

On derivative bounds for the rational quadratic Bézier paths

Bez, Helmut; Bez, Richard

DOI:

[10.1016/j.cagd.2012.12.003](https://doi.org/10.1016/j.cagd.2012.12.003)

Document Version

Publisher's PDF, also known as Version of record

Citation for published version (Harvard):

Bez, H & Bez, R 2013, 'On derivative bounds for the rational quadratic Bézier paths', *Computer Aided Geometric Design*, vol. 30. <https://doi.org/10.1016/j.cagd.2012.12.003>

[Link to publication on Research at Birmingham portal](#)

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

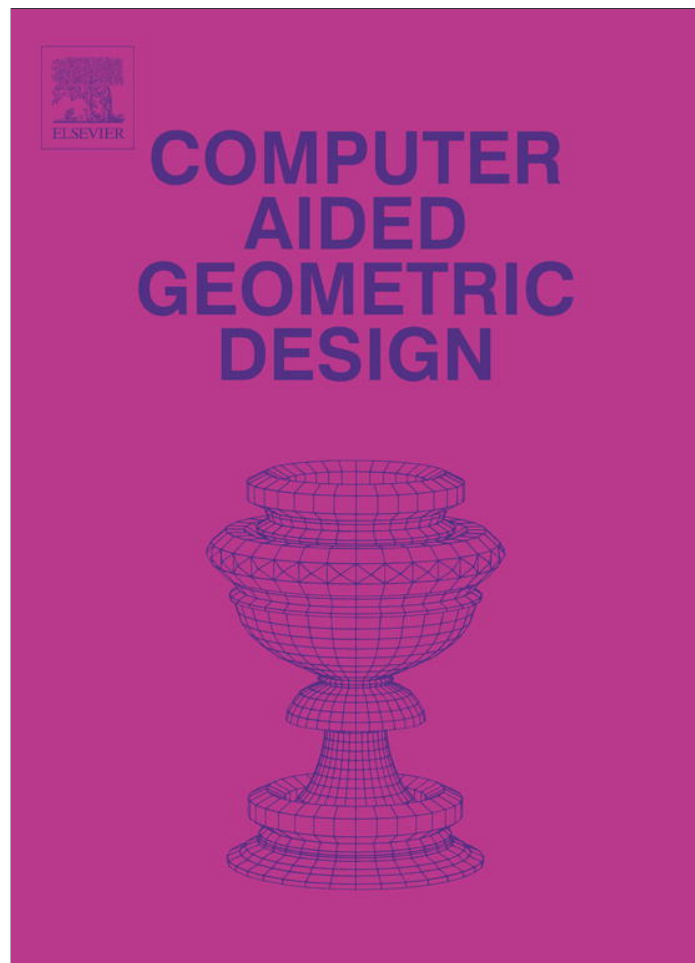
Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.



(This is a sample cover image for this issue. The actual cover is not yet available at this time.)

This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



Contents lists available at SciVerse ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd



On derivative bounds for the rational quadratic Bézier paths [☆]



H.E. Bez ^{a,*}, N. Bez ^b

^a Department of Computer Science, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK

^b School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

ARTICLE INFO

Article history:

Received 18 January 2012

Received in revised form 12 November 2012

Accepted 18 December 2012

Available online 20 December 2012

Keywords:

Derivative bounds

Quadratic Bézier paths

Rational parametrisation

Optimal parametrisation

ABSTRACT

New derivative bounds for the rational quadratic Bézier paths are obtained, both for particular weight vectors and for classes of equivalent parametrisations. A comprehensive analysis of our bounds against existing bounds is made.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

Let $V = \{(v_0, v_1, v_2): v_i \in \mathbb{R}^d\}$ and $\Omega = \{(w_0, w_1, w_2): w_i \in \mathbb{R}^+\}$, where d is a natural number and $\mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$. The rational quadratic Bézier path with vertices $v \in V$ and weights $w \in \Omega$ may be written as $\sigma[v, w]$ where

$$\sigma[v, w](t) = \frac{w_0(1-t)^2v_0 + 2w_1t(1-t)v_1 + w_2t^2v_2}{w_0(1-t)^2 + 2w_1t(1-t) + w_2t^2}$$

for $t \in [0, 1]$. In terms of derivative bounds, most previous work, and that of this paper, is concerned with uniform bounds on $\sigma'[v, w]$ of the tensor product form

$$|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Phi(w), \quad (1)$$

for all $(v, w) \in V \times \Omega$ and $t \in [0, 1]$. Here, $\Delta_1(v) = \max_{0 \leq j \leq 1} |v_j - v_{j+1}|$, $\Delta_2(v) = \max_{0 \leq i, j \leq 2} |v_i - v_j|$ and $\tau \in \{1, 2\}$. A fundamental problem is to obtain such bounds with the weight function $\Phi: \Omega \rightarrow \mathbb{R}^+$ as small as possible. We shall refer to (1) as a *pointwise* bound.

For a given pair $(v, w) \in V \times \Omega$ it is natural to determine the *invariant* bound $2\Delta_\tau(v)\tilde{\Phi}(w)$, associated with the pointwise bound $2\Delta_\tau(v)\Phi(w)$, where

$$\tilde{\Phi}(w) = \min\{\Phi(\tilde{w}): \tilde{w} \sim w\}.$$

Here, $\tilde{w} \sim w$ if there exists $\lambda \in \mathbb{R}^+$ such that

$$\tilde{w} = \text{diag}(1, \lambda, \lambda^2)w = (w_0, \lambda w_1, \lambda^2 w_2). \quad (2)$$

[☆] This paper has been recommended for acceptance by J. Peters.

* Corresponding author.

E-mail address: h.e.bez@lboro.ac.uk (H.E. Bez).

It is well known that the paths $\sigma[v, w]$ and $\sigma[v, \tilde{w}]$ parametrise the same curve if $\tilde{w} \sim w$. Writing

$$\mathcal{J}(w) = \frac{w_1}{(w_0 w_2)^{1/2}}$$

we have $\mathcal{J}(\tilde{w}) = \mathcal{J}(w)$ whenever $\tilde{w} \sim w$, and \mathcal{J} therefore parametrises the space of equivalence classes.

The purpose of this paper is to obtain new pointwise and invariant bounds for quadratic Bézier paths. Importantly, we also compare our bounds with existing bounds in a comprehensive way, clarifying the merits of each approach that has been taken, with a view to new developments in higher degree cases.

Derivative bounds for the rational Bézier paths were first obtained in Floater (1992); this work inspired a number of recent papers in which improvements on the bounds were sought (see Wang et al., 1997 for corresponding results for the rectangular Bézier surface patches). Improved bounds for Bézier paths were obtained by Hermann (1999), for the important quadratic and cubic cases and when $\tau = 1$. Hermann's approach is to employ a Möbius transformation, to normalise the path form, and capitalise on the induced symmetry to efficiently analyse the maximum value of the derivative of the normalised form using elementary calculus. Several authors have made use of an alternative, more algebraic, approach of degree-elevation and convexity, possibly combined with the de Casteljau algorithm. Such an approach was used in Selimovic (2005) for paths of arbitrary degree and $\tau \in \{1, 2\}$. Selimovic's bounds were improved upon by Zhang and Ma (2006), for degree less than seven when $\tau = 1$, and for arbitrary degree when $\tau = 2$.

In certain cases, we provide definitive comparisons of pointwise bounds. In particular, we shall prove that the pointwise bounds obtained by Zhang and Ma, for the quadratic case and $\tau \in \{1, 2\}$, are improved upon by our bounds obtained by degree-elevating to the quartic case (the lowest possible degree). We also show that Hermann's pointwise bound is superior to the bound given by Zhang and Ma in the quadratic case when $\tau = 1$. See the forthcoming Theorems 4.1 and 4.3. In most cases, however, it is not possible to find a simple characterisation of the weight space where one pointwise bound is superior to another – but in several such cases, we provide a definitive comparison of the associated invariant bounds. To summarise in an over-simplified manner, we shall see that the invariant bounds obtained from Hermann's approach are not easy to lower using the algebraic arguments mentioned above; see Theorems 4.2 and 4.4 for some precise statements. In order to be in a position to make such comparisons, we follow the approach of Hermann to obtain a new pointwise bound for the quadratic case when $\tau = 2$ (see Theorem 2.1). It is also necessary to provide a theorem which establishes the full scope of the degree-elevation and convexity approach; in Section 3, we prove such a general theorem which confirms that bounds obtained from degree-elevating and convexity improve as the degree increases, as one would expect. This general result is applied in the quadratic case; furthermore, we obtain the associated invariant bounds when the degree is elevated to four and five, albeit for “small” values of the invariant \mathcal{J} in the latter case; see Theorem 3.3. This is a pertinent case to consider because all existing bounds analysed in this paper are “sharp” for “large” values of the invariant \mathcal{J} . In Section 4 we make this precise and moreover conclude that the invariant bounds arising from Hermann's approach are essentially sharp as the invariant \mathcal{J} approaches zero; these observations are also novel.

2. Bounds from Hermann's approach

A direct computation yields

$$\frac{1}{2}\sigma'[v, w](t) = \frac{w_0 w_1 (1-t)^2 (v_1 - v_0) + w_0 w_2 t(1-t)(v_2 - v_0) + w_1 w_2 t^2 (v_2 - v_1)}{(w_0 (1-t)^2 + 2w_1 t(1-t) + w_2 t^2)^2}. \quad (3)$$

For $\mu \in \mathbb{R}^+$ let $M_\mu : [0, 1] \rightarrow [0, 1]$ denote the Möbius transformation

$$M_\mu(t) = (\mu + (1 - \mu)t)^{-1}t.$$

Then $\sigma[v, w](M_\mu(t)) = \sigma[v, \tilde{w}](t)$ where $\tilde{w} = (w_0, \mu^{-1}w_1, \mu^{-2}w_2) \sim w$. Choosing $\mu = (w_2/w_0)^{1/2}$ and homogeneity has the effect of normalising the weight $w \mapsto (1, \mathcal{J}(w), 1)$. Using (3) and the triangle inequality we find that

$$\frac{|\sigma'[v, \tilde{w}](t)|}{2\Delta_\tau(v)} \leq \frac{\mathcal{J}(w)(1-t)^2 + (3-\tau)t(1-t) + \mathcal{J}(w)t^2}{((1-t)^2 + 2\mathcal{J}(w)t(1-t) + t^2)^2}. \quad (4)$$

The effect of this symmetrisation of the weight is that one may now easily compute *exactly* the maximum value over $[0, 1]$ of the rational function on the right-hand side of (4). Indeed, we may immediately restrict our attention to $t \in [0, \frac{1}{2}]$ by invariance under $t \mapsto 1-t$, and furthermore

$$\frac{\mathcal{J}(w)(1-t)^2 + (3-\tau)t(1-t) + \mathcal{J}(w)t^2}{((1-t)^2 + 2\mathcal{J}(w)t(1-t) + t^2)^2} = \frac{\mathcal{J}(w)(1-2s) + (3-\tau)s}{(1-2s + 2\mathcal{J}(w)s)^2},$$

where $s = t(1-t) \in [0, \frac{1}{4}]$. It is precisely this reduction in the degree of the variable that permits a straightforward exact computation of the maximum value. Elementary considerations using calculus show that

$$\max_{s \in [0, 1/4]} \left| \frac{\mathcal{J}(w)(1-2s) + (3-\tau)s}{(1-2s + 2\mathcal{J}(w)s)^2} \right| = H_\tau(\mathcal{J}(w)), \quad (5)$$

where $H_1(\mathcal{J}) = \max\{\mathcal{J}, 2(1 + \mathcal{J})^{-1}\}$ and

$$H_2(\mathcal{J}) = \begin{cases} (1 + 2\mathcal{J})(1 + \mathcal{J})^{-2} & \text{for } \mathcal{J} \in (0, C_0), \\ \frac{1}{8}(1 - 2\mathcal{J})^2(\mathcal{J} - 1)^{-1}(1 - 2\mathcal{J}^2)^{-1} & \text{for } \mathcal{J} \in [C_0, C_1], \\ \mathcal{J} & \text{for } \mathcal{J} \in (C_1, \infty). \end{cases}$$

Here, $C_0 = \frac{1}{12}(1 + \sqrt{73})$, $C_1 = \frac{1}{4}(1 + \sqrt{5})$ and note that $0 < C_0 < C_1 < 1$. Setting

$$\Theta_\tau(w) = H_\tau(\mathcal{J}(w)) \max\left\{\left(\frac{w_2}{w_0}\right)^{\frac{1}{2}}, \left(\frac{w_0}{w_2}\right)^{\frac{1}{2}}\right\},$$

via the chain rule, we have shown the following, due to Hermann (1999) when $\tau = 1$ using the above, and a new pointwise bound when $\tau = 2$.

Theorem 2.1. For each $(v, w) \in V \times \Omega$, $\tau \in \{1, 2\}$ and $t \in [0, 1]$,

$$|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Theta_\tau(w).$$

It is clear from (5) that $\Theta_2(w) \leq \Theta_1(w)$ for each $w \in \Omega$. Thus, our new bound in Theorem 2.1 for $\tau = 2$ is a strict improvement on what one obtains from the $\tau = 1$ bound obtained by Hermann combined with the triviality $\Delta_1(v) \leq \Delta_2(v)$. An obvious advantage of the above approach, where a Möbius mapping is used to transform an arbitrary $w \in \Omega$ into a weight vector whose components are functions of the invariant $\mathcal{J}(w)$, is that the corresponding invariant bound is trivial to compute. In the case of the pointwise bounds in Theorem 2.1, we have $\tilde{\Theta}_\tau(w) = H_\tau(\mathcal{J}(w))$ since, trivially,

$$\min_{\lambda \in \mathbb{R}^+} \max\left\{\lambda\left(\frac{w_2}{w_0}\right)^{\frac{1}{2}}, \lambda^{-1}\left(\frac{w_0}{w_2}\right)^{\frac{1}{2}}\right\} = 1$$

which is attained at $\lambda = (w_0/w_2)^{1/2}$. As we shall see in the sequel, the invariant bounds $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are not easy to lower.¹

3. New bounds from degree-elevation and convexity

If $n \in \mathbb{N}$ and $\alpha_j, \beta_j \in \mathbb{R}^+$ for each $0 \leq j \leq n$, then we have the following convexity inequality

$$\frac{\sum_{j=0}^n \alpha_j}{\sum_{j=0}^n \beta_j} \leq \max_{0 \leq j \leq n} \frac{\alpha_j}{\beta_j}, \quad (6)$$

which has been used by a number of authors in obtaining certain pointwise bounds. We use (6) in our subsequent theorem, which verifies the expected fact that the bounds obtained by degree-elevation improve as the degree increases. Although the focus of the current paper is the quadratic case, we state our result in general since we have not been able to find this in the literature. We use the notation $B_j^{(n)}(t) = (1-t)^{n-j}t^j$ and adopt the convention that $\binom{n}{j} = 0$ for $j < 0$ and $j > n$.

Theorem 3.1. Suppose $a_0, \dots, a_\ell, b_0, \dots, b_m \in \mathbb{R}^+$, where $\ell, m \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ with $n \geq \max\{\ell, m\}$,

$$\frac{\sum_{j=0}^\ell a_j B_j^{(\ell)}(t)}{\sum_{j=0}^m b_j B_j^{(m)}(t)} = \frac{\sum_{j=0}^n \alpha_j^{(n)} B_j^{(n)}(t)}{\sum_{j=0}^n \beta_j^{(n)} B_j^{(n)}(t)} \leq \max_{0 \leq j \leq n} \left\{ \frac{\alpha_j^{(n)}}{\beta_j^{(n)}} \right\}, \quad (7)$$

where, for each $0 \leq j \leq n$, $\alpha_j^{(n)} = \sum_{k=0}^\ell \binom{n-\ell}{j-k} a_k$ and $\beta_j^{(n)} = \sum_{k=0}^m \binom{n-m}{j-k} b_k$. Furthermore, the sequence $(\Phi_n)_{n \geq \max\{\ell, m\}}$ given by $\Phi_n = \max_{0 \leq j \leq n} \{\alpha_j^{(n)} / \beta_j^{(n)}\}$ is non-increasing and hence convergent.

Proof. The formulae for $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ are well known and the bound in (7) follows from (6). To see that (Φ_n) is non-increasing, first note that $\alpha_0^{(n+1)} / \beta_0^{(n+1)} = \alpha_0^{(n)} / \beta_0^{(n)}$ and $\alpha_{n+1}^{(n+1)} / \beta_{n+1}^{(n+1)} = \alpha_n^{(n)} / \beta_n^{(n)}$. Now fix $1 \leq j \leq n$. Then

$$\alpha_j^{(n+1)} = \sum_{k=0}^\ell \binom{n+1-\ell}{j-k} a_k = \sum_{k=0}^\ell \binom{n-\ell}{j-k} a_k + \sum_{k=0}^\ell \binom{n-\ell}{j-k-1} a_k$$

¹ Of course, there is the potential cost of estimating the maximum value of a product of functions by the product of the maximum values of each function. This provides scope for improvement for “small” values of $\mathcal{J}(w)$.

and similarly

$$\beta_j^{(n+1)} = \sum_{k=0}^m \binom{n-m}{j-k} b_k + \sum_{k=0}^m \binom{n-m}{j-k-1} b_k.$$

Using (6) it follows that

$$\frac{\alpha_j^{(n+1)}}{\beta_j^{(n+1)}} \leq \max \left\{ \frac{\sum_{k=0}^{\ell} \binom{n-\ell}{j-k} a_k}{\sum_{k=0}^m \binom{n-m}{j-k} b_k}, \frac{\sum_{k=0}^{\ell} \binom{n-\ell}{j-k-1} a_k}{\sum_{k=0}^m \binom{n-m}{j-k-1} b_k} \right\} = \max \left\{ \frac{\alpha_j^{(n)}}{\beta_j^{(n)}}, \frac{\alpha_{j-1}^{(n)}}{\beta_{j-1}^{(n)}} \right\} \leq \Phi_n.$$

Taking a maximum over j , it follows that $\Phi_{n+1} \leq \Phi_n$ as claimed. \square

Using (3), the triangle inequality and Theorem 3.1 we obtain new pointwise bounds. In particular, with $(\ell, m) = (2, 4)$, and the inputs

$$(a_0, a_1, a_2) = (w_0 w_1, (3 - \tau) w_0 w_2, w_1 w_2) \quad (8)$$

and

$$(b_0, b_1, b_2, b_3, b_4) = (w_0^2, 4w_0 w_1, 4w_1^2 + 2w_0 w_2, 4w_1 w_2, w_2^2),$$

we obtain the following decreasing sequences of pointwise bounds.

Theorem 3.2. For each $(v, w) \in V \times \Omega$, $\tau \in \{1, 2\}$, $t \in [0, 1]$ and $n \geq 4$,

$$|\sigma'[v, w](t)| \leq 2\Delta_{\tau}(v)\Phi_{\tau,n}(w),$$

where $\Phi_{\tau,n}(w)$ is equal to

$$\max_{0 \leq j \leq n} \frac{\binom{n-2}{j} w_0 w_1 + (3 - \tau) \binom{n-2}{j-1} w_0 w_2 + \binom{n-2}{j-2} w_1 w_2}{\binom{n-4}{j} w_0^2 + 4 \binom{n-4}{j-1} w_0 w_1 + 2 \binom{n-4}{j-2} (2w_1^2 + w_0 w_2) + 4 \binom{n-4}{j-3} w_1 w_2 + \binom{n-4}{j-4} w_2^2}$$

and satisfies $\Phi_{\tau,n+1}(w) \leq \Phi_{\tau,n}(w)$.

The decreasing sequences of bounds, of Theorems 3.1 and 3.2, provide a means of investigating the limits of the degree-elevation approach to the determination of bounds. However, for reasons that will become clear, we consider the cases $n \in \{4, 5\}$ separately and derive the corresponding invariant bounds (for “small” values of the invariant $\mathcal{J}(w)$ only in the latter case) which are new.

Theorem 3.3. For each $w \in \Omega$, $\tau \in \{1, 2\}$,

$$\tilde{\Phi}_{\tau,4}(w) = \max \left\{ \mathcal{J}(w), \frac{3 - \tau + 2\mathcal{J}(w)}{4\mathcal{J}(w)}, \frac{3 - \tau + \mathcal{J}(w)}{1 + 2\mathcal{J}(w)^2} \right\}$$

and, for $\mathcal{J}(w)^2 < \frac{1}{6}(5 - 3\tau + \sqrt{9\tau^2 - 48\tau + 70})$,

$$\tilde{\Phi}_{\tau,5}(w) = \max \left\{ \mathcal{J}(w), \frac{3 - \tau + 3\mathcal{J}(w)}{1 + 4\mathcal{J}(w)}, \frac{3(3 - \tau) + 4\mathcal{J}(w)}{2 + 4\mathcal{J}(w) + 4\mathcal{J}(w)^2} \right\}.$$

Proof. For $n \in \{4, 5\}$, $0 \leq j \leq n$, $x_0, x_1 \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$, let $\phi_{\tau,n,j}(x_0, x_1, \lambda)$ be given by

$$\frac{\binom{n-2}{j} \frac{1}{\lambda x_1} + (3 - \tau) \binom{n-2}{j-1} + \binom{n-2}{j-2} \lambda x_0}{\binom{n-4}{j} \frac{1}{\lambda^2 x_0 x_1} + 4 \binom{n-4}{j-1} \frac{1}{\lambda x_1} + 2 \binom{n-4}{j-2} (2 \frac{x_0}{x_1} + 1) + 4 \binom{n-4}{j-3} \lambda x_0 + \binom{n-4}{j-4} \lambda^2 x_0 x_1}. \quad (9)$$

Since $\Phi_{\tau,n}(\tilde{w}) = \max_{0 \leq j \leq n} \phi_{\tau,n,j}(\frac{w_1}{w_0}, \frac{w_2}{w_1}, \lambda)$, where $\tilde{w} = (w_0, \lambda w_1, \lambda^2 w_2)$, we wish to calculate

$$\min_{\lambda \in \mathbb{R}^+} \max_{0 \leq j \leq n} \phi_{\tau,n,j}(x_0, x_1, \lambda). \quad (10)$$

Note that we have the following useful symmetry property

$$\phi_{\tau,n,j}(x_0, x_1, \lambda) = \phi_{\tau,n,n-j}\left(\frac{1}{x_1}, \frac{1}{x_0}, \frac{1}{\lambda}\right). \quad (11)$$

Moreover, when $\lambda = (x_0 x_1)^{-1/2}$ we have $\phi_{\tau,n,j}(x_0, x_1, \lambda) = \phi_{\tau,n,n-j}(x_0, x_1, \lambda)$. It is easy to prove that for any $x_0, x_1 \in \mathbb{R}^+$, $\phi_{\tau,n,j}(x_0, x_1, \cdot)$ is increasing for $j \in \{0, 1\}$, or equivalently using (11), $\phi_{\tau,n,j}(x_0, x_1, \cdot)$ is decreasing for $j \in \{n, n-1\}$. Therefore, if $j \in \{0, 1\}$ then

$$\max\{\phi_{\tau,n,j}(x_0, x_1, \lambda), \phi_{\tau,n,n-j}(x_0, x_1, \lambda)\}$$

is decreasing for $\lambda \in (0, (x_0 x_1)^{-1/2})$ and increasing for $\lambda \in ((x_0 x_1)^{-1/2}, \infty)$. When $n = 4$, the remaining function $\phi_{\tau,4,2}(x_0, x_1, \lambda)$ is also decreasing for $\lambda \in (0, (x_0 x_1)^{-1/2})$ and increasing for $\lambda \in ((x_0 x_1)^{-1/2}, \infty)$. The claimed expression for $\tilde{\Phi}_{\tau,4}$ follows. When $n = 5$, the restriction $\frac{x_0}{x_1} < \frac{1}{6}(5 - 3\tau + \sqrt{9\tau^2 - 48\tau + 70})$ implies $\phi_{\tau,5,2}(x_0, x_1, \cdot)$ is increasing, and consequently $\phi_{\tau,5,3}(x_0, x_1, \cdot)$ is decreasing. The claimed formula for $\tilde{\Phi}_{\tau,5}$ now follows as above for $n = 4$. \square

The proof above demonstrates that for $n \in \{4, 5\}$ the quantity in (10) is attained at $\lambda = (x_0 x_1)^{-1/2}$. However for $n \geq 6$ this is not necessarily the case because of the following.

Proposition 3.4. Suppose $w \in \Omega$, $\tau \in \{1, 2\}$ and n is an even integer greater than or equal to 6. Then there exists a neighbourhood $\mathcal{N}_{\tau,n}$ of zero such that whenever $\mathcal{J}(w) \in \mathcal{N}_{\tau,n}$ the mapping $\lambda \mapsto \Phi_{\tau,n}(w_0, \lambda w_1, \lambda^2 w_2)$ is not minimised at $\lambda = (w_2/w_0)^{1/2}$.

We do not give a full proof of Proposition 3.4 here and simply indicate why it is true. Firstly, as in the above, let $\phi_{\tau,n,j}(x_0, x_1, \lambda)$ be given by the expression in (9). A direct argument using calculus shows that $\phi_{\tau,n,n/2}(x_0, x_1, \lambda)$ has a global maximum, as function of λ , which is uniquely attained at $\lambda = (x_0 x_1)^{-1/2}$, provided that $n \geq 6$ and x_0/x_1 is sufficiently small. Now for each j , with $x_0 = w_1/w_0$ and $x_1 = w_2/w_1$, we have that $\phi_{\tau,n,j}(x_0, x_1, \frac{1}{(x_0 x_1)^{1/2}})$ is equal to

$$\frac{\binom{n-2}{j}\mathcal{J}(w) + (3-\tau)\binom{n-2}{j-1} + \binom{n-2}{j-2}\mathcal{J}(w)}{\binom{n-4}{j} + 4\binom{n-4}{j-1}\mathcal{J}(w) + 2\binom{n-4}{j-2}(2\mathcal{J}(w)^2 + 1) + 4\binom{n-4}{j-3}\mathcal{J}(w) + \binom{n-4}{j-4}}.$$

If $\mathcal{J}(w) \in \mathcal{N}_{\tau,n}$ is sufficiently small, by continuity and since

$$\max_{0 \leq j \leq n} \left[\binom{n-4}{j} + 2\binom{n-4}{j-2} + \binom{n-4}{j-4} \right]^{-1} \binom{n-2}{j-1}$$

is uniquely attained at $j = n/2$, it follows that $\phi_{\tau,n,j}(x_0, x_1, \frac{1}{(x_0 x_1)^{1/2}})$ is uniquely maximised when $j = n/2$. Since

$$\Phi_{\tau,n}(w_0, \lambda w_1, \lambda^2 w_2) = \max_{0 \leq j \leq n} \phi_{\tau,n,j}(x_0, x_1, \lambda)$$

it follows, again by continuity, that this cannot be minimised when $\lambda = (x_0 x_1)^{-1/2}$, as claimed.

We remark that when the invariant $\mathcal{J}(w)$ is “large” (i.e. the complementary case to Proposition 3.4), all of the invariant bounds considered in this paper cannot be improved as we demonstrate at the end of the subsequent section.

4. A comparison and evaluation of bounds

We begin with $\tau = 1$ and note that, for each $(v, w) \in V \times \Omega$ and $t \in [0, 1]$, the bound $|\sigma'[v, w](t)| \leq 2\Delta_1(v)\Lambda_1(w)$ was proved in Zhang and Ma (2006), where² $\Lambda_1(w) = \max\{\frac{w_0}{w_1}, \frac{w_1}{w_2}, \frac{w_1}{w_0}, \frac{w_2}{w_1}\}$. The following theorem shows that the pointwise bounds Θ_1 (due to Hermann) and $\Phi_{1,4}$ (of Theorem 3.2) are superior to Λ_1 .

Theorem 4.1. If $w \in \Omega$ then $\Theta_1(w) \leq \Lambda_1(w)$ and $\Phi_{1,4}(w) \leq \Lambda_1(w)$.

Before offering some remarks and proof of this, we also note the following comparison of the invariant bounds determined by Θ_1 , $\Phi_{1,4}$, $\Phi_{1,5}$ and Λ_1 .

Theorem 4.2. If $w \in \Omega$ then $\tilde{\Lambda}_1(w) = \max\{\mathcal{J}(w), \mathcal{J}(w)^{-1}\}$, and

$$\tilde{\Theta}_1(w) \leq \tilde{\Phi}_{1,4}(w) \leq \tilde{\Lambda}_1(w).$$

If, in addition $\mathcal{J}(w)^2 < \frac{1}{6}(2 + \sqrt{31})$, then we have

$$\tilde{\Theta}_1(w) \leq \tilde{\Phi}_{1,5}(w) \leq \tilde{\Phi}_{1,4}(w) \leq \tilde{\Lambda}_1(w).$$

Given Theorem 4.1 it is natural to compare Θ_1 and $\Phi_{1,4}$ (and, of course, $\Phi_{1,n}$ for $n \geq 5$ in light of Theorem 3.2). We note that it is possible to find weights $w \in \Omega$ for which $\Phi_{1,4}(w) < \Theta_1(w)$, however, a full characterisation of such weights

² The commonly used notation in the literature for $\Lambda_1(w)$ is $\max\{\omega, \omega^{-1}\}$, where $\omega = \max_i \frac{w_i}{w_{i+1}}$; this notation is misleading because the two quantities are different.

is not easy to describe. We also remark that Theorem 4.2 highlights that the invariant bounds are significantly easier to compare. As $\mathcal{J}(w)$ approaches zero, notice that the invariant bounds $\tilde{\Lambda}_1(w)$ and $\tilde{\Phi}_{1,4}(w)$ blow up to infinity. Elevating the degree once more removes this singularity from the invariant bound – as is evident from the expression for $\tilde{\Phi}_{1,5}(w)$ in Theorem 3.3.

Proof of Theorem 4.1. Observe that

$$\Lambda_1(w) = \max\{\mathcal{J}(w), \mathcal{J}(w)^{-1}\} \max\left\{\left(\frac{w_2}{w_0}\right)^{\frac{1}{2}}, \left(\frac{w_0}{w_2}\right)^{\frac{1}{2}}\right\},$$

and since $2(1 + \mathcal{J})^{-1} \leq \mathcal{J}^{-1}$ for $\mathcal{J} \in (0, 1]$, it follows that $\Theta_1(w) \leq \Lambda_1(w)$. To see that $\Phi_{1,4}(w) \leq \Lambda_1(w)$, note that

$$\Phi_{1,4}(w) = \max\left\{\frac{w_1}{w_0}, \frac{w_1}{w_2}, \frac{1}{2}\left(1 + \frac{w_2}{w_1}\right), \frac{1}{2}\left(1 + \frac{w_0}{w_1}\right), \frac{w_0 w_1 + 4w_0 w_2 + w_1 w_2}{4w_1^2 + 2w_0 w_2}\right\}$$

and so it clearly suffices to check that

$$\frac{w_0 w_1 + 4w_0 w_2 + w_1 w_2}{4w_1^2 + 2w_0 w_2} \leq \max\left\{\frac{w_0}{w_1}, \frac{w_1}{w_2}, \frac{w_1}{w_0}, \frac{w_2}{w_1}\right\}.$$

Equivalently, by setting $x_0 = w_1/w_0$ and $x_1 = w_2/w_1$, we show that

$$\frac{1}{x_1} + 4 + x_0 \leq 2\left(2\frac{x_0}{x_1} + 1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\} \quad (12)$$

for all $x_0, x_1 \in \mathbb{R}^+$. Note that (12) is obvious when $x_0 \geq x_1$. When $1 \leq x_0 \leq x_1$ we have

$$\frac{1}{x_1} + 4 + x_0 \leq 5 + x_0 \leq 4x_0 + 2x_1 = 2\left(2\frac{x_0}{x_1} + 1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\}$$

and using the symmetry $(x_0, x_1) \mapsto (1/x_1, 1/x_0)$ it follows that (12) holds for $x_0 \leq x_1 \leq 1$ as well. For the remaining case $x_0 \leq 1 \leq x_1$, first assume $x_1 \leq 1/x_0$. Since $x_0 + 1/x_0 \geq 2$ it follows that

$$\frac{1}{x_1} + 4 + x_0 \leq \frac{1}{x_1} + \left(\frac{3}{x_1} + \frac{2}{x_0}\right) = 2\left(2\frac{x_0}{x_1} + 1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\}.$$

Similarly, when $x_1 \geq 1/x_0$ we use $x_1 + 1/x_1 \geq 2$ to obtain

$$\frac{1}{x_1} + 4 + x_0 \leq (3x_0 + 2x_1) + x_0 = 2\left(2\frac{x_0}{x_1} + 1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\},$$

which completes our proof of (12). \square

Proof of Theorem 4.2. It is straightforward to check that

$$\min_{\lambda \in \mathbb{R}^+} \max\{\lambda x_0, \lambda x_1, (\lambda x_0)^{-1}, (\lambda x_1)^{-1}\} = \max\left\{\left(\frac{x_0}{x_1}\right)^{\frac{1}{2}}, \left(\frac{x_1}{x_0}\right)^{\frac{1}{2}}\right\}$$

which is attained when $\lambda = (x_0 x_1)^{-1/2}$. Consequently we obtain the claimed formula for $\tilde{\Lambda}_1(w)$ by taking $x_0 = w_1/w_0$ and $x_1 = w_2/w_1$. By Theorems 3.2 and 4.1, it remains to show that $\tilde{\Theta}_1(w) \leq \tilde{\Phi}_{1,5}(w)$ whenever $\mathcal{J}(w) < 1$; that is,

$$\frac{2}{1 + \mathcal{J}} \leq \max\left\{\frac{2 + 3\mathcal{J}}{1 + 4\mathcal{J}}, \frac{3 + 2\mathcal{J}}{1 + 2\mathcal{J} + 2\mathcal{J}^2}\right\}$$

whenever $\mathcal{J} < 1$. This follows because the quadratic $\mathcal{J} \mapsto 2\mathcal{J}^2 - \mathcal{J} - 1$ is negative for $\mathcal{J} \in (0, 1)$. \square

We conclude this section with some comparisons regarding the case $\tau = 2$. The best pointwise bounds appear to be due to Zhang and Ma (2006), who followed a number of authors in using the de Casteljau algorithm for rational Bézier curves, due to Farin (1983). In the quadratic case, the bound in Zhang and Ma (2006) follows from

$$|\sigma'[v, w](t)| \leq 2\Delta_2(v) \frac{(w_0(1-t) + w_1 t)(w_1(1-t) + w_2 t)}{(w_0(1-t)^2 + 2w_1 t(1-t) + w_2 t^2)^2}. \quad (13)$$

In particular, considering separately the cases

$$w_1(1-t) + w_2 t \geq w_0(1-t) + w_1 t \quad \text{and} \quad w_1(1-t) + w_2 t \leq w_0(1-t) + w_1 t \quad (14)$$

and using Theorem 3.1, one obtains $|\sigma'[v, w](t)| \leq 2\Delta_2(v)\Lambda_2(w)$, which is due to Zhang and Ma (2006). Here, $\Lambda_2(w) = \max\{\frac{w_1}{w_0}, \frac{w_1}{w_2}, \frac{1}{2}(1 + \frac{w_2}{w_1}), \frac{1}{2}(1 + \frac{w_0}{w_1})\}$. We note that a better bound is easily obtained by slightly modifying the above argument. In particular, bypassing considerations like (14), and using Theorem 3.1 on the right-hand side of (13) already improves the result in Zhang and Ma (2006). However, this may be bettered still; from this approach, the input coefficient vector for the numerator would be

$$(w_0w_1, w_0w_2 + w_1^2, w_1w_2)$$

which has first and third components equal, and second component *greater* than, the respective components of the input in (8) which led to Theorem 3.2 (the denominators are, of course, the same). We conclude that the approach based on the de Casteljau algorithm in Zhang and Ma (2006) does not appear to yield better results than Theorem 3.2. In particular, we have shown the following.

Theorem 4.3. For each $w \in \Omega$, $\Phi_{2,4}(w) \leq \Lambda_2(w)$.

We remark that $\Lambda_2(w)$ (and hence $\Phi_{2,n}(w)$ for each $n \geq 4$) beats the bound $\Theta_2(w)$, of Theorem 2.1, for weights w in a non-trivial, but difficult to describe, subset of Ω . At the level of the invariant bounds, the picture is clearer and we have the following analogue of Theorem 4.2.

Theorem 4.4. Let $w \in \Omega$. Then $\tilde{\Lambda}_2(w) = \max\{\mathcal{J}(w), \frac{1}{2}(1 + \mathcal{J}(w)^{-1})\}$ and

$$\tilde{\Theta}_2(w) \leq \tilde{\Phi}_{2,4}(w) \leq \tilde{\Lambda}_2(w).$$

If, in addition $\mathcal{J}(w)^2 < \frac{1}{6}(\sqrt{10} - 1)$, then we have

$$\tilde{\Theta}_2(w) \leq \tilde{\Phi}_{2,5}(w) \leq \tilde{\Phi}_{2,4}(w) \leq \tilde{\Lambda}_2(w).$$

Proof. One can compute $\tilde{\Lambda}_2$ in a similar way that we computed $\tilde{\Lambda}_1$ in the proof of Theorem 4.2; we omit the details. Next, straightforward considerations yield $\tilde{\Theta}_2(w) = \tilde{\Phi}_{2,4}(w) = \mathcal{J}(w)$ whenever $\mathcal{J}(w) \in [C_1, \infty)$. For $\mathcal{J}(w) \in (0, C_0]$, we have $\tilde{\Theta}_2(w) \leq \tilde{\Phi}_{2,4}(w)$ since $(2\mathcal{J} + 1)(1 + \mathcal{J})^{-2} \leq \frac{1}{2} + \frac{1}{4}\mathcal{J}^{-1}$ for all $\mathcal{J} \in (0, 1)$. Finally, for $\mathcal{J} \in (C_0, C_1)$ we have $\frac{1}{8}(2\mathcal{J} - 1)^2(1 - \mathcal{J})^{-1}(2\mathcal{J}^2 - 1)^{-1} \leq \frac{1}{2} + \frac{1}{4}\mathcal{J}^{-1}$ because the quartic $\mathcal{J} \mapsto -8\mathcal{J}^4 + 12\mathcal{J}^2 - 3\mathcal{J} - 2$ is positive on (C_0, C_1) . Therefore $\tilde{\Theta}_2(w) \leq \tilde{\Phi}_{2,4}(w)$ whenever $\mathcal{J}(w) \in (C_0, C_1)$, and consequently for all weights. It remains to show that whenever $\mathcal{J}(w)^2 < \frac{1}{6}(\sqrt{10} - 1)$ we have $\tilde{\Theta}_2(w) \leq \tilde{\Phi}_{2,5}(w)$. One can easily check that for such weights w ,

$$\tilde{\Theta}_2(w) = \frac{1 + 2\mathcal{J}(w)}{(1 + \mathcal{J}(w))^2} \quad \text{and} \quad \tilde{\Phi}_{2,5}(w) = \frac{3 + 4\mathcal{J}(w)}{2 + 4\mathcal{J}(w) + 4\mathcal{J}(w)^2}$$

in which case the desired inequality holds because the cubic $\mathcal{J} \mapsto 4\mathcal{J}^3 + \mathcal{J}^2 - 2\mathcal{J} - 1$ is negative for $\mathcal{J} > 0$ with $\mathcal{J}^2 < \frac{1}{6}(\sqrt{10} - 1)$. \square

We note that invariant bounds have also been obtained in Zheng (2005) which correspond to the uniform pointwise bounds $|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Upsilon_\tau(w)$, where $\Upsilon_\tau(w) = (\max_i w_i / \min_j w_j)^{3-\tau}$, obtained by Floater (1992). Zheng proved that $\tilde{\Upsilon}_\tau(w) = \max\{\mathcal{J}(w)^{3-\tau}, \mathcal{J}(w)^{\tau-3}\}$. These invariant bounds are out-performed by the Zhang and Ma bounds, $\tilde{\Lambda}_\tau(w)$, $\tau \in \{1, 2\}$, and hence by all other invariant bounds of their respective type considered in this paper. Zheng (2005) also considered the invariant bounds from the general degree bounds obtained in Floater (1992), however, except for the quadratic case, Zheng does not provide an explicit formula for these invariant bounds. We point out that in Bez and Bez (2012b) we establish certain invariant bounds which are explicit and improve upon Zheng's bounds in the general degree case.

We conclude this section by providing certain sharpness considerations in the case $\tau = 1$. Analogous conclusions are possible when $\tau = 2$; we leave the details to the reader. Observe that we always have $|\sigma'[v, w](0)| = 2\frac{w_1}{w_0}|v_1 - v_0|$ and $|\sigma'[v, w](1)| = 2\frac{w_1}{w_2}|v_2 - v_1|$, so if $|v_1 - v_0| = |v_2 - v_1|$ then

$$\max_{t \in [0, 1]} |\sigma'[v, w](t)| \geq 2\Delta_1(v) \frac{w_1}{\min\{w_0, w_2\}}. \quad (15)$$

When $\mathcal{J}(w) \geq 1$ each of the upper bounds $\Theta_1(w)$, $\Phi_{1,n}(w)$, $\Lambda_1(w)$ coincides with $\frac{w_1}{\min\{w_0, w_2\}}$, which shows sharpness in this pointwise sense. Moreover, $\frac{w_1}{\min\{w_0, w_2\}} \geq \mathcal{J}(w)$ and therefore, by (15), for weights with $\mathcal{J}(w) \geq 1$, the invariant bounds $\tilde{\Theta}_1(w)$, $\tilde{\Phi}_{1,n}(w)$, $\tilde{\Lambda}_1(w)$ coinciding with $\mathcal{J}(w)$, are sharp. Importantly, the invariant bound $\tilde{\Theta}_1(w)$ is also sharp at $\mathcal{J}(w) = 0$; to show this we require the following lemma.

Lemma 4.5. We have $\min_{a \in (0, \infty)} \max_{t \in [0, 1]} \phi(a, t) = 1$, where

$$\phi(a, t) = at(1 - t)((1 - t)^2 + at^2)^{-2}.$$

Proof. Use $\partial_j \phi$ to denote the j th partial derivative of ϕ for $j \in \{1, 2\}$. For each $a > 0$ there exists a unique point $t(a) \in (0, 1)$ such that $\partial_2 \phi(a, t(a)) = 0$. Note that $t(1) = \frac{1}{2}$, $a \mapsto t(a)$ is decreasing and $\max_{t \in [0, 1]} \phi(a, t) = \phi(a, t(a))$. Thus, it suffices to prove that $\min_{a \in (0, \infty)} \phi(a, t(a)) = \phi(1, t(1))$. This follows from the mean value theorem and the fact that $\partial_1 \phi(a, t(a))$ is positive for $a > 1$ and negative for $a < 1$. \square

At $\mathcal{J}(w) = 0$ we³ have $w = (w_0, 0, w_2)$. If v is such that $v_1 - v_0$ and $v_2 - v_1$ are parallel unit vectors, then $\Delta_1(v) = 1$ and the invariant bound $\tilde{\Theta}_1(w)$ gives $\min_{\tilde{w} \sim w} \max_{t \in [0, 1]} |\sigma'[v, w](t)| \leq 4$. Writing $a = w_2/w_0$ we have from (3) that

$$|\sigma'[v, w](t)| = \frac{2at(1-t)}{((1-t)^2 + at^2)^2} |v_2 - v_0| = \frac{4at(1-t)}{((1-t)^2 + at^2)^2},$$

and from Lemma 4.5 it follows that $\min_{\tilde{w} \sim w} \max_{t \in [0, 1]} |\sigma'[v, w](t)| = 4$, which occurs at $\tilde{w} = (1, 0, 1)$. Hence $\tilde{\Theta}_1(w)$ is sharp at $\mathcal{J}(w) = 0$.

5. Conclusions

In this paper, we have established clear and comprehensive comparisons of both pointwise and invariant bounds for quadratic rational Bézier paths. A new bound Θ_2 is derived, following the approach in Hermann (1999); moreover, we demonstrated that the invariant bounds $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are currently the best known and are difficult to beat using the approach of degree raising and convexity. This is because after the initial normalisation of the weights, the arguments leading to the bounds Θ_1 and Θ_2 cannot possibly be improved. However, in the cubic case, after an initial normalisation of the weights, the argument in Hermann (1999) is less tight. Indeed, in significant regions of the invariant space, we have shown that the associated invariant bound arising from

$$\max \left\{ \frac{w_0}{w_1}, \frac{w_1}{w_2}, \frac{w_2}{w_3}, \frac{w_1}{w_0}, \frac{w_2}{w_1}, \frac{w_3}{w_2} \right\}$$

(which comes from a degree-elevation and convexity argument) is smaller than that obtained from Hermann's cubic bound (see Bez and Bez, 2012a). This highlights the potential for obtaining further improvements on the invariant bounds for rational Bézier paths of degree 3 and above using degree-elevation and convexity; progress in this direction has already been made in Bez and Bez (2012b).

Acknowledgement

The authors wish to thank the London Mathematical Society for partially funding this work under grant SC7-1011-15.

References

- Bez, H.E., Bez, N., 2012a. On derivative bounds for the rational cubic Bézier paths. Preprint.
- Bez, H.E., Bez, N., 2012b. New minimal bounds for the derivatives of rational Bézier paths and rational rectangular Bézier surfaces. Preprint.
- Farin, G., 1983. Algorithms for rational Bézier curves. *Computer Aided Design* 15, 73–77.
- Floater, M.S., 1992. Derivatives of rational Bézier curves. *Computer Aided Geometric Design* 9, 161–174.
- Hermann, T., 1999. On the derivatives of second and third degree rational Bézier curves. *Computer Aided Geometric Design* 16, 157–163.
- Selimovic, I., 2005. New bounds on the magnitude of the derivative of rational Bézier curves and surfaces. *Computer Aided Geometric Design* 22, 321–326.
- Wang, G.J., Sederberg, T.W., Saito, T., 1997. Partial derivatives of rational Bézier surfaces. *Computer Aided Geometric Design* 14, 377–381.
- Zhang, R.J., Ma, W., 2006. Some improvements on the derivative bounds of rational Bézier curves and surfaces. *Computer Aided Geometric Design* 23, 563–572.
- Zheng, J., 2005. Minimizing the maximal ratio of weights of a rational Bézier curve. *Computer Aided Geometric Design* 22, 275–280.

³ Strictly speaking, the weight vector $(w_0, 0, w_2)$ does not belong to the weight space Ω under analysis in this paper; however, for such sharpness considerations in the limiting case of the invariant, it makes sense to allow such weights.