Example 6:

$$\varphi(t) = \left( \frac{(t^2 - 1)(t^4 - 1 + 9t^2)}{9t^2(t^2 + 1)}, \frac{-(t^8 - 2t^6 + 2t^2 - 1 - 54t^4)}{27(t^2 + 1)t^3} \right)$$

Example 7:

$$\varphi(t) = (-83t^{23} + 98t^{20} - 48t^{18} - 19t^{13} + 62t^{11} + 37t^8, -13 - 64t^{27} + 64t^{25} - 90t^{22} - 60t^{12} - 34t^2)$$

Example 8:

$$\varphi(t) = \left( \frac{9 + 85t^6 + 80t^5 + 90t^3 + 74t^2 + 27t}{5 - 91t^6 + 81t^5 + 65t^4 - 12t^2 + 78t}, \frac{-56 - 5t^6 + 36t^5 - 8t^4 + 30t^3 - 3t}{-79 - 70t^5 + 42t^4 + 9t^3 - 21t^2 - 27t} \right)$$

Example 9:

$$\varphi(t) = (t^{17} + 80 - 20t^5 - 4t^4 - 89t^3 - 77t^2 + 69t, t^{17} - 64 - 33t^6 + 21t^4 - 35t^3 + 97t^2 + 30t)$$

## Appendix II: Parametrizations of the space curves used in the experimentation

For each example, will use the notation  $\varphi(t) = (x(t), y(t), z(t))$ .

Example 1:

$$x(t) = \frac{36t(-1 - 98t - 3954t^2 - 78868t^3 - 726692t^4 - 1092840t^5 + 31242296t^6 + 193263952t^7)}{q(t)}$$

$$y(t) = \frac{-648t^2(1 + 84t + 2940t^2 + 54880t^3 + 549996t^4 + 2492112t^5 + 2385712t^6)}{q(t)}$$

$$z(t) = \frac{36t(476t^3 + 426t^2 + 42t + 1)}{64660t^4 + 10976t^3 + 1176t^2 + 56t + 1},$$

with  $q(t) = 112t + 5488t^2 + 153664t^3 + 2741608t^4 + 33057472t^5 + 272552896t^6 + 1419416320t^7 + 4180915600t^8 + 1$

Example 2:

$$x(t) = \frac{7 + 33t^{10} + 80t^9 - 57t^7 + 88t^3 + 75t^2}{5t^{10} + 61t^8 + 8t^7 + 71t^6 - 16t^5 + 37t}$$

$$y(t) = \frac{18t^8 + 28t^7 + 58t^5 + 69t^4 + 8t^3 + 4t}{5t^{10} + 61t^8 + 8t^7 + 71t^6 - 16t^5 + 37t}$$

$$z(t) = \frac{-94t^9 - 59t^5 + 16t^4 - 82t^3 + 69t^2 - t}{5t^{10} + 61t^8 + 8t^7 + 71t^6 - 16t^5 + 37t}$$

Example 3:

$$\varphi(t) = \left( \frac{t^{20} + t - 1}{t^2 + 1}, \frac{t^{21} - 2}{t^2 + 1}, \frac{t^5 + 1}{t^2 + 1} \right)$$

Example 4:

$$\varphi(t) = \left( \frac{t^2 + 1}{t^4 + 1}, \frac{1}{t^3}, t^2 \right)$$

Example 5:

$$x(t) = \frac{(t - 1)^4(1 + 4t + 7t^2)}{1 - 4t + 17t^2 - 5t^6 - 13t^4 + 20t^5 + 48t^3}$$

$$y(t) = \frac{(1 - 4t + 22t^2 - 4t^3 + t^4)(1 + t)^2}{1 - 4t + 17t^2 - 5t^6 - 13t^4 + 20t^5 + 48t^3}$$

$$z(t) = \frac{(1 - 4t + 22t^2 - 4t^3 + t^4)(1 + t)^2}{1 - 4t + 17t^2 - 5t^6 - 13t^4 + 20t^5 + 48t^3}$$

Example 6:

$$\varphi(t) = \left( \frac{1 - 3t^2}{(t^2 + 1)^2}, \frac{(1 - 3t^2)t}{(t^2 + 1)^2}, \frac{(1 - 3t^2)t^3}{(t^2 + 1)^4} \right)$$

Example 7:

$$x(t) = \frac{87 - 7t^4 + 22t^3 - 55t^2 - 94t}{-73 - 56t^4 - 62t^2 + 97t}$$

$$y(t) = \frac{-82 - 4t^4 - 83t^3 - 10t^2 + 62t}{-73 - 56t^4 - 62t^2 + 97t}$$

$$z(t) = \frac{-82 - 4t^4 - 83t^3 - 10t^2 + 62t}{-73 - 56t^4 - 62t^2 + 97t}$$

Example 8:

$$x(t) = 91 + 11t^{12} - 49t^{10} - 47t^7 + 40t^6 - 81t$$

$$y(t) = -28t^{12} + 16t^{10} + 30t^8 - 27t^5 - 15t^3 - 59t^2$$

$$z(t) = 53 + 43t^{10} + 92t^9 - 91t^6 - 88t^3 - 48t$$

Example 9:

$$x(t) = -90t^{16} + 81t^8 + 65t^6 - 12t^5 + 78t^4 + 5t^3$$

$$y(t) = -70t^{16} + 42t^{15} + 9t^{12} - 21t^9 - 27t^8 - 79t^5$$

$$z(t) = 62 - 14t^{14} + 83t^{12} - 96t^7 - 8t^3 - 54t^2$$