A Müntz Type Theorem for a Family of Corner Cutting Schemes

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Abstract

By identifying a family of corner cutting schemes as a dimension elevation process of Gelfond-Bézier curves, we give a Müntz type condition for the convergence of the generated control polygons to the underlying curve. The surprising emergence of the Müntz condition in the problem raises the question of a possible connection between the density questions of nested Chebyshev spaces and the convergence of the corresponding dimension elevation algorithms.

Keywords: Corner cutting schemes, Bézier curves, Gelfond-Bézier curves, Müntz spaces, density of Müntz spaces.

1. Introduction

Let n be a fixed positive integer and let $0 < r_1 < r_2 < ...r_n < r_{n+1}... < r_m < ...$ be an infinite strictly increasing sequence of positive real numbers. Given a polygon $(P_0, P_1, ..., P_n)$ in $\mathbb{R}^s, s \geq 1$, we apply the following corner cutting scheme : For $i = 0, 1, ..., n$, we set $P_i^0 = P_i$ and for $j = 1, 2, ...,$ we construct iteratively new polygons $(P_0^j, P_1^j, ..., P_{n+j}^j)$ using the inductive rule

$$P_0^j = P_0^{j-1} \quad P_{n+j}^j = P_{n+j-1}^{j-1} \tag{1}$$

and for $i = 1, ..., n + j - 1$

$$P_i^j = \frac{r_i}{r_{n+j}} P_{i-1}^{j-1} + \left(1 - \frac{r_i}{r_{n+j}}\right) P_i^{j-1} \tag{2}$$

Figure 1 shows the first iteration of the corner cutting scheme on a planar polygon (P_0, P_1, P_2, P_3) for $n = 3$ and positive real numbers $0 < r_1 < r_2 < r_3 < r_4$. In the case the real numbers r_i are given by $r_i = i$ for every index i, then we recognize the degree elevation algorithm of Bézier curves, and in which it is well known that the control polygons of the elevated degree converges to the underlying Bézier curve. Consider, now, the case in which $r_i = i$ for $i = 1, ..., n$

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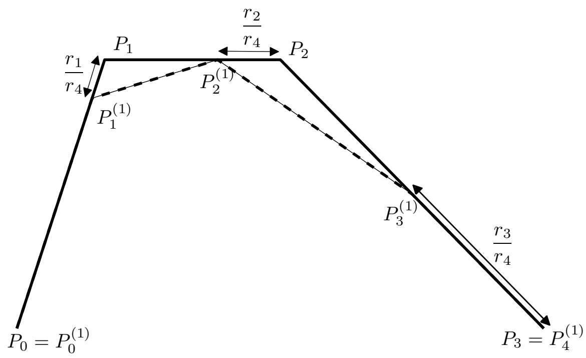

Figure 1: The first iteration of the corner cutting scheme (1) and (2) for the parameters $n = 3, r_1 = 2, r_2 = 5, r_3 = 7$ and $r_4 = 14$.

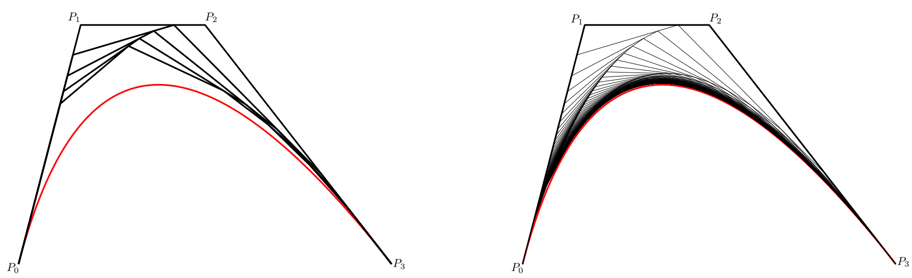

Figure 2: The sequence of polygons generated by the corner cutting scheme (1) and (2) and parameters $n = 3, r_1 = 1, r_2 = 2, r_3 = 3$ and $r_j = 2j$ for $j \geq 4$. (left, four iterations of the scheme; right, 100 iterations of the scheme). The red curve is the Bézier curve associated with the control polygon (P_0, P_1, P_2, P_3).

and $r_i = 2i$ for $i > n$. Figure 2 (left) shows the generated polygons from the scheme (1) and (2) from four iterations, while Figure 2 (right) shows the generated polygons from 100 iterations. The figure suggests the convergence of the generated polygons to the Bézier curve with control points $(P_0, P_1, ...P_n)$. Consider, now, the case in which $r_i = i$ for $i = 1, ..., n$, while $r_i = i^2$ for $i > n$. Figure 3 (left) shows the generated polygons from four iterations, while Figure 3 (right) shows the obtained polygons after 100 iterations. It is clear from the figure that the limiting polygon does not converge to the Bézier curve with control points $(P_0, P_1, ..., P_n)$. As we will exhibit in this work, the main difference between the example of Figure 2 and the one of Figure 3 is the fact that in the former we have $\sum_{i=1}^{\infty} 1/r_i = \infty$, while in the latter we have $\sum_{i=1}^{\infty} 1/r_i < \infty$. We will show, as a particular case of our main result, that if $r_i = i$ for $i = 1, ..., n$, and $\lim_{s\to\infty} r_s = \infty$, then the limiting polygon generated from the corner cutting scheme (1) and (2) converges to the Bézier curve with control points $(P_0, P_1, ..., P_n)$ if and only if the real numbers r_i satisfy $\sum_{i=1}^{\infty} 1/r_i = \infty$. The emergence of the limiting polygon as a Bézier curve in the case $r_i = i$ for $i = 1, ..., n$ can be hinted to as follows : the linear space formed by the monomials with exponents the numbers $r_i = i$ for $i = 1, ..., n$ and extended by a constant is given by $E = span(1, t, t^2, ..., t^n)$; which is the linear space of

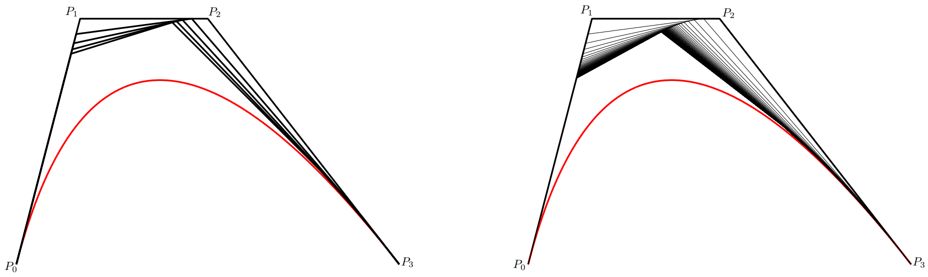

Figure 3: The sequence of polygons generated by the corner cutting scheme (1) and (2) and parameters $n = 3, r_1 = 1, r_2 = 2, r_3 = 3$ and $r_j = j^2$ for $j \geq 4$. (left, four iterations of the scheme; right, 100 iterations of the scheme). The red curve is the Bézier curve associated with the control polygon (P_0, P_1, P_2, P_3).

polynomial of degree n. The space E has a special basis (the Bernstein basis) in which the notion of Bézier curve can be defined. For general real numbers $r_i, i = 1, ..., n$, and imitating the previous construction, we obtain the Müntz space $F = span(1, t^{r_1}, t^{r_2}, ..., t^{r_n})$. The linear space F also possess a special basis (the Gelfond-Bernstein basis) first defined by Hirschman and Widder [6] and extended by Gelfond [5], which is in a certain sense a generalization of the Bernstein basis to the Müntz space F (in the case $r_i = i, i = 1, ..., n$, the Gelfond-Bernstein basis coincide with the Bernstein basis). Using the Gelfond-Bernstein basis, we can canonically define the notion of Gelfond-Bézier curve with control points $(P_0, P_1, ..., P_n)$. Now, consider, for example, the limiting polygon of the corner cutting scheme (1) and (2) for the case $n = 3$ and in which $r_1 = 2, r_2 = 4, r_3 = 5$ and $r_i = 2i$ for $i > 3$. Figure 4 shows the generated polygons from 100 iterations and also shows the Gelfond-Bézier curve associated with the Müntz space $F = span(1, t^{r_1}, t^{r_2}, t^{r_3}) = span(1, t^2, t^4, t^5)$ and control polygon (P_0, P_1, P_2, P_3). The figure suggests that the limiting polygon converges to the Gelfond-Bézier curve. Therefore, The main objective of this paper is to, effectively, prove the following

Theorem 1. *Let n be a fixed positive integer and let $0 < r_1 < r_2 < ...r_n < r_{n+1} < ... < r_m < ...$ be an infinite strictly increasing sequence of positive real numbers such that $\lim_{s\to\infty} r_s = \infty$. Then the limiting polygon generated from a polygon $(P_0, P_1, ..., P_n)$ in $\mathbb{R}^s, s \geq 1$ using the corner cutting scheme (1) and (2) converges (pointwise and uniformly) to the Gelfond-Bézier curve associated with the Müntz space $(1, t^{r_1}, t^{r_2}, ..., t^{r_n})$ and control polygon $(P_0, P_1, ..., P_n)$ if and only if the real numbers r_i satisfy the condition*

$$\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty \tag{3}$$

Let us contrast our theorem with the celebrated original Müntz theorem on the density of Müntz spaces [10]

Theorem 2. (Müntz Theorem) *Let $0 < r_1 < r_2 < ...r_n < r_{n+1} < ... < r_m < ...$ be an infinite strictly increasing sequence of positive real numbers such that $\lim_{s\to\infty} r_s = \infty$. The Müntz space $span(1, t^{r_1}, ..., t^{r_m}, ...)$ is a dense subset of $C([0,1])$ (the linear space of continuous functions on $[0,1]$ endowed with the*

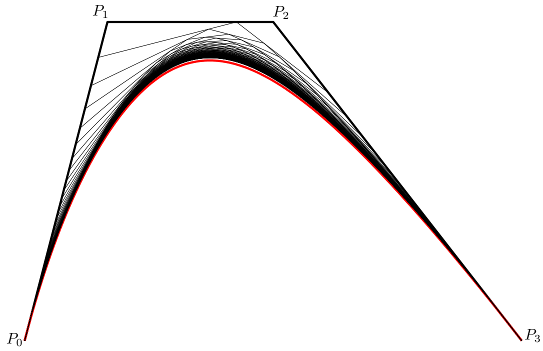

Figure 4: The sequence of polygons generated from 100 iterations of the corner cutting scheme (1) and (2) and parameters $n = 3, r_1 = 2, r_2 = 4, r_3 = 14$ and $r_j = 2j + 10$ for $j \geq 4$. The red curve is the Gelfond-Bézier curve associated with the Müntz space $span(1, t^2, t^4, t^14)$ and control polygon (P_0, P_1, P_2, P_3)

uniform norm) if and only if

$$\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty$$

The emergence of the Müntz condition (3) in both of Theorem 1 and Theorem 2 is rather surprising and may suggest a deep relation between the problem of density in Müntz spaces and the convergence of corner cutting schemes. In fact, as we will show in section 2, the corner cutting scheme (1) and (2) can be interpreted as a dimension elevation algorithm of Gelfond-Bézier curves. Therefore, Theorem 1 can be restated as claiming that under the condition that the sequence $0 < r_1 < r_2 ... < r_n < ...$ satisfies $\lim_{s \to \infty} = \infty$, the density of the Müntz space $span(1, t^{r_1}, ..., t^{r_m}, ...)$ is equivalent to the convergence of the dimension elevation algorithm of Gelfond-Bézier curves to the underlying curve. We can push this analogy even further as follows : It has been shown in [1] that the Gelfond-Bernstein bases are limit of the Chebyshev-Bernstein bases of Müntz spaces over an interval $[a, 1]$ as a goes to zero. From this property, it is not hard to show that the conditions of Theorem 1 are sufficient for the convergence of the dimension elevation algorithm of a Chebyshev-Bézier curve in Müntz spaces to the underlying curve. As the Chebyshev-Bernstein bases over an interval $[a, b]$ can be defined for any linear space $E = span(1, u_1, ..., u_m)$ such that the space $DE = span(u'_1, ..., u'_m)$ is an extended Chebyshev space of order m over the interval $[a, b]$ [11, 8], we can ask for the following more general question : Let n be a fixed positive integer and let $u_1, u_2, ..., u_n, ..., u_m, ...$ be an infinite sequence of C^∞ functions over an interval $[a, b]$ such that for every $k \geq 1$, the space $E_k = span(1, u_1, u_2, ..., u_k)$ is such that $DE_k = span(u'_1, u'_2 ..., u'_k)$ is an extended Chebyshev space of order k over the interval $[a, b]$. For any function $F \in span(1, u_1, u_2, ..., u_n)$ with control polygon $(P_0, ..., P_n)$ over the interval $[a, b]$, we can define the control polygons of the dimension elevation algorithm [9] with respect to the nested spaces $E_n \subset E_{n+1} \subset .... \subset E_m \subset ...$, the question

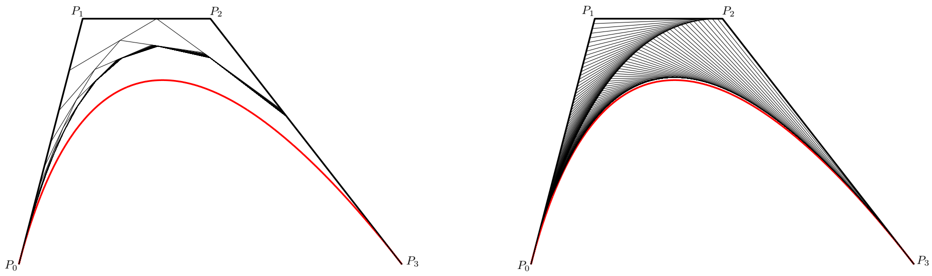

Figure 5: The sequence of polygons generated from 100 iterations of the corner cutting scheme (1) and (2) and parameters $n = 3, r_1 = 1, r_2 = 2, r_3 = 3$ and for the left figure we have $r_j = 5 - \frac{1}{j}$ for $j \geq 4$ and for the right figure, $r_j = 50 - \frac{1}{j}$ for $j \geq 4$. The blue curve is the Bézier curve associated with the control polygon (P_0, P_1, P_2, P_3).

is then

Is there a connection between the density of the space $E_\infty = span(1, u_1, u_2, ..., u_n, ..., u_m, ...)$ as a subset of $C([a,b])$ and the convergence of the associated dimension elevation algorithm to the underlying curve ? (Q)

A hypothesis of equivalence is ruled out by the following fact: the condition $\lim_{s\to\infty} r_s = \infty$ can be dropped in Müntz theorem 2, however, such condition is necessary in Theorem 1. For example, Figure 5 (left) shows the limiting polygon for the case $n = 3$, $r_1 = 1, r_2 = 2, r_3 = 3$ and $r_j = 5 - \frac{1}{j}$ for $j > 3$. The limiting polygon does not converge to the Bézier curve with control polygon (P_0, P_1, P_2, P_3). As it will be clear within this work, the main reason for the non-convergence of the dimension elevation algorithm to the underlying curve in this case is the fact that the set of control points $(\eta_i^m)_{0\leq i\leq m}$ of the function t^{r_1} with respect to the Müntz space $span(1, t^{r_1}, t^{r_2}, ..., t^{r_m})$ does not form a dense subset of the interval $[0, 1]$ as m goes to infinity. It is interesting to note that for example when $n = 3$ and the real number r_i are given by $r_1 = 1, r_2 = 2, r_3 = 3$ and $r_i = 50 - \frac{1}{i}$, the limiting polygon is very close to the underlying curve and yet does not converge to the curve, as shown in Figure 5 (right).

Regarding question (Q), we conjecture the following scenario : *If for any fixed positive integer n, the dimension elevation algorithm with respect to the nested spaces $E_n \subset E_{n+1} \subset .... \subset E_m \subset ...$ over an interval $[a, b]$ converges to the underlying Chebyshev-Bézier curve then the space E_∞ is dense as a subset of $C([a, b])$.*

The proof of Theorem 1 consists in first showing, in section 2, that the corner cutting scheme (1) and (2) can be interpreted as a dimension elevation algorithm of the Gelfond-Bézier curves. This allows us, in section 3, through a refinement of the elegant method of Prautzsch and Kobbelt [12] to prove the theorem by induction on the fixed integer n.

2. Gelfond-Bézier curves

Let f be a smooth real function defined on an interval I. For any real numbers $x_0 \leq x_1 \leq ... \leq x_n$ in the interval I, the divided difference $[x_0, ..., x_n]f$

of the function f supported at the point $x_i, i = 0, ..., n$ is recursively defined by $[x_0]f = f(x_0)$ and

$$[x_0, x_1, ..., x_n]f = \frac{[x_1, ..., x_n]f - [x_0, x_1, ..., x_{n-1}]f}{x_n - x_0} \quad \text{if} \quad n > 0. \qquad (4)$$

If some of the x_i coincide, then the divided difference $[x_0, ..., x_n]f$ is defined as the limit of (4) when the distance of the x_i become arbitrary small. A simple inductive argument show that when the x_i are pairwise distinct then we have

$$[x_0, ..., x_n]f = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)} = \frac{\begin{vmatrix} 1 & x_0 & \dots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & \dots & x_1^{n-1} & f(x_1) \\ \dots & \dots & \dots & \dots & \\ 1 & x_n & \dots & x_n^{n-1} & f(x_n) \end{vmatrix}}{V(x_0, x_1, ..., x_n)}, \qquad (5)$$

where $V(x_0, ..., x_n)$ is the Vandermonde determinant. Note that by (4) the divided difference $[x_0, x_1, ..., x_n]f$ is symmetric in the arguments $x_0, x_1, ...x_n$. Consider, now, the function $f_t(x) = t^x$, where t is viewed as a parameter. For a sequence $\Lambda = (0 = r_0, r_1, ..., r_n)$ of strictly increasing real numbers, we denote by E_Λ the Müntz space $E_\Lambda = span(t^{r_0}, t^{r_1}, ..., t^{r_n})$.

Definition 1. *For a sequence $\Lambda = (0 = r_0, r_1, ..., r_n)$ of strictly increasing real numbers, the Gelfond-Bernstein basis of the Müntz space E_Λ with respect to the interval $[0, 1]$ is defined by*

$$H_{k,\Lambda}^n(t) = (-1)^{n-k} r_{k+1}...r_n [r_k, ..., r_n] f_t \quad \text{for} \quad k = 0, ..., n-1$$

and

$$H_{n,\Lambda}^n(t) = t^{r_n}.$$

The determinant representation of the divided differences (5) shows that for $k = 0, ..., n-1$, the Gelfond-Bernstein basis can be expressed as

$$H_{k,\Lambda}^n(t) = \frac{r_{k+1} r_{k+2} ... r_n}{V(r_k, r_{k+1}, ..., r_n)} \begin{vmatrix} t^{r_k} & 1 & r_k & \dots & r_k^{n-k-1} \\ t^{r_{k+1}} & 1 & r_{k+1} & \dots & r_{k+1}^{n-k-1} \\ \dots & \dots & \dots & \dots & \\ t^{r_n} & 1 & r_n & \dots & r_n^{n-k-1} \end{vmatrix}. \qquad (6)$$

Formula (6) reiterate the fact that every function $H_{k,\Lambda}^n$, $k = 0, ..., n$ is an element of the space E_Λ. The Gelfond-Bernstein basis possesses several properties that are similar to the classical Bernstein basis over the interval $[0, 1]$. For the sequence $\Lambda = (0, 1, 2, ..., n)$, the Gelfond-Bernstein basis coincide with the classical Bernstein basis. Moreover, for any sequence $\Lambda = (0 = r_0, r_1, ..., r_n)$ of strictly increasing real numbers, and for any $k = 0, ..., n$, we have [1, 7]

$$0 \leq H_{k,\Lambda}^n(t) \leq 1 \quad \text{for any} \quad t \in [0, 1] \qquad (7)$$

and for any $t \in [0, 1]$, we have

$$\sum_{k=0}^{n} H_{k,\Lambda}^n(t) = 1. \qquad (8)$$

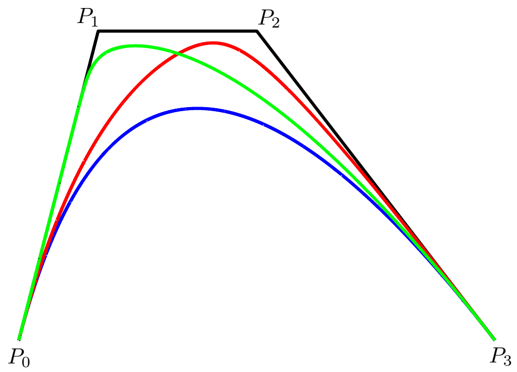

Figure 6: Gelfond-Bézier curves associated with the control polygon (P_0, P_1, P_2, P_3) and Müntz spaces : blue curve $span(1, t, t^2, t^3)$, red curve $span(1, t, t^2, t^{20})$, green curve $span(1, t^2, t^{50}, t^{100})$.

Moreover, the Gelfond-Bernstein basis is totally positive on [0,1], i.e. for any sequence $0 \leq t_0 < t_1 < ... < t_n \leq 1$, the matrix $(H^n_{k,\Lambda}(t_j))_{0\leq k,j\leq n}$ is totally positive (i.e. all its minors are nonnegative). This property gives rise to the so-called variation diminishing property of Gelfond-Bézier curve, i.e. given a Gelfond-Bézier curve Γ with parametrization

$$P(t) = \sum_{k=0}^{n} H^n_{k,\Lambda}(t)P_i \quad \text{with} \quad P_i \in \mathbb{R}^s, s \geq 1, \tag{9}$$

the number of intersections of any hyperplane in $\mathbb{R}^s$ with Γ does not exceed the number of intersection of the hyperplane with the control polygon $(P_0, P_1, ..., P_n)$. Note also that for the Gelfond-Bézier curve in (9), we have $P(0) = P_0$ and $P(1) = P_n$. Figure 6 shows examples of Gelfond-Bézier curves associated with a single control polygon (P_0, P_1, P_2, P_3) and different sequences Λ. For a more thorough study of Gelfond-Bézier curves, we refer to [1].

Lemma 1. *Let $\Lambda_1 = (0 = r_0, r_1, ..., r_n)$ and $\Lambda_2 = (0 = r_0, r_1, ..., r_n, r_{n+1})$ be two sequences of strictly increasing real numbers. Then, for $k = 0, ..., n$*

$$H^n_{k,\Lambda_1}(t) = \frac{r_{n+1} - r_k}{r_{n+1}} H^{n+1}_{k,\Lambda_2}(t) + \frac{r_{k+1}}{r_{n+1}} H^{n+1}_{k+1,\Lambda_2}(t). \tag{10}$$

Proof. From the definition of the Gelfond-Bernstein basis, the right hand side of equation (10) is given by (for $k \leq n - 1$)

$$(-1)^{n-k} r_k r_{k+1} ... r_n \left([r_{k+1}, ..., r_{n+1}]f_t - (r_{n+1} - r_k)[r_k, ..., r_{n+1}]f_t\right). \tag{11}$$

From the definition of the divided difference, we have

$$[r_{k+1}, ..., r_{n+1}]f_t - [r_k, ..., r_n]f_t = (r_{n+1} - r_k)[r_k, ..., r_{n+1}]f_t.$$

Inserting the last equation into (11) conclude the proof of the lemma for $k \leq n - 1$. For $k = n$, the left hand side of (10) is equal to

$$t^{r_{n+1}} - (r_{n+1} - r_n)[r_n, r_{n+1}]f_t = t^{r_n} = H^n_{n,\Lambda_1}(t).$$

$\square$

Let Λ_1 and Λ_2 be the two sequences given in Lemma 1, and let P be an element of the Müntz space E_{Λ_1}. As $E_{\Lambda_1} \subset E_{\Lambda_2}$, the function P can be expressed in both of the Gelfond-Bernstein bases associated with the two spaces as

$$P(t) = \sum_{k=0}^{n} H_{k,\Lambda_1}^{n}(t) P_k = \sum_{k=0}^{n+1} H_{k,\Lambda_2}^{n+1}(t) \tilde{P}_k. \tag{12}$$

Using Lemma 1 to detect the coefficients of $H_{k,\Lambda_2}^{n+1}(t)$ in the expansion (12), we readily find

Corollary 1. *The Gelfond-Bézier points $\tilde{P}_k$ in (12) are related to the Gelfond-Bézier points P_k by the relations*

$$\tilde{P}_0 = P_0, \quad \tilde{P}_{n+1} = P_n, \tag{13}$$

and for $k = 1, 2, ..., n$

$$\tilde{P}_k = \frac{r_k}{r_{n+1}} P_{k-1} + \left(1 - \frac{r_k}{r_{n+1}}\right) P_k. \tag{14}$$

Note that equations (13) and (14) describe the first iteration of the corner cutting scheme (1) and (2). Therefore, the corner cutting scheme can be can interpreted as an iterative dimension elevation of the Gelfond-Bézier curve P with respect to the nested Müntz spaces $E_{\Lambda_n} \subset E_{\Lambda_{n+1}} \subset ...E_{\Lambda_m} \subset ...$, where Λ_k refers to the sequence $\Lambda_k = (0 = r_0, r_1, ..., r_k)$.

For later use, we will need the following two lemmas, in which we omit the proofs as they can be readily obtained from the determinant representation (6) of the Gelfond-Bernstein bases.

Lemma 2. *Let $H_{k,\Lambda}^{n}, k = 0, ..., n$ be the Gelfond-Bernstein basis associated with the sequence $\Lambda = (0 = r_0, r_1, r_2, ..., r_n)$. Then, for $k = 0, ..., n$ we have*

$$tH_{k,\Lambda}^{n}(t) = \frac{\prod_{j=k+1}^{n} r_j}{\prod_{j=k+1}^{n} (r_j + 1)} H_{k+1,\Lambda_1}^{n+1}(t),$$

where $\Lambda_1 = (0 = r_0, 1, r_1 + 1, r_2 + 1, ..., r_n + 1)$

Lemma 3. *Let α be a positive real number, and let $H_{k,\Lambda_1}^{n}, k = 0, ..., n$ (resp. $H_{k,\Lambda_2}^{n}, k = 0, ..., n$) be the Gelfond-Bernstein basis associated with the sequence $\Lambda_1 = (0 = r_0, r_1, r_2, ..., r_n)$ (resp. $\Lambda_2 = (0 = \alpha r_0, \alpha r_1, \alpha r_2, ..., \alpha r_n)$. Then, for $k = 0, ..., n$, we have*

$$H_{k,\Lambda_1}^{n}(t^\alpha) = H_{k,\Lambda_2}^{n}(t).$$

3. The convergence of the dimension elevation algorithm

The fundamental idea for the proof of Theorem 1 is essentially simple, and can be viewed as a refinement of the method of Prautzsch and Kobbelt [12]. However, in practice, the simplicity of the idea is overshadowed by the complexity of the technical details. Therefore, to exhibit the fundamental idea of the proof, we will first use it for the proof the classical fact that the control polygons of the degree elevation of a Bézier curve converge to the underlying curve.

Let P be a polynomial of degree n represented in the Bernstein basis ,over the interval $[0,1]$, of degree $m > n$ as

$$P(t) = \sum_{i=0}^{m} b_i^m B_i^m(t). \tag{15}$$

By induction on n, we will prove that

$$\max_i |b_i^m - P(\frac{i}{m})| = O(\frac{1}{m}). \tag{16}$$

For $n \leq 1$, we have $b_i^m = P(i/m)$, since the Bernstein-Bézier representation has linear precision. Now, let us assume that (16) hold for polynomials of degree $n-1$. Given a polynomial P of degree n with the Bernstein representation (15), we consider the polynomial Q defined by

$$Q(t) = P(t) - \frac{t}{n} P'(t).$$

The polynomial Q is of degree $n-1$. Moreover, from (15), we have

$$Q(t) = \sum_{i=0}^{m} b_i^m B_i^m(t) - \frac{t}{n} \sum_{i=0}^{m-1} c_i^{m-1} B_i^{m-1}(t), \tag{17}$$

where we have written the derivative P' as

$$P'(t) = \sum_{i=0}^{m-1} c_i^{m-1} B_i^{m-1}(t).$$

We can give explicit expressions for the coefficients c_i^{m-1} but, as we will see, such expressions will not be needed. Now, using the fact that

$$t B_i^{m-1}(t) = \frac{i+1}{m} B_{i+1}^m(t)$$

we obtain from (17)

$$Q(t) = b_0^m B_0^m(t) + \sum_{i=1}^{m} \left(b_i^m - \frac{i}{mn} c_{i-1}^{m-1} \right) B_i^m(t).$$

The induction hypothesis on Q shows that

$$\max_i \left| \left(P(\frac{i}{m}) - b_i^m \right) - \frac{i}{mn} \left(P'(\frac{i}{m}) - c_{i-1}^{m-1} \right) \right| = O(\frac{1}{m}).$$

The last equation leads to (using the fact that $i \leq m$)

$$\max_i |P(\frac{i}{m}) - b_i^m| = O(\frac{1}{m}) + \frac{1}{n} \max_i |P'(\frac{i}{m}) - c_{i-1}^{m-1}|. \tag{18}$$

Now, the polynomial P' is also of degree $n-1$ and therefore, we can apply the induction hypothesis on P', namely we have

$$\max_i |P'(\frac{i}{m-1}) - c_i^{m-1}| = O(\frac{1}{m-1}).$$

We have

$$|P'(\frac{i}{m}) - c_{i-1}^{m-1}| \leq |P'(\frac{i}{m}) - P'(\frac{i-1}{m-1})| + |P'(\frac{i-1}{m-1}) - c_{i-1}^{m-1}|.$$

Therefore, we have

$$\max_i |P'(\frac{i}{m}) - c_{i-1}^{m-1}| \leq |\frac{i}{m} - \frac{i-1}{m-1}|E + O(\frac{1}{m}) = O(\frac{1}{m}),$$

where $E = \max_{[0,1]} |P''(x)|$. Inserting the last equation into (18) conclude the proof of (16).

Notations: In order to apply the previous idea to the case of arbitrary Müntz spaces, we will first set some notations. We define the following difference operator Δ acting on sequences as follows : If $\Lambda_m = (0 = r_0, r_1, r_2, ..., r_m)$ is a sequence of strictly increasing real numbers, then

$$\Delta\Lambda_m = (0 = r_0, r_2 - r_1, r_3 - r_1, ..., r_m - r_1).$$

For $1 \leq k \leq m-1$, we define the sequence $\Delta^k\Lambda_m$ iteratively using the equation

$$\Delta^k\Lambda_m = \Delta(\Delta^{k-1}\Lambda_m) \quad \text{with} \quad \Delta^0\Lambda_m = \Lambda_m$$

Therefore, we have

$$\Delta^k\Lambda_m = (0 = r_0, r_{k+1} - r_k, r_{k+2} - r_k, ..., r_m - r_k).$$

Now, for $i = 0, ..., m-k$, we denote by $\eta_i^{(k)}(\Lambda_m)$ the ith control point of the function $t^{r_{k+1}-r_k}$ with respect to the Müntz space $E_{\Delta^k\Lambda_m}$, namely, we have

$$\eta_0^{(k)}(\Lambda_m) = 0; \quad \eta_{m-k}^{(k)}(\Lambda_m) = 1$$

and for $i = 1, ..., m-k-1$

$$\eta_i^{(k)}(\Lambda_m) = \prod_{j=i+k+1}^{m} \left(1 - \frac{r_{k+1} - r_k}{r_j - r_k}\right) \tag{19}$$

We will adopt the convention that if $i < 0$, then $\eta_i^{(k)}(\Lambda_0) = 0$ and also write $\eta_i^{(0)}(\Lambda_m)$ simply as $\eta_i(\Lambda_m)$. We have

Theorem 3. *Let $\Lambda_n = (0 = r_0, r_1, ..., r_n)$ be a sequence of strictly increasing real numbers and let $\Lambda_m = (0 = r_0, r_1, ..., r_n, ..., r_m)$ be a longer sequence of strictly increasing numbers. Let P be an element of the Müntz space E_{Λ_n} written in the Gelfond-Bernstein bases of E_{Λ_n} and E_{Λ_m} as*

$$P(t) = \sum_{i=0}^{n} p_i H_{i,\Lambda_n}^n(t) = \sum_{i=0}^{m} b_i^m H_{i,\Lambda_m}^m(t).$$

Then, there exist $(n-1)$ constants C_k depending only on the function P and the finite parameters $r_1, ..., r_n$, such that

$$\left|P(\eta_i(\Lambda_m)^{1/r_1}) - b_i^m\right| \leq \sum_{k=0}^{n-2} C_k \left|\eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}}\right| \tag{20}$$

for all $i = 0, ..., m$. We adopt the convention that $\sum_{k=0}^{-1} = 0$.

Proof. We will proceed by induction on n. For $n = 1$ and as $\eta_i(\Lambda_m)$ are the control points of the function t^{r_1} with respect to the Müntz space E_{Λ_m} over the interval $[0, 1]$, we have $P(\eta_i(\Lambda_m)^{1/r_1}) - b_i^m = 0$ and the conclusion follows. Let us assume the claim of the theorem is true for any element of a Müntz space of order less or equal to $n - 1$. Let P be an element of the space E_{Λ_n} written in the Gelfond-Bernstein bases of E_{Λ_n} and E_{Λ_m} as

$$P(t) = \sum_{i=0}^{n} p_i H_{i,\Lambda_n}^n(t) = \sum_{i=0}^{m} b_i^m H_{i,\Lambda_m}^m(t). \tag{21}$$

Let us denote by $\bar{\Lambda}_n$ and $\bar{\Lambda}_m$ the sequences $\bar{\Lambda}_n = (0 = r_0, 1, r_2/r_1, ..., r_n/r_1)$ and $\bar{\Lambda}_m = (0 = r_0, 1, r_2/r_1, ..., r_n/r_1, ..., r_m/r_1)$. To the function P in (21), we associate the function $\bar{P}$ in the space $E_{\bar{\Lambda}_n}$ defined as

$$\bar{P}(t) = \sum_{i=0}^{n} p_i H_{i,\bar{\Lambda}_n}^n(t). \tag{22}$$

As the corner cutting scheme (1) and (2) associated with a sequence $S = (r_1, ..., r_l)$ is invariant by a multiplication of every elements of S by the same scalar, we necessarily have

$$\bar{P}(t) = \sum_{i=0}^{m} b_i^m H_{i,\bar{\Lambda}_m}^m(t).$$

Consider the following function Q defined by

$$Q(t) = \bar{P}(t) - \frac{r_1}{r_n} t \bar{P}'(t).$$

It can be readily checked that the function Q is an element of the Müntz space of order $n - 1$, $E_{\Pi_{n-1}}$ where $\Pi_{n-1} = (0 = r_0, 1, r_2/r_1, ..., r_{n-1}/r_1)$. Therefore, we can apply the induction hypothesis to the function Q. Before we apply such induction, let us first express the function Q in the Gelfond-Bernstein basis $H_{i,\bar{\Lambda}_m}^m$. We have

$$Q(t) = \sum_{i=0}^{m} b_i^m H_{i,\bar{\Lambda}_m}^m(t) - \frac{r_1}{r_n} t \sum_{i=0}^{m-1} c_i^{m-1} H_{i,\Gamma_{m-1}}^{m-1}(t), \tag{23}$$

where we have denoted $\bar{P}'$ as

$$\bar{P}'(t) = \sum_{i=0}^{m-1} c_i^{m-1} H_{i,\Gamma_{m-1}}^{m-1}(t), \tag{24}$$

where $\bar{H}_{i,\Gamma_{m-1}}^{m-1}$ is the Gelfond-Bernstein basis with respect to the sequence $\Gamma_{m-1} = (0 = r_0, (r_2/r_1) - 1, (r_3/r_1) - 1, ..., (r_m/r_1) - 1)$. From Lemma 2, we have

$$t H_{i,\Gamma_{m-1}}^{m-1}(t) = \prod_{j=i+2}^{m} \left(1 - \frac{r_1}{r_j}\right) H_{i+1,\bar{\Lambda}_m}^m(t) = \eta_{i+1}(\bar{\Lambda}_m) H_{i+1,\bar{\Lambda}_m}^m(t).$$

Therefore, from (23), we have

$$Q(t) = b_0^m H^m_{0,\bar{\Lambda}_m}(t) + \sum_{i=1}^{m} \left(b_i^m - \frac{r_1}{r_n} \eta_i(\bar{\Lambda}_m) c_{i-1}^{m-1} \right) H^m_{i,\bar{\Lambda}_m}(t).$$

The last equation shows that we have applied upon Q a dimension elevation from the Müntz space associated with the sequence Π_{n-1} to the Müntz space associated with the sequence $\bar{\Lambda}_m$. Moreover, it can easily be checked that $\eta_i^{(k)}(\bar{\Lambda}_m) = \eta_i^{(k)}(\Lambda_m)$ for any i and k. Therefore, the induction hypothesis and the expression of Q show that there exist $(n-2)$ constant C_k depending only on the polynomial Q and the parameters $r_1, ..., r_{n-1}$ such that for $i = 0, ..., m$ we have

$$\left| \left(\bar{P}(\eta_i(\Lambda_m)) - b_i^m\right) - \frac{r_1}{r_n}\eta_i(\Lambda_m)\left(\bar{P}'(\eta_i(\Lambda_m))) - c_{i-1}^{m-1}\right) \right| \leq$$
$$\sum_{k=0}^{n-3} C_k \left| \eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}} \right|.$$

Therefore, we have

$$\left|\bar{P}(\eta_i(\Lambda_m)) - b_i^m\right| \leq \sum_{k=0}^{n-3} C_k \left| \eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}} \right| + \left| \frac{r_1}{r_n}\eta_i(\Lambda_m)\left(\bar{P}'(\eta_i(\Lambda_m)) - c_{i-1}^{m-1}\right) \right|. \tag{25}$$

Now the function $\bar{P}'$ is an element of the Müntz space of order $n-1$, $E_{\Gamma_{n-1}}$, where $\Gamma_{n-1} = (0 = r_0, (r_2/r_1) - 1, (r_3/r_1) - 1, ..., (r_n/r_1) - 1)$. The expression (24), shows that we have applied upon $\bar{P}'$ a dimension elevation from the Müntz space associated with the sequence Γ_{n-1} to the Müntz space associated with the sequence Γ_{m-1}. Therefore, again by the induction hypothesis, there exist $(n-2)$ constants E_k depending only on $\bar{P}'$ and the parameters $r_1, ..., r_n$ such that

$$\left| \bar{P}'(\eta_i(\Gamma_{m-1})^{\frac{r_1}{r_2-r_1}}) - c_i^{m-1} \right| \leq \sum_{k=0}^{n-3} E_k \left| \eta_{i-k}^{(k)}(\Gamma_{m-1}) - \left(\eta_{i-(k+1)}^{(k+1)}(\Gamma_{m-1})\right)^{\frac{r_{k+2}-r_{k+1}}{r_{k+3}-r_{k+2}}} \right|$$

It can be easily shown that $\eta_i^{(k)}(\Gamma_{m-1}) = \eta_i^{(k+1)}(\Lambda_m)$. Therefore, we have

$$\left| \bar{P}'(\eta_i^{(1)}(\Lambda_m)^{\frac{r_1}{r_2-r_1}}) - c_i^{m-1} \right| \leq \sum_{k=0}^{n-3} E_k \left| \eta_{i-k}^{(k+1)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+2)}(\Lambda_m)\right)^{\frac{r_{k+2}-r_{k+1}}{r_{k+3}-r_{k+2}}} \right|.$$

Moreover, we have

$$\left|\bar{P}'(\eta_i(\Lambda_m)) - c_{i-1}^{m-1}\right| \leq \left| \bar{P}'(\eta_i(\Lambda_m)) - \bar{P}'((\eta_{i-1}^{(1)}(\Lambda_m))^{\frac{r_1}{r_2-r_1}}) \right| +$$
$$\left| \bar{P}'((\eta_{i-1}^{(1)}(\Lambda_m))^{\frac{r_1}{r_2-r_1}}) - c_{i-1}^{m-1} \right|.$$

Therefore,

$$\left|\bar{P}'(\eta_i(\Lambda_m)) - c_{i-1}^{m-1}\right| \leq \left| \eta_i(\Lambda_m) - (\eta_{i-1}^{(1)}(\Lambda_m))^{\frac{r_1}{r_2-r_1}}) \right| C +$$
$$\sum_{k=1}^{n-2} E_{k-1} \left| \eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}} \right|$$

where $C = max_{[0,1]}|\bar{P}''(x)|$. Inserting the last inequality into (25) and using the obvious fact that $|(\eta_i(\Lambda_m))| \leq 1$, show that there exist $(n-2)$ constants L_i depending only on the polynomial $\bar{P}$ and the real values $r_1, ..., r_n$ such that

$$\left|\bar{P}(\eta_i(\Lambda_m)) - b_i^m\right| \leq \sum_{k=0}^{n-2} L_k \left|\eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}}\right|. \quad (26)$$

In view of Lemma 3 and equation (22), we have

$$\bar{P}(\eta_i(\Lambda_m)) = P(\eta_i(\Lambda_m)^{1/r_1}).$$

Inserting the last equation into (26) conclude the proof of the theorem. □

The following lemma is implicit in [7], and even more explicit in [3], as our hypothesis are different from the ones taken in the latter and for the seek of completeness, we will include it proof.

Lemma 4. *Let γ be a strictly positive and let a_j and b_j, $j = 1, 2...$ be sequences of real numbers in $]0,1[$ such that*

$$\lim_{j\to\infty} \frac{\ln b_j}{\ln a_j} = \gamma. \quad (27)$$

Define $A_i(m) = \prod_{j=i+1}^m a_j$ and $B_i(m) = \prod_{j=i+1}^m b_j$ $(i < m)$ and let us assume that for any fixed i, we have

$$\lim_{m\to\infty} A_i(m) = 0 \quad \text{and} \quad \lim_{m\to\infty} B_i(m) = 0. \quad (28)$$

Then

$$\lim_{m\to\infty} (A_i(m)^\gamma - B_i(m)) = 0 \quad \text{uniformly in} \quad i.$$

Proof. We should prove that for every $\epsilon_1 > 0$, there exist an m_0 such that for all $m \geq m_0$ we have

$$|A_i(m)^\gamma - B_i(m)| < \epsilon_1 \quad \text{for} \quad i = 1, 2, ..., m.$$

Let us fix an $\epsilon_1 > 0$ and select an $\epsilon \in]0,1[$ such that

$$1 - \epsilon^\epsilon < \epsilon_1, \quad \epsilon^\gamma < \epsilon_1; \quad \epsilon < \gamma \quad \text{and} \quad \epsilon^{\gamma-\epsilon} < \epsilon_1.$$

Condition (27) shows that there exists a j_0 such that for any $j \geq j_0$, we have

$$\gamma - \epsilon < \frac{log\, b_j}{log\, a_j} < \gamma + \epsilon. \quad (29)$$

Since $log\, a_j < 0$, (29) imply that for any $j \geq j_0$, we have

$$a_j^{\gamma+\epsilon} < b_j < a_j^{\gamma-\epsilon}.$$

The last equation shows that for any $j \geq j_0$ and for any $m \geq j_0$ we have

$$A_j(m)^{\gamma+\epsilon} \leq B_j(m) \leq A_j(m)^{\gamma-\epsilon}.$$

We can rephrase the last assertion as follow : There exists an $m_0 = j_0$ such that for any $m \geq m_0$, we have

$$A_j(m)^{\gamma+\epsilon} \leq B_j(m) \leq A_j(m)^{\gamma-\epsilon}, \quad \text{for} \quad j = j_0, j_0+1, ..., m. \tag{30}$$

Let us fix $m \geq m_0 = j_0$. If for a certain index $j \geq j_0$, we have $A_j(m)^\gamma \geq B_j(m)$ and $A_j(m) < \epsilon$, then, we have

$$0 \leq A_j(m)^\gamma - B_j(m) < \epsilon^\gamma < \epsilon_1.$$

If for a certain index $j \geq j_0$, we have $A_j(m)^\gamma \geq B_j(m)$ and $A_j(m) \geq \epsilon$, then from (30) and using the fact that $A_j(m) < 1$, we have

$$0 \leq A_j(m)^\gamma - B_j(m) \leq A_j(m)^\gamma - A_j(m)^{\gamma+\epsilon} = A_j(m)^\gamma(1 - A_j(m)^\epsilon) < 1 - \epsilon^\epsilon < \epsilon_1.$$

If for a certain index $j \geq j_0$, we have $A_j(m)^\gamma \leq B_j(m)$ and $A_j(m) < \epsilon$, then, from (30), we have

$$0 \leq B_j(m) - A_j(m)^\gamma \leq A_j(m)^{\gamma-\epsilon} - A_j(m)^\gamma \leq \epsilon^{\gamma-\epsilon} < \epsilon_1.$$

Finally, if for a certain index $j \geq j_0$, we have $A_j(m)^\gamma \leq B_j(m)$ and $A_j(m) \geq \epsilon$, then, from (30), we have

$$0 \leq B_j(m) - A_j(m)^\gamma \leq 1 - A_j^\gamma \leq 1 - \epsilon^\epsilon < \epsilon_1.$$

As we have exhausted all the possible cases on the behavior of a pair of numbers $A_j(m)$ and $B_j(m)$ for a certain index $j \geq j_0$, the conclusion of theses cases show that for any $m \geq m_0 = j_0$, we have

$$|A_j(m)^\gamma - B_j(m)| < \epsilon_1 \quad \text{for} \quad j = j_0, j_0+1, ..., m. \tag{31}$$

Now condition (28), shows in particular that for any $j < j_0$, there exists an $M_0(j)$ such that for any $m \geq M_0(j)$, we have

$$|A_j(m)^\gamma - B_j(m)| < \epsilon_1.$$

As we have a finite set of $M_0(j), j = 1, ..., j_0 - 1$, if we denote by $L_0 = \max_{j=1,...,j_0-1}(M_0(j))$, then for any $m \geq L_0$, we have

$$|A_j(m)^\gamma - B_j(m)| < \epsilon_1 \quad \text{for} \quad j = 1, 2, ..., j_0 - 1. \tag{32}$$

Therefore, by taking $M_0 = max(m_0, L_0)$ and in view of (31) and (32), we have for any $m \geq M_0$

$$|A_j(m)^\gamma - B_j(m)| < \epsilon_1 \quad \text{for} \quad j = 1, 2, ..., m.$$

□

From the last lemma, we can prove the following

Theorem 4. *Let $\Lambda_\infty = (0 = r_0, r_1, ..., r_n, ..., r_m, ...)$ be an infinite sequence of strictly increasing real numbers such that*

$$\lim_{s\to\infty} r_s = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{r_i} = \infty. \tag{33}$$

For any integer m, we denote by Λ_m the subsequence of Λ_∞ given by $\Lambda_m = (0 = r_0, r_1, ..., r_m)$. Let P be an element of the Müntz space E_{Λ_n} written in the Gelfond-Bernstein bases of E_{Λ_n} and E_{Λ_m} ($m \geq n$) as

$$P(t) = \sum_{i=0}^{n} p_i H_{i,\Lambda_n}^{n}(t) = \sum_{i=0}^{m} b_i^m H_{i,\Lambda_m}^{m}(t).$$

Then

$$\lim_{m\to\infty} P(\eta_i(\Lambda_m)^{1/r_1}) - b_i^m = 0 \quad \text{uniformly in} \quad i. \tag{34}$$

Proof. In view of Theorem 3, we need only to show that under the conditions (33), the right hand side of (20) converges to zero as m goes to infinity, uniformly in i. As we have finite terms in the sum in (20), we will only need to show that for any fixed k such that $0 \leq k \leq n-2$, we have

$$\lim_{m\to\infty} \eta_{i-k}^{(k)}(\Lambda_m) - \left(\eta_{i-(k+1)}^{(k+1)}(\Lambda_m)\right)^{\frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}}} = 0 \quad \text{uniformly in} \quad i. \tag{35}$$

Let us first deal with the indices i such that $i > k+1$, in this case, if we denote by

$$a_j = \left(1 - \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}}\right) \quad \text{and} \quad b_j = \left(1 - \frac{r_{k+1}-r_k}{r_j - r_k}\right); \quad j > k+2,$$

then, according to (19), and imitating the notations of the Lemma 4, we have

$$\eta_{i-k}^{(k)}(\Lambda_0) = \prod_{j=i+1}^{m} b_j = B_i(m) \quad \text{and} \quad \eta_{i-(k+1)}^{(k+1)}(\Lambda_0) = \prod_{j=i+1}^{m} a_j = A_i(m).$$

The fact that the sequence $(r_k)_{0\leq k\leq\infty}$ is strictly increasing, shows that a_j and b_j are elements of the interval $]0,1[$ ($j > k+2$). Moreover, as $\lim_{j\to\infty} r_j = \infty$ and using the l'Hospital's rule, show that

$$\lim_{j\to\infty} \frac{\ln b_j}{\ln a_j} = \frac{r_{k+1}-r_k}{r_{k+2}-r_{k+1}} = \gamma > 0.$$

To prove that for any fixed $i > k+1$, we have $\lim_{m\to\infty} A_i(m) = 0$, we proceed as follows : Since $a_j \in]0,1[$, we have

$$\frac{1}{a_j} \geq 1 + \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}} \quad \text{and then} \quad \frac{1}{A_i(m)} \geq \prod_{j=i+1}^{m} \left(1 + \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}}\right). \tag{36}$$

Moreover,

$$\sum_{j=i+1}^{m} \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}} = (r_{k+2}-r_{k+1}) \sum_{j=i+1}^{m} \frac{1}{r_j - r_{k+1}} \geq (r_{k+2}-r_{k+1}) \sum_{j=i+1}^{m} \frac{1}{r_j}.$$

Therefore, the conditions (33) show that

$$\lim_{m\to\infty} \sum_{j=i+1}^{m} \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}} = \infty \quad \text{thus} \quad \lim_{m\to\infty} \prod_{j=i+1}^{m} \left(1 + \frac{r_{k+2}-r_{k+1}}{r_j - r_{k+1}}\right) = \infty,$$

which by (36) conclude that $\lim_{m\to\infty} A_i(m) = 0$. Similar treatments for b_j show that for any fixed $i > k+1$, $\lim_{m\to\infty} B_i(m) = 0$. Therefore, applying Lemma 4 (after an obvious shift of indices) shows the convergence of (34) uniformly in $i > k+1$. For $i < k+1$, then the term in (35) is zero and if $i = k+1$ the term in (35) is $\eta_1^{(k)}(\Lambda_m)$, which converges to zero as m goes to infinity. Then, using the trick of finitude as at the end of the proof of Lemma 4, conclude the proof. □

In order to conclude the proof of the main Theorem 1 using Theorem 4, we need to show that the point set $D_m = \{\eta_i(\Lambda_m)^{1/r_1}, i = 0, ..., m\}$ form a dense subset of the interval $[0,1]$ as m goes to infinity. For this aim, we need the following result proven by Hirschman and Widder [6] and Gelfond [5].

Theorem 5. **(Hirschman-Widder [6], Gelfond [5])**
Let $\Lambda_\infty = (0 = r_0, r_1, ..., r_n, ..., r_l, ...)$ be an infinite sequence of strictly increasing real numbers such that

$$\lim_{s\to\infty} r_s = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.$$

To every continuous function f in the interval $[0,1]$, we associate the operator $B_n^{\Lambda_\infty}(f)$ defined as

$$B_m^{\Lambda_\infty}(f)(x) = \sum_{i=0}^{m} f(\eta_i(\Lambda_m)^{1/r_1}) H_{i,\Lambda_m}^m(x),$$

where $\Lambda_m = (0 = r_0, r_1, ..., r_m)$. Then the sequences $B_m^{\Lambda_\infty}(f)$ is uniformly convergent with limit f as m goes to infinity.

Using the last theorem, we can now prove the following

Proposition 1. *Let $\Lambda_\infty = (0 = r_0, r_1, ..., r_n, ..., r_l, ...)$ be an infinite sequence of strictly increasing real numbers such that*

$$\lim_{s\to\infty} r_s = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.$$

Denote by D_m the point set $D_m = \{\eta_i(\Lambda_m)^{1/r_1}, i = 0, ..., m\}$. Then, as m goes to infinity, the point set D_m form a dense subset of the interval $[0,1]$.

Proof. Let ϵ be a strictly positive real number and let x_0 be a real number in the interval $[0,1]$. Consider the continuous piecewise linear function f defined as

$$f(x) = \begin{cases} \frac{-x}{x_0} + 1 & \text{if} \quad 0 \le x \le x_0 \\ \frac{x}{1-x_0} - \frac{x_0}{1-x_0} & \text{if} \quad x_0 \le x \le 1. \end{cases}$$

As the function f is continuous, then by Theorem 5, there exist an m_0 such that for any $m \ge m_0$, we have

$$|f(x_0) - \sum_{i=0}^{m} f(\eta_i(\Lambda_m)^{1/r_1}) H_{i,\Lambda_m}^m(x_0)| < \epsilon.$$

Evaluating f in the last equation leads to

$$\begin{aligned}
&\frac{1}{x_0}\sum_{\eta_i(\Lambda_m)^{1/r_1}\leq x_0}(x_0-\eta_i(\Lambda_m)^{1/r_1})H^m_{i,\Lambda_m}(x_0)+\\
&\frac{1}{1-x_0}\sum_{\eta_i(\Lambda_m)^{1/r_1}>x_0}(\eta_i(\Lambda_m)^{1/r_1}-x_0)H^m_{i,\Lambda_m}(x_0)<\epsilon.
\end{aligned}\tag{37}$$

Now, if $|x_0-\eta_i(\Lambda_m)^{1/r_1}|>\epsilon$ for $i=1,2,...,m$, then, using the fact that $\sum_{i=0}^m H^m_{i,\Lambda_m}(x_0)=1$, shows that the left hand side expression of (37) is also strictly greater than ϵ, leading to a contradiction. Therefore, for any $\epsilon>0$ and any $x_0\in[0,1]$, there exists an m_0 such that for any $m\geq m_0$, there exists an $i\leq m$ such that $|x_0-\eta_i(\Lambda_m)^{1/r_1}|<\epsilon$. □

As this point, we are ready to prove Theorem 1.

Proof of Theorem 1 : Let us prove the theorem when the sequence $\Lambda_\infty=(0=r_0,r_1,...,r_n,....,r_l,...)$ of strictly increasing real numbers satisfy the conditions (33). In this case, if we denote by P the Gelfond-Bézier curve with control points $(P_0,P_1,...,P_n)$, we have to show that given a point $x\in[0,1]$ and a sequence of real numbers $\eta_{i(x)}(\Lambda_m)^{1/r_1}$ that converges to x as m goes to infinity (this is possible thanks to the density proposition 1), the points $b^m_{i(x)}$ converges to $P(x)$ as m goes to infinity. As the function P is continuous, $P(\eta_{i(x)}(\Lambda_m)^{1/r_1})$ converges to $P(x)$. Therefore, for any $\epsilon>0$, there exists an M_0, such that for any $m\geq M_0$, we have

$$|P(x)-P(\eta_{i(x)}(\Lambda_m)^{1/r_1})|<\epsilon.$$

Moreover, from Theorem 4, there exists an M_1 such that for any $m\geq M_1$

$$|b^m_{i(x)}-P(\eta_{i(x)}(\Lambda_m)^{1/r_1})|<\epsilon.$$

Therefore, $|P(x)-b^m_{i(x)}|<2\epsilon$ for any $m\geq max(M_0,M_1)$, thereby proving the pointwise convergence of the dimension elevated control polygons to the Gelfond-Bézier curve. The convergence is uniform as we have

$$\max_x|P(x)-b^m_{i(x)}|\leq\max_x|P(x)-P(\eta_{i(x)}(\Lambda_m)^{1/r_1})|+\max_{i(x)}|P(\eta_{i(x)}(\Lambda_m)^{1/r_1})-b^m_{i(x)}|$$

The function P is continuous in the compact interval $[0,1]$, thus

$$\max_x|P(x)-P(\eta_{i(x)}(\Lambda_m)^{1/r_1})|\to 0\quad\text{as } m \text{ goes to infinity,}$$

and Theorem 4 shows that

$$\max_{i(x)}|P(\eta_{i(x)}(\Lambda_m)^{1/r_1})-b^m_{i(x)}|\to 0\quad\text{as } m \text{ goes to infinity.}$$

This conclude the proof of the if part of the theorem. To prove the only if part of the theorem, we proceed by contradiction. Let us assume that the real number r_i satisfy

$$\sum_{i=1}^\infty\frac{1}{r_i}<\infty.$$

Without loss of generality, we can take the case in which the control points $(P_0, P_1, ..., P_n)$ are real numbers such that $P_0 = 0$, $P_1 = 1$ and $P_1 < P_2 < ... < P_n$. In this case, for any m, we have $b_0^m = 0$ and b_1^m is a strictly decreasing function of m that converges to a strictly positive number $0 < \delta < 1$

$$\delta = \lim_{m\to\infty} b_1^m = \prod_{i=2}^{\infty} \left(1 - \frac{r_1}{r_j}\right).$$

In this case we would have a gap between the point zero and δ, namely, if we take $x = \delta/2$ than for any $m \geq 0$, we have

$$|b_i^m - x| > \frac{1}{4} \quad \text{for} \quad i = 1, ..., m$$

and therefore, the limiting control polygon does not converge pointwise to the Gelfond-Bézier curve.

4. Discussion

In the following, we give a list of directions for future research as well as some open problems.

1- The corner cutting scheme (1) and (2) can be generalized so as to describe the dimension elevation algorithm of rational Gelfond-Bézier curves. In this case, weighted corner-cutting schemes are derived and the methods developed in this work can contribute to the study of the convergence of these new family of corner cutting schemes.

2- We can study the limiting polygon of the corner cutting scheme in case we relax the hypothesis of strictly increasing sequences $0 < r_1 < r_2 < ... < r_m < ...$ to only increasing sequences $0 < r_1 \leq r_2 \leq .... \leq r_m \leq ....$ In this case, the Gelfond-Bézier curves involve logarithmic functions [5], namely, if we rewrite the exponent $(r_1, r_2, ..., r_n) = (\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_m)$ where the real number $\tilde{r}_i$ are distinct and if we denote by $m_j, j = 1, ..., m$ the number of indices $i = 0, ..., n$ for which $r_i = \tilde{r}_i$, then the space $span(1, t^{r_1}, t^{r_2}, ..., t^{r_n}) = span(1, x^{r_j} (\ln x)^i; j = 0, 1, ..m; i = 0, 1, ..., m_j - 1)$. The results of this work could be extended to this case by a limiting process.

3- The Gelfond-Bézier curves are too "degenerate" at the origin to study the dimension elevation algorithm in case we impose no condition of monotonicity on the real numbers r_i. For instance, if we consider the case $n = 3$, $r_1 = 1, r_2 = 2, r_3 = 3$ and $r_j = 1/j$, for $j > 3$ and we start with a control polygon (P_0, P_1, P_2, P_3) then the dimension elevated control polygon to the order m is not obtained by a corner cutting scheme similar to (1) and (2) but instead the algorithm collapses the first $m-3$ control points to P_0 while the remaining control points are given by (P_1, P_2, P_3) [1]. However, if we consider the dimension elevation algorithm of Gelfond-Bézier curves far from the origin, i.e. over an interval $[a, 1]$ with $a > 0$, then the Gelfond-Bernstein basis coincide with the Chebyshev-Bernstein basis [1], the degeneracy at the origin disappear and the algorithm leads to a family of corner cutting schemes without imposing any condition of monotonicity on the real numbers r_i. Unfortunately, such family of corner cutting schemes involves rather complicated coefficients expressed in term of Schur functions [2]. It will be interesting to find, for the far from the origin case, conditions on the real number r_i for the convergence of the dimension elevation algorithm to the underlying curve. In the theory of Müntz spaces over an interval $[a,1]$ with $a>0$, and in which we impose no condition on the real numbers r_i (beside that they are pairwise distinct), then the corresponding Müntz space is a dense subset of $C([a,1])$ if and only if the real numbers r_i satisfy the so-called full Müntz condition [4]

$$\sum_{r_k \neq 0} \frac{1}{|r_k|} = \infty. \tag{38}$$

The question is then does the surprising emergence of the Müntz condition in Theorem 1 for the real numbers r_i with the condition of the theorem, repeat itself for the full Müntz condition (38) for the far from the origin case.

4- It is not difficult to show that with the conditions of Theorem 1, the conditions (33) are sufficient for the uniform convergence of the dimension elevation algorithm of Chebyshev-Bézier curve of the associated Müntz space to the underlying curve (the proof will be published elsewhere). However, the pointwise convergence is more involved and the question of whether the Müntz condition is necessary prove to be interesting.

5- It is probably a difficult problem to study the rate of convergence of the corner cutting scheme (1) and (2) in case the real numbers r_i satisfy the condition of Theorem 1. Adapting the method of Prautzsch and Kobbelt [12] to this problem shows for example that if the numbers r_i are integers and that there exists a constant K such that $r_j \leq K + j$ for all $j \geq 1$ then the rate of convergence of the corner cutting scheme is in $O(\frac{1}{m})$.

6- It may happen that studying the limiting polygon of the corner cutting scheme (1) and (2) is richer under the condition

$$\sum_{k=1}^{\infty} \frac{1}{r_k} < \infty, \tag{39}$$

in analogy with the problem of studying the uniform closure of the Müntz space $E_\infty = span(1, t^{r_1}, ..., t^{r_n}, ...)$ under the condition (39).

Acknowledgment : This work was partially supported by the MEXT Global COE project. Osaka University, Japan.

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