Using implicit equations of parametric curves and surfaces without computing them: polynomial algebra by values

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Abstract

The availability of the implicit equation of a plane curve or of a 3D surface can be very useful in order to solve many geometric problems involving the considered curve or surface: for example, when dealing with the point position problem or answering intersection questions. On the other hand, it is well known that in most cases, even for moderate degrees, the implicit equation is either difficult to compute or, if computed, the high degree and the big size of the coefficients makes extremely difficult its use in practice.

We will show that, for several problems involving plane curves, 3D surfaces and some of their constructions (for example, offsets), it is possible to use the implicit equation (or, more precisely, its properties) without needing to explicitly determine it. We replace the computation of the implicit equation by the evaluation of the considered parameterizations in a set of points and its use, in order to deal with the considered geometric problems, is translated into one or several generalized eigenvalue problems on matrix pencils (depending again on several evaluations of the considered parameterizations).

This is the so called "Polynomial Algebra by Values" approach where the huge polynomial equations coming from Elimination Theory (e.g., using resultants) are replaced by big structured and sparse numerical matrices. For these matrices there are well known numerical techniques allowing to provide the results we need to answer the geometric questions on the considered curves and surfaces.

Key words: Bézout matrix of two polynomials, Offsets, Topology computations, Computations in the Lagrange Basis, Intersection problems for curves and surfaces.

Introduction

Using algebraic and symbolic technics for dealing with the so called position and intersection problems for algebraic curves and surfaces in Computer Aided Design is not new. Resultants and other notions coming from Elimination Theory and Algebraic Geometry are very useful to determine implicit equations, in order to compute intersection points and curves, etc. However, when the degrees of the polynomials involved in the curve or surface representation become high, the polynomials produced have a very high degree and often, their coefficients have huge size. Thus, using them becomes a complicated task. Moreover, in many cases, it is required to compute the determinant of a polynomial matrix, that from the symbolic and numerical point of views is not an easy and safe (regarding stability and accuracy) task.

The main purpose of this paper is to show how to use the properties of the objects coming from Elimination Theory to deal with several cases of the point position and intersection problems for curves and surfaces, avoiding the computation of the determinant of any polynomial matrix and requiring only the knowledge of the evaluation at several nodes of the polynomials involved in the curves and surfaces representation. To avoid the computation of the determinant of a polynomial matrix when dealing with intersection problems between curves was already introduced in [29] and [30]: the polynomial matrix is replaced by a matrix pencil whose generalized eigenvalues are the roots of the determinant of the considered polynomial matrix (this is called the linearization of the polynomial matrix).

We show here how to use this approach when the curves and surfaces are given by values and how to deal also with offsets (for parametric curves) and including also topological questions. For example, we will show how to compute the topology of the offset of a given parametric plane curve when this curve is presented by values: the offset of a parametric plane curve is a real algebraic plane curve (in general not rational) whose defining polynomial has degree much higher than the degree of the original curve, and it is usually quite dense. We are replacing the polynomial equations with huge coefficients or high degree coming from Elimination Theory (when using, for example, resultants) by big, structured and sparse numerical matrices. For these matrices there are well known numerical technics allowing to provide the results we need to answer the geometric questions on the considered curves and surfaces.

The methods to be presented here are based on the possibility of performing many operations on polynomials, such as differentiation or Bézout matrices construction, working directly with Lagrange interpolation data, and avoiding the transformation into monomial basis, which is a source of numerical instability. In addition, the special structure of the nullspace of the Bézout matrix, and the ability of computing the roots of the determinant of a Bézout matrix whose coefficients are polynomials by using generalized eigenvalues, will play an important role.

The paper is organized in the following way. The first section introduces the basic tools for dealing with the Polynomial Algebra by Values approach: how to compute the Bézout matrix of two polynomials presented in the Lagrange Basis (i.e., by values) and how to determine the companion matrix pencil for a polynomial matrix when this matrix is also presented in the Lagrange Basis. The second section summarizes the results in [10] in order to compute the topology of the curve $f(x, y) = 0$ when it is presented by values. The third section shows how to solve “by values” six geometric problems for plane curves: the point position problem for parametric plane curves and their offsets, the intersection of two parametric curves, the self-intersections of a parametric curve, the intersections of a parametric curve and the offsets to another curve, and the determination of the topology of the offset to a given parametric plane curve. The fourth section shows how to solve “by values” the point position and intersection problems for ruled and ringed surfaces, and surfaces of revolution, and how to determine the intersection curve between one these surfaces and an arbitrary parametric surface. The last section introduces briefly some topics that, in our opinion, deserve special attention to improve the algorithms introduced here.

Basic notation

In what follows, vectors and matrices are denoted by bold letters. We denote the vector space of polynomials of degree at most n by $\mathbb{P}_n$. For bivariate polynomials of degree at most m in the first variable, and degree at most n in the second, the associated vector space is denoted by $\mathbb{P}_{m,n}$. For a bivariate polynomial $p(z, t)$, $\deg_z(p(z, t))$ (resp. $\deg_t(p(z, t))$) denotes the degree of p when considered as a polynomial in z (resp. t) with polynomial coefficients in t (resp. z).

1 Polynomial Algebra by values: the Bézout matrix and the linearization of matrix polynomials in the Lagrange basis

In this section, we present the main mathematical tools which help us to replace the algebraic manipulation of polynomials (and their roots) by computations with numerical matrices (and their eigenvalues). This is a well-studied approach in the familiar monomial basis. Here, the main focus is on providing the necessary tools to carry out such an approach when the monomial description of the polynomials is not available or is costly to be computed. Instead, we have, or can deduce, the evaluations of the given polynomials at some sample points as well as degree bounds for the polynomials.

To replace a given polynomial problem by values with a so-called generalized eigenvalue problem by values, we introduce two main matrix constructions: a symmetric definition of the Bézout matrix from Elimination Theory and a pair of matrices associated with such a Bézout matrix, known as the companion matrix pencil.

1.1 The Bézout matrix of two polynomials

Standard definitions of the Bézout matrix make explicit reference to the monomial or power basis (see, e.g., [5]). However, all constructions of the Bézout matrix make use of the so-called Cayley quotient which makes no reference to any particular basis in which the given polynomials are represented.

Definition 1 *Let $p(t)$ and $q(t)$ be two univariate polynomials, and assume that $n = \max(\deg(p(t)), \deg(q(t)))$. The Cayley quotient of $p(t)$ and $q(t)$ is the function $C_{p,q}$ of degree at most $n-1$ defined by*

$$C_{p,q}(t,z) = \frac{p(t)q(z) - p(z)q(t)}{t - z}. \tag{1}$$

If $\Phi(t) = [\phi_1(t), \ldots, \phi_n(t)]^T$ is a polynomial basis for $\mathbb{P}_{n-1}$, the Bézout matrix in the basis Φ is the $n \times n$ symmetric matrix $\mathbf{B}$ such that

$$C_{p,q}(t,z) = \Phi(t)^T \, \mathbf{B} \, \Phi(z). \tag{2}$$

The proof of the following lemma can be found in [4].

Lemma 1 *Let $p(t), q(t) \in \mathbb{P}_n$ such that $n = \max(\deg(p(t)), \deg(q(t)))$. Suppose that $t^* \in \mathbb{C}$ is a common zero of $p(t)$ and $q(t)$. If $\mathbf{B}$ is the Bézout matrix of $p(t)$ and $q(t)$ in the basis Φ of $\mathbb{P}_{n-1}$, then $\Phi(t^*)$ is a null vector of $\mathbf{B}$.*

Remark 1 *When $d = \deg(p) > \deg(q)$, the principal coefficient a_d of $p(t)$ appears as a factor of every element of the first row of the Bézout matrix $\mathbf{B}$, so it will be a factor of $\det(\mathbf{B})$. If $p(t)$ is expressed in the Lagrange basis, as in definition 2, its principal coefficient can be computed using the formula*

$$a_d = \sum_{i=1}^{d+1} \omega_i(\tau) p_i.$$

If the polynomials are bivariate, $p(s,t)$ and $q(s,t)$, the entries of the Bézout *matrix of* p *and* q*, with respect to* t*,* $\mathbf{B}(s)$*, will be polynomials in* s*. If* $\deg_t(p) > \deg_t(q)$ *the principal coefficient* $a_d(s)$ *of* $p(s,t)$*, as a polynomial in* t*, will be a factor of* $\det(\mathbf{B}(s))$*. Some authors ([22],[39]) use a different definition of the Bézout matrix, with the same properties of Lemma 1, which is not symmetric, and whose determinant has not the factor* a_d *when* $\deg_t(p) > \deg_t(q)$*, (see [12], page 237, and Example 1).*

Example 1 *Let* $p(s,t) = (s^3-6)t^4 + (-12+s^6)t^2 - 9 + 2s^3$*, and* $q(s,t) = (-s+2)t^3 + st^2 - 4t + s^2$*.*

Then the Bézout matrix with respect to t*,* $\mathbf{B}(s)$*, takes the following form*

$$\begin{bmatrix} -\left(s^3-6\right)\left(-s+2\right) & -\left(s^3-6\right)s & 4\,s^3-24 & -\left(s^3-6\right)s^2 \\ -\left(s^3-6\right)s & \dots & \dots & \dots \\ 4\,s^3-24 & \dots & \dots & \dots \\ -\left(s^3-6\right)s^2 & \dots & \dots & \dots \end{bmatrix}$$

If we call $\hat{\mathbf{B}}(s)$ *the non-symmetric version of the Bézout matrix, it turns out that* $\det(\mathbf{B}(s)) = (s^3-6)\det(\hat{\mathbf{B}}(s))$*.*

The construction of the Bézout matrix using the Cayley quotient in the monomial basis (i.e., power basis) is classic (see [5]). In this work, we do not intend to give the details of such a well-known derivation of the Bézout matrix. Our primary focus will be on the application of the Bézout matrix in analyzing several geometric problems involving curves and surfaces which are described by their values at some given nodes (i.e., in the Lagrange polynomial basis).

The construction of the Bézout matrix associated with a pair of univariate polynomials specified in the Lagrange polynomial basis (i.e. given by values) has been introduced and fully studied in [35] and [36].

1.2 The Bézout matrix in the Lagrange basis

There are several applications of Bézout matrices for *bivariate* polynomials given by samples in each variable (see for example [4]). The natural basis is the *tensor product* of the Lagrange basis in each variable. Before discussing some applications we review some facts about the Lagrange interpolation problem. We consider the so-called barycentric representation of the Lagrange polynomial basis which is known to have numerical advantages over the familiar standard definition (see [6,32]).

Definition 2
Let $\tau = (\tau_1, \dots, \tau_{d+1}) \in \mathbb{C}^{d+1}$ *be a vector whose numerical entries are all distinct. We define*

$$\ell(t;\tau) = \prod_{j=1}^{d+1}(t-\tau_j)\,, \quad \omega_i(\tau) = \left[\prod_{\substack{j=1\\ j\neq i}}^{d+1}(\tau_i-\tau_j)\right]^{(-1)}, \tag{3}$$

where the $\omega_i(\tau)$ *are called the barycentric weights. Then, the associated barycentric Lagrange polynomials are given by*

$$L_i(t;\tau) = \frac{\omega_i(\tau)\ell(t;\tau)}{t-\tau_i}\,, \quad (1 \leq i \leq d+1)\,, \tag{4}$$

and if $p(t) \in \mathbb{P}_d$,

$$p(t) = \ell(t;\tau)\sum_{i=1}^{d+1}\frac{\omega_i(\tau)p_i}{t-\tau_i}\,, \tag{5}$$

where $p_i = p(\tau_i), 1 \leq i \leq d+1$. *Moreover, we call*

$$\mathbf{L}(t;\tau) = [L_1(t;\tau), \dots, L_{d+1}(t;\tau)]^T \in [\mathbb{P}_d]^{d+1} \tag{6}$$

the Lagrange polynomial basis.

Definition 3 *A polynomial* $p(t) \in \mathbb{P}_d$ *is said to be given by values if for a set of* $d+1$ *nodes* $(\tau_1, \dots, \tau_{d+1})$, *the values* $p_i = p(\tau_i), 1 \leq i \leq d+1$, *are known.*

A bivariate polynomial $p(s,t) \in \mathbb{P}_{m,n}$ *is said to be given by values if for a rectangular grid* $\{(\sigma_i, \tau_j) : 1 \leq i \leq m+1, 1 \leq j \leq n+1\}$, *the values* $p_{i,j} = p(\sigma_i, \tau_j), 1 \leq i \leq m+1, 1 \leq j \leq n+1$, *are known.*

Remark 2 *For computing purposes, once the nodes* $(\tau_1, \dots, \tau_{d+1})$ *are known or chosen, the barycentric weights are computed only once. If a polynomial* $p(t)$ *is given by values, it can easily be evaluated at a particular value of* $t = t_0$, *by evaluating the polynomials of the Lagrange basis* $L_i(t_0;\tau), 1 \leq i \leq d+1$. *The whole evaluation of* $p(t_0)$ *requires* $2d+1$ *additions/substractions and* d^2+d *products. Note that* $p(t)$ *is not constructed by interpolation for this evaluation.*

Using barycentric representation of the Lagrange polynomial basis, we now introduce the Bézout matrix by values. The following results has been proved in [36].

Theorem 1 *Let* $p(t)$ *and* $q(t) \in \mathbb{P}_d$, *such that* $d = \max(\deg(p(t)), \deg(q(t)))$. *Suppose* $\tau = (\tau_1, \dots, \tau_{d+1}) \in \mathbb{C}^{d+1}$ *consists of distinct numerical values. Let* $\mathbf{p} = (p_1, \dots, p_{d+1})$ *and* $\mathbf{q} = (q_1, \dots, q_{d+1})$ *be numerical data such that* $p(\tau_i) = p_i$ *and* $q(\tau_i) = q_i, 1 \leqslant i \leqslant d+1$. *Let* $\tilde{\tau} = (\tau_1, \dots, \tau_d) \in \mathbb{C}^d$ *consist of all the nodes in* τ *except* τ_{d+1}. *Let* $p'_i = p'(\tau_i)$ *and* $q'_i = q'(\tau_i)$ *denote the values of* *the derivatives of $p(t)$ and $q(t)$ ($1 \leqslant i \leqslant d$). Then, the Bézout matrix in the Lagrange basis $\mathbf{L}(t;\tilde{\tau}) \in [\mathbb{P}_{d-1}]^d$ is the $d \times d$ matrix $\mathbf{Bez}_{p,q} = [b_{i,j}]$, with entries given by*

$$b_{i,j} = \frac{p_i q_j - p_j q_i}{\tau_i - \tau_j}, \quad 1 \le i \le d,\ 1 \le j \le d,\ i \neq j, \tag{7}$$

$$b_{i,i} = p_i'\, q_i - p_i\, q_i'\,, \quad 1 \le i \le d. \tag{8}$$

Corollary 4 *With the notation in Theorem 1,*

$$C_{p,q}(t,z) = \mathbf{L}(t;\tilde{\tau})^T\, \mathbf{Bez}_{p,q}\, \mathbf{L}(z;\tilde{\tau}). \tag{9}$$

To compute the diagonal entries of the Bézout matrix in the Lagrange basis in Theorem 1, we need to have a systematic way of computing all the derivatives of the polynomial interpolant on the given sample points. From [2,37], we have that

$$p_i' = \frac{1}{\omega_i(\tau)} \sum_{\substack{j=1 \\ j \neq i}}^{d+1} \frac{\omega_j(\tau)(p_j - p_i)}{\tau_i - \tau_j}, \tag{10}$$

with $1 \le i \le d+1$.

Remark 3 *Note that τ_{d+1} is needed for computing p_i' and q_i' (despite (7) and (8) do not show τ_{d+1} explicitly). However, $\deg_z(C_{p,q}) = \deg_t(C_{p,q}) \le d-1$, and this explains the need for introducing $\tilde{\tau}$. In what follows, $\tilde{\tau}$ will denote the set of nodes τ with the last element removed.*

The common roots of two polynomials $p(t)$ and $q(t)$, can be obtained from the nullspace of the $\mathbf{Bez}_{p,q}$, using a method called *taking moments.* In the case of a simple common root, the nullspace has dimension 1, and the method works according to the following results, whose proofs are presented in [10].

Lemma 2 *Let $p(t), q(t) \in \mathbb{P}_d$, such that $d = \max(\deg(p(t)), \deg(q(t)))$. We assume that $p(t)$ and $q(t)$ are specified in the Lagrange basis, by data τ, $\mathbf{p}$, and $\mathbf{q}$, as in Theorem 1. Suppose that $t^* \in \mathbb{C}$ is a simple common zero of $p(t)$, and $q(t)$. If $\mathbf{Bez}_{p,q}$ is the Bézout matrix in the Lagrange basis $\mathbf{L}(t;\tilde{\tau}) \in [\mathbb{P}_{d-1}]^d$, then $\mathbf{L}(t^*;\tilde{\tau})$ is a null vector of $\mathbf{Bez}_{p,q}$.*

Theorem 2 *Let $p(t)$ and $q(t)$ be univariate polynomials as described and specified in Lemma 2. To compute t^*, the simple common root of $p(t)$, and $q(t)$, we may use any vector $\mathbf{U} = [u_1, u_2, \ldots, u_d]^T$ in the nullspace of the corresponding Bézout matrix in the Lagrange basis $\mathbf{L}(t;\tilde{\tau}) \in [\mathbb{P}_{d-1}]^d$. This is done by means of a procedure that, in what follows, will be referred to as "taking moments".*

That is,

$$t^* = \frac{\sum_{i=1}^{d} \tau_i u_i}{\sum_{i=1}^{d} u_i}. \tag{11}$$

For the case of a multiple common root of $p(t)$ and $q(t)$, with multiplicity k, we use the especial structure of the nullspace of the Bézout matrix that has been presented in [24], adapted to the Lagrange basis case and using Vandermonde matrices. This leads to a linear system of k equations and k unknowns, from which the common root is obtained efficiently. For full details, see [10].

1.3 Companion matrix pencils by values

As already mentioned, for the solution of some of the geometric problems addressed in this paper we consider two bivariate polynomials $p(s,t), q(s,t)$ (see, for instance section 3.3), and we need to determine the roots of the determinant of the Bézout matrix, with respect to one of the variables, say t, of p and q. This Bézout matrix is a polynomial matrix in s. In what follows, we will call the roots of $\det(\mathbf{B}(s))$ the *polynomial eigenvalues* of $\mathbf{B}(s)$. To avoid the computation of the determinant we apply a method of linearization. Before, we recall the definition of generalized eigenvalue, which is a well known concept in the field of Numerical Linear Algebra, and then we show how to construct appropriate companion matrix pencils in the Lagrange basis.

Definition 5 *Given a pair of matrices* $\mathbf{M}, \mathbf{N}$ *of size* $n \times n$*, a number* $\lambda \in \mathbb{C}$ *is called a generalized eigenvalue of* $(\mathbf{M}, \mathbf{N})$ *if there exists a non null vector* $\mathbf{v}$ *of dimension* n *such that*

$$\mathbf{M}\mathbf{v} = \lambda \mathbf{N}\mathbf{v}.$$

The vector $\mathbf{v}$ *is called a generalized eigenvector, and the pair* $(\mathbf{M}, \mathbf{N})$ *is called a matrix pencil.*

The generalized eigenvalues are the roots of $\det(\mathbf{M} - \lambda\mathbf{N})$*. If* $\deg(\det(\mathbf{M} - \lambda\mathbf{N})) < n$*, it is said that* ∞ *is a generalized eigenvalue with multiplicity* $n - \deg(\det(\mathbf{M} - \lambda\mathbf{N}))$.

To learn more about generalized eigenvalues see [11] and [27].

For a given matrix polynomial $\mathbf{B}(s)$ specified in the Lagrange basis $\mathbf{L}(s;\tau)$ we can construct a block matrix pencil $(\mathbf{C}_0, \mathbf{C}_1)$ whose finite generalized eigenvalues agree with the polynomial eigenvalues of $\mathbf{B}(s)$. The process of transforming the matrix polynomial $\mathbf{B}(s)$ to the matrix pencil $(\mathbf{C}_0, \mathbf{C}_1)$ is called the linearization of $\mathbf{B}(s)$, for reasons that will be clear in Theorem 3 below.

Definition 6 *Consider a square matrix polynomial of size $r \times r$ and degree d,*

$$\mathbf{B}(s) = \sum_{j=1}^{d+1} \mathbf{B}_j L_j(s;\tau)\,, \tag{12}$$

where, for $1 \leqslant i \leqslant d+1$, $\mathbf{B}_i$ is a $r \times r$ constant matrix such that $\mathbf{B}(\tau_i) = \mathbf{B}_i$. The companion pencil for the matrix polynomial $\mathbf{B}(s)$ is given by the pair of matrices $(\mathbf{C}_0, \mathbf{C}_1)$ of size $r(d+2) \times r(d+2)$, defined by:

$$\mathbf{C}_0 = \begin{pmatrix} \tau_1\mathbf{I} & & & & \mathbf{B}_1 \\ & \tau_2\mathbf{I} & & & \mathbf{B}_2 \\ & & \ddots & & \vdots \\ & & & \tau_{d+1}\mathbf{I} & \mathbf{B}_{d+1} \\ -\omega_1\mathbf{I} & -\omega_2\mathbf{I} & \cdots & -\omega_{d+1}\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{C}_1 = \begin{pmatrix} \mathbf{I} & & & & \\ & \mathbf{I} & & & \\ & & \ddots & & \\ & & & \mathbf{I} & \\ & & & & \mathbf{0} \end{pmatrix}, \tag{13}$$

where $\mathbf{I}$ and $\mathbf{0}$ are the $r \times r$ identity and zero matrices, respectively.

A proof of the following theorem can be found in [9].

Theorem 3 *If $(\mathbf{C}_0, \mathbf{C}_1)$ is the companion matrix pencil associated to the polynomial matrix $\mathbf{B}(s)$, given in definition 6, then*

$$\det \mathbf{B}(s) = \det\left(s\mathbf{C}_1 - \mathbf{C}_0\right).$$

If $\deg(\mathbf{B}(s)) = d$, and the size of $\mathbf{B}(s)$ is r, then $\deg(\det \mathbf{B}(s)) \leq dr$.

The solutions of $\det(s\mathbf{C}_1 - \mathbf{C}_0) = 0$ are the *generalized eigenvalues* of the matrix pencil $(\mathbf{C}_0, \mathbf{C}_1)$. The multiplicity of ∞ as a generalized eigenvalue for the matrix pencil $(\mathbf{C}_0, \mathbf{C}_1)$ is bigger or equal than $2r$, and it is equal to the multiplicity of 0 as a generalized eigenvalue of the matrix pencil $(\mathbf{C}_1, \mathbf{C}_0)$.

Remark 4 *In the present paper, we do not discuss algorithms for computing generalized eigenvalues; we assume that robust software exists to solve this problem whenever they occur. See [11,2,3,9,8] for specific details.*

2 Topology determination of a real algebraic plane curve $f(x, y) = 0$ when f is presented by values

The idea of solving a polynomial eigenvalue problem by values corresponding to a polynomial matrix given by values, presented in Section 1, can be used to develop a new family of methods for determining the topology of a real algebraic plane curve presented either parametrically or defined by its implicit equation. In [10], we give full details of the theory and of the algorithms for solving such problems. Here, we only give a brief overview of how to determine the topology of $f(x, y) = 0$ by values which will be used in the sections that follow. As a preliminary step, we present a method for solving a system of two bivariate polynomial equations, given by values.

2.1 Bivariate polynomial solver

Our methodology to compute the common roots of a pair of bivariate polynomials $p(s, t)$ and $q(s, t)$ in $\mathbb{P}_{m,n}$, specified by values, has two principal steps: the elimination and the eigenvalue computation. The algorithm we provide here for solving such a polynomial problem is therefore a hybrid of resultant-based and eigenvalue techniques. First, we eliminate the variable t, constructing the Bézout matrix $\mathbf{B}(s)$, with respect to t, of $p(s, t)$ and $q(s, t)$ (as defined in Theorem 1). Then, the finite generalized eigenvalues of the companion pencil of Definition 6 are the roots of the determinant of $\mathbf{B}(s)$. If s^* is a generalized eigenvalue, then there exists t^* such that $p(s^*, t^*) = 0$ and $q(s^*, t^*) = 0$, whence t^* can be recovered from the nullspace of $\mathbf{B}(s^*)$ by "taking moments" according to Theorem 2.

2.2 Computing the topology of $f(x, y) = 0$

Let us assume that $f(x, y)$ is a squarefree polynomial. The typical strategy for computing the topology of $f(x, y) = 0$, with f presented in any given basis, is to determine the number of intersections of the curve with the vertical lines $\{(x, y) \in \mathbb{R}^2 : x = \textit{const.}\}$, (see, for instance [21] or [14]). This number may change at the critical points, see the following definition.

Definition 7 *Consider the algebraic curve $\mathcal{C}$ defined by $f(x, y) = 0$. Let $f_x(x, y)$ and $f_y(x, y)$ denote the partial derivatives of $f(x, y)$ with respect to x and y, respectively.*

A point (α, β) is a critical point of $\mathcal{C}$ if $f(\alpha, \beta) = 0$, and $f_y(\alpha, \beta) = 0$.

A point (α, β) is a singular point of $\mathcal{C}$ if $f(\alpha, \beta) = 0$, $f_x(\alpha, \beta) = 0$, and $f_y(\alpha, \beta) = 0$.

A point (α, β) is a regular point of $\mathcal{C}$ if $f(\alpha, \beta) = 0$, and $f_y(\alpha, \beta) \neq 0$.

The output of the algorithm is a graph representing the topology of the curve, together with the data generated during the process, which includes the coordinates of the critical points. The graph in figure 1 represents the topology of a curve of degree 8 in x and y; the circled dots are the critical points. The typical method consists of three main steps:

(1) Find the x–coordinates of the critical points of $f(x, y) = 0$; suppose there are r of them, and let us call them $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. The vertical lines $\{(x, y) \in \mathbb{R}^2 : x = \alpha_i\}$ are called critical lines.
(2) For each $\alpha_i, 1 \leq i \leq r$, find the y–coordinates of the critical points and of the other points on the curve on the critical line $\{(x, y) \in \mathbb{R}^2 : x = \alpha_i\}$.
(3) For each point on a critical line, determine the number of segments of the curve connecting the point to points on the closest critical line, to the left side and to the right side. If the point is regular, it will be exactly one curve segment on each side; otherwise, the number could be zero or bigger than one.

Definition 8 *Let $f(x, y)$ be a squarefree polynomial. The real algebraic plane curve defined by f is in generic position if the following two conditions are satisfied:*

(a) The leading coefficient of f with respect to y (which is a polynomial in x) has no real roots.
(b) For every $\alpha \in \mathbb{R}$ the number of distinct complex roots of

$$f(\alpha, y) = 0, \; f_x(\alpha, y) = 0,$$

is 0 *or* 1.

When the curve $f(x, y) = 0$ is in generic position there are no vertical asymptotes, and there is exactly one critical point of the curve on each critical line. If the curve is in generic position the algorithms are greatly improved by adapting the previous strategy in the following way:

(2') For each α_i, find the y–coordinate β_{α_i} of the unique critical point over α_i, and find the other points of the curve in the same critical line, which are regular.
(3') For each pair α_i, α_{i+1}, choose an intermediate value $\alpha_i < \alpha_i^* < \alpha_{i+1}$, find the curve points with $x-$coordinate equal to α_i^*, and determine the edges of the topological graph representing the curve segments.

Assuming that $f(x, y)$ is a polynomial in $\mathbb{P}_{m,n}$ described by values, it is possible to compute the evaluation of $f(x, y)$ and its partial derivatives with respect to y at any given point and this data is all what we need to determine the topology of $f(x, y) = 0$. Next we describe very briefly the approach introduced in [10] to determine the topology of $f(x, y) = 0$, assuming also that $f(x, y) = 0$ is in generic position.

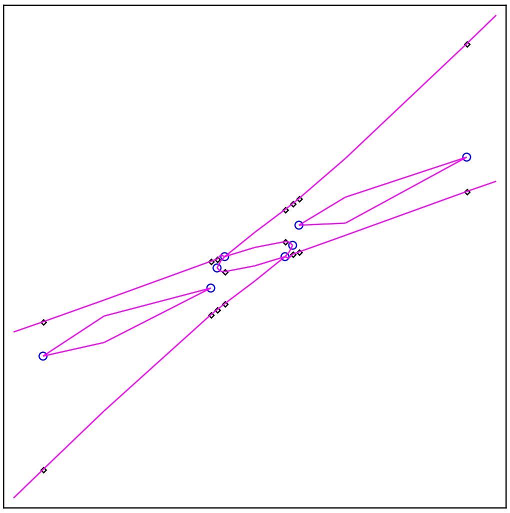

Fig. 1. Graph representing the topology of an implicit curve of degree 8.

The x–coordinates of the critical points of the curve $f(x, y) = 0$ (step (1)) are determined by computing the real generalized eigenvalues of the matrix pencil associated with $\mathbf{B}(x)$ where $\mathbf{B}(x)$ is the Bézout matrix of $f(x, y)$ and $f_y(x, y)$ with respect to y (with dimension $n - 1$).

For each such eigenvalue α (step (2')), by using the vectors in a basis of Nullspace($\mathbf{B}(\alpha)$), the corresponding y–coordinate β_α of the critical point can be obtained by "taking moments"(see Theorem 2). Since $f(x, y)$ is in generic position, the rest of the roots of $f(\alpha, y)$ are simple. If β_α has multiplicity k as a root of $f(\alpha, y)$, one can obtain the Lagrange representation of the polynomial (see [21,10])

$$F^\alpha(y) = \frac{f(\alpha, y)}{(y - \beta_\alpha)^k} ,$$

and then easily compute the real roots of $F^\alpha(y)$.

The same approach as above can be used in order to compute the simple roots of $f(\alpha^*, y)$ where α^* is an arbitrary real number between two consecutive x–coordinates of critical points (step (3')). It is worth to mention here that, due to the generic position hypothesis, the way the edges connecting the computed points are determined follows from an easy combinatorial reasoning (see [21,20]). The topology graph in figure 1 has been obtained from the implicit equation of a curve of degree 8 (in each variable), given by values.

3 Solving by values some geometric problems for algebraic plane curves

In this section we present methods for the solution of the following important geometric problems for algebraic plane curves, assuming that they are given by values:

- The point position problem for a parametric plane curve.
- The point position problem for the d-offset to a parametric plane curve.
- Computing the intersection of two algebraic plane curves.
- Computing the self-intersections of a parametric plane curve.
- Computing the intersection of a parametric plane curve and the d-offset to another parametric plane curve.
- Determining the topology of the d-offset to a parametric plane curve.

3.1 Solving by values the point position problem for parametric plane curves

Given $(\xi, \eta) \in \mathbb{R}^2$, we wish to determine if (ξ, η) lies on a prescribed algebraic curve $\gamma \subset \mathbb{R}^2$, when γ is a parametric curve presented by values.

Suppose that the explicit parametrization $(x, y) = (P(t), Q(t))$ describing γ is known only by the data τ, $\mathbf{P}$, $\mathbf{Q} \in \mathbb{R}^{n+1}$ such that $(P_i, Q_i) = (P(\tau_i), Q(\tau_i))$, $1 \le i \le n+1$, where $P(t)$ and $Q(t)$ are assumed to be polynomials of degree at most n. As such, the data $\mathbf{P}$ and $\mathbf{Q}$ are the coefficients of $P(t)$ and $Q(t)$, respectively, using the Lagrange basis $\mathbf{L}(t; \tau)$ in definition 2.

Defining $p(t) := P(t) - \xi$ and $q(t) := Q(t) - \eta$, the coefficients of the polynomials $p(t)$ and $q(t)$ in the Lagrange basis $\mathbf{L}(t; \tau)$ are $\mathbf{p} := (P_1 - \xi, \ldots, P_{n+1} - \xi)$ and $\mathbf{q} := (Q_1 - \eta, \ldots, Q_{n+1} - \eta)$. By definition of $p(t)$ and $q(t)$, $(\xi, \eta) \in \gamma$ iff the polynomials $p(t)$ and $q(t)$ admit a common root. According to Lemma 1, $p(t)$ and $q(t)$ admit a common root only when the associated Bézout matrix admits nontrivial null vectors. Thus, we can set up $\mathbf{Bez}_{p,q} = \mathbf{Bez}_{P(t)-\xi, Q(t)-\eta}$ in the Lagrange basis $\mathbf{L}(t; \tilde{\tau})$ as in Theorem 1, and determine its nullspace. That is, working *directly* from the prescribed data (ξ, η), τ, $\mathbf{P}$, and $\mathbf{Q}$, we can determine whether the given point (ξ, η) lies on the prescribed curve γ.

It is possible that the real point $(\xi, \eta) = (P(t^*), Q(t^*)) \in \mathbb{R}^2$ might lie on the curve γ for some complex argument $t^* \in \mathbb{C} \backslash \mathbb{R}$. Such a point is called a *geometric extraneous component* (see [17]). For most curve intersection problems, we discard such solutions, keeping only intersections with real arguments.

Whenever we need to determine the nullspace of $\mathbf{Bez}_{p,q}$ or its rank (for instance, when applying Theorem 2), we use Singular Value Decomposition

(SVD). Computing the smallest singular value of $\mathbf{B}$ gives the distance to the nearest singular *unstructured* matrix (i.e., if $\mathbf{B}$ is symmetric or has some other particular structure, then the nearest singular matrix need not have that same structure), whence it is only a lower bound on the distance to the nearest polynomially parametrized curve that goes through (ξ, η), or to the nearest point $(\hat{\xi}, \hat{\eta})$ which is on the curve γ. Therefore, this method is reliable only in deciding when (ξ, η) is not on the curve. However, in practice, we expect that (as in the Example 2, below) a small singular value will indicate that the point is close to one that is truly on the curve.

3.1.1 Extension to Rational Parametrization

We can generalize the method previously presented from polynomial curves to rational curves. Let γ be a planar curve with parametrization $(x, y) = (P(t), Q(t))/R(t)$ where P, Q, and the common denominator R are all polynomial functions with $n = \max(\deg(P), \deg(Q), \deg(R))$. The functions are not known but the vectors $\mathbf{P} = (P_1, \ldots, P_{n+1})$, $\mathbf{Q} = (Q_1, \ldots, Q_{n+1})$, and $\mathbf{R} = (R_1, \ldots, R_{n+1})$ specify the values of the functions P, Q, and R, respectively, at the values $\tau = (\tau_1, \ldots, \tau_{n+1})$, such that $R(\tau_i) \neq 0$, for $1 \leq i \leq n+1$.

To determine whether (ξ, η) lies on the rational curve γ, the procedure is as before, with a minor alteration; we define $p(t) := P(t) - \xi R(t)$ and $q(t) := Q(t) - \eta R(t)$ and compute the corresponding Bézout matrix $\mathbf{Bez}_{p,q}$ in the basis $\mathbf{L}(t; \tilde{\tau})$. Again, Lemma 1 implies that at any simple common zero t^* of p and q, the vector $\mathbf{L}(t^*; \tilde{\tau})$ is a null vector of $\mathbf{Bez}_{p,q}$. Taking moments gives the desired value of t^* (see Theorem 2).

3.2 Solving by values the point position problem for the offset to a parametric plane curve

The d-offset to a plane curve γ is the locus of the points which are at constant distance $d > 0$ from the curve along its normal line (see [25,28,31,16,38]). For each t, there are two points in the offset at distance d from $\gamma(t)$, one on each side of γ. The offset to an algebraic curve is an algebraic curve, but it is not rational in general.

Given $(\xi, \eta) \in \mathbb{R}^2$, we wish to determine if it lies on one of the offsets $\gamma_d \subset \mathbb{R}^2$, $d > 0$, to a prescribed parametric plane curve, given in the Lagrange basis. If the point (ξ, η) lies on γ_d, the offset to γ at distance d, then there exists $t \in \mathbb{R}$ such that

$$F(\xi, \eta; t) = 0\,, \qquad G(\xi, \eta; t) = 0,$$

where

$$\begin{aligned} F(\xi,\eta;t) &:= (\xi - P(t))^2 + (\eta - Q(t))^2 - d^2, \\ G(\xi,\eta;t) &:= P'(t)(\xi - P(t)) + Q'(t)(\eta - Q(t)). \end{aligned} \tag{14}$$

Assuming that $\gamma := \{(P(t), Q(t)) \colon t \in \mathbb{R}\}$ the required data is, in this case, $(P_i, Q_i) = (P(\tau_i), Q(\tau_i))$, $1 \leq i \leq 2n+1$, where $n = \max(\deg(P), \deg(Q))$. The computation of the Bézout matrix $\mathbf{Bez}_{F,G}$ in the basis $\mathbf{L}(t;\tilde{\tau})$, with $\tau = (\tau_1, \ldots, \tau_{2n+1})$, together with its singular values provides the needed information to determine if $(\xi,\eta) \in \gamma_d$. If this is the case then the computation of the nullspace of $\mathbf{Bez}_{F,G}$ allows also to determine the footpoints in γ for (ξ,η) (i.e. the point in γ producing (ξ,η) in γ_d).

Example 2 *Let $\gamma = (P(t), Q(t))$ be the curve with $n = 3$, defined by the data*

k	1	2	3	4	5	6	7
τ_k	-3	-2	-1	0	1	2	3
$P(\tau_k)$	-39	-6	3	0	-3	6	39
$Q(\tau_k)$	-48	-15	0	3	0	-3	0

Only four nodes are needed here to deal with γ but we need $2n+1$ values of $P(t)$ and $Q(t)$ for working with the offsets of γ. In order to determine if the point $A = (0,0)$ lies on γ_1 (the offset at distance $d = 1$), we compute the matrix $\mathbf{Bez}_{F(A;t),G(A;t)}$

$$\begin{pmatrix} -6010226 & -709236 & 21828 & 6904 & 5178 & 124884 \\ -709236 & -82322 & 4452 & 1446 & 964 & 12654 \\ 21828 & 4452 & 214 & -48 & -24 & 268 \\ 6904 & 1446 & -48 & 46 & 0 & 534 \\ 5178 & 964 & -24 & 0 & -170 & 1068 \\ 124884 & 12654 & 268 & 534 & 1068 & -5906 \end{pmatrix}$$

by using the formula in Theorem 1 applied to $F(A;t)$ and $G(A;t)$ (only the evaluations of $P(t)$ and $Q(t)$ and their derivatives at the nodes τ_k are needed here). Since the smallest singular value of $\mathbf{Bez}_{F(A;t),G(A;t)}$ is 54.4184723 we conclude that $A \notin \gamma_1$.

The same approach used before but applied to the point $B = (0.5, 2.065403766)$ makes the smallest singular value of $\mathbf{Bez}_{F(B;t),G(B;t)}$ to be $2.227512514 \cdot 10^{-8}$ concluding that $B \in \gamma_1$. The computation of the nullspace of $\mathbf{Bez}_{F(B;t),G(B;t)}$ (of dimension 1) and "taking moments" according to Theorem 2 provides the *value of t, t^*, generating the footpoint $C = (-0.1376920296, 2.970197663)$ in γ after determining $P(t^*)$ and $Q(t^*)$.*

Figure 2 shows the curve γ and the offset γ_1, together with the points A, B and C.

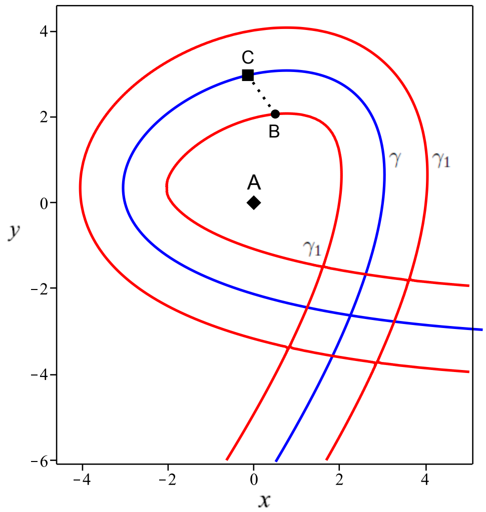

Fig. 2. Curves γ and two-sided offset γ_1, together with the points $A \notin \gamma_1$, $B \in \gamma_1$ and its footpoint $C \in \gamma$.

3.3 The intersection problem for parametric plane curves

We suppose that the curves γ and $\hat{\gamma}$ have parametric representations $(x, y) = (P(t), Q(t))$ and $(x, y) = (\hat{P}(s), \hat{Q}(s))$, respectively. We assume that all the parametric functions are polynomials, and that both $n = \max(\deg(P), \deg(Q))$ and $\hat{n} = \max(\deg(\hat{P}), \deg(\hat{Q}))$ are known. The functions P and Q are specified in the Lagrange basis $\mathbf{L}(t;\tau)$ by their values $\mathbf{P} = (P_1, \ldots, P_{n+1})$ and $\mathbf{Q} = (Q_1, \ldots, Q_{n+1})$ at some distinct parameter values $\tau = (\tau_1, \ldots, \tau_{n+1})$. Similarly, $\hat{P}$, and $\hat{Q}$ are known by their values $\hat{\mathbf{P}} = (\hat{P}_1, \ldots, \hat{P}_{\hat{n}+1})$ and $\hat{\mathbf{Q}} = (\hat{Q}_1, \ldots, \hat{Q}_{\hat{n}+1})$, at distinct parameter values $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_{\hat{n}+1})$.

In order to determine if there are any intersections between the curves γ and $\hat{\gamma}$, and if so, to compute them, we have to solve the system of equations $P(t) = \hat{P}(s)$ and $Q(t) = \hat{Q}(s)$.

Following the standard practice in the monomial basis, we use Theorem 1 to form the Bézout matrix for the polynomials $P(t) - \hat{P}(s)$ and $Q(t) - \hat{Q}(s)$ in the Lagrange basis $\mathbf{L}(t;\tilde{\tau})$ (treating s as a symbolic parameter):

$$\mathbf{B}(s) = \mathbf{Bez}_{P(t)-\hat{P}(s),\, Q(t)-\hat{Q}(s)} = \sum_{k=1}^{\hat{n}+1} B_k L_k(s;\sigma), \tag{15}$$

where the coefficients are the $\hat{n}+1$ constant matrices of size $n \times n$:

$$B_k := \mathbf{Bez}_{P(t)-\hat{P}_k,\, Q(t)-\hat{Q}_k}\,, \quad 1 \le k \le \hat{n}+1. \tag{16}$$

The motivation for the preceding construction is as follows. Each point of intersection (ξ, η) of γ and $\hat{\gamma}$ gives rise to a pair of numerical values s^* and t^* such that $(\xi, \eta) = (P(t^*), Q(t^*)) = (\hat{P}(s^*), \hat{Q}(s^*))$. Since $P(t)-\xi$ and $Q(t)-\eta$ have a common root then $\mathbf{Bez}_{P(t)-\xi,\, Q(t)-\eta}$, computed in the basis $\mathbf{L}(t; \tilde{\tau})$, is singular: the matrix $\mathbf{B}(s^*)$ is singular so s^* is a polynomial eigenvalue of $\mathbf{B}(s)$. Thus, having computed the coefficients of $\mathbf{B}(s)$ in the basis $\mathbf{L}(s; \sigma)$, the corresponding polynomial eigenvalues are candidates for parameter values s^* where the curves γ and $\hat{\gamma}$ cross.

3.3.1 Computing the intersection between γ and $\hat{\gamma}$ by using the implicit equation of γ

We consider the previous intersection problem using a slightly different approach. First, we construct a symbolic $n \times n$ Bézout matrix $\mathbf{B}(x, y)$ in the parameters x and y by using the data associated with the curve γ. Next, we substitute $x = \hat{P}(s)$ and $y = \hat{Q}(s)$ in $\mathbf{B}(x, y)$ to obtain a matrix polynomial in s. This matrix polynomial plays the same role as the implicit equation associated with γ; in particular, the point (ξ, η) lies on γ if $\mathbf{B}(\xi, \eta)$ is a singular matrix.

Specifically, we define the $n \times n$ Bézout matrix $\mathbf{B}(x, y)$ by

$$\mathbf{B}(x, y) := \mathbf{Bez}_{P(t)-x,\, Q(t)-y}\,, \tag{17}$$

in the Lagrange basis $\mathbf{L}(t; \tilde{\tau})$, treating x and y as symbolic parameters. It is easy to prove that the entries of $\mathbf{B}(x, y)$ are linear in x and y; in particular, for the off–diagonal entries where $i \neq j$,

$$\begin{aligned}[\mathbf{B}(x, y)]_{i,j} = \left(\frac{Q(\tau_j) - Q(\tau_i)}{\tau_i - \tau_j}\right) x + \left(\frac{P(\tau_j) - P(\tau_i)}{\tau_i - \tau_j}\right) y \\ + \left(\frac{P(\tau_i)Q(\tau_j) - P(\tau_j)Q(\tau_i)}{\tau_i - \tau_j}\right),\end{aligned}$$

and for the diagonal entries,

$$[\mathbf{B}(x, y)]_{i,i} = (P'(\tau_i)Q(\tau_i) - P(\tau_i)Q'(\tau_i)) - Q'(\tau_i)x - P'(\tau_i)y\ .$$

Thus, $\mathbf{B}(x, y) = x\,\mathbf{U} + y\,\mathbf{V} + \mathbf{W}$ where $\mathbf{U} = -\mathbf{Bez}_{Q(t),\,1}$, $\mathbf{V} = -\mathbf{Bez}_{1,\,P(t)}$ and $\mathbf{W} = \mathbf{Bez}_{P(t),\,Q(t)}$, with all the Bézout matrices computed in the Lagrange basis $\mathbf{L}(t; \tilde{\tau})$.

Having computed the Bézout matrices $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$, the intersection problem is solved by the substitution $x = \hat{P}(s)$ and $y = \hat{Q}(s)$ in $\mathbf{B}(x, y)$ to obtain the matrix polynomial in s, $\mathbf{B}(s) := \mathbf{B}(\hat{P}(s), \hat{Q}(s))$. As before, the intersection points of γ and $\hat{\gamma}$ are polynomial eigenvalues of $\mathbf{B}(s)$, so the solution of the associated polynomial eigenvalue problem leads to the potential intersections. If needed, the value of t giving the intersection point provided by s^* can be determined by computing the nullspace of $\mathbf{B}(s^*)$ and by "taking moments" (see Theorem 2).

3.4 Computing the self–intersections points of γ

The previous approach can be used to determine the self–intersection points of a given curve γ: it is enough to define $(\hat{P}(s), \hat{Q}(s)) := (P(s), Q(s))$. However, instead of solving by values the system of equations $P(t) = P(s)$ and $Q(t) = Q(s)$, we analyze the following system of equations

$$\begin{aligned} F(t,s) &:= \tfrac{P(t)-P(s)}{t-s} = 0, \\ G(t,s) &:= \tfrac{Q(t)-Q(s)}{t-s} = 0, \end{aligned} \qquad (18)$$

because $\det(\mathbf{B}(s))$ would be identically zero if we had chosen to define $\mathbf{B}(s) = \mathbf{Bez}_{P(t)-P(s),\,Q(t)-Q(s)}$. Observe that $F(t, s)$ and $G(t, s)$ are the Cayley quotient of $P(t)$ and 1, and $Q(t)$ and 1, respectively. If $\max(\deg(P), \deg(Q)) = n$ then $\max(\deg(F), \deg(G)) = n-1$. The resultant of $F(t, s)$ and $G(t, s)$, with respect to t, is called the D-resultant or Taylor resultant of $P(t)$ and $Q(t)$ (see [15] and [1]). In addition, we suppose that the parametrization of the curve is proper [33,34]. Recall that a parametrization is said to be proper if it is injective for almost all the points of the curve, which implies that there is at most a finite number of points of the curve generated by more than one value of the parameter t. If the parametrization is not proper, then $\mathbf{B}(s)$ is identically zero.

As a consequence of the special form of $F(t, s)$ and $G(t, s)$, the degree of $\mathbf{B}(s)$ is $n - 2$.

Proposition 9 *Let F and G be defined by equations (18), where P and Q are univariate polynomials, with $n = \max(\deg(P), \deg(Q))$. If $\mathbf{B}(s) = \mathbf{Bez}_{F,G}$, then $\deg(\mathbf{B}(s)) = n - 2$.*

Proof. If $P(t) = \sum_{j=0}^{n} a_j t^j$, and $Q(t) = \sum_{j=0}^{n} b_j t^j$, then $F(t, s) = \sum_{j=0}^{n-1} A_j t^j$, $G(t, s) = \sum_{j=0}^{n-1} B_j t^j$, where $A_j = \sum_{i=0}^{n-j-1} a_{j+i+1} s^i$, $B_j = \sum_{i=0}^{n-j-1} b_{j+i+1} s^i$.

Apply the following formula from [26], page 276, for the coefficients C_{kl} of the

Bézout matrix in the power basis:

$$C_{kl} = D_{0,k+\ell+1} + D_{1,k+\ell} + \cdots + D_{k,\ell+1}, \quad D_{u,v} = A_{n-v}B_{n-u} - A_{n-u}B_{n-v}.$$

The proposition follows after a lengthy computation. $\square$

By hypothesis, we only have the vectors $\mathbf{P} = (P_1, \ldots, P_{n+1})$, $\mathbf{Q} = (Q_1, \ldots, Q_{n+1})$ specifying the values of the polynomials $P(t)$ and $Q(t)$ at the values $\tau = (\tau_1, \ldots, \tau_{n+1})$. Then

$$\mathbf{B}(s) = \mathbf{Bez}_{F,G} = \sum_{k=1}^{n-1} \mathbf{B}_k L_k(s; \tilde{\tilde{\tau}}), \quad \tilde{\tilde{\tau}} = (\tau_1, \ldots, \tau_{n-1}),$$

where $\mathbf{B}_k = \mathbf{B}(\tau_k)$, $1 \leq k \leq n-1$, (see Definition 6). Since $F(\tau_k, \tau_k) = P'(\tau_k)$ and $F'(\tau_k, \tau_k) = P''(\tau_k)/2$ (the same for $G(t,s)$), and every $\mathbf{B}_k$ is symmetric, according to Theorem 1, the entries of $\mathbf{B}_k$ are given by the following equalities, for $1 \leq i, j \leq n-1$:

$$\begin{aligned}
b_{i,j} &= \frac{[P(\tau_i) - P(\tau_k)][Q(\tau_j) - Q(\tau_k)] - [P(\tau_j) - P(\tau_k)][Q(\tau_i) - Q(\tau_k)]}{(\tau_i - \tau_k)(\tau_j - \tau_k)(\tau_i - \tau_j)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } i \neq j\,, i \neq k\,, j \neq k\,, \\
b_{i,j} &= \frac{-P'(\tau_k)[Q(\tau_j) - Q(\tau_k)] + Q'(\tau_k)[P(\tau_j) - P(\tau_k)]}{(\tau_k - \tau_j)^2}, \quad \text{if } i \neq j\,, i = k\,, \\
b_{i,i} &= \frac{P'(\tau_i)[Q(\tau_i) - Q(\tau_k)] - Q'(\tau_i)[P(\tau_i) - P(\tau_k)]}{(\tau_i - \tau_k)^2}, \quad \text{if } i \neq k\,, \\
b_{i,i} &= \frac{P''(\tau_k)Q'(\tau_k) - P'(\tau_k)Q''(\tau_k)}{2}, \quad \text{if } i = k\,.
\end{aligned}$$

Consequently the values of the first and second derivatives of $P(t)$ and $Q(t)$ at $\tilde{\tilde{\tau}}$ are required. For this purpose, equation (10) is applied.

Example 3 *Let γ be the curve used in the Example 2. In order to determine the self–intersections points of γ, since $n = 3$, we need four nodes which will be $\tau = (\tau_1, \tau_3, \tau_4, \tau_5)$. First we determine the matrix $\mathbf{B}(s)$ which happens to be of degree 1. Next we determine the matrix pencil associated to $\mathbf{B}(s)$ which requires only two nodes; we consider for example (τ_3, τ_4). Then the matrix pencil whose generalized eigenvalues are going to provide the parameter values* *for the self–intersection points is*

$$\mathbf{C}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & -42 & -27 \\ 0 & -1 & 0 & 0 & -27 & -18 \\ 0 & 0 & 0 & 0 & -27 & -18 \\ 0 & 0 & 0 & 0 & -18 & -15 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Its generalized eigenvalues are

$$s_1 = \frac{1+\sqrt{37}}{4}, \; s_2 = \frac{1-\sqrt{37}}{4}$$

together with ∞ *with multiplicity* 4*. Evaluating* $(P(s), Q(s))$ *in* s_1 *(or* s_2*) we get the unique self–intersection point of* γ*:* $(2.249999988, -2.624999998)$ *. Figure 3 shows the curve* γ *together with its self–intersection point.*

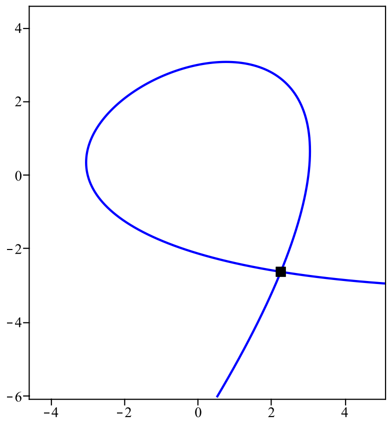

Fig. 3. Curve γ together with its self–intersection point.

We should mention that second derivatives must be avoided if possible. More concretely, if it is not difficult to obtain values of $P(t)$ and $Q(t)$ at different sets of nodes, we can compute the matrix $\mathbf{B}(s)$ without computing second derivatives. Suppose that for two disjoint sets of nodes, $\tau = (\tau_1 < \ldots < \tau_{n+1})$ and $\sigma = (\sigma_1 < \ldots < \sigma_{n-1})$ (see Remark 3), we have the values of $P(t)$ and $Q(t)$. Then we can compute $\mathbf{B}(s)$ in the following way:

$$\mathbf{B}(s) = \mathbf{Bez}_{F,G} = \sum_{k=1}^{n-1} \mathbf{B}_k L_k(s; \sigma),$$

with $\mathbf{B}(\sigma_k) = \mathbf{B}_k$, $1 \leq k \leq n-1$. Thus, the entries of $\mathbf{B}_k$ are given, for

$1 \le i, j \le n-1$, by

$$b_{i,j} = \frac{[P(\tau_i) - P(\sigma_k)][Q(\tau_j) - Q(\sigma_k)] - [P(\tau_j) - P(\sigma_k)][Q(\tau_i) - Q(\sigma_k)]}{(\tau_i - \sigma_k)(\tau_j - \sigma_k)(\tau_i - \tau_j)}, \quad \text{if } i \neq j,$$

$$b_{i,i} = \frac{P'(\tau_i)(Q(\tau_i) - Q(\sigma_k)) - Q'(\tau_i)(P(\tau_i) - P(\sigma_k))}{(\tau_i - \sigma_k)^2}, \quad \text{otherwise.} \tag{19}$$

Example 4 *We consider the same example as before but we determine the matrix pencil associated to* $\mathbf{B}(s)$ *from a new set of nodes disjoint with* τ, $\sigma = (\tau_2, \tau_6)$. *In this case, the matrix pencil is*

$$\mathbf{C}_0 = \begin{bmatrix} -2 & 0 & 0 & 0 & -57 & -36 \\ 0 & -2 & 0 & 0 & -36 & -21 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 & -9 \\ 1/4 & 0 & -1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & -1/4 & 0 & 0 \end{bmatrix}, \mathbf{C}_1 = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

Obviously, its generalized eigenvalues are the same as before.

In the following example we obtain the self-intersection at a triple point applying the formula (19) to a parametric curve of degree 4.

Example 5 *Let* $(P(t), Q(t))$ *be a curve such that* $\deg(P) \le 4, \deg(Q) \le 4$. *To find the self-intersections we need the values of* P *and* Q *at two disjoint sets of nodes* τ *and* σ, *with five and three elements, respectively. We use the following,*

i	1	2	3	4	5
τ_i	-1.5	-0.5	0.5	1.5	2.5
$P(\tau_i)$	-3.875	2.375	1.625	-0.125	3.125
$Q(\tau_i)$	14.125	-0.875	2.125	-0.875	14.125

k	1	2	3
σ_k	0	1	2
$P(\sigma_k)$	2.5	0.5	0.5
$Q(\sigma_k)$	1	1	1

Using the formula (19) we obtain the matrices of size 3, $\mathbf{B}_k, 1 \le k \le 3$. *Then, the companion matrix pencil is of size 12, with* ∞ *as eigenvalue of multiplicity* 6, *and the eigenvalues* $-1, 1$ *and* 2, *each with multiplicy* 2. *This three values of* t *correspond to the same point* $(0.5, 1)$. *The data corresponds to the curve in figure 4*

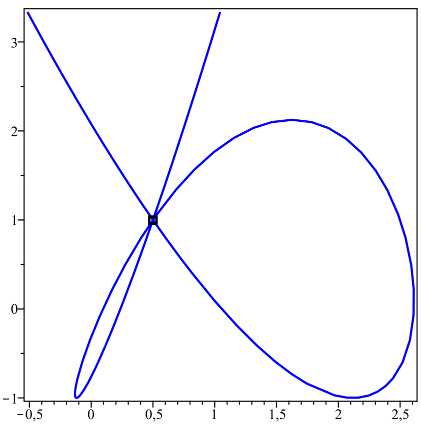

Fig. 4. Curve of degree 4 with triple self–intersection point.

If γ is presented by a rational parameterization $(P(t)/R(t), Q(t)/R(t))$ then, to compute the self–intersection points of γ, the same strategy is to be applied but using now the following equations

$$F(t,s) := \frac{P(t)R(s)-P(s)R(t)}{t-s} = 0$$
$$G(t,s) := \frac{Q(t)R(s)-Q(s)R(t)}{t-s} = 0$$

instead of those in equation (18). As in section 3.1.1, we must avoid to use nodes that anihilate $R(t)$. This is again the D-resultant or Taylor resultant of $(P(t)/R(t), Q(t)/R(t))$, according to the definition presented in [23].

3.5 Computing the intersection between γ_d and $\hat{\gamma}$.

Assume that the curves γ and $\hat{\gamma}$ are given by values, as in section 3.3. To determine the intersection points between $\hat{\gamma}$ and γ_d, the offset at distance d from γ, we need to solve by values the system of equations $F(s,t) = 0$ and $G(s,t) = 0$ where:

$$\begin{aligned} F(s,t) &:= (\hat{P}(s) - P(t))^2 + (\hat{Q}(s) - Q(t))^2 - d^2, \\ G(s,t) &:= P'(t)(\hat{P}(s) - P(t)) + Q'(t)(\hat{Q}(s) - Q(t)) \ . \end{aligned} \tag{20}$$

We start by computing the Bézout matrix of $F(s,t)$ and $G(s,t)$, with respect to t

$$\mathbf{B}(s) = \mathbf{Bez}_{F(s,t),\,G(s,t)},$$

which is a polynomial matrix in s obtained by evaluating $F(s,t)$ and $G(s,t)$ in τ, according to the formula in Theorem 1. Next, the polynomial eigenvalues of $\mathbf{B}(s)$ are obtained through the computation of the generalized eigenvalues of the pencil in (13), requiring the evaluation of $\mathbf{B}(s)$ in $\hat{\sigma}$. The footpoints in γ of the intersection points in γ_d can be determined after computing the nullspace of $\mathbf{B}(s^*)$, for each polynomial eigenvalue s^* of $\mathbf{B}(s)$, by "taking moments" according to Theorem 2.

Example 6 *Let γ be the curve used in the Example 2 and $\hat{\gamma} = (\hat{P}(s), \hat{Q}(s))$ be the curve defined by $\hat{n} = 3$ and the data*

k	1	2	3	4
σ_k	-1	-0.5	0	0.5
$\hat{P}(\sigma_k)$	0	1.125	1	0.375
$\hat{Q}(\sigma_k)$	2	0	0	0.5

(21)

We want to determine the intersection points between γ_1 and $\hat{\gamma}$.

We start by computing by values the matrix $\mathbf{B}(s) = \mathbf{Bez}_{F(s,t),G(s,t)}$. Since the degrees of F and G, with respect to t, are bounded by 6 we obtain that $\mathbf{B}(s)$ is a 6×6 matrix of degree 9 whose entries depend on $\hat{P}(s)$ and $\hat{Q}(s)$. We need 10 nodes to represent $\mathbf{B}(s)$ in the Lagrange Basis in order to formulate the corresponding generalized eigenvalue problem, and the new set of s nodes is chosen to be $\{-2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5\}$. The values of $\hat{P}(s)$ and $\hat{Q}(s)$ at the new nodes are determined by using the values in (21).

The linearization of $\mathbf{B}(s)$ produce a matrix pencil of dimension 66 whose real generalized eigenvalues are:

$$s_1 = 1.765213073, \quad s_2 = 1.548450950,$$
$$s_3 = 1.535848157, \quad s_4 = 1.255746313,$$
$$s_5 = -0.996791271, \quad s_6 = -1.205950879.$$

Each s_i produces a different intersection point $(\hat{P}(s_i), \hat{Q}(s_i))$ between γ_1 and $\hat{\gamma}$ that can be visualized in Figure 5. Computing the nullspace of $\mathbf{B}(s_i)$ and "taking moments" produce the values of t corresponding to the footpoints in γ for the intersection points in γ_1.

3.6 Computing by values the topology of the offset to a given plane curve

Following standard terminology on offsets, we shall call the initial curve the *generator curve*. We start presenting the method for a generator curve parametrized by polynomials, and then we extend it to the case where the parametrization uses rational functions.

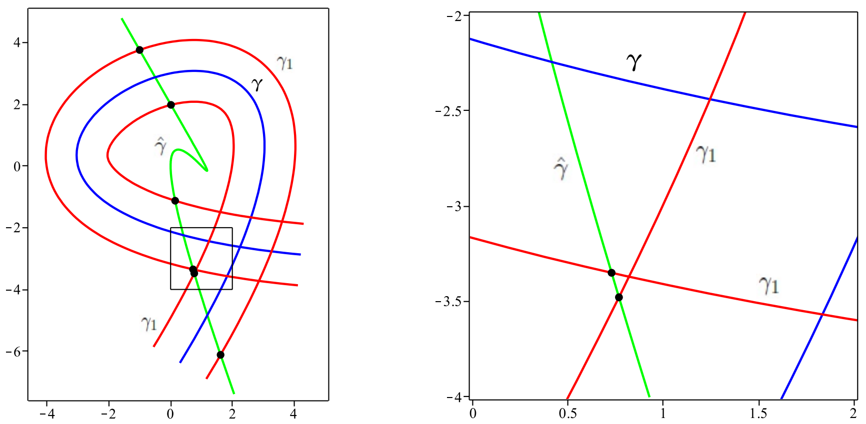

Fig. 5. Curves γ, γ_1 and $\hat{\gamma}$, together with the intersection points (left). Zoom area where two intersection points are very close to a self–intersection of γ_1 (right).

For algebraic properties of offsets the reader is referred to the papers [16] and [38], and the references therein.

3.6.1 Polynomial case

Assume that the generator curve, γ, is parametrized by $(P(t), Q(t))$ where $P(t)$ and $Q(t)$ are polynomials in $\mathbb{R}[t]$ with degrees bounded by m_1 and m_2, respectively, and let $n = \max(m_1, m_2)$. A point $(x, y) \in \mathbb{R}^2$ is in the offset γ_d to distance $d > 0$ of $(P(t), Q(t))$ if there exists $t \in \mathbb{R}$ such that (see [16])

$$\begin{aligned} F(x, y; t) &:= (x - P(t))^2 + (y - Q(t))^2 - d^2 &&= 0, \\ G(x, y; t) &:= P'(t)(x - P(t)) + Q'(t)(y - Q(t)) &&= 0. \end{aligned} \tag{22}$$

Note that γ_d is not a rational curve in general, but it is always an algebraic curve. The degrees of F and G, with respect to t, are $2n$ and $2n-1$, respectively, whence the degree of the implicit equation of γ_d is $\delta(\gamma) = 4n - 2$ (see [16], Corollary 2.3).

Assume we know the values of $P(t)$ and $Q(t)$ at a collection $\tau = (\tau_1 < \tau_2 < \ldots < \tau_{2n+1})$ of real values of the parameter t. Let $\mathbf{B}(x, y)$ be the Bézout matrix of $F(x, y; t)$ and $G(x, y; t)$, with respect to t, computed in the Lagrange basis $\mathbf{L}(t; \tilde{\tau})$ (here $\tilde{\tau} = (\tau_1 < \tau_2 < \ldots < \tau_{2n})$),

$$\mathbf{B}(x, y) = \begin{pmatrix} b_{1,1}(x, y) & \ldots & b_{1,2n}(x, y) \\ \vdots & & \vdots \\ b_{1,2n}(x, y) & \ldots & b_{2n,2n}(x, y) \end{pmatrix}.$$

According to Theorem 1, the entries in $\mathbf{B}(x, y)$ are computed as:

$$b_{i,j} = \frac{F(x, y; \tau_i)G(x, y; \tau_j) - F(x, y; \tau_j)G(x, y; \tau_i)}{\tau_i - \tau_j},$$

$$b_{i,i} = \left.\frac{\mathrm{d}F(x, y; t)}{\mathrm{d}t}\right|_{t=\tau_i} G(x, y; \tau_i) - F(x, y; \tau_i)\left.\frac{\mathrm{d}G(x, y; t)}{\mathrm{d}t}\right|_{t=\tau_i},$$

requiring only the evaluations of $P(t)$ and $Q(t)$. The derivatives of F and G at $t = \tau_i$ depend only on the derivatives of $P(t)$ and $Q(t)$ at $t = \tau_i$ and these are computed by using formula (10). Notice that

$$\begin{aligned}\mathbf{B}(x, y) = {} & U_1x^3 + U_2x^2y + U_3xy^2 + U_4y^3 + \\ & + V_1x^2 + V_2xy + V_3y^2 + W_1x + W_2y + W_3,\end{aligned}$$

where the matrices U_i, V_j, and W_k are of size $2n$ and depend on t only.

The implicit equation of the offset curve γ_d is

$$f(x, y) = \det(\mathbf{B}(x, y)) = 0. \tag{23}$$

In order to determine its topology we adapt the method presented in [10] that was summarized in Section 2. To accomplish this goal we need the description of $f(x, y)$ “á la Lagrange”: choose a collection of $\delta(\gamma) + 1$ real numbers as y–nodes, and a collection of $2\delta(\gamma) + 1$ real numbers as x–nodes:

$$\begin{aligned}\sigma &:= (\sigma_1 < \sigma_2 < \ldots < \sigma_{\delta(\gamma)+1}),\\ \rho &:= (\rho_1 < \rho_2 < \ldots < \rho_{2\delta(\gamma)+1}).\end{aligned}$$

The corresponding description by values of $f(x, y)$ is given by

$$f(\rho_k, \sigma_i) = \det(\mathbf{B}(\rho_k, \sigma_i)),$$

with $1 \leq k \leq 2\delta(\gamma) + 1$ and $1 \leq i \leq \delta(\gamma) + 1$.

Example 7 *Let γ be the curve used in Example 2. We want to determine the topology of the offset curve at distance $d = 1$. The polynomial $f(x, y)$ in equation (23) is of degree 10 in x and y. The matrices $(\mathbf{C}_0, \mathbf{C}_1)$ in the companion matrix pencil have size 220. The topology graph is displayed in figure 6.*

There are two extraneous isolated points (see Figure 7) with coordinates

$$(-1.968105008837924, 0.308333749431888),$$

$$(1.553604112963281, 1.632520632499762).$$

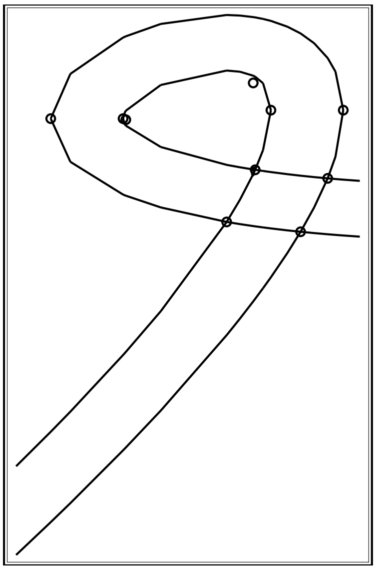

Fig. 6. The topology of the offset at distance 1 in Example 7.

These points are solutions of the implicit equation corresponding to complex and non real values of the parameter t producing real points (see [17] and section 3.1).

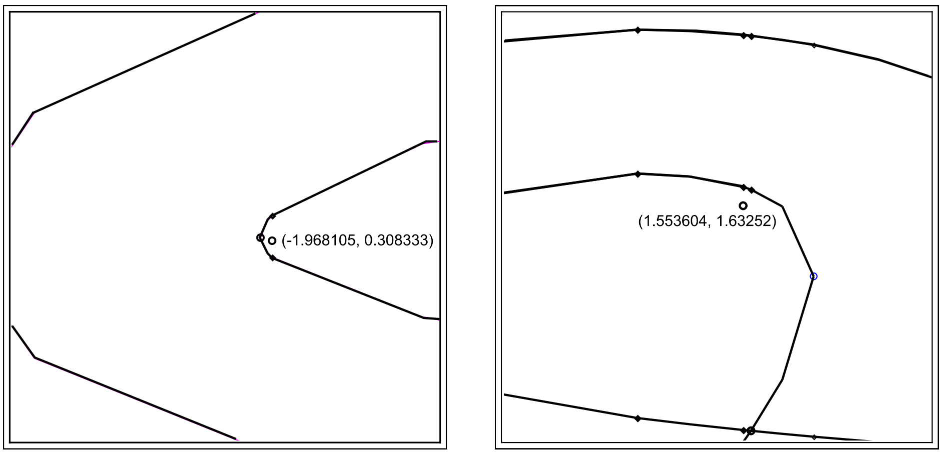

Fig. 7. Zoom of extraneous isolated points in the offset curve of Example 7.

The method explained in this section produces as output, in addition to the graph representing the topology of the offset curve, the list of the critical points, which includes the singular points. This can be very useful for offset trimming, and for plotting purposes.

3.6.2 Rational case

Assume now that the generator curve is parametrized by

$$\left(\frac{P(t)}{R(t)}, \frac{Q(t)}{R(t)}\right),$$

where $P(t)$, $Q(t)$ and $R(t)$ are polynomials in $\mathbb{P}_n$, and $\gcd(P(t), Q(t), R(t)) = 1$. The equations for the d-offset are in this case

$$\begin{aligned} F(x, y; t) &:= (R(t)x - P(t))^2 + (R(t)x - Q(t))^2 - d^2R(t)^2 = 0, \\ G(x, y; t) &:= (R(t)P'(t) - R'(t)P(t))(R(t)x - P(t)) \\ &\qquad\qquad + (R(t)Q'(t) - R'(t)Q(t))(R(t)y - Q(t)) = 0. \end{aligned}$$

In this case, $3n + 1$ nodes τ_i are necessary, and the degree of the offset is bounded by $6n - 4$. In addition, we must ask that $R(\tau_i) \neq 0$ and the values $R(\tau_i)P'(\tau_i) - R'(\tau_i)P(\tau_i)$, $R(\tau_i)Q'(\tau_i) - R'(\tau_i)Q(\tau_i)$, cannot be simultaneously zero. Note that the number of values of t not satisfying these conditions is finite.

Once these conditions are verified, we construct $\mathbf{B}(x, y)$, the Bézout matrix of $F(x, y; t)$ and $G(x, y; t)$ (with respect to t), analogously to the previous case. The implicit equation of the offset curve will be defined again by the equation (23). The remaining steps of the method are similar, using the new bound $\delta(\gamma) \leq 6n - 4$.

4 Solving by values the point position and intersection problems for ruled and ringed surfaces, and surfaces of revolution

The point position problem for a surface asks if a given point $(\xi, \eta, \zeta) \in \mathbb{R}^3$ lies on a prescribed rational surface $\mathcal{T}$. We analyze here how to solve this problem when the surface is presented by values and $\mathcal{T}$ is either a surface of revolution, a ruled surface or a ringed surface. We discuss also how to intersect a space curve or a rational surface with these type of surfaces when all objects are presented by values.

4.1 Is a point (ξ, η, ζ) in a surface of revolution? How to intersect a surface of revolution with a space curve

Let $\mathcal{S}$ be a surface of revolution with parametric equations:

$$S(s,u) = \left(\varphi(s)\frac{1-u^2}{1+u^2}, \varphi(s)\frac{2u}{1+u^2}, \psi(s)\right), \tag{24}$$

where the parametric equations of the generating curve

$$\varphi(s) = \frac{n_\varphi(s)}{d_\varphi(s)}, \qquad \psi(s) = \frac{n_\psi(s)}{d_\psi(s)},$$

are rational functions, whose numerators and denominators are known by values.

A point $(\xi, \eta, \zeta) \in \mathbb{R}^3$ is in $\mathcal{S}$, if and only there exists $s \in \mathbb{R}$ such that (see [7,18])

$$\begin{aligned} \xi^2 + \eta^2 &= \varphi^2(s) = \left(\frac{n_\varphi(s)}{d_\varphi(s)}\right)^2, \\ \zeta \;=\; \psi(s) &= \frac{n_\psi(s)}{d_\psi(s)}, \end{aligned}$$

whence,

$$\begin{aligned} f(\xi,\eta;s) &:= d_\varphi^2(s)(\xi^2+\eta^2) - n_\varphi^2(s) = 0, \\ g(\zeta;s) &:= \qquad\qquad d_\psi(s)\zeta - n_\psi(s) = 0. \end{aligned} \tag{25}$$

If the degrees of $n_\varphi(s)$, $d_\varphi(s)$, $n_\psi(s)$ and $d_\psi(s)$ are bounded by n, then, by computing the Bézout matrix $\mathbf{B}$ of $f(\xi,\eta;s)$ and $g(\eta;s)$ with respect to the s-nodes $\sigma = (\sigma_1, \ldots, \sigma_{2n+1})$, we can determine if (ξ,η,ζ) lies on $\mathcal{S}$ or not: it is enough to determine the singular values of $\mathbf{B}$. The nodes must be chosen such that $d_\varphi(\sigma_i) \neq 0$, $d_\psi(\sigma_i) \neq 0$, for $1 \leq i \leq 2n+1$.

If we want to intersect $\mathcal{S}$ with a space curve $\mathcal{C}$ given by the parameterization $(P(t)/D(t), Q(t)/D(t), R(t)/D(t))$ (with $P(t)$, $Q(t)$, $R(t)$ and $D(t)$ polynomials also known by values) then it is enough to consider the equations (see [18])

$$\begin{aligned} \overline{f}(t;s) &:= d_\varphi^2(s)\left(P(t)^2 + Q(t)^2\right) - n_\varphi^2(s)D(t)^2 = 0, \\ \overline{g}(t;s) &:= \qquad\qquad\qquad d_\psi(s)R(t) - n_\psi(s)D(t) = 0. \end{aligned}$$

Evaluating $\overline{f}(t;s)$ and $\overline{g}(t;s)$ in the s-nodes in σ allows to determine $\mathbf{B}(t) = \mathbf{Bez}_{\overline{f}(t;s),\overline{g}(t;s)}$. Using now the t-nodes and the values of $P(t)$, $Q(t)$, $R(t)$ and $D(t)$ we solve the corresponding polynomial eigenvalue problem for $\mathbf{B}(t)$. For each eigenvalue t^*, we obtain the corresponding value of s^* by taking moments (see Theorem 2).

If $\deg_s(\overline{f}) \neq \deg_s(\overline{g})$, and $a(t)$ is the principal coefficient of the polynomial with higher degree ($\overline{f}$ or $\overline{g}$) as a polynomial in s, then some generalized eigenvalues can be produced by the roots of $a(t)$ (see Remark 1). Thus, we check whether each (t^*, s^*) is a true real solution of $\overline{f}(t; s) = 0, \overline{g}(t; s) = 0$ (i.e., an intersection point between $\mathcal{S}$ and $\mathcal{C}$) or not (see the Example 8).

Example 8 *Let S be the surface of revolution parametrized by (24), with*

$$\varphi = \frac{2s^2 + 4}{s^2 + 1}, \quad \psi = s^3,$$

and the space curve

$$\mathcal{C}(t) = \left(\frac{t^2 + t - 3}{t - 2}, \frac{-t^3 - 1}{t - 2}, \frac{2t^2 + 3t - 5}{t - 2} \right).$$

In this example, $\deg_t(\overline{f}) + \deg_t(\overline{g}) = 8$, and $\max(\deg_s(\overline{f}), \deg_s(\overline{g})) = 4$, so we chose nodes $(-4, -3, -2, -1, 0, 1, 2, 3, 4)$ for t, and $(-2, -1, 0, 1, 2)$ for s. The size of the companion matrix pencil is 32. There are four generalized real eigenvalues:

$$[-2.58974079576, -2.09173222969, 0.918128416693, 1.13850625658].$$

Only the first and the fourth eigenvalues give real intersection points; the other two are roots of the principal coefficient of $\overline{f}$. The corresponding points, lifted using the parametrization of $\mathcal{C}(t)$, are:

$$\begin{array}{l} (-0.24337247857285, -3.5663807492767, -0.14037663995668), \\ (\ 0.65618264955832,\ 2.8737618116269,\ -1.1699583079099\ ). \end{array}$$

See the figure 8.

4.2 Is a point (ξ, η, ζ) in a ruled surface? How to intersect a ruled surface with a space curve

Consider the ruled surface $\mathcal{R}$ given by:

$$\mathbf{R}(u, s) = \mathbf{C}(u) + s\,\mathbf{a}(u),$$

where $\mathbf{C} = (C_x, C_y, C_z)$ and $\mathbf{a} = (a_x, a_y, a_z)$ are vector valued functions with rational components, whose numerators and denominators are known by values. Assuming that $\mathbf{a}$ is not identically zero, and without loss of generality,

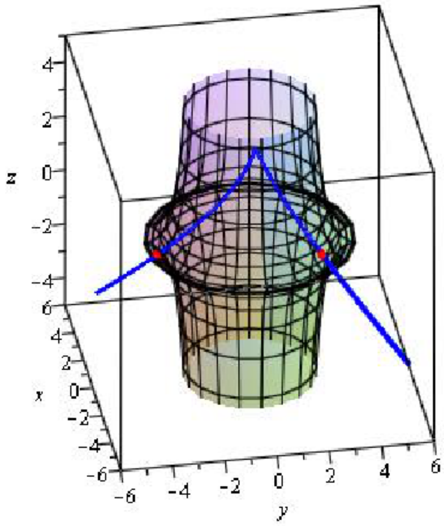

Fig. 8. Intersection of a surface of revolution and a space curve.

that $a_z(u) \not\equiv 0$, it follows that

$$f(\xi, \zeta; u) := a_z(u)(\xi - C_x(u)) - (\zeta - C_z(u))a_x(u) = 0,$$
$$g(\eta, \zeta; u) := a_z(u)(\eta - C_y(u)) - (\zeta - C_z(u))a_y(u) = 0.$$

If the degrees of $\mathbf{C}(u)$ and $\mathbf{a}(u)$ are bounded by n, then by computing the Bézout matrix $\mathbf{B}$ of $f(\xi, \zeta; u)$ and $g(\eta, \zeta; u)$ with respect to the u-nodes $\sigma = (\sigma_1, \ldots, \sigma_{2n+1})$, we can determine if (ξ, η, ζ) lies on $\mathcal{R}$ or not: it is enough to determine the singular values of $\mathbf{B}$.

If we want to intersect $\mathcal{R}$ with a space curve $\mathcal{C}$ given by the parameterization $(P(t)/D(t), Q(t)/D(t), R(t)/D(t))$ (with $P(t)$, $Q(t)$, $R(t)$ and $D(t)$ polynomials, also known by values) then it is enough to consider the equations

$$\overline{f}(t; u) := a_z(u)(P(t) - D(t)C_x(u)) - (R(t) - D(t)C_z(u))a_x(u) = 0,$$
$$\overline{g}(t; u) := a_z(u)(Q(t) - D(t)C_y(u)) - (R(t) - D(t)C_z(u))a_y(u) = 0.$$

Evaluating $\overline{f}(t; u)$ and $\overline{g}(t; u)$ in the u–nodes in σ allows to determine $\mathbf{B}(t) = \mathbf{Bez}_{\overline{f}(t;u), \overline{g}(t;u)}$. Using now the t–nodes and the values of $P(t)$, $Q(t)$, $R(t)$ and $D(t)$ we solve the corresponding polynomial eigenvalue problem for $\mathbf{B}(t)$ whose real and finite generalized eigenvalues produce the values of t at the intersection points between $\mathcal{R}$ and $\mathcal{C}$.

As in section 4.1, if the $\deg_u(\overline{f}) \neq \deg_u(\overline{g})$, there might be eigenvalues produced by the roots of the principal coefficient of the polynomial with highest degree in s, (see Remark 1). Thus, we must check that each obtained point belongs to the surface $\mathcal{R}$.

Remark 5 *The same approach used in this section to solve the point position problem for surfaces of revolution and ruled surfaces (or for intersecting them with space curves) can be applied to rational ringed surfaces. A rational ringed surface $\mathcal{A}$ is determined by the directrix*

$$\mathbf{C}(u) = \left(\frac{C_1(u)}{C_0(u)}, \frac{C_2(u)}{C_0(u)}, \frac{C_3(u)}{C_0(u)}\right),$$

the normal

$$\mathbf{N}(u) = \left(\frac{N_1(u)}{N_0(u)}, \frac{N_2(u)}{N_0(u)}, \frac{N_3(u)}{N_0(u)}\right),$$

and the radius

$$r(u) = \frac{r_1(u)}{r_0(u)},$$

where $C_i(u)$, $N_i(u)$ and $r_i(u)$ are polynomials. A point (ξ, η, ζ) lies on $\mathcal{A}$ if and only if there exists $u \in \mathbb{R}$ such that

$$\left(\xi - \frac{C_1(u)}{C_0(u)}\right)\frac{N_1(u)}{N_0(u)} + \left(\eta - \frac{C_2(u)}{C_0(u)}\right)\frac{N_2(u)}{N_0(u)} + \left(\zeta - \frac{C_3(u)}{C_0(u)}\right)\frac{N_3(u)}{N_0(u)} = 0,$$

$$\left(\xi - \frac{C_1}{C_0}\right)^2 + \left(\eta - \frac{C_2(u)}{C_0(u)}\right)^2 + \left(\zeta - \frac{C_3(u)}{C_0(u)}\right)^2 = \left(\frac{r_1(u)}{r_0(u)}\right)^2.$$

4.3 Computing by values the section of a parametric surface

By using an example we analyze how to determine the topology of the intersection between a parametric surface and a plane, by using only the evaluation of the equations in the parameterization.

The parametric equations of the bicubic surface $\mathcal{B}$ are:

$$\begin{aligned}
x(u,v) &= 3v(v-1)^2 + (u-1)^3 + 3u,\\
y(u,v) &= 3u(u-1)^2 + v^3 + 3v,\\
z(u,v) &= -3u(u^2-5u+5)v^3 - 3(u^3+6u^2-9u+1)v^2\\
&\qquad + v(6u^3+9u^2-18u+3) - 3u(u-1).
\end{aligned}$$

The implicit equation of $\mathcal{B}$ has the following structure:

$$\mathcal{H}_{\mathcal{B}}(x,y,z) = z^9 + \sum_{i=1}^{9} r_i(x,y) z^{9-i}.$$

The technique presented in [19] determines the coefficients of $\mathcal{H}_{\mathcal{B}}(x,y,z)$ in non expanded form: for example, the first two coefficients in $\mathcal{H}_{\mathcal{B}}(x,y,z)$ are:

$$
\begin{aligned}
\mathbf{r}_1(x,y) &= -\tfrac{233469x}{2048}+\tfrac{188595y}{2048}-\tfrac{112832595}{262144}-\tfrac{81x^2}{64}+\tfrac{135xy}{32}-\tfrac{81y^2}{64}, \\
\mathbf{r}_2(x,y) &= -\tfrac{54187594407}{16777216}x^2+\tfrac{48101467761}{8388608}xy-\tfrac{38812918311}{16777216}y^2 \\
&\quad -\tfrac{20972672709381}{536870912}x+\tfrac{17975329363179}{536870912}y+\tfrac{1215}{2048}x^3y-\tfrac{4779}{4096}x^2y^2 \\
&\quad +\tfrac{14456151}{65536}x^2y+\tfrac{1215}{2048}xy^3-\tfrac{13181049}{65536}xy^2-\tfrac{729}{8192}x^4-\tfrac{4105971}{65536}x^3 \\
&\quad -\tfrac{729}{8192}y^4+\tfrac{3129597}{65536}y^3-\tfrac{226569919823911171}{137438953472}+\tfrac{1}{2}\mathbf{r}_1(x,y)^2.
\end{aligned}
$$

showing that $\mathbf{r}_1(x,y)$ appears again in $\mathbf{r}_2(x,y)$. The same happens with many other subexpressions in the remaining coefficients of the implicit equation of the surface $\mathcal{B}$.

If we want to intersect $\mathcal{B}$ with another parametric surface $\mathcal{A}$ given by the parameterization

$$x = \frac{X(s,t)}{W(s,t)}, \quad y = \frac{Y(s,t)}{W(s,t)}, \quad z = \frac{Z(s,t)}{W(s,t)},$$

then we must analyze the real algebraic plane curve defined by

$$f(s,t) = \mathcal{H}_{\mathcal{B}}\left(\frac{X(s,t)}{W(s,t)}, \frac{Y(s,t)}{W(s,t)}, \frac{Z(s,t)}{W(s,t)}\right) = 0. \tag{26}$$

If we apply to this problem the approach introduced in this paper, we avoid the expansion of the expression in (26), which is probably very complicated, and we avoid computing the discriminant of $f(s,t)$ with respect to t, which will be a polynomial in s with a very high degree.

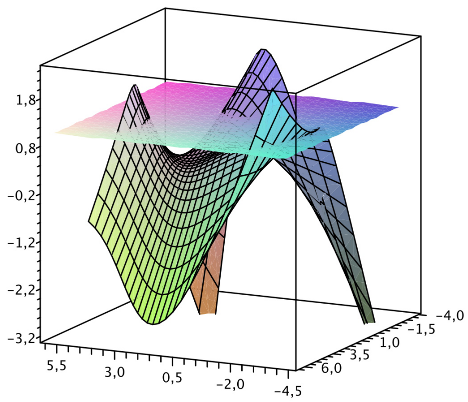

Fig. 9. The surface $\mathcal{B}$ and the plane $z = 1$

Thus, computing the intersection between $\mathcal{B}$ and the plane $z = 1$ (see Figure

9) requires the topological analysis of the curve

$$g(x,y) = \mathcal{H}_{\mathcal{B}}(x,y,1) = 0,$$

which is a polynomial of degree 18 in x and of degree 18 in y. In this case, the Bézout matrix $\mathbf{B}(x)$ of $g(x,y)$ and $g_y(x,y)$ (with respect to y) has dimension 18 and degree 38 (as a polynomial matrix in x). Thus, the x–coordinates of the critical points for $g(x,y) = 0$ are the real and finite generalized eigenvalues of the matrix pencil determined by $\mathbf{B}(x)$ by using Definition 6 (with dimension equal to 684).

For this concrete example, the discriminant of $g(x,y)$ (with respect to y) has degree 234 and 22 different real roots. The approach here presented produces the topology presented in Figure 10 (by using `Maple`) where several isolated points can be easily visualized (corresponding to extraneous components, since they appear as real points of $\mathcal{B}$ attained when x and y take complex and non real values, but producing a point with real coordinates; see [17]).

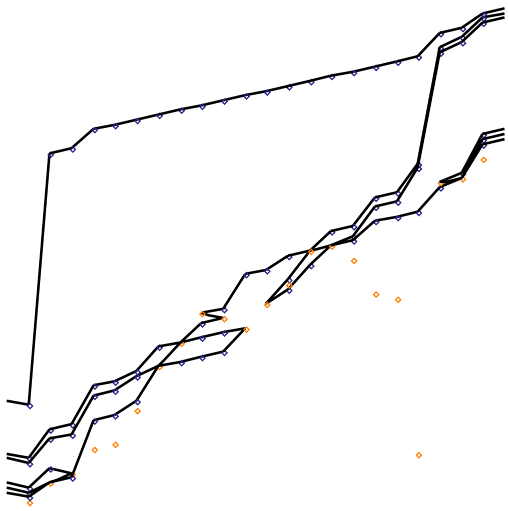

Fig. 10. The topology of the intersection curve of the surface $\mathcal{B}$ and the plane $z = 1$, from section 4.3 (see figure 9).

4.4 *Computing by values the intersection of a parametric surface and a structured surface*

We show here how to analyze the intersection of a parametric surface $\mathcal{T}$ given by the parameterization

$$x = \frac{X(u,v)}{W(u,v)}, \quad y = \frac{Y(u,v)}{W(u,v)}, \quad z = \frac{Z(u,v)}{W(u,v)},$$

and a structured surface $\mathcal{S}$ (i.e., of revolution, ruled or ringed). We analyze only the case of $\mathcal{S}$ being a surface of revolution since the other two cases are very similar to the case we are going to consider here.

If $\mathcal{S}$ is the surface of revolution in (24) (replacing u by t) then, according to the equations in (25), $(u, v) \in \mathbb{R}^2$ produces an intersection point between $\mathcal{T}$ and $\mathcal{S}$ if and only if there exists $s \in \mathbb{R}$ such that $f(u, v; s) = 0$ and $g(u, v; s) = 0$ where

$$
\begin{aligned}
f(u, v; s) &:= d_\varphi^2(s)\,(X(u, v)^2 + Y(u, v)^2) - n_\varphi^2(s) W(u, v)^2,\\
g(u, v; s) &:= d_\psi(s) Z(u, v) - n_\psi(s) W(u, v).
\end{aligned}
$$

Let n be an upper bound for the degrees of $n_\varphi(s)$, $d_\varphi(s)$, $n_\psi(s)$ and $d_\psi(s)$ and $\sigma = (\sigma_1 < \sigma_2 < \ldots < \sigma_{2n+1})$ a collection of $2n + 1$ nodes (since the degree in s of $f(u, v; s)$ and $g(u, v; s)$ is bounded by $2n$). Then the Bézout matrix of $f(u, v; s)$ and $g(u, v; s)$, $\mathbf{B}(u, v) = \mathbf{Bez}_{f(u,v;s),\, g(u,v;s)}$ (with respect to s) is determined by evaluating $n_\varphi(s)$, $d_\varphi(s)$, $n_\psi(s)$ and $d_\psi(s)$ in σ. The matrix $\mathbf{B}(u, v)$ depends on $X(u, v)$, $Y(u, v)$, $Z(u, v)$ and $W(u, v)$ and can be evaluated easily by evaluating these polynomials.

The curve defined by the implicit equation $h(u, v) = \det(\mathbf{B}(u, v)) = 0$ is a representation of the intersection curve between $\mathcal{T}$ and $\mathcal{S}$ in the uv domain, that must be lifted to $\mathbb{R}^3$ by using the parametrization of $\mathcal{T}$. A first step into this direction is to determine the topology of $h(u, v) = 0$ and this is accomplished by using the method described in Section 2. If m_1, m_2 are bounds for the degrees in u and v of $\det(\mathbf{B}(u, v))$, respectively, then we need $2m_1 + 1$ nodes for u and $m_2 + 1$ for v

$$
\begin{aligned}
\tau &:= (\tau_1 < \tau_2 < \ldots < \tau_{2m_1+1}),\\
\rho &:= (\rho_1 < \rho_2 < \ldots < \rho_{m_2+1}),
\end{aligned}
$$

to get the corresponding Lagrange description of $h(u, v)$, given by

$$
h(\tau_i, \rho_k) = \det(\mathbf{B}(\tau_i, \rho_k)),
$$

with $1 \leq i \leq 2m_1 + 1$ and $1 \leq i \leq m_2 + 1$. This is all the data we need in order to determine by values the topology of $h(u, v) = 0$ by using the method presented in Section 2 (see [10] for more details).

In the case we are considering here, of a surface of revolution, we can take

$$
\begin{aligned}
m_1 &= 2n \max(2\deg_u(X(u, v)), 2\deg_u(Y(u, v)), 2\deg_u(W(u, v)), \deg_u(Z(u, v))),\\
m_2 &= 2n \max(2\deg_v(X(u, v)), 2\deg_v(Y(u, v)), 2\deg_v(W(u, v)), \deg_v(Z(u, v))).
\end{aligned}
$$

Example 9 *Consider the torus given as a surface of revolution, with para-*

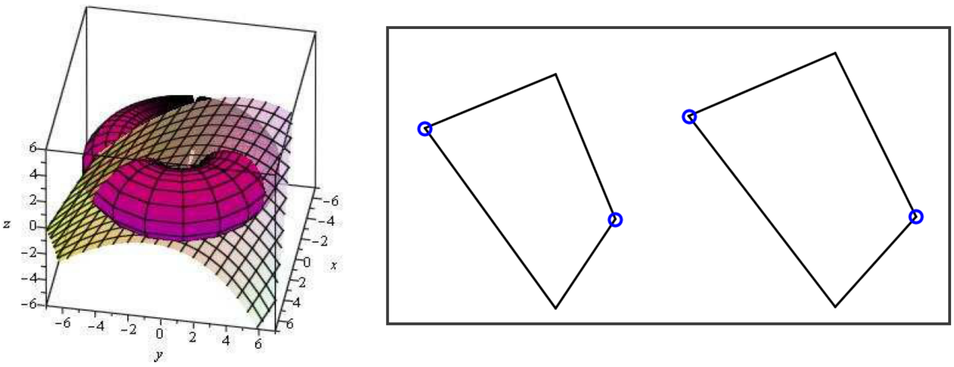

Fig. 11. The topology of $\mathcal{S} \cap \mathcal{T}$ in example 9.

metrization

$$\mathcal{S}(s,t) = \left(\varphi(s)\frac{1-t^2}{1+t^2}, \varphi(s)\frac{2t}{1+t^2}, \psi(s) \right), \tag{27}$$

where

$$\varphi(s) = \frac{s^2+6}{s^2+1}, \quad \psi(s) = \frac{4s}{s^2+1}. \tag{28}$$

We compute the topology of the intersection curve of $\mathcal{S}$ with the following ruled surface:

$$\mathcal{T}(u,v) = (u^2, 3u+v+1, 2v),$$

by using the method of Section 2. The degrees of the bivariate function $h(u,v) = \det(\mathbf{B}(u,v))$ are $\deg_u(h) = 8$, and $\deg_v(h) = 4$, so we use 17 nodes for u and 5 nodes for v. The algebraic curve defined by $h(u,v) = 0$ has the following four critical points:

$$\begin{array}{l}
(-1.9406563931821103492, 0.29394335943779257963), \\
(-0.58539520958661413941, -0.35709273559911053741), \\
(-0.064196535862312867021, 0.37138871973471026866), \\
(\,1.5556926523873762013, -.34217143375501427059),
\end{array}$$

which are obtained by solving a generalized eigenvalues problem with companion matrix pencil of size 72, and taking moments. The figure 11 shows the surfaces and the graph representing the topology of the curve $h(u,v) = 0$ in the uv-plane.

As part of the output of the method used to determine the topology of the intersection curve, we get the coordinates and parameter values of the critical points, the simple points in the critical lines, and the curve points in intermediate non-critical lines (see Section 2). This information can be very useful, for example, as a starting point for plotting the intersection curve.

5 Further work

We are currently considering how to extend this approach to answer questions involving the elimination of two parameters from three equations. This is the case, for example when dealing with the question: is $(\xi, \eta) \in \gamma_d$ when $\gamma := \{(\alpha, \beta) \in \mathbb{R}^2 \colon f(\alpha, \beta) = 0\}$? If the considered real algebraic plane curve γ is presented by its implicit equation $f(x, y) = 0$ then a point (ξ, η) lies on γ_d, the offset to γ at distance d, if and only if there exists $(x, y) \in \mathbb{R}^2$ such that

$$\begin{aligned} (\xi - x)^2 + (\eta - y)^2 - d^2 &= 0, \\ f(x, y) &= 0, \\ f_x(x, y)(\xi - x) - f_y(x, y)(\eta - y) &= 0, \end{aligned} \tag{29}$$

where $f_x(x, y)$ and $f_y(x, y)$ denote the partial derivatives of f, with respect to x and y, respectively. An approach based on the so-called Dixon resultant (see [13]), can provide a matrix formulation by values, allowing to eliminate two parameters from three equations.

It is worth studying the possibility of constructing by values other versions of the Bézout matrix, such as the one mentioned in Remark 1, together with the analysis of the advantages and disadvantages of working with such a version of Bézout matrix.

Numerical aspects deserve some further consideration; one problem is related to the proper application of numerical technics to take full advantage of the matrix structure, and another is how to proceed with singular situations when dealing with the computed generalized eigenvalues (how to deal with clusters of generalized eigenvalues is a relevant question here).

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