

The Argyris isogeometric space on unstructured multi-patch planar domains

Mario Kapl^a, Giancarlo Sangalli^{b,c}, Thomas Takacs^d

^a*Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Austria*

^b*Dipartimento di Matematica “F. Casorati”, Università degli Studi di Pavia, Italy*

^c*Istituto di Matematica Applicata e Tecnologie Informatiche “E. Magenes” (CNR), Italy*

^d*Institute of Applied Geometry, Johannes Kepler University Linz, Austria*

Abstract

Multi-patch spline parametrizations are used in geometric design and isogeometric analysis to represent complex domains. We deal with a particular class of C^0 planar multi-patch spline parametrizations called analysis-suitable G^1 (AS- G^1) multi-patch parametrizations (cf. [10]). This class of parametrizations has to satisfy specific geometric continuity constraints, and is of importance since it allows to construct, on the multi-patch domain, C^1 isogeometric spaces with optimal approximation properties. It was demonstrated in [22] that AS- G^1 multi-patch parametrizations are suitable for modeling complex planar multi-patch domains.

In this work, we construct a basis, and an associated dual basis, for a specific C^1 isogeometric spline space \mathcal{A} over a given AS- G^1 multi-patch parametrization. We call the space \mathcal{A} the Argyris isogeometric space, since it is C^1 across interfaces and C^2 at all vertices and generalizes the idea of Argyris finite elements (see [1]) to tensor-product splines. The considered space \mathcal{A} is a subspace of the entire C^1 isogeometric space \mathcal{V}^1 , which maintains the reproduction properties of traces and normal derivatives along the interfaces. Moreover, it reproduces all derivatives up to second order at the vertices. In contrast to \mathcal{V}^1 , the dimension of \mathcal{A} does not depend on the domain parametrization, and \mathcal{A} admits a basis and dual basis which possess a simple explicit representation and local support.

We conclude the paper with some numerical experiments, which exhibit the optimal approximation order of the Argyris isogeometric space \mathcal{A} and demonstrate the applicability of our approach for isogeometric analysis.

Keywords: Isogeometric Analysis, Argyris isogeometric space, analysis-suitable G^1 parametrization, planar multi-patch domain

1. Introduction

Multi-patch spline parametrizations are a powerful tool in computer-aided geometric design for modeling complex domains (cf. [13, 17]). In the framework of isogeometric analysis (IGA) (cf. [4, 11, 18]) the underlying spline spaces of these parametrizations are used to define (smooth) discretization spaces for numerically solving partial differential equations (PDEs) over the multi-patch domains. When solving a fourth order PDE, such as the biharmonic equation, e.g. [3, 10, 20, 23, 35], the Kirchhoff-Love plate/shell problem, e.g. [2, 5, 25, 26, 27], or the Cahn-Hilliard

equation, e.g. [14, 15, 30], by means of its weak formulation using a standard Galerkin projection, approximating functions of global C^1 -smoothness are needed.

In this work we construct an isogeometric space with global C^1 regularity over an unstructured multi-patch parametrization. Our construction takes inspiration from the Argyris finite element [1], which has been the progenitor of all the C^1 triangular finite elements. The original Argyris construction uses polynomials of total degree 5 and the following degrees-of-freedom: The function values, first and second derivatives at the element vertices, and the normal derivatives at the element edge midpoints. These degrees-of-freedom determine the trace at the edges (a polynomial of degree 5 on each edge) and the normal derivatives at the edges (a polynomial of degree 4), and by that, determine the polynomial on the whole triangle (see Figure 1(a)). C^1 regularity follows from the gluing of the normal derivative at the edges.

After Argyris, other C^1 constructions on triangular meshes have been proposed, in the context of finite elements (see for example the book [9]). The Argyris triangular element has been used only recently for surface parametrizations, in [19]. However, splines on triangular meshes are commonly used in geometric design, often with C^2 smoothness at the mesh vertices, see the book [29] for references.

In the finite element literature, the results on C^1 quadrilateral elements are restricted to meshes of structured rectangles, where the first and most well-known construction is the Bogner-Fox-Schmit element, introduced in [8]. Isogeometric analysis has been reinvigorating this interest: The recent papers [10, 20, 21, 23] and the book [6] shed light on the conditions of global C^1 regularity for isogeometric spaces on unstructured quadrilateral meshes, showing at the same time that these spaces are difficult to characterize. This motivates our present work: We design a basis and the related degrees-of-freedom for a subspace of the complete C^1 isogeometric space, mimicking the Argyris construction. The space we propose is therefore named *Argyris isogeometric space* and denoted by \mathcal{A} .

Consider a quadrilateral element $\bar{\Omega}^{(i)} = \mathbf{F}^{(i)}([0, 1]^2)$, i.e., given by a bilinear mapping of the reference square element. The lowest-degree polynomial version of \mathcal{A} contains functions φ_h such that $\varphi_h \circ \mathbf{F}^{(i)}$ is a (bi)quintic polynomial, and $(\nabla \varphi_h \cdot \mathbf{d}) \circ \mathbf{F}^{(i)}$ is a quartic polynomial on the edges of the reference element. There are two main differences with respect to the original Argyris triangle. The first is obvious: $\varphi_h|_{\bar{\Omega}^{(i)}}$ is not polynomial since $\mathbf{F}^{(i)}$ is not linear, in general. The second is technical: \mathbf{d} is not the normal unitary vector to the edges of $\bar{\Omega}^{(i)}$, it is instead a suitable non-constant direction, shared with the adjacent quadrilateral and dependent on it. The degrees-of-freedom are indeed:

- 24 (vertex) degrees-of-freedom giving the value, first and second derivatives at each vertex, fully determining the trace of φ_h at the boundary of the element,
- 4 (edge) degrees-of-freedom, one per edge, that with the previous information fully determine $(\nabla \varphi_h \cdot \mathbf{d}) \circ \mathbf{F}^{(i)}$ as a quartic polynomial on each edge,
- 4 (interior) degrees-of-freedom that, with the trace and directional derivative given at the boundary, fully determine φ_h .

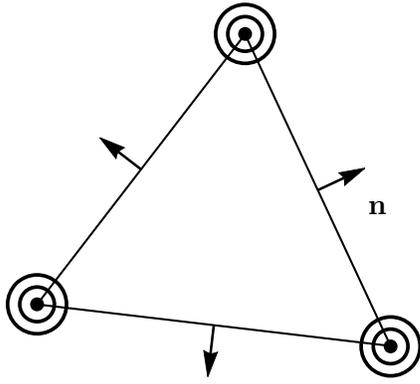
The total number of degrees-of-freedom is 32 and they are depicted in Figure 1(c). However the condition that $(\nabla\varphi_h \cdot \mathbf{d}) \circ \mathbf{F}^{(i)}$ is a quartic polynomial on each edge is responsible for 4 additional scalar constraints due to degree elevation to quintic polynomials. The degrees-of-freedom together with the constraints give 36 conditions, which matches the dimension of the (bi)quintic polynomial space. We remark that the boundary degrees-of-freedom above correspond to the ones of the original Argyris triangular element, with \mathbf{d} replacing \mathbf{n} (as for the Argyris triangular element, these edge degrees-of-freedom depend on the mesh and do not admit a universal definition on the reference element). Interior degrees-of-freedom appear also in the higher-degree Argyris triangular element, see Figure 1(b). Our construction is generalizable to any degree $p \geq 5$, see Figure 1(d).

The construction above further extends in two ways. The first is that, instead of a quadrilateral element, we can allow a Bézier patch, that is $\mathbf{F}^{(i)}$ can be a p -degree polynomial. However there are compatibility conditions between adjacent patch parametrizations that guarantee that the Argyris isogeometric space does not get overconstrained: We need the $\mathbf{F}^{(i)}$ to form an *analysis-suitable* G^1 multi-patch parametrization (see [10]). The second extension is from a polynomial space (on each patch) to a tensor-product spline space. Not only is this important in isogeometric analysis, it also allows a degree reduction. Indeed we can construct an Argyris isogeometric patch from (bi)cubic C^1 continuous splines, see Figure 1(e).

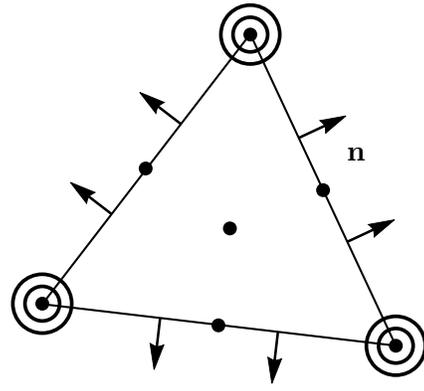
In this paper we show that the Argyris isogeometric space is well defined by constructing a suitable basis and dual basis possessing desirable properties, such as local support and an explicit representation, which can be evaluated and manipulated easily.

A key ingredient of our approach is the determination of the compatibility conditions that the parametrizations $\mathbf{F}^{(i)}$ of the patches need to fulfill in order to guarantee that the Argyris isogeometric space possesses optimal approximation order. This is based on the mentioned work [10], where the class of AS- G^1 multi-patch parametrizations has been introduced. The paper [22] has then shown numerically that this class of parametrizations enables the geometric design of complex planar multi-patch domains. The construction can be extended to surfaces, and the C^1 isogeometric spaces constructed on AS- G^1 multi-patch parametrizations exhibit optimal convergence under h -refinement.

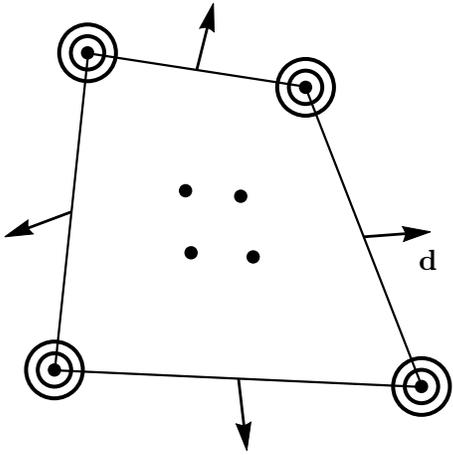
Piecewise bilinear multi-patch parametrizations are a subclass of the class of AS- G^1 multi-patch parametrizations and were considered to generate a C^1 basis in [6, 20, 23]. The focus therein is however to characterize the full C^1 isogeometric space, which we denote \mathcal{V}^1 in this paper. The papers [20, 23] study \mathcal{V}^1 for uniform spline functions of degree 3. The work [6] focuses on Bézier polynomials of degree 4 and 5 and generates basis functions by means of minimal determining sets (cf. [28]) for the involved Bézier coefficients. These approaches can be extended to mapped piecewise bilinear multi-patch parametrizations, which are also AS- G^1 and allow to model certain domains with curved boundaries and interfaces, see [20, 23]. Still, more general AS- G^1 multi-patch parametrizations, such as domains with smooth boundaries, cannot be handled. In [21] an explicit basis construction was given allowing non-uniform isogeometric spline functions of arbitrary degree $p \geq 3$ and regularity (with regularity r up to $p - 2$), but on a two-patch geometry, with AS- G^1 parametrization.



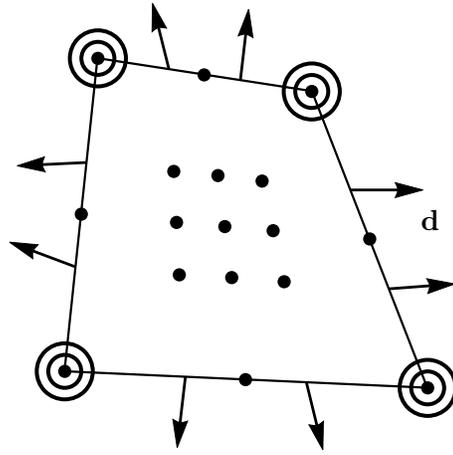
(a) Triangular element for degree 5.



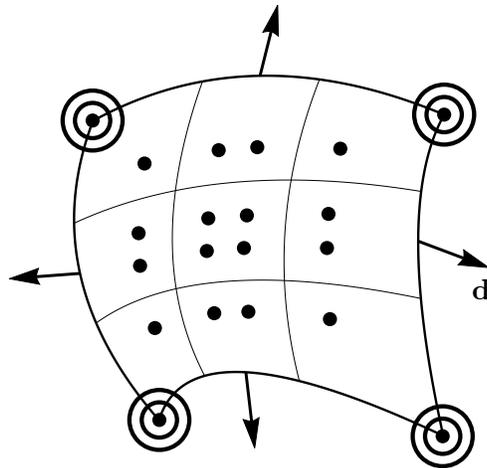
(b) Triangular element for degree 6.



(c) Quadrilateral element for bidegree 5.



(d) Quadrilateral element for bidegree 6.



(e) Isogeometric element for bidegree 3.

Figure 1: Argyris-type finite elements, with associated vertex, edge and interior degrees-of-freedom. Note that for the quadrilateral and isogeometric elements the derivative in normal direction \mathbf{n} has to be replaced by the derivative in a transversal direction \mathbf{d} (see Definition 20).

Unlike the Argyris isogeometric space \mathcal{A} , the dimension of the full C^1 space, that is \mathcal{V}^1 , depends on the domain parametrization, see [21]. In fact \mathcal{A} possesses a simpler structure than \mathcal{V}^1 , but maintains its reproduction properties for traces and normal derivatives along the interfaces. In the present work we only provide numerical evidence of the optimal approximation properties of \mathcal{A} , postponing the mathematical analysis to a further work.

In our setting, and in the papers [6, 7, 10, 20, 21, 23, 31], the C^1 isogeometric functions are defined over a domain given by a multi-patch parametrization which is not C^1 at the patch interfaces. However there is another possibility, that is, when the multi-patch parametrization is C^1 everywhere except in the vicinity of an extraordinary vertex, where the parametrization is singular. C^1 isogeometric spaces in this case are constructed and studied in [24, 32, 33, 34, 39, 40, 41]. A special case is when the parametrization is polar at the extraordinary vertex, see [36, 37, 38].

The remainder of the paper is organized as follows. Section 2 presents some basic definitions and notations which are used throughout the paper. This includes the presentation of the spline spaces and of the multi-patch domain parametrizations as well as the local and global indexing for the patches, edges and vertices that we use. In Section 3 we recall the concept of AS- G^1 multi-patch parametrizations and the framework of C^1 isogeometric spline spaces over this class of multi-patch parametrizations. Section 4 describes the construction of a basis and of its associated dual basis for the Argyris isogeometric space \mathcal{A} . In Section 5 we perform L^2 -approximation over different AS- G^1 parametrizations to demonstrate the potential of the Argyris isogeometric space \mathcal{A} for applications in IGA. After the concluding remarks in Section 6, we deliver technical proofs in Appendix A, and describe in Appendix B and Appendix C some extensions of our construction.

2. Preliminaries

We describe the general notation as well as the multi-patch framework, which will be considered and used throughout the paper. First, in Sections 2.1 and 2.2 we introduce the general notation, the uni- and bivariate B-spline spaces and bases as well as the multi-patch domain we consider. Then, we recall in Section 2.3 the standard global-to-local index mapping of mesh objects within our framework. Finally, we introduce in Section 2.4 a specific local reparametrization, which will simplify the definitions of the smooth basis functions in Section 4.

2.1. Basic notation and spline spaces

We consider an open domain $\Omega \subset \mathbb{R}^2$, connected and regular, and $\Gamma = \partial\Omega$ being its boundary. If $\omega \subset \Omega$ is a manifold of dimension 0 (a point) or 1 (a line), we denote by $C^k(\omega)$ the set of piecewise smooth functions defined on Ω for which the k -order derivatives are continuous at each point of ω .

We denote by $\mathcal{S}_h^{p,r}$ the spline space of degree p and continuity C^r on the parameter domain $[0, 1]$, constructed from an open knot vector with n non-empty knot-spans (i.e., elements), then having mesh size $h = 1/n$. We restrict here to uniform knot spans for simplicity, see [21] and Appendix B for the generalization. The multiplicity of the interior knots is $p - r$.

Definition 1 (Univariate B-spline basis). Given a integers $p \geq 1$, $r \leq p$, and $n \geq 1$, we denote by $\{b_j\}_{j \in \{0,1,\dots,N-1\}}$, with $N = (p-r)(n-1) + p + 1$, the standard B-spline basis for the C^r univariate p -degree polynomial space $\mathcal{S}_h^{p,r}$ on $[0, 1]$ with uniform mesh of mesh size $h = 1/n$.

The tensor-product spline space on the parameter (reference) domain $[0, 1]^2$ is $\mathcal{S}_h^{\mathbf{p},\mathbf{r}} = \mathcal{S}_h^{p,r} \otimes \mathcal{S}_h^{p,r}$, where $\mathbf{p} = (p, p)$ and $\mathbf{r} = (r, r)$ indicate double indices which we assume, for the sake of simplicity, to be the same in the the two directions.

Remark 1. For any \mathbf{r} , $\mathcal{S}_1^{\mathbf{p},\mathbf{r}}$ is the space of polynomials of degree \mathbf{p} .

Definition 2 (Tensor-product B-spline basis). We denote the standard tensor-product B-spline basis of the space $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$ by $\{b_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{I}}$, with $b_{\mathbf{j}}(\xi_1, \xi_2) = b_{j_1}(\xi_1)b_{j_2}(\xi_2)$, where $\mathbf{j} = (j_1, j_2)$ and $\mathbb{I} = \{0, \dots, N-1\} \times \{0, \dots, N-1\}$.

Assumption 3 (Minimum regularity within the patches). We assume $r \geq 1$, that is, $\mathcal{S}_h^{\mathbf{p},\mathbf{r}} \subset C^1([0, 1]^2)$.

2.2. Multi-patch domain parametrization

We consider a multi-patch domain parametrization, composed of four-sided subdomains, called patches, interfaces between those patches as well as vertices, where several interfaces meet. The index sets containing all patches, all edges and all vertices will be denoted by \mathcal{I}_Ω , \mathcal{I}_Σ and $\mathcal{I}_\mathcal{X}$, respectively. Moreover, we will have $\mathcal{I}_\Sigma = \mathcal{I}_\Sigma^\circ \cup \mathcal{I}_\Sigma^\Gamma$, where \mathcal{I}_Σ° collects all indices representing the patch interfaces and the indices in $\mathcal{I}_\Sigma^\Gamma$ represent all boundary edges. Similarly, we will have $\mathcal{I}_\mathcal{X} = \mathcal{I}_\mathcal{X}^\circ \cup \mathcal{I}_\mathcal{X}^\Gamma$, where the indices in $\mathcal{I}_\mathcal{X}^\circ$ represent all interior vertices and the ones in $\mathcal{I}_\mathcal{X}^\Gamma$ represent all boundary vertices. To avoid confusion, we will denote all index sets of patches, interfaces and vertices with a calligraphic \mathcal{I} , and all index sets of basis functions with a double struck \mathbb{I} .

Assumption 4 (Multi-patch domain Ω). The domain Ω is the image of a regular multi-patch spline parametrization, i.e.,

$$\overline{\Omega} = \bigcup_{i \in \mathcal{I}_\Omega} \overline{\Omega^{(i)}}, \quad (1)$$

where $\{\Omega^{(i)}\}_{i \in \mathcal{I}_\Omega}$ is a regular and disjoint partition, without hanging nodes, and each $\Omega^{(i)}$ is an open spline patch,

$$\mathbf{F}^{(i)} : [0, 1]^2 \rightarrow \overline{\Omega^{(i)}} \subset \mathbb{R}^2, \quad (2)$$

where $\mathbf{F}^{(i)} \in \mathcal{S}_h^{\mathbf{p},\mathbf{r}} \times \mathcal{S}_h^{\mathbf{p},\mathbf{r}}$ are non-singular and orientation-preserving, i.e., for all $i \in \mathcal{I}_\Omega$ and for all $(\xi_1, \xi_2) \in [0, 1]^2$, it holds

$$\det \begin{bmatrix} \partial_1 \mathbf{F}^{(i)}(\xi_1, \xi_2) & \partial_2 \mathbf{F}^{(i)}(\xi_1, \xi_2) \end{bmatrix} > 0. \quad (3)$$

The domain Ω is partitioned into the union of patches, interfaces and interior vertices

$$\Omega = \left(\bigcup_{i \in \mathcal{I}_\Omega} \Omega^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\Sigma^\circ} \Sigma^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\mathcal{X}^\circ} \mathbf{x}^{(i)} \right).$$

The boundary Γ of the domain is given by the collection of boundary edges and boundary vertices

$$\Gamma = \left(\bigcup_{i \in \mathcal{I}_\Sigma^\Gamma} \Sigma^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\chi^\Gamma} \mathbf{x}^{(i)} \right).$$

2.3. Global to local index conversion for edges and vertices

Each edge and vertex of the multi-patch partition corresponds to a *global index* $i \in \mathcal{I}_\Sigma$ or $i \in \mathcal{I}_\chi$, respectively. In order to relate the edges and vertices to the patch parametrizations, each global index will be associated with a set of *local indices*, depending on the patches that share the edge or vertex.

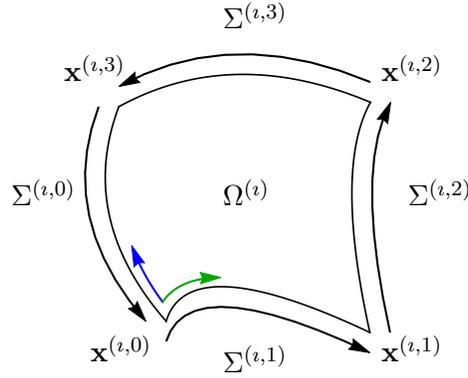


Figure 2: Local indexing of edges and vertices for the patch $\Omega^{(i)}$. The parametrizations (4) induce a counterclockwise orientation of the edges. Here, the green and blue arrows represent the local coordinate system for the parameters ξ_1 and ξ_2 , respectively.

The local index, which we define in the following, is in fact a multi-index (i, κ) , comprised of a patch index $i \in \mathcal{I}_\Omega$ and a local numbering $\kappa \in \{0, \dots, 3\}$.

Remark 2. We always denote with i or i_k dependent (secondary) indices, which depend on a (primary) index i , e.g. $\Omega^{(i_1)}$, $\Omega^{(i_2)}$ being the patches sharing an interface $\Sigma^{(i)}$.

The four edges of each patch $\Omega^{(i)}$ are indexed as follows:

$$\begin{aligned} \Sigma^{(i,0)} &= \{\mathbf{F}^{(i)}(0, 1 - \xi) : \xi \in]0, 1[\}, & \Sigma^{(i,1)} &= \{\mathbf{F}^{(i)}(\xi, 0) : \xi \in]0, 1[\}, \\ \Sigma^{(i,2)} &= \{\mathbf{F}^{(i)}(1, \xi) : \xi \in]0, 1[\}, & \Sigma^{(i,3)} &= \{\mathbf{F}^{(i)}(1 - \xi, 1) : \xi \in]0, 1[\}; \end{aligned} \quad (4)$$

for its four vertices we set:

$$\begin{aligned} \mathbf{x}^{(i,0)} &= \{\mathbf{F}^{(i)}(0, 0)\}, & \mathbf{x}^{(i,1)} &= \{\mathbf{F}^{(i)}(1, 0)\}, \\ \mathbf{x}^{(i,2)} &= \{\mathbf{F}^{(i)}(1, 1)\}, & \mathbf{x}^{(i,3)} &= \{\mathbf{F}^{(i)}(0, 1)\}; \end{aligned}$$

see Figure 2. Here and in what follows, the local coordinate system is depicted with green and blue arrows, corresponding to the ξ_1 - and ξ_2 -parameter directions, respectively.

The global to local index conversion for the edges is defined as follows.

Assumption 5 (Global to local index conversion for edges). For each global index $i \in \mathcal{I}_\Sigma^\circ$ there exists a set $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_1, \kappa_1), (\iota_2, \kappa_2)\}$, with $\iota_1, \iota_2 \in \mathcal{I}_\Omega$, $\iota_1 \neq \iota_2$, and $\kappa_1, \kappa_2 \in \{0, 1, 2, 3\}$, and

$$\Sigma^{(i)} = \Sigma^{(\iota_1, \kappa_1)} = \Sigma^{(\iota_2, \kappa_2)} \subset \Omega. \quad (5)$$

For each global index $i \in \mathcal{I}_\Sigma^\Gamma$ we have $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_1, \kappa_1)\}$, with $\iota_1 \in \mathcal{I}_\Omega$ and $\kappa_1 \in \{0, 1, 2, 3\}$, and

$$\Sigma^{(i)} = \Sigma^{(\iota_1, \kappa_1)} \subset \Gamma. \quad (6)$$

Similarly, we can define the global to local index conversion for vertices.

Assumption 6 (Global to local index conversion for vertices). For each $i \in \mathcal{I}_\mathcal{X}$ there exists a set $\mathcal{T}_{\mathbf{x}^{(i)}} = \{(\iota_2, \kappa_2), \dots, (\iota_{2\nu}, \kappa_{2\nu})\}$, with $\iota_2, \dots, \iota_{2\nu} \in \mathcal{I}_\Omega$, being ν different patch indices, and $\kappa_2, \dots, \kappa_{2\nu} \in \{0, 1, 2, 3\}$, where

$$\mathbf{x}^{(i)} = \mathbf{x}^{(\iota_2, \kappa_2)} = \dots = \mathbf{x}^{(\iota_{2\nu}, \kappa_{2\nu})}. \quad (7)$$

Here, ν is the *patch valence* of the vertex $\mathbf{x}^{(i)}$.

Note that we only consider even sub-indices for all patches. This is because later we introduce odd sub-indices for the adjacent interfaces (see Figure 3 and Definition 8). The patch valence coincides with the *edge valence* for interior vertices (this is the classical notion of valence). For boundary vertices the edge valence is larger by one than the patch valence.

Note that the inverse mapping from local to global indices is unique in the following sense: Each local edge index is associated to a unique global index of an interface or a boundary edge, i.e., for each $\iota_1 \in \mathcal{I}_\Omega$ and for each $\kappa_1 \in \{0, 1, 2, 3\}$ there exists exactly one $i \in \mathcal{I}_\Sigma$, such that $(\iota_1, \kappa_1) \in \mathcal{T}_{\Sigma^{(i)}}$. Moreover, for each $\iota_2 \in \mathcal{I}_\Omega$ and for each $\kappa_2 \in \{0, 1, 2, 3\}$ there exists exactly one $i \in \mathcal{I}_\mathcal{X}$, such that $(\iota_2, \kappa_2) \in \mathcal{T}_{\mathbf{x}^{(i)}}$.

2.4. Parametrization in standard form for edges and vertices

In order to simplify the construction of smooth basis functions across interfaces and vertices, we assume that the patch parametrizations are given in *standard form*, as depicted in Figure 3. This is obviously not always the case, but can be achieved by reparametrization, as we will demonstrate in Lemma 1.

For an interface in standard form, we assume that the two neighboring patches meet in a certain way, as given in the following definition.

Definition 7. Given an interface $\Sigma^{(i)}$, for $i \in \mathcal{I}_\Sigma^\circ$, and let ι_1, ι_2 be the corresponding patch indices as in Assumption 5. We have given a *parametrization in standard form* for the interface $\Sigma^{(i)}$, if

$$\mathbf{F}^{(\iota_1)}(0, \xi) = \mathbf{F}^{(\iota_2)}(\xi, 0), \quad \text{for all } \xi \in [0, 1]. \quad (8)$$

This corresponds to a configuration as depicted in Figure 3 (left). Similarly, for a boundary edge $\Sigma^{(i)}$, with $i \in \mathcal{I}_\Sigma^\Gamma$, we say that a parametrization in standard form is given if $\Sigma^{(i)} = \{\mathbf{F}^{(\iota_1)}(0, \xi) : \xi \in]0, 1[\}$.

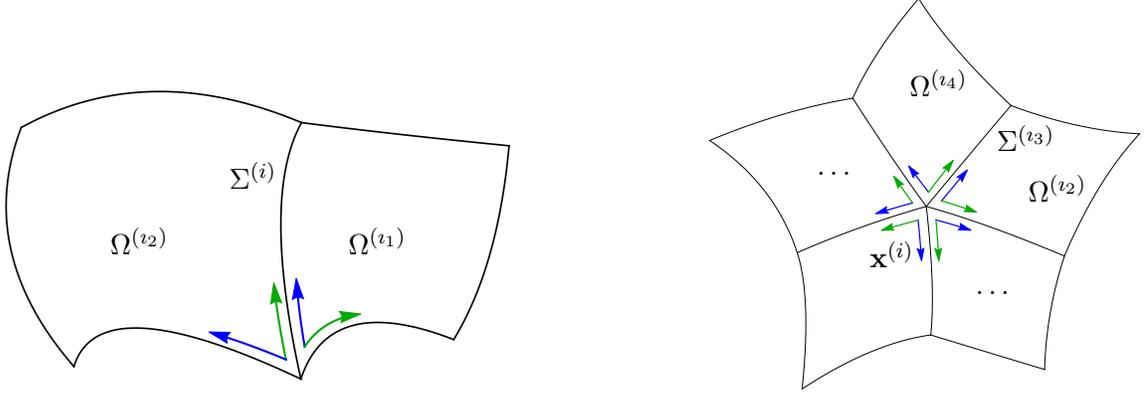


Figure 3: Parametrization in standard form for a given interface $\Sigma^{(i)}$ (left) and vertex $\mathbf{x}^{(i)}$ (right). The local coordinates are depicted in a similar fashion as in Figure 2.

For a vertex in standard form, we assume that the vertex is enclosed by edges and patches

$$\Sigma^{(\iota_1)}, \Omega^{(\iota_2)}, \Sigma^{(\iota_3)}, \Omega^{(\iota_4)}, \dots, \Sigma^{(\iota_{2\nu-1})}, \Omega^{(\iota_{2\nu})}, \Sigma^{(\iota_{2\nu+1})}$$

in counterclockwise order and that the patches are parametrized in such a way that the vertex is at the origin, as depicted in Figure 3 (right). This is detailed in the following definition.

Definition 8. Given a vertex $\mathbf{x}^{(i)}$, for $i \in \mathcal{I}_{\mathcal{X}}$, and let $\iota_2, \dots, \iota_{2\nu}$ be the corresponding patch indices as in Assumption 6. We have given a *parametrization in standard form* for the vertex $\mathbf{x}^{(i)}$, if

$$\mathbf{F}^{(\iota_{2\ell})}(0, \xi) = \mathbf{F}^{(\iota_{2\ell+2})}(\xi, 0), \quad \text{for all } \xi \in [0, 1], \quad (9)$$

where $\ell = 1, \dots, \nu$ for interior vertices while $\ell = 1, \dots, \nu - 1$ for boundary vertices. We have $\overline{\Sigma^{(\iota_{2\ell+1})}} = \overline{\Omega^{(\iota_{2\ell})}} \cap \overline{\Omega^{(\iota_{2\ell+2})}}$, with $\ell = 0, \dots, \nu$ for interior vertices, and with $\ell = 1, \dots, \nu - 1$ for boundary vertices. For interior vertices we have $\iota_1 = \iota_{2\nu+1}$, and for boundary vertices we set instead $\overline{\Sigma^{(\iota_1)}} = \Gamma \cap \overline{\Omega^{(\iota_2)}}$ as well as $\overline{\Sigma^{(\iota_{2\nu+1})}} = \overline{\Omega^{(\iota_{2\nu})}} \cap \Gamma$.

In the case of Definition 8, all interfaces are in standard form. The even/odd indexing for patches/edges will come handy in Section 4.

When considering a single edge $\Sigma^{(i)}$ or a single vertex $\mathbf{x}^{(i)}$, it is not restrictive to assume that the given parametrizations are in standard form, as stated in the following lemma.

Lemma 1 (Reparametrizations that yield a standard form). *Let $\mathbf{r} : [0, 1]^2 \rightarrow [0, 1]^2$ with $\mathbf{r}(\xi_1, \xi_2) = (1 - \xi_2, \xi_1)$. Given an interface $\Sigma^{(i)}$, for $i \in \mathcal{I}_{\Sigma}^{\circ}$, and $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_1, \kappa_1), (\iota_2, \kappa_2)\}$, then $\mathbf{F}^{(\iota_1)} \circ \mathbf{r}^{\kappa_1}$ and $\mathbf{F}^{(\iota_2)} \circ \mathbf{r}^{\kappa_2-1}$ are suitable reparametrizations of the adjacent patches $\Omega^{(\iota_1)}$ and $\Omega^{(\iota_2)}$ that yield a parametrization in standard form for the interface $\Sigma^{(i)}$. Similarly, $\mathbf{F}^{(\iota_1)} \circ \mathbf{r}^{\kappa_1}$ yields a parametrization in standard form for a boundary edge $\Sigma^{(i)}$ with $i \in \mathcal{I}_{\Sigma}^{\Gamma}$.*

Given a vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_{\mathcal{X}}$, and $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_2, \kappa_2), \dots, (\iota_{2\nu}, \kappa_{2\nu})\}$. Assuming that the patches are arranged in counterclockwise order, then $\mathbf{F}^{(\iota_{2\ell})} \circ \mathbf{r}^{\kappa_{2\ell}}$, for $\ell = 1, \dots, \nu$, are suitable reparametrizations of the adjacent patches $\Omega^{(\iota_{2\ell})}$ that yield a parametrization in standard form for the vertex $\mathbf{x}^{(i)}$.

Here, \mathbf{r} corresponds to the $\pi/2$ counterclockwise rotation map in the parameter domain. We denote by $\mathbf{F}^{(i)} \circ \mathbf{r}^\kappa$ the reparametrization of the patch $\Omega^{(i)}$ (with parametrization $\mathbf{F}^{(i)}$) by a rotation with an angle of $\kappa\pi/2$, see Figure 4.

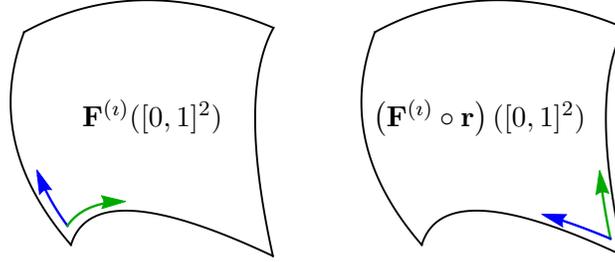


Figure 4: Visualization of a reparametrization for a given patch $\Omega^{(i)}$. The local coordinates are depicted in a similar fashion as in Figure 2.

A proof of Lemma 1 is straightforward and will be omitted here. Actually, for each interface there exist two parametrizations in standard form, depending on the order of the indices ι_1 and ι_2 , (changing the order of the patches simply changes the orientation of the interface with respect to the two patches). Similarly, for an interior vertex the choice of the indices is not unique; moreover, in case of an interior vertex, all sub-indices are considered to be modulo (2ν) , e.g., $\iota_0 = \iota_{2\nu}$.

3. C^1 isogeometric spaces

In this section we define C^0 and then C^1 multi-patch isogeometric spaces, discuss the relation to geometric continuity G^1 of the graph parametrization, and the notion of AS- G^1 continuity of the multi-patch parametrization.

3.1. Isogeometric spaces

Definition 9 (Isogeometric spaces). We define the C^0 isogeometric space as

$$\mathcal{V}^0 = \left\{ \varphi_h \in C^0(\overline{\Omega}) \mid \text{for all } i \in \mathcal{I}_\Omega, f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)} \in \mathcal{S}_h^{\mathbf{p}, \mathbf{r}} \right\}, \quad (10)$$

and the C^1 isogeometric space as

$$\mathcal{V}^1 = \mathcal{V}^0 \cap C^1(\overline{\Omega}). \quad (11)$$

3.2. C^1 regularity of isogeometric functions and G^1 graph regularity

The graph $\Phi \subset \Omega \times \mathbb{R}$ of an isogeometric function $\varphi_h : \Omega \rightarrow \mathbb{R}$ is naturally split into patches $\Phi^{(i)}$ having the parametrizations

$$\begin{bmatrix} \mathbf{F}^{(i)} \\ f_h^{(i)} \end{bmatrix} : [0, 1]^2 \rightarrow \Phi^{(i)}, \quad (12)$$

where $f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)}$.

The C^1 continuity at an interface $\Sigma^{(i)}$ is, by definition, the G^1 (geometric) continuity of its graph parametrization, as stated in the next proposition.

Proposition 1. *Under Assumption 3, an isogeometric function $\varphi_h \in \mathcal{V}^0$ belongs to \mathcal{V}^1 if and only if the parametrization (12) of its graph is G^1 continuous at the interfaces $\bar{\Sigma}^{(i)}$, for all $i \in \mathcal{I}_\Sigma^\circ$.*

For a discussion and generalizations of the equivalence above see [10, 16, 23]. The definition of G^1 continuity, in our context, is detailed below.

Definition 10 (G^1 continuity at $\Sigma^{(i)}$). Consider an interface $\Sigma^{(i)}$, with $i \in \mathcal{I}_\Sigma^\circ$. Assume $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_1, 0), (\iota_2, 1)\}$, that is, the adjacent patches have parametrizations $\mathbf{F}^{(\iota_1)}$ and $\mathbf{F}^{(\iota_2)}$ in standard form for $\Sigma^{(i)}$ (see Section 2.4). The graph parametrization (12) is said to be G^1 at $\bar{\Sigma}^{(i)}$ if there exist functions $\alpha^{(i, \iota_1)} : [0, 1] \rightarrow \mathbb{R}$, $\alpha^{(i, \iota_2)} : [0, 1] \rightarrow \mathbb{R}$ and $\beta^{(i)} : [0, 1] \rightarrow \mathbb{R}$ such that for all $\xi \in [0, 1]$,

$$\alpha^{(i, \iota_1)}(\xi)\alpha^{(i, \iota_2)}(\xi) > 0 \quad (13)$$

and

$$\alpha^{(i, \iota_1)}(\xi) \begin{bmatrix} \partial_2 \mathbf{F}^{(\iota_2)}(\xi, 0) \\ \partial_2 f_h^{(\iota_2)}(\xi, 0) \end{bmatrix} + \alpha^{(i, \iota_2)}(\xi) \begin{bmatrix} \partial_1 \mathbf{F}^{(\iota_1)}(0, \xi) \\ \partial_1 f_h^{(\iota_1)}(0, \xi) \end{bmatrix} + \beta^{(i)}(\xi) \begin{bmatrix} \partial_2 \mathbf{F}^{(\iota_1)}(0, \xi) \\ \partial_2 f_h^{(\iota_1)}(0, \xi) \end{bmatrix} = \mathbf{0}. \quad (14)$$

In the framework of Definition 10, it is useful to introduce functions $\beta^{(i, \iota_1)}$ and $\beta^{(i, \iota_2)}$ such that

$$\beta^{(i)}(\xi) = \alpha^{(i, \iota_1)}(\xi)\beta^{(i, \iota_2)}(\xi) + \alpha^{(i, \iota_2)}(\xi)\beta^{(i, \iota_1)}(\xi). \quad (15)$$

We call the functions $\alpha^{(i, \iota_1)}$, $\alpha^{(i, \iota_2)}$, $\beta^{(i, \iota_1)}$ and $\beta^{(i, \iota_2)}$ the *gluing data* for the interface $\Sigma^{(i)}$.

In the context of IGA, the multi-patch domain parametrizations $\mathbf{F}^{(i)}$ are considered given (at least, at each linearization step) and determine the gluing data above, while $f_h^{(i)}$ are the unknowns which need to fulfill the gluing condition (third equation of (14)) in order to represent C^1 isogeometric functions. This is stated in the next result which combines Proposition 1 and Definition 10.

Proposition 2. *For each $i \in \mathcal{I}_\Sigma^\circ$, assuming $\mathcal{T}_{\Sigma^{(i)}} = \{(\iota_1, 0), (\iota_2, 1)\}$ (i.e., $\mathbf{F}^{(\iota_1)}$ and $\mathbf{F}^{(\iota_2)}$ in standard form for $\Sigma^{(i)}$), let $\alpha^{(i, \iota_1)}$, $\alpha^{(i, \iota_2)}$ and $\beta^{(i)}$ such that for all $\xi \in [0, 1]$ it holds $\alpha^{(i, \iota_1)}(\xi)\alpha^{(i, \iota_2)}(\xi) > 0$ and*

$$\alpha^{(i, \iota_1)}(\xi)\partial_2 \mathbf{F}^{(\iota_2)}(\xi, 0) + \alpha^{(i, \iota_2)}(\xi)\partial_1 \mathbf{F}^{(\iota_1)}(0, \xi) + \beta^{(i)}(\xi)\partial_2 \mathbf{F}^{(\iota_1)}(0, \xi) = \mathbf{0}. \quad (16)$$

Let $\varphi_h \in \mathcal{V}^0$. Then $\varphi_h \in \mathcal{V}^1$ if and only if for all $i \in \mathcal{I}_\Sigma^\circ$ and $\xi \in [0, 1]$, it holds

$$\alpha^{(i, \iota_1)}(\xi)\partial_2 f_h^{(\iota_2)}(\xi, 0) + \alpha^{(i, \iota_2)}(\xi)\partial_1 f_h^{(\iota_1)}(0, \xi) + \beta^{(i)}(\xi)\partial_2 f_h^{(\iota_1)}(0, \xi) = 0, \quad (17)$$

where $f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)}$.

Proof. Due to the regularity condition (3), for each $\xi \in [0, 1]$, (16) are two linearly independent equations for $[\alpha^{(i, \iota_1)}(\xi), \alpha^{(i, \iota_2)}(\xi), \beta^{(i)}(\xi)]$, whose solutions are, for completeness,

$$\begin{aligned} \alpha^{(i, \iota_1)}(\xi) &= \gamma(\xi) \det \begin{bmatrix} \partial_1 \mathbf{F}^{(\iota_1)}(0, \xi) & \partial_2 \mathbf{F}^{(\iota_1)}(0, \xi) \end{bmatrix}, \\ \alpha^{(i, \iota_2)}(\xi) &= \gamma(\xi) \det \begin{bmatrix} \partial_1 \mathbf{F}^{(\iota_2)}(\xi, 0) & \partial_2 \mathbf{F}^{(\iota_2)}(\xi, 0) \end{bmatrix}, \\ \beta^{(i)}(\xi) &= \gamma(\xi) \det \begin{bmatrix} \partial_2 \mathbf{F}^{(\iota_2)}(\xi, 0) & \partial_2 \mathbf{F}^{(\iota_1)}(0, \xi) \end{bmatrix}, \end{aligned} \quad (18)$$

where $\gamma(\xi)$ is arbitrary and we have used $\partial_2 \mathbf{F}^{(\nu_1)}(0, \xi) = \partial_1 \mathbf{F}^{(\nu_2)}(\xi, 0)$.

Assume now φ_h is C^1 , that is, (14) holds, by Proposition 1. The gluing data allowed in (14) are already determined by (16), as given in (18). The third equation of (14) is the same as (17).

Conversely, if (17) holds, together with (16) it yields (14), that is, φ_h is C^1 . \square

What this proposition shows, is that the gluing data are completely determined (up to a common factor $\gamma(\xi)$) by the patch parametrizations $\mathbf{F}^{(\nu_1)}$, $\mathbf{F}^{(\nu_2)}$. The C^1 condition on the isogeometric function φ_h is then a linear constraint on the functions $f_h^{(\nu_1)}$, $f_h^{(\nu_2)}$ in the parameter domain. The proof above also shows that one can select the gluing data such that $\alpha^{(i, \nu_1)}, \alpha^{(i, \nu_2)} \in \mathcal{S}_h^{2p-1, r-1}$ and $\beta^{(i)} \in \mathcal{S}_h^{2p, r}$. A special case is when the gluing data are polynomial functions. This happens in particular (but not only) for Bézier patches, which is often the case in literature. In this situation, since $\alpha^{(i, \nu_1)}$, $\alpha^{(i, \nu_2)}$ and $\beta^{(i)}$ are determined up to a common multiplicative function, it is not restrictive to assume the following.

Assumption 11 (Simplification of gluing data). If $\alpha^{(i, \nu_1)}$ and $\alpha^{(i, \nu_2)}$ are polynomial functions, we assume that they are relatively prime (i.e., $\deg(\gcd(\alpha^{(i, \nu_1)}, \alpha^{(i, \nu_2)})) = 0$).

Note also that the choice of $\beta^{(i, \nu_1)}$ and $\beta^{(i, \nu_2)}$ is not unique.¹ One can show that there exist piecewise polynomial functions $\beta^{(i, \nu_1)}$ and $\beta^{(i, \nu_2)}$ that satisfy equation (15).

3.3. Analysis-suitable G^1 condition

In order to ensure optimal reproduction properties for the trace and normal derivative along the interfaces of the isogeometric space \mathcal{V}^1 , we introduce an additional condition on the geometry parametrization, as in [10, 21], stated as a condition on the gluing data $\alpha^{(i, \nu_1)}$, $\alpha^{(i, \nu_2)}$, $\beta^{(i, \nu_1)}$ and $\beta^{(i, \nu_2)}$.

Definition 12 (Analysis-suitable G^1 parametrization). The parametrizations $\mathbf{F}^{(\nu_1)}$ and $\mathbf{F}^{(\nu_2)}$ are *analysis-suitable G^1* at the interface $\Sigma^{(i)}$ (in short, AS- $G^1(\Sigma^{(i)})$ or AS- G^1) if there exist gluing data $\alpha^{(i, \nu_1)}$, $\alpha^{(i, \nu_2)}$, $\beta^{(i, \nu_1)}$ and $\beta^{(i, \nu_2)}$ that are linear polynomials and such that (14)–(15) hold.

The class of AS- G^1 parametrizations contains, but is not restricted to, bilinear parametrizations, see [10, 22]. Note that this is a non-trivial requirement. However, it was shown in [22] that many multi-patch geometries can be reparametrized to satisfy the AS- G^1 constraints, motivating the following assumption.

Assumption 13. We assume that for all interfaces $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^o$, the parametrizations $\mathbf{F}^{(\nu_1)}$ and $\mathbf{F}^{(\nu_2)}$ are analysis-suitable G^1 .

¹Even if not necessary in this paper, from the practical point of view it is advisable to select stable gluing data e.g. by minimizing $\|\alpha^{(i, \nu_1)} - 1\|_{L^2(0,1)}^2 + \|\alpha^{(i, \nu_2)} - 1\|_{L^2(0,1)}^2$ as well as $\|\beta^{(i, \nu_1)}\|_{L^2(0,1)}^2 + \|\beta^{(i, \nu_2)}\|_{L^2(0,1)}^2$. In case of parametric continuity, i.e., $\beta^{(i)} \equiv 0$ and $\alpha^{(i, \nu_1)} = \alpha^{(i, \nu_2)}$, this implies $\beta^{(i, \nu_1)} \equiv \beta^{(i, \nu_2)} \equiv 0$ and $\alpha^{(i, \nu_1)} = \alpha^{(i, \nu_2)} \equiv 1$.

4. The Argyris isogeometric space $\mathcal{A} \subset \mathcal{V}^1$

Unlike \mathcal{V}^0 , the isogeometric space \mathcal{V}^1 has a complex structure. Every interface generates certain C^1 constraints, which are usually not independent of each other. At every interior vertex, several interfaces meet, which may lead to possibly conflicting constraints. The geometry needs to satisfy additional conditions there. In the context of computer-aided geometric design these are sometimes called vertex enclosure constraints, see e.g. [17]. In addition to this issue concerning vertices, the dimension of \mathcal{V}^1 depends also on the domain parametrization even for the simplest configuration of two bilinear patches, as shown in [21, 23]. For this reason, instead of dealing with \mathcal{V}^1 itself, we introduce a suitable subspace $\mathcal{A} \subset \mathcal{V}^1$ which is simpler to construct and has a dimension which is independent of the geometry parametrization. The space \mathcal{A} is named Argyris isogeometric space since it represents an extension of the classical Argyris finite element space, as explained in the Introduction.

4.1. Description and properties of \mathcal{A}

The functions spanning the space \mathcal{A} are standard isogeometric functions within the patches, possess a special structure at the patch interfaces and edges (which is motivated by [10]) and are C^2 at all vertices.

Precisely, we define \mathcal{A} as the (direct) sum of interior, edge and vertex components:

$$\mathcal{A} = \left(\bigoplus_{i \in \mathcal{I}_\Omega} \mathcal{A}_{\Omega^{(i)}}^\circ \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\Sigma} \mathcal{A}_{\Sigma^{(i)}}^\circ \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\mathcal{X}} \mathcal{A}_{\mathbf{x}^{(i)}} \right). \quad (19)$$

The patch-interior basis functions spanning $\mathcal{A}_{\Omega^{(i)}}^\circ$ have their support entirely contained in one patch. They are taken as those functions supported on the patch $\Omega^{(i)}$ which have vanishing function values and gradients at the entire boundary of the patch.

The edge-interior basis functions spanning $\mathcal{A}_{\Sigma^{(i)}}^\circ$ have their support contained in an (h -dependent) neighborhood of the edge $\Sigma^{(i)}$. They span function values and normal derivatives along the interface and have vanishing derivatives up to second order at the endpoints of the interface (vertices of the multi-patch domain). They are thus supported in at most two patches.

The vertex basis functions spanning $\mathcal{A}_{\mathbf{x}^{(i)}}$ are supported within an (h -dependent) neighborhood of the vertex $\mathbf{x}^{(i)}$. There are exactly six such functions per vertex and they span the function value and all derivatives up to second order at the vertex. Hence, they are C^2 at the vertex by definition, see Proposition 4.

The precise definitions of the different types of basis functions will be given in the three sections below, more precisely in Definitions 16, 18 and 21, for the patch-interior, edge-interior and vertex basis, respectively.

Note that for splines of maximal smoothness $r = p - 1$ within the patches there is no convergence of the approximation error under h -refinement even on a simple bilinear two-patch domain, see [10]. Hence, we have the following request.

Assumption 14 (Maximal regularity within the patches). We assume $r \leq p - 2$.

The assumption above is needed to allow h -refinement (see [10]). Moreover, the split (19) itself is well defined if the spline spaces are sufficiently refined, as stated in the next assumption.

Assumption 15 (Minimal mesh resolution within the patches). We assume $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$ has $n \geq \frac{4-r}{p-r-1}$ elements per direction, that is $h \leq \frac{p-r-1}{4-r}$.

Remark 3. A Bézier patch (i.e., $h = 1$) fulfills Assumption 15 and, trivially, Assumption 14, for degree $p \geq 5$. This is the case considered in [6, 31]. However, we study the subspace $\mathcal{A} \subset \mathcal{V}^1$.

Remark 4. The dimensions of the subspaces in (19) do not depend on the geometry and satisfy

$$\dim(\mathcal{A}_{\Omega^{(i)}}^\circ) = ((p-r)(n-1) + p-3)^2,$$

for each $i \in \mathcal{I}_\Omega$,

$$\dim(\mathcal{A}_{\Sigma^{(i)}}^\circ) = 2(p-r-1)(n-1) + p-9,$$

for each $i \in \mathcal{I}_\Sigma$, as well as

$$\dim(\mathcal{A}_{\mathbf{x}^{(i)}}) = 6,$$

for each $i \in \mathcal{I}_\mathcal{X}$, cf. Definition 16, 18, 19 and Lemma 3. Hence, the dimension of \mathcal{A} is completely determined by the degree p , the regularity r , the number of elements in each direction n , as well as the number of patches, edges and vertices, via

$$\dim(\mathcal{A}) = |\mathcal{I}_\Omega| \cdot ((p-r)(n-1) + p-3)^2 + |\mathcal{I}_\Sigma| \cdot (2(p-r-1)(n-1) + p-9) + |\mathcal{I}_\mathcal{X}| \cdot 6.$$

4.2. The patch-interior function space $\mathcal{A}_{\Omega^{(i)}}^\circ$

For each patch $\Omega^{(i)}$ with $i \in \mathcal{I}_\Omega$, we define a function space $\mathcal{A}_{\Omega^{(i)}}^\circ$ as the span of all basis functions supported in $\Omega^{(i)}$, which have vanishing function value and vanishing gradients at the patch boundary $\partial\Omega^{(i)}$. For the following definition, recall that $b_{\mathbf{j}}$, for $\mathbf{j} \in \mathbb{I}$, is the standard tensor-product B-spline basis for the spline space $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$.

Definition 16 (Patch-interior basis). Let $i \in \mathcal{I}_\Omega$, then we define

$$\mathcal{A}_{\Omega^{(i)}} = \text{span} \left\{ B_{\mathbf{j}}^{(i)} : \overline{\Omega}^{(i)} \rightarrow \mathbb{R} \text{ such that } B_{\mathbf{j}}^{(i)} \circ \mathbf{F}^{(i)} = b_{\mathbf{j}}, \text{ for } \mathbf{j} \in \mathbb{I} \right\} \quad (20)$$

and

$$\mathcal{A}_{\Omega^{(i)}}^\circ = \text{span} \left\{ B_{\mathbf{j}}^{(i)} : \overline{\Omega}^{(i)} \rightarrow \mathbb{R} \text{ such that } B_{\mathbf{j}}^{(i)} \circ \mathbf{F}^{(i)} = b_{\mathbf{j}}, \text{ for } \mathbf{j} \in \mathbb{I}_{\Omega^{(i)}}^\circ \right\} \quad (21)$$

with $\mathbb{I}_{\Omega^{(i)}}^\circ = \{2, \dots, N-3\}^2 \subset \mathbb{I}$.

With a little abuse of notation, we consider the functions of $\mathcal{A}_{\Omega^{(i)}}^\circ$ as defined on the whole $\overline{\Omega}$ by extending to zero outside $\overline{\Omega}^{(i)}$. We easily have then $\mathcal{A}_{\Omega^{(i)}}^\circ \subseteq \mathcal{V}^1$ and in particular

$$\mathcal{A}_{\Omega^{(i)}}^\circ = \left\{ \varphi_h \in \mathcal{V}^1 \text{ such that } \varphi_h(\mathbf{x}) = 0 \text{ and } \nabla \varphi_h(\mathbf{x}) = \mathbf{0}, \text{ for all } \mathbf{x} \in \overline{\Omega} \setminus \Omega^{(i)} \right\}.$$

We also define straightforwardly a projection operator onto the subspace $\mathcal{A}_{\Omega^{(i)}}^\circ$.

Definition 17 (Patch-interior dual basis and projector). Let $\{\lambda_{\mathbf{j}} = \lambda_{j_1} \otimes \lambda_{j_2}\}_{\mathbf{j} \in \mathbb{I}}$ be a dual basis for the basis $\{b_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{I}}$. For each $i \in \mathcal{I}_{\Omega}$, we define the projector $\Pi_{\mathcal{A}_{\Omega^{(i)}}^{\circ}} : L^2(\Omega) \rightarrow \mathcal{V}^1$ such that

$$\Pi_{\mathcal{A}_{\Omega^{(i)}}^{\circ}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\Omega^{(i)}}^{\circ}} \Lambda_{\mathbf{j}}(\varphi) \mathbf{B}_{\mathbf{j}}^{(i)},$$

where $\Lambda_{\mathbf{j}}(\varphi) = \lambda_{\mathbf{j}}(\varphi \circ \mathbf{F}^{(i)})$.

4.3. The edge function space $\mathcal{A}_{\Sigma^{(i)}}^{\circ}$

In this section, we consider first the most interesting case of interior edges, that is when $\Sigma^{(i)}$, for $i \in \mathcal{I}_{\Sigma}^{\circ}$, is an interface between the patches $\Omega^{(i_1)}$ and $\Omega^{(i_2)}$. The extension to boundary edges is straightforward and will be discussed briefly after Definition 18. The parametrizations $\mathbf{F}^{(i_1)}$, $\mathbf{F}^{(i_2)}$ are assumed to be in standard form for $\Sigma^{(i)}$, see Section 2.4. The first step is the definition of a space over $\overline{\Omega}^{(i_1)} \cup \overline{\Omega}^{(i_2)}$, which contains isogeometric functions fulfilling the C^1 constraints at the interface $\Sigma^{(i)}$. We define $\mathcal{A}_{\Sigma^{(i)}}$ as the direct sum of two subspaces $\mathcal{A}_{\Sigma^{(i)},0}$ and $\mathcal{A}_{\Sigma^{(i)},1}$, following the construction in [10]. The spaces span the function values and the cross derivative values along the interface, respectively². The number of elements in the spaces $\mathcal{S}_h^{p,r+1}$ and $\mathcal{S}_h^{p-1,r}$ considered below is the same and is denoted by n .

Definition 18 (Basis at the interfaces). Let $\Sigma^{(i)}$, for $i \in \mathcal{I}_{\Sigma}^{\circ}$, be an interface in standard form. Consider the univariate spline spaces $\mathcal{S}^+ = \mathcal{S}_h^{p,r+1}$ and $\mathcal{S}^- = \mathcal{S}_h^{p-1,r}$, with bases $\{b_j^+\}_{j \in \mathbb{I}^+}$, and $\{b_j^-\}_{j \in \mathbb{I}^-}$, respectively, where $\mathbb{I}^{\pm} = \{0, \dots, N^{\pm} - 1\}$ with $N^- = (p - r - 1)(n - 1) + p$ and $N^+ = N^- + 1 = (p - r - 1)(n - 1) + p + 1$. Recall that $\{b_j\}_{j \in \{0,1,\dots,N-1\}}$ is the standard univariate B-spline basis for $\mathcal{S}_h^{p,r}$, where $N = (p - r)(n - 1) + p + 1$.

We define

$$\mathcal{A}_{\Sigma^{(i)},0} = \text{span} \left\{ \overline{\mathbf{B}}_{(j_1,0)}^{(i)} : \overline{\Omega}^{(i_1)} \cup \overline{\Omega}^{(i_2)} \rightarrow \mathbb{R}, \text{ for } j_1 \in \mathbb{I}^+ \right\}, \quad (22)$$

with

$$\begin{aligned} \overline{\mathbf{B}}_{(j_1,0)}^{(i)} \circ \mathbf{F}^{(i_1)}(\xi_1, \xi_2) &= \overline{f}_{(j_1,0)}^{(i,\nu_1)} = b_{j_1}^+(\xi_2) c_0(\xi_1) - \beta^{(i,\nu_1)}(\xi_2) (b_{j_1}^+)'(\xi_2) c_1(\xi_1), \\ \overline{\mathbf{B}}_{(j_1,0)}^{(i)} \circ \mathbf{F}^{(i_2)}(\xi_1, \xi_2) &= \overline{f}_{(j_1,0)}^{(i,\nu_2)} = b_{j_1}^+(\xi_1) c_0(\xi_2) - \beta^{(i,\nu_2)}(\xi_1) (b_{j_1}^+)'(\xi_1) c_1(\xi_2), \end{aligned} \quad (23)$$

where $c_0 = b_0 + b_1$ and $c_1 = \frac{h}{p} b_1$, and

$$\mathcal{A}_{\Sigma^{(i)},1} = \text{span} \left\{ \overline{\mathbf{B}}_{(j_1,1)}^{(i)} : \overline{\Omega}^{(i_1)} \cup \overline{\Omega}^{(i_2)} \rightarrow \mathbb{R}, \text{ for } j_1 \in \mathbb{I}^- \right\}, \quad (24)$$

with

$$\begin{aligned} \overline{\mathbf{B}}_{(j_1,1)}^{(i)} \circ \mathbf{F}^{(i_1)}(\xi_1, \xi_2) &= \overline{f}_{(j_1,1)}^{(i,\nu_1)} = \alpha^{(i,\nu_1)}(\xi_2) b_{j_1}^-(\xi_2) b_1(\xi_1), \\ \overline{\mathbf{B}}_{(j_1,1)}^{(i)} \circ \mathbf{F}^{(i_2)}(\xi_1, \xi_2) &= \overline{f}_{(j_1,1)}^{(i,\nu_2)} = -\alpha^{(i,\nu_2)}(\xi_1) b_{j_1}^-(\xi_1) b_1(\xi_2). \end{aligned} \quad (25)$$

We define

$$\mathcal{A}_{\Sigma^{(i)}} = \mathcal{A}_{\Sigma^{(i)},0} \oplus \mathcal{A}_{\Sigma^{(i)},1} = \text{span} \left\{ \overline{\mathbf{B}}_{\mathbf{j}}^{(i)} : \overline{\Omega}^{(i_1)} \cup \overline{\Omega}^{(i_2)} \rightarrow \mathbb{R}, \text{ for } \mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}} \right\}, \quad (26)$$

²The complete C^1 space is slightly larger for certain configurations, see Appendix C and [21] for a construction of the complete basis.

with $\mathbb{I}_{\Sigma^{(i)}} = (\mathbb{I}^+ \times \{0\}) \cup (\mathbb{I}^- \times \{1\})$.

Finally we define the space $\mathcal{A}_{\Sigma^{(i)}}^\circ$ as

$$\mathcal{A}_{\Sigma^{(i)}}^\circ = \text{span} \left\{ \bar{\mathbf{B}}_{\mathbf{j}}^{(i)} : \bar{\Omega}^{(i_1)} \cup \bar{\Omega}^{(i_2)} \rightarrow \mathbb{R}, \text{ for } \mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}}^\circ \right\}, \quad (27)$$

with $\mathbb{I}_{\Sigma^{(i)}}^\circ = \{\mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}} : j_1 + j_2 \geq 3 \text{ and } j_1 \leq N^- - 3\}$.

Remark 5. *Assumption 14 guarantees that \mathcal{S}^+ and \mathcal{S}^- are proper spline spaces, that is, piecewise polynomials when $h < 1$ (see also Theorem 1 of [10]). Assumption 15 guarantees that the space $\mathcal{A}_{\Sigma^{(i)}}^\circ$ is nonempty ($N^- \geq 5$).*

For completeness, we give now the definition of the basis at the boundary edge, which is just a simplification of the previous one.

Definition 19 (Basis at the boundary edges). Let $\Sigma^{(i)}$, for $i \in \mathcal{I}_\Sigma^\Gamma$, be a boundary edge in standard form. With the same notation as in Definition 18, we define

$$\mathcal{A}_{\Sigma^{(i)},0} = \text{span} \left\{ \bar{\mathbf{B}}_{(j_1,0)}^{(i)} : \bar{\Omega}^{(i_1)} \rightarrow \mathbb{R}, \text{ for } j_1 \in \mathbb{I}^+ \right\}, \quad (28)$$

with

$$\bar{\mathbf{B}}_{(j_1,0)}^{(i)} \circ \mathbf{F}^{(i_1)}(\xi_1, \xi_2) = \bar{f}_{(j_1,0)}^{(i,i_1)} = b_{j_1}^+(\xi_2)c_0(\xi_1), \quad (29)$$

and

$$\mathcal{A}_{\Sigma^{(i)},1} = \text{span} \left\{ \bar{\mathbf{B}}_{(j_1,1)}^{(i)} : \bar{\Omega}^{(i_1)} \rightarrow \mathbb{R}, \text{ for } j_1 \in \mathbb{I}^- \right\}, \quad (30)$$

with

$$\bar{\mathbf{B}}_{(j_1,1)}^{(i)} \circ \mathbf{F}^{(i_1)}(\xi_1, \xi_2) = \bar{f}_{(j_1,1)}^{(i,i_1)} = b_{j_1}^-(\xi_2)b_1(\xi_1). \quad (31)$$

We set

$$\mathcal{A}_{\Sigma^{(i)}} = \mathcal{A}_{\Sigma^{(i)},0} \oplus \mathcal{A}_{\Sigma^{(i)},1} = \text{span} \left\{ \bar{\mathbf{B}}_{\mathbf{j}}^{(i)} : \bar{\Omega}^{(i_1)} \rightarrow \mathbb{R}, \text{ for } \mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}} \right\}, \quad (32)$$

with $\mathbb{I}_{\Sigma^{(i)}} = (\mathbb{I}^+ \times \{0\}) \cup (\mathbb{I}^- \times \{1\})$, and

$$\mathcal{A}_{\Sigma^{(i)}}^\circ = \text{span} \left\{ \bar{\mathbf{B}}_{\mathbf{j}}^{(i)} : \bar{\Omega}^{(i_1)} \rightarrow \mathbb{R}, \text{ for } \mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}}^\circ \right\}, \quad (33)$$

with $\mathbb{I}_{\Sigma^{(i)}}^\circ = \{\mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}} : j_1 + j_2 \geq 3 \text{ and } j_1 \leq N^- - 3\}$.

As for the patch-interior space, we can extend the functions of $\mathcal{A}_{\Sigma^{(i)}}$ and $\mathcal{A}_{\Sigma^{(i)}}^\circ$ to zero. Remarkably, we have the following two inclusions.

Lemma 2. *We have*

$$\mathcal{A}_{\Sigma^{(i)}} \subset C^1(\Sigma^{(i)}).$$

Proof. By construction, the pair of functions (23) fulfills condition (17), that is for all $\xi \in [0, 1]$,

$$\alpha^{(i,i_1)}(\xi)\partial_2 \bar{f}_{(j_1,0)}^{(i,i_2)}(\xi, 0) + \alpha^{(i,i_2)}(\xi)\partial_1 \bar{f}_{(j_1,0)}^{(i,i_1)}(0, \xi) + \beta^{(i)}(\xi)\partial_2 \bar{f}_{(j_1,0)}^{(i,i_1)}(0, \xi) = 0,$$

The same holds for the pair of functions (25): For all $\xi \in [0, 1]$,

$$\alpha^{(i,i_1)}(\xi)\partial_2 \bar{f}_{(j_1,1)}^{(i,i_2)}(\xi, 0) + \alpha^{(i,i_2)}(\xi)\partial_1 \bar{f}_{(j_1,1)}^{(i,i_1)}(0, \xi) + \beta^{(i)}(\xi)\partial_2 \bar{f}_{(j_1,1)}^{(i,i_1)}(0, \xi) = 0.$$

The statement follows thanks to Proposition 2, see [10] for more details. \square

Proposition 3. *We have*

$$\mathcal{A}_{\Sigma^{(i)}}^{\circ} \subset \mathcal{V}^1.$$

Proof. One can easily characterize $\mathcal{A}_{\Sigma^{(i)}}^{\circ}$ as the subset of functions of $\mathcal{A}_{\Sigma^{(i)}}$ having null value, gradient and Hessian at the edge endpoints. Then, by construction, the functions in $\mathcal{A}_{\Sigma^{(i)}}^{\circ}$ have vanishing trace and derivative at $\Sigma^{(i')}$, for all $i' \in \mathcal{I}_{\Sigma}$, $i' \neq i$. The statement follows. \square

We further define a dual basis and projection operator onto $\mathcal{A}_{\Sigma^{(i)}}^{\circ}$.

Definition 20 (Edge-interior dual basis and projector). Let $\{\lambda_j^+\}_{j \in \mathbb{I}^+}$ be a dual basis for $\{b_j^+\}_{j \in \mathbb{I}^+}$ and $\{\lambda_j^-\}_{j \in \mathbb{I}^-}$ a dual basis for $\{b_j^-\}_{j \in \mathbb{I}^-}$. We define

$$\begin{aligned} \bar{\Lambda}_{(j_1,0)}^{(i)}(\varphi) &= \lambda_{j_1}^+ \left(\varphi \circ \mathbf{F}^{(t_1)}(0, \bullet) \right), \\ \bar{\Lambda}_{(j_1,1)}^{(i)}(\varphi) &= \lambda_{j_1}^- \left(\frac{h}{p} (\nabla \varphi) \circ \mathbf{F}^{(t_1)}(0, \bullet) \cdot \mathbf{d}^{(i)}(\bullet) \right), \end{aligned}$$

where $\mathbf{d}^{(i)}$ is the transversal vector

$$\mathbf{d}^{(i)}(\xi) = \frac{1}{\alpha^{(i,t_1)}(\xi)} \left(\partial_1 \mathbf{F}^{(t_1)}(0, \xi) + \beta^{(i,t_1)}(\xi) \partial_2 \mathbf{F}^{(t_1)}(0, \xi) \right).$$

We define the projector $\Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}} : C^1(\Sigma^{(i)}) \rightarrow \mathcal{A}_{\Sigma^{(i)}}^{\circ}$ such that

$$\Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}}^{\circ}} \bar{\Lambda}_{\mathbf{j}}^{(i)}(\varphi) \bar{\mathbf{B}}_{\mathbf{j}}^{(i)}. \quad (34)$$

Remark 6. *The projector $\Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}}$ inherits its properties (such as the locality of the support) from the basis functions $\bar{\mathbf{B}}_{\mathbf{j}}^{(i)}$ and from the univariate dual bases $\{\lambda_j^+\}_{j \in \mathbb{I}^+}$ and $\{\lambda_j^-\}_{j \in \mathbb{I}^-}$.*

Remark 7. *One can define $\Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}}$ beyond $C^1(\Sigma^{(i)})$, e.g. $H^{3/2+\epsilon}(\Omega)$, for $\epsilon > 0$, suffices.*

4.4. The vertex function space $\mathcal{A}_{\mathbf{x}^{(i)}}$

Let $i \in \mathcal{I}_{\mathcal{X}}$, and $\mathbf{x}^{(i)}$ be a vertex with $\Sigma^{(t_1)}, \Omega^{(t_2)}, \Sigma^{(t_3)}, \dots, \Omega^{(t_{2\nu})}, \Sigma^{(t_{2\nu+1})}$ the sequence of edges and patches around $\mathbf{x}^{(i)}$ in counterclockwise order. Throughout this section, we always assume that the parametrizations $\mathbf{F}^{(t_2)}, \dots, \mathbf{F}^{(t_{2\nu})}$ are in standard form for the vertex $\mathbf{x}^{(i)}$, as stated in Section 2.4.

We define the vertex function space $\mathcal{A}_{\mathbf{x}^{(i)}}$ via a suitable projection operator.

Lemma 3. *There exists a function space $\mathcal{A}_{\mathbf{x}^{(i)}} \subset \mathcal{V}^1$ of dimension 6 and a suitable projector $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} : C^2(\mathbf{x}^{(i)}) \rightarrow \mathcal{A}_{\mathbf{x}^{(i)}} \subset \mathcal{V}^1$, such that for all $\varphi \in C^2(\mathbf{x}^{(i)})$ it holds*

$$\partial_{x_1}^{m_1} \partial_{x_2}^{m_2} (\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} \varphi)(\mathbf{x}^{(i)}) = \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \varphi(\mathbf{x}^{(i)}) \quad (35)$$

for $m_1, m_2 \geq 0$ and $m_1 + m_2 \leq 2$.

We will give a constructive proof of this lemma later. To do so, we need to establish some preliminary constructions. Having given such an interpolatory projector, we can now define a basis for the vertex space $\mathcal{A}_{\mathbf{x}^{(i)}}$.

Definition 21 (Basis at the vertices). Let $\mathcal{A}_{\mathbf{x}^{(i)}}$ and $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}$ be as in Lemma 3. Let

$$\sigma = \left(\frac{h}{p\nu} \sum_{\ell=1}^{\nu} \|\nabla \mathbf{F}^{(\iota_{2\ell})}(0, 0)\| \right)^{-1};$$

then

$$\mathcal{A}_{\mathbf{x}^{(i)}} = \text{span} \left\{ \mathring{\mathbf{B}}_{\mathbf{j}}^{(i)} : \mathbf{j} \in \mathbb{I}_{\mathbf{x}^{(i)}} \right\} \subset \mathcal{V}^1, \quad (36)$$

where $\mathring{\mathbf{B}}_{\mathbf{j}}^{(i)}$

$$\mathring{\mathbf{B}}_{\mathbf{j}}^{(i)} = \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} \left(\sigma^{j_1+j_2} \frac{(x_1 - x_1^{(i)})^{j_1}}{j_1!} \frac{(x_2 - x_2^{(i)})^{j_2}}{j_2!} \right), \quad (37)$$

and $\mathbb{I}_{\mathbf{x}^{(i)}} = \{\mathbf{j} = (j_1, j_2) : 0 \leq j_1, j_2 \text{ and } j_1 + j_2 \leq 2\}$.

Remark 8. The factor $\sigma^{j_1+j_2}$ in (37) is introduced to guarantee a uniform scaling, in L^∞ , of the basis functions. Indeed the vertex basis functions fulfill

$$\partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \left(\mathring{\mathbf{B}}_{\mathbf{j}}^{(i)} \right) (\mathbf{x}^{(i)}) = \sigma^{j_1+j_2} \delta_{j_1}^{m_1} \delta_{j_2}^{m_2}$$

for $0 \leq m_1 \leq 2, 0 \leq m_2 \leq 2$ and $m_1 + m_2 \leq 2$, where δ_j^m is the Kronecker delta.

The construction is organized in two steps: In the first part we identify in each patch a set of functions $\mathbf{B}_{\mathbf{j}}^{(\ell)}$ and $\bar{\mathbf{B}}_{\mathbf{j}}^{(\ell)}$ with specific interpolation properties at the vertex $\mathbf{x}^{(i)}$; then we combine the functions above to define global C^1 isogeometric functions that allow interpolation up to second order derivatives at the vertex $\mathbf{x}^{(i)}$.

Definition 22. Let k be an even index. We define the space $\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}$ of dimension 4 as

$$\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)} = \text{span} \left\{ \mathbf{B}_{\mathbf{j}}^{(\iota_k)} : \mathbf{j} \in \{0, 1\}^2 \right\},$$

where $\mathbf{B}_{\mathbf{j}}^{(\iota_k)}$ are given as in (20).

Lemma 4. For even k , there exists a unique projector

$$\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}} : C^2(\mathbf{x}^{(i)}) \rightarrow \mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)},$$

such that

$$\partial_1^{m_1} \partial_2^{m_2} \left(\left(\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}} \varphi \right) \circ \mathbf{F}^{(\iota_k)} \right) (0, 0) = \partial_1^{m_1} \partial_2^{m_2} \left(\varphi \circ \mathbf{F}^{(\iota_k)} \right) (0, 0) \quad (38)$$

for $0 \leq m_1 \leq 1, 0 \leq m_2 \leq 1$.

Proof. The existence of such an operator follows from the classical interpolation properties of the standard tensor-product B-spline basis. \square

Definition 23. Let k be an odd index. We define the space $\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}$ of dimension 5 as

$$\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)} = \text{span} \left\{ \overline{\mathbb{B}}_{\mathbf{j}}^{(i_k)} : 0 \leq j_1, 0 \leq j_2 \leq 1, j_1 + j_2 \leq 2 \right\},$$

where the functions $\overline{\mathbb{B}}_{\mathbf{j}}^{(i_k)}$ are given as in (23) and (25) if $\Sigma^{(i_k)}$ is an interface, or in (29) and (31) if $\Sigma^{(i_k)}$ is a boundary edge.

Lemma 5. *Let k be an odd index. There exists a unique projector*

$$\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}} : C^2(\mathbf{x}^{(i)}) \rightarrow \mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)},$$

such that if $i_{k-1} \in \mathcal{I}_\Omega$, for $0 \leq m_1 \leq 1, 0 \leq m_2 \leq 2$ and $m_1 + m_2 \leq 2$ it holds

$$\partial_1^{m_1} \partial_2^{m_2} \left((\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}} \varphi) \circ \mathbf{F}^{(i_{k-1})} \right) (0, 0) = \partial_1^{m_1} \partial_2^{m_2} \left(\varphi \circ \mathbf{F}^{(i_{k-1})} \right) (0, 0); \quad (39)$$

if $i_{k+1} \in \mathcal{I}_\Omega$, for $0 \leq m_1 \leq 2, 0 \leq m_2 \leq 1$ and $m_1 + m_2 \leq 2$ it holds

$$\partial_1^{m_1} \partial_2^{m_2} \left((\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}} \varphi) \circ \mathbf{F}^{(i_{k+1})} \right) (0, 0) = \partial_1^{m_1} \partial_2^{m_2} \left(\varphi \circ \mathbf{F}^{(i_{k+1})} \right) (0, 0). \quad (40)$$

A proof of this lemma can be found in Appendix A.

Definition 24 (Vertex projector). We define

$$\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} = \sum_{k=1}^{2\nu} (-1)^{k+1} \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}} = \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_1)}} - \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_2)}} \pm \dots + \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_{2\nu-1})}} - \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_{2\nu})}}. \quad (41)$$

With an abuse of notation, we assume $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}$ returns functions defined on the whole domain $\overline{\Omega}$, extending to zero outside the support.

Remark 9. *Since the projectors $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}}$ onto $\mathcal{A}_{\mathbf{x}^{(i)}}^{(i_k)}$ are defined by Hermite interpolation at the vertex $\mathbf{x}^{(i)}$, the projector $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}$ inherits its properties (such as the support size) from the basis functions $\mathbb{B}_{\mathbf{j}}^{(i_k)}$, for k even, and $\overline{\mathbb{B}}_{\mathbf{j}}^{(i_k)}$, for k odd.*

The projector $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}$ in Definition 24 will satisfy the properties required in Lemma 3. To complete the proof, we need one more preliminary result.

Lemma 6. *Let k be any even integer index and $\varphi \in C^2(\mathbf{x}^{(i)})$. Then*

$$\partial_1^{m_1} \partial_2^{m_2} \left((\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} \varphi) \circ \mathbf{F}^{(i_k)} \right) (0, 0) = \partial_1^{m_1} \partial_2^{m_2} \left(\varphi \circ \mathbf{F}^{(i_k)} \right) (0, 0) \quad (42)$$

for $m_1, m_2 \geq 0$ and $m_1 + m_2 \leq 2$. Moreover, if $\varphi \equiv 0$ then $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}} \varphi \equiv 0$.

This lemma states that the projector interpolates up to second order derivatives, when mapped into the parameter domain. A proof of this lemma can be found in Appendix A.

Proof of Lemma 3. Given $\varphi \in C^2(\mathbf{x}^{(i)})$, since $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}\varphi$ is supported in the region $\overline{\Omega}^{(i_2)} \cup \dots \cup \overline{\Omega}^{(i_{2\nu})}$, we need to consider the interfaces therein, that are $\overline{\Sigma}^{(i_1)}, \dots, \overline{\Sigma}^{(i_{2\nu-1})}$ for an interior vertex or $\overline{\Sigma}^{(i_3)}, \dots, \overline{\Sigma}^{(i_{2\nu-3})}$ for a boundary vertex. Then for an even index k , consider a generic $\overline{\Sigma}^{(i_{k+1})}$, with adjacent patches $\Omega^{(i_k)}$ and $\Omega^{(i_{k+2})}$. We have

$$\begin{aligned} \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}\varphi &= \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+1})}\varphi - \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_k)}\varphi + \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k-1})}\varphi, & \text{on the patch } \overline{\Omega}^{(i_k)}, \\ \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}\varphi &= \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+3})}\varphi - \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+2})}\varphi + \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+1})}\varphi, & \text{on the patch } \overline{\Omega}^{(i_{k+2})}. \end{aligned} \quad (43)$$

The term $-\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_k)}\varphi + \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k-1})}\varphi$ has vanishing trace and gradient at $\overline{\Sigma}^{(i_{k+1})}$. Indeed, adopting again the abbreviated notation as in the proof of Lemma 6, we have

$$(\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k-1})}\varphi) \circ \mathbf{F}^{(i_k)} - (\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_k)}\varphi) \circ \mathbf{F}^{(i_k)} = f_h^{(i_{k-1}, i_k)} - f_h^{(i_k, i_k)} \quad (44)$$

and, by plugging (Appendix A) and (Appendix A) into the right hand side of (44), it can be seen directly that the function and its gradient vanish at $(0, \xi)$, for all $\xi \in [0, 1]$. Similarly, $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+3})}\varphi - \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+2})}\varphi$ has vanishing trace and gradient at $\overline{\Sigma}^{(i_{k+1})}$. The only remaining term in (43) is $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}^{(i_{k+1})}\varphi$, which is $C^1(\overline{\Sigma}^{(i_{k+1})})$ by Lemma 2.

The interpolation property (35) follows from (42). Then, the dimension of the image of $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}$ follows from Lemma 6. \square

A dual vertex basis is straightforwardly derived by interpolation of derivatives up to second order.

Definition 25. The set $\{\check{\Lambda}_{\mathbf{j}}^{(i)}\}_{\mathbf{j} \in \mathbb{I}_{\mathbf{x}^{(i)}}$, with

$$\check{\Lambda}_{\mathbf{j}}^{(i)}(\varphi) = \frac{(\partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \varphi)(\mathbf{x}^{(i)})}{\sigma^{j_1+j_2}},$$

is a dual basis for $\{\check{\mathbf{B}}_{\mathbf{j}}^{(i)}\}_{\mathbf{j} \in \mathbb{I}_{\mathbf{x}^{(i)}}$.

We have by definition $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\mathbf{x}^{(i)}} \check{\Lambda}_{\mathbf{j}}^{(i)}(\varphi) \check{\mathbf{B}}_{\mathbf{j}}^{(i)}$.

4.5. Basis, dual basis and projector for the Argyris isogeometric space \mathcal{A}

To summarize the results of this section, the functions

- $\{\mathbf{B}_{\mathbf{j}}^{(i)}\}_{\mathbf{j} \in \mathbb{I}_{\Omega}^{\circ}}$, for $i \in \mathcal{I}_{\Omega}$ (as in Definition 16),
- $\{\overline{\mathbf{B}}_{\mathbf{j}}^{(i)}\}_{\mathbf{j} \in \mathbb{I}_{\Sigma}^{\circ}}$, for $i \in \mathcal{I}_{\Sigma}$ (as in Definitions 18 and 19), and
- $\{\check{\mathbf{B}}_{\mathbf{j}}^{(i)}\}_{\mathbf{j} \in \mathbb{I}_{\mathcal{X}}^{\circ}}$, for $i \in \mathcal{I}_{\mathcal{X}}$ (as in Definition 21),

form a basis for the space \mathcal{A} . The global projector

$$\Pi_{\mathcal{A}} : C^2(\Omega) \rightarrow \mathcal{A}$$

is defined via

$$\Pi_{\mathcal{A}}(\varphi) = \sum_{i \in \mathcal{I}_{\Omega}} \Pi_{\mathcal{A}_{\Omega^{(i)}}^{\circ}}(\varphi) + \sum_{i \in \mathcal{I}_{\Sigma}} \Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}}(\varphi) + \sum_{i \in \mathcal{I}_{\mathcal{X}}} \Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}(\varphi),$$

where

$$\Pi_{\mathcal{A}_{\Omega^{(i)}}^{\circ}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\Omega^{(i)}}^{\circ}} \Lambda_{\mathbf{j}}(\varphi) \mathbf{B}_{\mathbf{j}}^{(i)}$$

as in Definition 17,

$$\Pi_{\mathcal{A}_{\Sigma^{(i)}}^{\circ}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\Sigma^{(i)}}^{\circ}} \bar{\Lambda}_{\mathbf{j}}^{(i)}(\varphi) \bar{\mathbf{B}}_{\mathbf{j}}^{(i)}$$

as in Definition 20, and

$$\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}}(\varphi) = \sum_{\mathbf{j} \in \mathbb{I}_{\mathbf{x}^{(i)}}} \hat{\Lambda}_{\mathbf{j}}^{(i)}(\varphi) \hat{\mathbf{B}}_{\mathbf{j}}^{(i)}$$

as in Lemma 3 and Definition 25.

The following result follows directly from the definition of the space \mathcal{A} .

Proposition 4. *We have for $\varphi_h \in \mathcal{A}$, that $\varphi_h \in C^1(\Omega)$ and $\varphi_h \in C^2(\mathbf{x}^{(i)})$ for all $i \in \mathcal{I}_{\mathcal{X}}$.*

5. Numerical examples

We perform L^2 -approximation over two AS- G^1 multi-patch parametrizations to numerically show that the Argyris isogeometric space \mathcal{A} maintains the polynomial reproduction properties of the entire space \mathcal{V}^1 for the traces and normal derivatives along the interfaces, and that the space \mathcal{A} , being a subspace of \mathcal{V}^1 , produces relative L^2 errors of the same magnitude as the entire space \mathcal{V}^1 .

For this purpose, we consider the two AS- G^1 multi-patch parametrizations visualized in Fig. 5 (left). Both AS- G^1 geometries consist of single spline patches $\mathbf{F}^{(i)} \in \mathcal{S}_h^{\mathbf{p}, \mathbf{r}} \times \mathcal{S}_h^{\mathbf{p}, \mathbf{r}}$ with $\mathbf{p} = (3, 3)$, $\mathbf{r} = (1, 1)$ and $h = \frac{1}{2}$, and are generated by using the method presented in [22]. The construction of the AS- G^1 five patch parametrization was already demonstrated in [22, Example 1]. The AS- G^1 three-patch parametrization can be obtained in an analogous manner.

For both AS- G^1 geometries we generate a sequence of nested spaces \mathcal{A}_h and \mathcal{V}_h^1 for $\mathbf{p} = (3, 3)$ and $\mathbf{r} = (1, 1)$ by selecting the mesh size h as $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ and $\frac{1}{32}$. While the bases of the Argyris spaces \mathcal{A}_h are simply constructed as described in Section 4, the bases of the entire spaces \mathcal{V}_h^1 are obtained in the same way as in [22, Section 4.2] by means of the concept of minimal determining sets (cf. [28]). Note that in contrast to the basis functions of \mathcal{A}_h , the resulting basis functions of \mathcal{V}_h^1 are in general not locally supported and possess a support over at least one entire interface.

We use now the basis functions for the spaces \mathcal{A}_h and \mathcal{V}_h^1 to perform L^2 -approximation over the two AS- G^1 multi-patch parametrizations. Consider one of the spaces \mathcal{A}_h or \mathcal{V}_h^1 , and let $\phi_{\mathbf{j}}$ be the corresponding basis functions. The goal is to approximate the function

$$z : \Omega \rightarrow \mathbb{R}, \quad z(\mathbf{x}) = z(x_1, x_2) = 2 \cos(x_1) \sin(x_2), \quad (45)$$

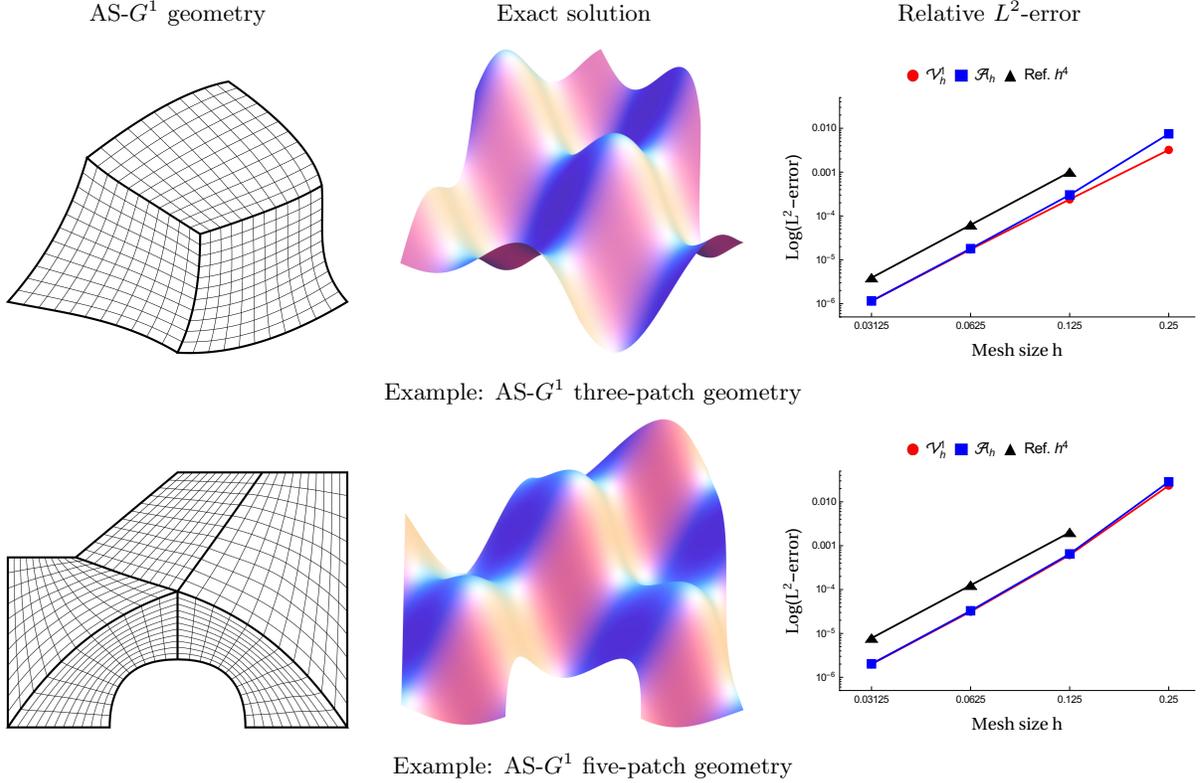


Figure 5: L^2 -projection over two AS- G^1 multi-patch parametrizations (left) by using the two different spaces \mathcal{A}_h and \mathcal{V}_h^1 (right) to approximate the function (45) (middle), cf. Table 1.

see Fig. 5 (middle), by the function

$$u_h(\mathbf{x}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{x}), \quad c_{\mathbf{j}} \in \mathbb{R},$$

via minimizing the term

$$\|u_h - z\|_{L^2}^2 = \int_{\Omega} (u_h(\mathbf{x}) - z(\mathbf{x}))^2 d\mathbf{x} \rightarrow \min_{c_{\mathbf{j}}}.$$

The isogeometric formulation of this linear problem was discussed in detail in [22, Section 4.2], and will be omitted here for the sake of brevity.

Table 1 and Fig. 5 (right) report the resulting relative L^2 -errors and the estimated convergence rates for the two spaces \mathcal{A}_h and \mathcal{V}_h^1 for the different mesh sizes h . The numerical results indicate for both spaces convergence rates of optimal order $\mathcal{O}(h^4)$ in the L^2 -norm and show that resulting relative L^2 -errors are of the same magnitude for the two spaces.

6. Conclusion

We presented for the class of AS- G^1 multi-patch parametrizations the construction of a basis and of an associated dual basis for the so-called Argyris isogeometric space \mathcal{A} , which generalizes the

		Subspace \mathcal{A}		Entire space \mathcal{V}^1		
AS- G^1 three-patch geometry (a)						
h	$\dim \mathcal{A}$	$\frac{\ u_h - z\ _{L^2}}{\ z\ _{L^2}}$	e.c.r. $\ \cdot\ _{L^2}$	$\dim \mathcal{V}^1$	$\frac{\ u_h - z\ _{L^2}}{\ z\ _{L^2}}$	e.c.r. $\ \cdot\ _{L^2}$
1/4	177	7.46e-03	-	222	3.21e-03	-
1/8	729	3.03e-04	4.62	822	2.4e-04	3.74
1/16	2985	1.8e-05	4.07	3174	1.73e-05	3.79
1/32	12105	1.15e-06	3.97	12486	1.14e-06	3.92
AS- G^1 five-patch geometry (b)						
h	$\dim \mathcal{A}$	$\frac{\ u_h - z\ _{L^2}}{\ z\ _{L^2}}$	e.c.r. $\ \cdot\ _{L^2}$	$\dim \mathcal{V}^1$	$\frac{\ u_h - z\ _{L^2}}{\ z\ _{L^2}}$	e.c.r. $\ \cdot\ _{L^2}$
1/4	291	2.86e-02	-	372	2.35e-02	-
1/8	1211	6.5e-04	5.46	1376	6.14e-04	5.25
1/16	4971	3.28e-05	4.31	5304	3.14e-05	4.29
1/32	20171	2.02e-06	4.02	20840	1.99e-06	3.98

Table 1: Resulting relative L^2 -errors with estimated convergence rates of the diagonally scaled mass matrices by performing L^2 -approximation over two AS- G^1 multi-patch geometries using the two C^1 spaces \mathcal{A} and \mathcal{V}^1 , cf. Fig. 5.

classical Argyris finite elements to multi-patch isogeometric spaces. It is a subspace of the entire C^1 isogeometric space \mathcal{V}^1 maintaining the polynomial reproduction properties of \mathcal{V}^1 for the traces and normal derivatives along the interfaces. This property of the subspace \mathcal{A} was shown numerically by performing L^2 -approximation over different AS- G^1 multi-patch parametrizations. The use of the Argyris space \mathcal{A} instead of the space \mathcal{V}^1 is advantageous since the subspace \mathcal{A} has a simpler structure and allows a uniform and simple construction of the basis functions independent of the AS- G^1 domain parametrization. The construction of the basis (and of its dual basis) is based on the decomposition of the space \mathcal{A} into the direct sum of three subspaces called the patch-interior, the edge and the vertex function space. The resulting basis and the dual basis have a simple form, since the single functions are locally supported and are explicitly given by closed form representations.

This paper presents the foundation for further studies of C^1 isogeometric spaces over AS- G^1 multi-patch parametrizations, by providing a basis and corresponding projectors. A first planned topic for future research is the theoretical investigation of the properties of the space \mathcal{A} , such as approximation error and stability estimates for h -refined meshes, which can be built upon a suitable dual basis. Moreover, one may also construct a basis forming a partition of unity, following the ideas presented in [12], based on local triangular Bézier surfaces at the vertices. We are also planning to extend the construction to surface domains and to use our approach to perform Kirchhoff-Love shell analysis for different linear and non-linear model configurations. Another challenging task will be the extension to volumetric domains. So far, no generalization of AS- G^1 parametrizations to volumetric domains is known.

Acknowledgments

G. Sangalli is member of the Gruppo Nazionale Calcolo Scientifico-Istituto Nazionale di Alta Matematica (GNCS-INDAM), and was partially supported by the European Research Council

through the FP7 Ideas Consolidator Grant *HIGEOM* n.616563. This support is gratefully acknowledged.

Appendix A. Proof of Lemma 5 and 6

Proof of Lemma 5. Let $\phi_{m_1, m_2} = \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \varphi(\mathbf{x}^{(i)})$. Assume $\iota_{k-1}, \iota_{k+1} \in \mathcal{I}_\Omega$ (this is not true in general for boundary vertices, if so the proof below simplifies in a trivial way). On each patch $\Omega^{(\iota_\ell)}$, $\ell = k-1$ or $\ell = k+1$, we can write the pull-back of φ and of the projection $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}} \varphi$ as

$$f^{(\iota_\ell)}(\xi_1, \xi_2) = \varphi \circ \mathbf{F}^{(\iota_\ell)}(\xi_1, \xi_2)$$

and

$$f_h^{(\iota_k, \iota_\ell)}(\xi_1, \xi_2) = (\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}} \varphi) \circ \mathbf{F}^{(\iota_\ell)}(\xi_1, \xi_2).$$

We have by definition, using the abbreviations $\nabla \phi = (\phi_{1,0}, \phi_{0,1})$ and

$$H\phi = \begin{pmatrix} \phi_{2,0} & \phi_{1,1} \\ \phi_{1,1} & \phi_{0,2} \end{pmatrix},$$

and using the chain rule of differentiation, that

$$\begin{aligned} f^{(\iota_\ell)}(0, 0) &= \phi_{0,0}, \\ \partial_1 f^{(\iota_\ell)}(0, 0) &= \nabla \phi \partial_1 \mathbf{F}^{(\iota_\ell)}(0, 0), \\ \partial_2 f^{(\iota_\ell)}(0, 0) &= \nabla \phi \partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0), \\ \partial_1^2 f^{(\iota_\ell)}(0, 0) &= (\partial_1 \mathbf{F}^{(\iota_\ell)}(0, 0))^T H\phi \partial_1 \mathbf{F}^{(\iota_\ell)}(0, 0) + \nabla \phi \partial_1 \partial_1 \mathbf{F}^{(\iota_\ell)}(0, 0), \\ \partial_1 \partial_2 f^{(\iota_\ell)}(0, 0) &= (\partial_1 \mathbf{F}^{(\iota_\ell)}(0, 0))^T H\phi \partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0) + \nabla \phi \partial_1 \partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0), \\ \partial_2^2 f^{(\iota_\ell)}(0, 0) &= (\partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0))^T H\phi \partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0) + \nabla \phi \partial_2 \partial_2 \mathbf{F}^{(\iota_\ell)}(0, 0). \end{aligned} \tag{A.1}$$

Consider the basis transformations from $\{b_0^+, b_1^+, b_2^+\}$ to $\{c_0^+, c_1^+, c_2^+\}$ and from $\{b_0^-, b_1^-\}$ to $\{c_0^-, c_1^-\}$, with

$$\partial_\xi^j c_i^+(0) = \delta_i^j \quad \text{for } j = 0, \dots, 2, \quad \text{and} \quad \partial_\xi^j c_i^-(0) = \delta_i^j \quad \text{for } j = 0, 1,$$

where δ_i^j is the Kronecker delta. Then, recalling (23) and (25), we can rewrite the functions $f_h^{(\iota_k, \iota_{k-1})}$ and $f_h^{(\iota_k, \iota_{k+1})}$ in terms of the new bases:

$$\begin{aligned} f_h^{(\iota_k, \iota_{k-1})}(\xi_1, \xi_2) &= \sum_{j=0}^2 d_{0,j} \left(c_j^+(\xi_2) c_0(\xi_1) - \beta^{(\iota_k, \iota_{k-1})}(\xi_2) (c_j^+)'(\xi_2) c_1(\xi_1) \right) \\ &+ \sum_{j=0}^1 d_{1,j} \alpha^{(\iota_k, \iota_{k-1})}(\xi_2) c_j^-(\xi_2) b_1(\xi_1), \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} f_h^{(\iota_k, \iota_{k+1})}(\xi_1, \xi_2) &= \sum_{j=0}^2 d_{0,j} \left(c_j^+(\xi_1) c_0(\xi_2) - \beta^{(\iota_k, \iota_{k+1})}(\xi_1) (c_j^+)'(\xi_1) c_1(\xi_2) \right) \\ &- \sum_{j=0}^1 d_{1,j} \alpha^{(\iota_k, \iota_{k+1})}(\xi_1) c_j^-(\xi_1) b_1(\xi_2). \end{aligned} \tag{A.3}$$

Considering (A.2), we then have

$$\begin{aligned}
f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= d_{0,0}, \\
\partial_2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= d_{0,1}, \\
\partial_2^2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= d_{0,2}, \\
\partial_1 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \frac{p}{h} \alpha^{(\iota_k, \iota_{k-1})}(0) d_{1,0} - \beta^{(\iota_k, \iota_{k-1})}(0) d_{0,1}, \\
\partial_1 \partial_2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \frac{p}{h} \alpha^{(\iota_k, \iota_{k-1})}(0) d_{1,1} - \beta^{(\iota_k, \iota_{k-1})}(0) d_{0,2} \\
&\quad + \frac{p}{h} (\alpha^{(\iota_k, \iota_{k-1})})'(0) d_{1,0} - (\beta^{(\iota_k, \iota_{k-1})})'(0) d_{0,1};
\end{aligned} \tag{A.4}$$

using the abbreviated notation

$$\mathbf{t}^{(\iota_k)}(\xi) = \partial_2 \mathbf{F}^{(\iota_{k-1})}(0, \xi) = \partial_1 \mathbf{F}^{(\iota_{k+1})}(\xi, 0)$$

and

$$\begin{aligned}
\mathbf{d}^{(\iota_k)}(\xi) &= \frac{1}{\alpha^{(\iota_k, \iota_{k-1})}(\xi)} (\partial_1 \mathbf{F}^{(\iota_{k-1})}(0, \xi) + \beta^{(\iota_k, \iota_{k-1})}(\xi) \partial_2 \mathbf{F}^{(\iota_{k-1})}(0, \xi)) \\
&= -\frac{1}{\alpha^{(\iota_k, \iota_{k+1})}(\xi)} (\partial_2 \mathbf{F}^{(\iota_{k+1})}(\xi, 0) + \beta^{(\iota_k, \iota_{k+1})}(\xi) \partial_1 \mathbf{F}^{(\iota_{k+1})}(\xi, 0))
\end{aligned}$$

we can determine all $d_{i,j}$ from the interpolation conditions

$$\begin{aligned}
f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= f^{(\iota_{k-1})}(0, 0), \\
\partial_2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \partial_2 f^{(\iota_{k-1})}(0, 0), \\
\partial_2^2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \partial_2^2 f^{(\iota_{k-1})}(0, 0), \\
\partial_1 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \partial_1 f^{(\iota_{k-1})}(0, 0), \\
\partial_1 \partial_2 f_h^{(\iota_k, \iota_{k-1})}(0, 0) &= \partial_1 \partial_2 f^{(\iota_{k-1})}(0, 0)
\end{aligned}$$

and both (A.1) and (A.4):

$$\begin{aligned}
d_{0,0} &= \phi_{0,0}, \\
d_{0,1} &= \nabla \phi \mathbf{t}^{(\iota_k)}(0), \\
d_{0,2} &= (\mathbf{t}^{(\iota_k)}(0))^T H \phi \mathbf{t}^{(\iota_k)}(0) + \nabla \phi (\mathbf{t}^{(\iota_k)})'(0), \\
\frac{p}{h} d_{1,0} &= \nabla \phi \mathbf{d}^{(\iota_k)}(0), \\
\frac{p}{h} d_{1,1} &= (\mathbf{t}^{(\iota_k)}(0))^T H \phi \mathbf{d}^{(\iota_k)}(0) + \nabla \phi (\mathbf{d}^{(\iota_k)})'(0).
\end{aligned} \tag{A.5}$$

Hence, there exists a projector $\Pi_{\mathcal{A}_{\mathbf{x}^{(\iota)}}^{(\iota_k)}}$ satisfying (39). We can reason similarly on (A.3), where we have

$$\begin{aligned}
f_h^{(\iota_k, \iota_{k+1})}(0, 0) &= d_{0,0}, \\
\partial_1 f_h^{(\iota_k, \iota_{k+1})}(0, 0) &= d_{0,1}, \\
\partial_1 \partial_1 f_h^{(\iota_k, \iota_{k+1})}(0, 0) &= d_{0,2}, \\
\partial_2 f_h^{(\iota_k, \iota_{k+1})}(0, 0) &= -\frac{p}{h} \alpha^{(\iota_k, \iota_{k+1})}(0) d_{1,0} - \beta^{(\iota_k, \iota_{k+1})}(0) d_{0,1}, \\
\partial_1 \partial_2 f_h^{(\iota_k, \iota_{k+1})}(0, 0) &= -\frac{p}{h} \alpha^{(\iota_k, \iota_{k+1})}(0) d_{1,1} - \beta^{(\iota_k, \iota_{k+1})}(0) d_{0,2} \\
&\quad - \frac{p}{h} (\alpha^{(\iota_k, \iota_{k+1})})'(0) d_{1,0} - (\beta^{(\iota_k, \iota_{k+1})})'(0) d_{0,1}.
\end{aligned} \tag{A.6}$$

In fact, after inserting (A.5) into (A.6) and simplifying, we obtain (A.1) for $\ell = k + 1$. Hence, the projector $\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_k)}}$ also satisfies (40), which concludes the proof. \square

Proof of Lemma 6. We use again the abbreviated notation $f^{(\iota_k)} = \varphi \circ \mathbf{F}^{(\iota_k)}$ and $f_h^{(\iota_\ell, \iota_k)} = (\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_\ell)}} \varphi) \circ \mathbf{F}^{(\iota_k)}$, where by definition $f_h^{(\iota_\ell, \iota_k)} = 0$ for $\ell \neq k - 1, k, k + 1$.

Then we have

$$(\Pi_{\mathcal{A}_{\mathbf{x}^{(i)}}^{(\iota_\ell)}} \varphi) \circ \mathbf{F}^{(\iota_k)} = f_h^{(\iota_{k-1}, \iota_k)} - f_h^{(\iota_k, \iota_k)} + f_h^{(\iota_{k+1}, \iota_k)}. \quad (\text{A.7})$$

The active degrees-of-freedom of three terms in (A.7) with respect to the underlying tensor-product spline space $\mathcal{S}_h^{\mathbf{P}, \mathbf{r}}$ are pictured in Figure A.6.

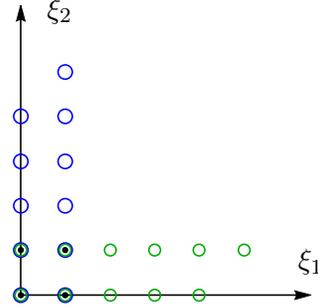


Figure A.6: Active degrees-of-freedom of the functions in (A.7) with respect to the tensor-product spline space $\mathcal{S}_h^{\mathbf{P}, \mathbf{r}}$: The degrees of freedom corresponding to $f_h^{(\iota_{k-1}, \iota_k)}$ depicted in green, corresponding to $f_h^{(\iota_k, \iota_k)}$ in black and corresponding to $f_h^{(\iota_{k+1}, \iota_k)}$ in blue.

By definition and thanks to the interpolation properties (38) and (39)–(40) we have:

$$\begin{aligned} f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= f^{(\iota_k)}(0, 0), \\ \partial_2 f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= \partial_2 f^{(\iota_k)}(0, 0), \\ \partial_2^2 f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= \partial_2^2 f^{(\iota_k)}(0, 0), \\ \partial_1 f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= \partial_1 f^{(\iota_k)}(0, 0), \\ \partial_1 \partial_2 f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= \partial_1 \partial_2 f^{(\iota_k)}(0, 0), \\ \partial_1^2 f_h^{(\iota_{k+1}, \iota_k)}(0, 0) &= 0, \end{aligned}$$

and

$$\begin{aligned} f_h^{(\iota_k, \iota_k)}(0, 0) &= f^{(\iota_k)}(0, 0), \\ \partial_1 f_h^{(\iota_k, \iota_k)}(0, 0) &= \partial_1 f^{(\iota_k)}(0, 0), \\ \partial_2 f_h^{(\iota_k, \iota_k)}(0, 0) &= \partial_2 f^{(\iota_k)}(0, 0), \\ \partial_1 \partial_2 f_h^{(\iota_k, \iota_k)}(0, 0) &= \partial_1 \partial_2 f^{(\iota_k)}(0, 0), \\ \partial_1^2 f_h^{(\iota_k, \iota_k)}(0, 0) &= 0, \\ \partial_2^2 f_h^{(\iota_k, \iota_k)}(0, 0) &= 0, \end{aligned}$$

and

$$\begin{aligned}
f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= f^{(\iota_k)}(0, 0), \\
\partial_1 f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= \partial_1 f^{(\iota_k)}(0, 0), \\
\partial_1^2 f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= \partial_1^2 f^{(\iota_k)}(0, 0), \\
\partial_2 f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= \partial_2 f^{(\iota_k)}(0, 0), \\
\partial_1 \partial_2 f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= \partial_1 \partial_2 f^{(\iota_k)}(0, 0), \\
\partial_2^2 f_h^{(\iota_{k-1}, \iota_k)}(0, 0) &= 0.
\end{aligned}$$

Using the above relations into (A.7) we get (42).

Finally, if $f^{(\iota_k)}$ vanishes, then all the derivatives above are null and, by definition, $f_h^{(\iota_{k-1}, \iota_k)}$, $f_h^{(\iota_k, \iota_k)}$ and $f_h^{(\iota_{k+1}, \iota_k)}$ are null. \square

Appendix B. Extension to non-uniform knots and (partially) matching meshes

Note that one can extend the presented construction easily to multi-patch domains with non-uniform meshes and partially matching interfaces. We will briefly sketch the necessary adaptations. We assume to have different spline spaces $\mathcal{S}^{(i)}$ for every patch $\Omega^{(i)}$ with $i \in \mathcal{I}_\Omega$. Every space satisfies

$$\mathcal{S}^{(i)} = \mathcal{S}_1^{(i)} \otimes \mathcal{S}_2^{(i)},$$

where $\mathcal{S}_k^{(i)}$ is a univariate spline space of degree p and regularity r , having $n_k^{(i)}$ distinct inner knots

$$0 < \eta_{k,1}^{(i)} < \eta_{k,2}^{(i)} < \dots < \eta_{k,n_k^{(i)}-1}^{(i)} < 1,$$

each with multiplicity $p - r$ and having 0 and 1 as boundary knots with multiplicity $p + 1$.

Having defined different spaces for every patch, we change Assumption 4 and Definition 9 and assume $\mathbf{F}^{(i)} \in \mathcal{S}^{(i)} \times \mathcal{S}^{(i)}$ as well as $f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)} \in \mathcal{S}^{(i)}$. Now, in order to have a sufficiently large C^0 isogeometric space along every interface, we need that the knot meshes are (partially) matching along all interfaces.

Assumption 26. Consider an interface $\Sigma^{(i)}$, with $i \in \mathcal{I}_\Sigma^c$. Assume $\mathbf{F}^{(\iota_1)}$, $\mathbf{F}^{(\iota_2)}$ are in standard form for $\Sigma^{(i)}$. Then the corresponding meshes are

- matching, i.e., $\mathcal{S}_2^{(\iota_1)} = \mathcal{S}_1^{(\iota_2)}$, or
- partially matching, i.e., $\mathcal{S}_2^{(\iota_1)} \subseteq \mathcal{S}_1^{(\iota_2)}$ or $\mathcal{S}_1^{(\iota_2)} \subseteq \mathcal{S}_2^{(\iota_1)}$.

Note that two meshes are matching along an interface, if the corresponding knots are the same. The meshes are partially matching along an interface, if the knots of one patch are a subset of the knots of the other.

The space \mathcal{A} can be constructed just as for uniform meshes. The patch-interior basis (Definition 16) needs no additional modification. The edge-interior basis (Definition 18) uses spaces \mathcal{S}^+ and \mathcal{S}^- , which are built from $\mathcal{S}_2^{(\iota_1)} \cap \mathcal{S}_1^{(\iota_2)}$ by reducing the knot multiplicity by one or reducing the polynomial degree by one, respectively. See [21] for a construction of the complete basis for non-uniform knots. The vertex basis (Definition 21) is defined as a linear combination of patch and edge contributions and can be constructed analogously.

Appendix C. Another approximating subspace $\tilde{\mathcal{A}} \subseteq \mathcal{V}^1$

Instead of the space \mathcal{A} given in (19), one can consider a slightly larger subspace $\tilde{\mathcal{A}} \subseteq \mathcal{V}^1$, which contains all isogeometric functions which are C^2 at the interior vertices and boundary vertices of valency $\nu \geq 3$ and are C^1 everywhere else. The space $\tilde{\mathcal{A}}$ is given by

$$\tilde{\mathcal{A}} = \left(\bigoplus_{i \in \mathcal{I}_\Omega} \tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\Sigma^\circ} \tilde{\mathcal{A}}_{\Sigma^{(i)}}^\circ \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_\chi^\circ \cup \tilde{\mathcal{I}}_\chi^\Gamma} \mathcal{A}_{\mathbf{x}^{(i)}} \right),$$

where the indices in $\tilde{\mathcal{I}}_\chi^\Gamma \subseteq \mathcal{I}_\chi^\Gamma$ represents all boundary vertices of valency $\nu \geq 3$. In contrast to (19), different patch interior spaces, denoted by $\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ$, and different edge function spaces, denoted by $\tilde{\mathcal{A}}_{\Sigma^{(i)}}^\circ$, are used to generate the space $\tilde{\mathcal{A}}$. A further difference is that the new edge function space $\tilde{\mathcal{A}}_{\Sigma^{(i)}}^\circ$ will be now only taken for all interfaces, and that the vertex function space $\mathcal{A}_{\mathbf{x}^{(i)}}$ have to be only selected for all interior vertices and for all boundary vertices of valency $\nu \geq 3$. Below, we will present the definitions of the spaces $\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ$ and $\tilde{\mathcal{A}}_{\Sigma^{(i)}}^\circ$, which will be similar to the ones for the spaces $\mathcal{A}_{\Omega^{(i)}}^\circ$ and $\mathcal{A}_{\Sigma^{(i)}}^\circ$, see Definition 16 and 18, respectively.

Before, we will need some additional assumptions and definitions. We assume that in case of $\beta^{(i)} \equiv 0$ for $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, the functions $\beta^{(i,\nu_1)}$ and $\beta^{(i,\nu_2)}$ are selected as $\beta^{(i,\nu_1)} \equiv \beta^{(i,\nu_2)} \equiv 0$. For each $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, let

$$z_\beta^{(i)} = \{\xi_0 \in h, \dots, (n-1)h \mid \beta^{(i)}(\xi_0) = 0\}, \quad h_\beta^{(i)} = \begin{cases} 0 & \text{if } \beta^{(i)} \equiv 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$d_\alpha^{(i)} = \max(\deg(\alpha^{(i,\nu_1)}), \deg(\alpha^{(i,\nu_2)})).$$

Since $\alpha^{(i,\nu_1)}$ and $\alpha^{(i,\nu_2)}$ are linear polynomials, and $\beta^{(i)}$ is a quadratic polynomial, we obtain that $d_\alpha^{(i)} \in \{0, 1\}$ and $z_\beta^{(i)} \in \{0, 1, 2, n\}$, cf. [21]. For each $\ell \in \{1, \dots, n-1\}$, we denote by $\mathcal{S}_{h,\ell}^{p,r}$ the univariate spline space of degree p on the parameter domain $[0, 1]$, constructed from the open knot vector with n non-empty knots spans with (mesh) size $h = 1/n$, where the inner knots ih , $i \in \{1, \dots, n-1\}$ with $i \neq \ell$, have multiplicity $p-r$, and the inner knot ℓh has multiplicity $p-r+1$. This means that functions of the space $\mathcal{S}_{h,\ell}^{p,r}$ are C^r on $[0, 1]$ except at the inner knot ℓh , where they are only C^{r-1} .

We first define the patch interior space $\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ \supseteq \mathcal{A}_{\Omega^{(i)}}^\circ$. In contrast to the space $\mathcal{A}_{\Omega^{(i)}}^\circ$, the isogeometric functions of the space $\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ$ need not have vanishing values and gradients at possible boundary edges of the multi-path domain Ω , but still have vanishing values and gradients at the patch interfaces.

Definition 27. Let $i \in \mathcal{I}_\Omega$, we define the space $\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ$ as

$$\tilde{\mathcal{A}}_{\Omega^{(i)}}^\circ = \text{span} \left\{ \mathbf{B}_j^{(i)} : \bar{\Omega}^{(i)} \rightarrow \mathbb{R} \text{ such that } \mathbf{B}_j^{(i)} \circ \mathbf{F}^{(i)} = b_j, \text{ for } \mathbf{j} \in \tilde{\mathbb{I}}_{\Omega^{(i)}}^\circ \right\}$$

where the index set $\tilde{\mathbb{I}}_{\Omega^{(i)}}^\circ$ takes all $\mathbf{j} \in \mathbb{I}$ which do not belong to the C^1 data of an interface $\Sigma^{(i)}$, with $i \in \mathcal{I}_\Sigma^\circ$, or to the C^2 data of a vertex $\mathbf{x}^{(i)}$, with $i \in \mathcal{I}_\chi^\circ \cup \tilde{\mathcal{I}}_\chi^\Gamma$.

The definition of the edge function space $\tilde{\mathcal{A}}_{\Sigma^{(i)}}^{\circ} \supseteq \mathcal{A}_{\Sigma^{(i)}}^{\circ}$ is based on the construction of the (entire) C^1 isogeometric space for AS- G^1 two-patch geometries presented in [21].

Definition 28. Let $\Sigma^{(i)}$, for $i \in \mathcal{I}_{\Sigma}^{\circ}$, be an interface in standard form. Consider the univariate spline spaces $\tilde{\mathcal{S}}^+ = \mathcal{S}_h^{p,r+h_{\beta}^{(i)}}$ and $\tilde{\mathcal{S}}^- = \mathcal{S}_h^{p-d_{\alpha}^{(i)},r}$, with bases $\{\tilde{b}_j^+\}_{j \in \tilde{\mathbb{I}}^+}$, and $\{\tilde{b}_j^-\}_{j \in \tilde{\mathbb{I}}^-}$, respectively, where $\tilde{\mathbb{I}}^{\pm} = \{0, \dots, \tilde{N}^{\pm} - 1\}$ with $\tilde{N}^+ = (p-r-h_{\beta}^{(i)})(n-1)+p+1$ and $\tilde{N}^- = (p-d_{\alpha}^{(i)}-r)(n-1)+p-d_{\alpha}^{(i)}+1$. For each $\ell \in \{1, \dots, n-1\}$, let $\tilde{b}_{\ell}^{\#}$ be a B-spline of the space $\mathcal{S}_{h,\ell}^{p,r}$ with the property $\tilde{b}_{\ell}^{\#}(\ell h) \neq 0$ which have vanishing derivatives up to second order at both interface vertices. We define the index set

$$\tilde{\mathbb{I}}^{\#} = \begin{cases} \emptyset & \text{if } z_{\beta}^{(i)} = 0 \text{ or } \beta \equiv 0, \\ \{\ell \in \{1, \dots, n-1\} \mid \beta^{(i)}(\ell h) = 0 \text{ and } \tilde{b}_{\ell}^{\#} \text{ exists}\} & \text{otherwise,} \end{cases}$$

and $\tilde{\mathbb{I}}_{\Sigma^{(i)}} = (\tilde{\mathbb{I}}^+ \times \{0\}) \cup (\tilde{\mathbb{I}}^- \times \{1\}) \cup (\tilde{\mathbb{I}}^{\#} \times \{2\})$. In addition, let $c_0 = b_0 + b_1$ and $c_1 = \frac{h}{p}b_1$. We define the space $\tilde{\mathcal{A}}_{\Sigma^{(i)}}$ as

$$\tilde{\mathcal{A}}_{\Sigma^{(i)}} = \text{span} \left\{ \tilde{\mathbf{B}}_{\mathbf{j}}^{(i)} : \bar{\Omega}^{(\iota_1)} \cup \bar{\Omega}^{(\iota_2)} \rightarrow \mathbb{R}, \text{ for } \mathbf{j} \in \tilde{\mathbb{I}}_{\Sigma^{(i)}} \right\},$$

where

$$\begin{aligned} \tilde{\mathbf{B}}_{(j_1,0)}^{(i)} \circ \mathbf{F}^{(\iota_1)}(\xi_1, \xi_2) &= \tilde{b}_{j_1}^+(\xi_2)c_0(\xi_1) - \beta^{(i,\iota_1)}(\xi_2)(\tilde{b}_{j_1}^+)'(\xi_2)c_1(\xi_1), \\ \tilde{\mathbf{B}}_{(j_1,0)}^{(i)} \circ \mathbf{F}^{(\iota_2)}(\xi_1, \xi_2) &= \tilde{b}_{j_1}^+(\xi_1)c_0(\xi_2) - \beta^{(i,\iota_2)}(\xi_1)(\tilde{b}_{j_1}^+)'(\xi_1)c_1(\xi_2), \end{aligned}$$

for $j_1 \in \tilde{\mathbb{I}}^+$,

$$\begin{aligned} \tilde{\mathbf{B}}_{(j_1,1)}^{(i)} \circ \mathbf{F}^{(\iota_1)}(\xi_1, \xi_2) &= \alpha^{(i,\iota_1)}(\xi_2)\tilde{b}_{j_1}^-(\xi_2)b_1(\xi_1), \\ \tilde{\mathbf{B}}_{(j_1,1)}^{(i)} \circ \mathbf{F}^{(\iota_2)}(\xi_1, \xi_2) &= -\alpha^{(i,\iota_2)}(\xi_1)\tilde{b}_{j_1}^-(\xi_1)b_1(\xi_2), \end{aligned}$$

for $j_1 \in \tilde{\mathbb{I}}^-$, and

$$\begin{aligned} \tilde{\mathbf{B}}_{(j_1,2)}^{(i)} \circ \mathbf{F}^{(\iota_1)}(\xi_1, \xi_2) &= \tilde{b}_{j_1}^{\#}(\xi_2)c_0(\xi_1) - \beta^{(i,\iota_1)}(\xi_2)(\tilde{b}_{j_1}^{\#})'(\xi_2)c_1(\xi_1) + \frac{\beta^{(i,\iota_1)}(j_1 h)}{\alpha^{(i,\iota_1)}(j_1 h)}\alpha^{(i,\iota_1)}(\xi_2)(\tilde{b}_{j_1}^{\#})'(\xi_2)c_1(\xi_1), \\ \tilde{\mathbf{B}}_{(j_1,2)}^{(i)} \circ \mathbf{F}^{(\iota_2)}(\xi_1, \xi_2) &= \tilde{b}_{j_1}^{\#}(\xi_1)c_0(\xi_2) - \beta^{(i,\iota_2)}(\xi_1)(\tilde{b}_{j_1}^{\#})'(\xi_1)c_1(\xi_2) + \frac{\beta^{(i,\iota_2)}(j_1 h)}{\alpha^{(i,\iota_2)}(j_1 h)}\alpha^{(i,\iota_2)}(\xi_1)(\tilde{b}_{j_1}^{\#})'(\xi_1)c_1(\xi_2), \end{aligned}$$

for $j_1 \in \tilde{\mathbb{I}}^{\#}$. Let then $\tilde{\mathcal{A}}_{\Sigma^{(i)}}^{\circ}$ be the subspace of $\tilde{\mathcal{A}}_{\Sigma^{(i)}}$, given by

$$\tilde{\mathcal{A}}_{\Sigma^{(i)}}^{\circ} = \{\varphi_h \in \tilde{\mathcal{A}}_{\Sigma^{(i)}} : \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \varphi_h(\mathbf{x}^{(\iota)}) = 0 \text{ for all } \iota \in \mathcal{I}_{\mathcal{X}}^{\circ} \cup \tilde{\mathcal{I}}_{\mathcal{X}}^{\Gamma} \text{ and } m_1, m_2 \geq 0 \text{ and } m_1 + m_2 \leq 2\}.$$

Remark 10. In contrast to the subspace \mathcal{A} , the dimension of the subspace $\tilde{\mathcal{A}}$ depends on the domain parametrization. Let $\hat{\mathcal{A}}_{\Omega^{(i)}}^{\circ} = \tilde{\mathcal{A}}_{\Omega^{(i)}}^{\circ}$ and let

$$\hat{\mathcal{A}}_{\Sigma^{(i)}}^{\circ} = \{\varphi_h \in \mathcal{A}_{\Sigma^{(i)}} : \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \varphi_h(\mathbf{x}^{(\iota)}) = 0 \text{ for all } \iota \in \mathcal{I}_{\mathcal{X}}^{\circ} \cup \tilde{\mathcal{I}}_{\mathcal{X}}^{\Gamma}, \text{ and } m_1, m_2 \geq 0 \text{ and } m_1 + m_2 \leq 2\}.$$

Then, the subspace

$$\hat{\mathcal{A}} = \left(\bigoplus_{i \in \mathcal{I}_{\Omega}} \hat{\mathcal{A}}_{\Omega^{(i)}}^{\circ} \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_{\Sigma}^{\circ}} \hat{\mathcal{A}}_{\Sigma^{(i)}}^{\circ} \right) \oplus \left(\bigoplus_{i \in \mathcal{I}_{\mathcal{X}}^{\circ} \cup \tilde{\mathcal{I}}_{\mathcal{X}}^{\Gamma}} \mathcal{A}_{\mathbf{x}^{(i)}} \right)$$

with $\mathcal{A} \subseteq \hat{\mathcal{A}} \subseteq \tilde{\mathcal{A}} \subseteq \mathcal{V}^1$, would be another choice of a C^1 isogeometric subspace, and its dimension is as the dimension of the subspace \mathcal{A} independent of the domain parametrization.

References

References

- [1] J. H. Argyris, I. Fried, and D. W. Scharpf. The TUBA family of plate elements for the matrix displacement method. *The Aeronautical Journal*, 72(692):701–709, 1968.
- [2] F. Auricchio, L. Beirão da Veiga, A. Buffa, C. Lovadina, A. Reali, and G. Sangalli. A fully ”locking-free” isogeometric approach for plane linear elasticity problems: a stream function formulation. *Comput. Methods Appl. Mech. Engrg.*, 197(1):160–172, 2007.
- [3] A. Bartezzaghi, L. Dedè, and A. Quarteroni. Isogeometric analysis of high order partial differential equations on surfaces. *Comput. Methods Appl. Mech. Engrg.*, 295:446 – 469, 2015.
- [4] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. Mathematical analysis of variational isogeometric methods. *Acta Numerica*, 23:157–287, 5 2014.
- [5] D. J. Benson, Y. Bazilevs, M.-C. Hsu, and T. J.R. Hughes. A large deformation, rotation-free, isogeometric shell. *Comput. Methods Appl. Mech. Engrg.*, 200(13):1367–1378, 2011.
- [6] M. Bercovier and T. Matskewich. *Smooth Bézier Surfaces over Unstructured Quadrilateral Meshes*. Lecture Notes of the Unione Matematica Italiana, Springer, 2017.
- [7] A. Blidia, B. Mourrain, and N. Villamizar. G^1 -smooth splines on quad meshes with 4-split macro-patch elements. *Computer Aided Geometric Design*, 52–53:106–125, 2017.
- [8] F. K. Bogner, R. L. Fox, and L. A. Schmit. The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulae. In *Proc. Conf. Matrix Methods in Struct. Mech., AirForce Inst. of Tech., Wright Patterson AF Base, Ohio*, 1965.
- [9] Philippe G Ciarlet. The finite element method for elliptic problems. *Classics in applied mathematics*, 40:1–511, 2002.
- [10] A. Collin, G. Sangalli, and T. Takacs. Analysis-suitable G^1 multi-patch parametrizations for C^1 isogeometric spaces. *Computer Aided Geometric Design*, 47:93 – 113, 2016.
- [11] J. A. Cottrell, T.J.R. Hughes, and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. John Wiley & Sons, Chichester, England, 2009.
- [12] P. Dierckx. On calculating normalized Powell-Sabin B-splines. *Computer Aided Geometric Design*, 15(1):61–78, 1997.
- [13] G. Farin. *Curves and Surfaces for Computer-Aided Geometric Design*. Academic Press, 1997.
- [14] H. Gómez, V. M Calo, Y. Bazilevs, and T. J.R. Hughes. Isogeometric analysis of the Cahn–Hilliard phase-field model. *Comput. Methods Appl. Mech. Engrg.*, 197(49):4333–4352, 2008.

- [15] H. Gomez, V. M. Calo, and T. J. R. Hughes. Isogeometric analysis of Phase–Field models: Application to the Cahn–Hilliard equation. In *ECCOMAS Multidisciplinary Jubilee Symposium: New Computational Challenges in Materials, Structures, and Fluids*, pages 1–16. Springer Netherlands, 2009.
- [16] D. Groisser and J. Peters. Matched G^k -constructions always yield C^k -continuous isogeometric elements. *Computer Aided Geometric Design*, 34:67 – 72, 2015.
- [17] J. Hoschek and D. Lasser. *Fundamentals of computer aided geometric design*. A K Peters Ltd., Wellesley, MA, 1993.
- [18] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39-41):4135–4195, 2005.
- [19] Gašper Jaklič and Tadej Kanduč. Hermite parametric surface interpolation based on Argyris element. *Computer Aided Geometric Design*, 56:67–81, 2017.
- [20] M. Kapl, F. Buchegger, M. Bercovier, and B. Jüttler. Isogeometric analysis with geometrically continuous functions on planar multi-patch geometries. *Comput. Methods Appl. Mech. Engrg.*, 316:209 – 234, 2017.
- [21] M. Kapl, G. Sangalli, and T. Takacs. Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations. *Computer Aided Geometric Design*, 52–53:75 – 89, 2017.
- [22] M. Kapl, G. Sangalli, and T. Takacs. Construction of analysis-suitable G^1 planar multi-patch parameterizations. *Computer-Aided Design*, 97:41 – 55, 2018.
- [23] M. Kapl, V. Vitrih, B. Jüttler, and K. Birner. Isogeometric analysis with geometrically continuous functions on two-patch geometries. *Comput. Math. Appl.*, 70(7):1518 – 1538, 2015.
- [24] K. Karčiauskas, T. Nguyen, and J. Peters. Generalizing bicubic splines for modeling and IGA with irregular layout. *Computer-Aided Design*, 70:23–35, 2016.
- [25] J. Kiendl, Y. Bazilevs, M.-C. Hsu, R. Wüchner, and K.-U. Bletzinger. The bending strip method for isogeometric analysis of Kirchhoff-Love shell structures comprised of multiple patches. *Comput. Methods Appl. Mech. Engrg.*, 199(35):2403–2416, 2010.
- [26] J. Kiendl, K.-U. Bletzinger, J. Linhard, and R. Wüchner. Isogeometric shell analysis with Kirchhoff-Love elements. *Comput. Methods Appl. Mech. Engrg.*, 198(49):3902–3914, 2009.
- [27] J. Kiendl, M.-Ch. Hsu, M. C. H. Wu, and A. Reali. Isogeometric Kirchhoff–Love shell formulations for general hyperelastic materials. *Comput. Methods Appl. Mech. Engrg.*, 291:280 – 303, 2015.

- [28] M.-J. Lai and L. L. Schumaker. *Spline functions on triangulations*, volume 110 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
- [29] M.-J.-Jun Lai and L. L. Schumaker. *Spline functions on triangulations*. Cambridge University Press, 2007.
- [30] J. Liu, L. Dedè, J. A. Evans, M. J. Borden, and T. J. R. Hughes. Isogeometric analysis of the advective Cahn–Hilliard equation: Spinodal decomposition under shear flow. *Journal of Computational Physics*, 242:321 – 350, 2013.
- [31] B. Mourrain, R. Vidunas, and N. Villamizar. Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology. *Computer Aided Geometric Design*, 45:108 – 133, 2016.
- [32] T. Nguyen, K. Karčiauskas, and J. Peters. A comparative study of several classical, discrete differential and isogeometric methods for solving poisson’s equation on the disk. *Axioms*, 3(2):280–299, 2014.
- [33] T. Nguyen, K. Karčiauskas, and J. Peters. C^1 finite elements on non-tensor-product 2d and 3d manifolds. *Applied Mathematics and Computation*, 272:148 – 158, 2016.
- [34] M.A. Scott, R.N. Simpson, J.A. Evans, S. Lipton, S.P.A. Bordas, T.J.R. Hughes, and T.W. Sederberg. Isogeometric boundary element analysis using unstructured t-splines. *Comp. Methods Appl. Mech. Engrg.*, 254:197 – 221, 2013.
- [35] A. Tagliabue, L. Dedè, and A. Quarteroni. Isogeometric analysis and error estimates for high order partial differential equations in fluid dynamics. *Computers & Fluids*, 102:277 – 303, 2014.
- [36] T. Takacs. Construction of smooth isogeometric function spaces on singularly parameterized domains. In *International Conference on Curves and Surfaces*, pages 433–451. Springer, 2014.
- [37] T. Takacs and B. Jüttler. H^2 regularity properties of singular parameterizations in isogeometric analysis. *Graphical Models*, 74(6):361–372, 2012.
- [38] D. Toshniwal, H. Speleers, R. Hiemstra, and T. J. R. Hughes. Multi-degree smooth polar splines: A framework for geometric modeling and isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 316:1005–1061, 2017.
- [39] D. Toshniwal, H. Speleers, and T. J. R. Hughes. Analysis-suitable spline spaces of arbitrary degree on unstructured quadrilateral meshes. Technical Report 16, Institute for Computational Engineering and Sciences (ICES), 2017.
- [40] D. Toshniwal, H. Speleers, and T. J. R. Hughes. Smooth cubic spline spaces on unstructured quadrilateral meshes with particular emphasis on extraordinary points: Geometric design and isogeometric analysis considerations. *Comput. Methods Appl. Mech. Engrg.*, 327:411–458, 2017.

- [41] M. Wu, B. Mourrain, A. Galligo, and B. Nkonga. Hermite type spline spaces over rectangular meshes with complex topological structures. *Communications in Computational Physics*, 21(3):835–866, 2017.