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Mapping rational rotation-minimizing frames from polynomial curves on to rational curves

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Abstract

Given a polynomial space curve $\mathbf{r}(\xi)$ that has a rational rotation–minimizing frame (an RRMF curve), a methodology is developed to construct families of rational space curves $\tilde{\mathbf{r}}(\xi)$ with the same rotation–minimizing frame as $\mathbf{r}(\xi)$ at corresponding points. The construction employs the dual form of a rational space curve, interpreted as the edge of regression of the envelope of a family of osculating planes, having normals in the direction $\mathbf{u}(\xi) = \mathbf{r}'(\xi) \times \mathbf{r}''(\xi)$ and distances from the origin specified in terms of a rational function $f(\xi)$ as $f(\xi)/\|\mathbf{u}(\xi)\|$. An explicit characterization of the rational curves $\tilde{\mathbf{r}}(\xi)$ generated by a given RRMF curve $\mathbf{r}(\xi)$ in this manner is developed, and the problem of matching initial and final points and frames is shown to impose only linear conditions on the coefficients of $f(\xi)$, obviating the non–linear equations (and existence questions) that arise in addressing this problem with the RRMF curve $\mathbf{r}(\xi)$. Criteria for identifying low–degree instances of the curves $\tilde{\mathbf{r}}(\xi)$ are identified, by a cancellation of factors common to their numerators and denominators, and the methodology is illustrated by a number of computed examples.

Keywords: rotation–minimizing frames; rational curves; edge of regression; rational functions; polynomial factorization; spatial motion interpolants.

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1 Introduction

An *adapted* orthonormal frame $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ on a space curve $\mathbf{r}(\xi)$ consists of the unit tangent $\mathbf{t}(\xi) = \mathbf{r}'(\xi)/\|\mathbf{r}'(\xi)\|$ and unit vectors $\mathbf{u}(\xi), \mathbf{v}(\xi)$ that span the curve normal plane. The frame angular velocity $\boldsymbol{\omega}$ specifies its variation along the curve through the relations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{t}, \quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{u}, \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{v},$$

where s denotes the curve arc length. The frame $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ is said to be a *rotation–minimizing frame* (RMF) — or *Bishop frame* [3] — if its angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$, i.e., the normal–plane vectors $\mathbf{u}$ and $\mathbf{v}$ exhibit no rotation about the tangent $\mathbf{t}$.

Rotation–minimizing frames have diverse applications in robotics, computer animation, 5–axis CNC machining, swept surface constructions, spatial motion planning, and related fields. Since the construction of RMFs on general polynomial/rational curves amounts [26] to integrating a first–order differential equation, many numerical approximation schemes have been formulated [16, 22, 23, 24, 25, 27, 29, 30, 31, 32]. On the other hand, there has been increasing interest in the identification of curves that admit *exact* RMFs [4, 5, 8, 11, 12, 13, 17, 18, 19, 21] — i.e., the frame vectors $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ have a rational dependence on the parameter ξ, and the angular velocity satisfies $\boldsymbol{\omega}(\xi) \cdot \mathbf{t}(\xi) \equiv 0$ (see [9] for a review). Such curves are necessarily *Pythagorean–hodograph* (PH) *curves* [7], since only PH curves have rational unit tangents. These studies have focused primarily on *polynomial* PH curves, although the paper [1] constructs rational curves with rational RMFs by applying Möbius transformations in $\mathbb{R}^3$ (composed of inversions in planes and spheres) to piecewise planar PH cubics, and thereby solves a G^1 Hermite interpolation problem.

The present study adopts a different approach to the construction of rational curves with rational RMFs, which exploits the hodograph structure $\mathbf{r}'(\xi)$ of polynomial PH curves with rational rotation–minimizing frames (or *RRMF curves*). Namely, the vector $\mathbf{u}(\xi) = \mathbf{r}'(\xi) \times \mathbf{r}''(\xi)$ is employed to define a (non–unit) binormal vector for a space curve $\tilde{\mathbf{r}}(\xi)$. The family $\Pi(\xi)$ of osculating planes of $\tilde{\mathbf{r}}(\xi)$ can then be defined by introducing a support function $f(\xi)$ such that $f(\xi)/\|\mathbf{u}(\xi)\|$ specifies their distances from the origin. The envelope of the planes $\Pi(\xi)$ defines the *tangent developable* of the rational curve $\tilde{\mathbf{r}}(\xi)$, which may be recovered as the *edge of regression* [28] (or cuspidal edge) of the tangent developable.

The rational curve $\tilde{\mathbf{r}}(\xi)$ has exactly the same RMF $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ as the polynomial RRMF curve $\mathbf{r}(\xi)$, and an explicit parameterization of it may be derived in terms of $\mathbf{u}(\xi)$ and $f(\xi)$. Moreover, we show that these new curves $\tilde{\mathbf{r}}(\xi)$ greatly simplify the interpolation of initial and final points and frames, because of their linear dependence on $f(\xi)$ — with polynomial RRMF curves this is a non–linear problem, and for the quintic RRMF curves it is known [13] that solutions do not exist for all instances of the boundary data.

The plan for the remainder of this paper is as follows. First, some relevant properties of the polynomial RRMF curves are briefly summarized in Section 2. These are employed in Section 3 to develop an explicit parameterization for rational curves $\tilde{\mathbf{r}}(\xi)$ that have the same (rational) RMF as a given polynomial RRMF curve $\mathbf{r}(\xi)$, and some key features of this representation are discussed. Section 4 identifies conditions under which low–degree curves

$\tilde{\mathbf{r}}(\xi)$ can be obtained by cancellation of factors common to the numerator and denominator, through appropriate choices of $f(\xi)$. Based upon the linear dependence of $\tilde{\mathbf{r}}(\xi)$ on $f(\xi)$, the extreme simplicity with which given initial and final points can be interpolated, $\tilde{\mathbf{r}}(0) = \mathbf{p}_i$ and $\tilde{\mathbf{r}}(1) = \mathbf{p}_f$, is then illustrated in Section 5. In Section 6 we describe some computed examples, to illustrate implementation of the method. Finally, Section 7 summarizes the key findings of the present study, and suggests avenues for further investigation.

2 Rational RMFs on polynomial PH curves

Recall [6, 10] that a polynomial PH space curve $\mathbf{r}(\xi)$ may be constructed from a quaternion polynomial

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k} \tag{1}$$

by integration of the hodograph specified by the product[1]

$$\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi), \tag{2}$$

where $\mathcal{A}^*(\xi) = u(\xi) - v(\xi)\,\mathbf{i} - p(\xi)\,\mathbf{j} - q(\xi)\,\mathbf{k}$ is the conjugate of $\mathcal{A}(\xi)$. If $\mathcal{A}(\xi)$ is of degree m, and specified in Bernstein form on $\xi \in [\,0,1\,]$ as

$$\mathcal{A}(\xi) = \sum_{i=0}^{m} \mathcal{A}_i \binom{m}{i} (1-\xi)^{m-i}\xi^i, \tag{3}$$

it generates the spatial PH curve of odd degree $n = 2m+1$ defined (up to an integration constant) by

$$\mathbf{r}(\xi) = \int \mathbf{r}'(\xi)\,\mathrm{d}\xi = \int \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)\,\mathrm{d}\xi. \tag{4}$$

The parametric speed of $\mathbf{r}(\xi)$ — the derivative $\mathrm{d}s/\mathrm{d}\xi$ of the curve arc length s with respect to the parameter ξ — is $\sigma(\xi) = |\mathcal{A}(\xi)|^2 = u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi)$.

Every spatial PH curve is equipped with a rational adapted orthonormal frame — the *Euler–Rodrigues frame* (ERF) — defined [5] by

$$(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi)) = \frac{(\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi), \mathcal{A}(\xi)\,\mathbf{j}\,\mathcal{A}^*(\xi), \mathcal{A}(\xi)\,\mathbf{k}\,\mathcal{A}^*(\xi))}{|\mathcal{A}(\xi)|^2}\,, \tag{5}$$

wherein $\mathbf{e}_1(\xi)$ is the curve tangent, and $\mathbf{e}_2(\xi)$, $\mathbf{e}_3(\xi)$ span the normal plane. This is not (in general) an RMF, since it has [9] the angular velocity component

$$\omega_1 = \boldsymbol{\omega}\cdot\mathbf{e}_1 = \frac{2(uv' - u'v - pq' + p'q)}{\sigma^2}$$

[1]Calligraphic characters such as $\mathcal{A}$ denote quaternions, with scalar and vector parts $\mathrm{scal}(\mathcal{A})$ and $\mathrm{vect}(\mathcal{A})$. The choice of the unit vector $\mathbf{i}$ in (2) is merely conventional — any other unit vector may be used instead, corresponding to a change of coordinates.

in the direction of the tangent $\mathbf{e}_1 = \mathbf{t}$. However, if we introduce new normal–plane vectors $\mathbf{u}(\xi), \mathbf{v}(\xi)$ defined by rotating $\mathbf{e}_2(\xi), \mathbf{e}_3(\xi)$ about $\mathbf{e}_1(\xi)$ by the angle function

$$\theta(\xi) = -\,2\arctan\frac{b(\xi)}{a(\xi)}$$

for relatively prime polynomials $a(\xi), b(\xi)$ — i.e., we set

$$\left[\begin{array}{c}\mathbf{u}(\xi)\\ \mathbf{v}(\xi)\end{array}\right] = \frac{1}{a^2(\xi)+b^2(\xi)}\left[\begin{array}{cc} a^2(\xi)-b^2(\xi) & -2\,a(\xi)b(\xi)\\ 2\,a(\xi)b(\xi) & a^2(\xi)-b^2(\xi)\end{array}\right]\left[\begin{array}{c}\mathbf{e}_2(\xi)\\ \mathbf{e}_3(\xi)\end{array}\right], \tag{6}$$

this rotation induces an angular velocity $\mathrm{d}\theta/\mathrm{d}s = 2(a'b-ab')/\sigma\,(a^2+b^2)$ of $\mathbf{e}_2$, $\mathbf{e}_3$ about $\mathbf{e}_1 = \mathbf{t}$. Consequently, $\mathrm{d}\theta/\mathrm{d}s$ precisely cancels the ERF angular velocity component ω_1, and $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ is a rational RMF, if relatively prime polynomials $a(\xi), b(\xi)$ exist that satisfy [21] the condition

$$\frac{uv'-u'v-pq'+p'q}{u^2+v^2+p^2+q^2} = \frac{ab'-a'b}{a^2+b^2}. \tag{7}$$

The existence of polynomial PH space curves with *rational* RMFs (or *RRMF curves*) satisfying (7) has been characterized [4, 5, 8, 12, 18, 19] by systems of algebraic constraints on the coefficients $\mathcal{A}_0,\ldots,\mathcal{A}_m$ of the quaternion polynomial (3). A survey of these results may be found in [9], and a comprehensive theory of RRMF curves was formulated in [11]. The simplest instances are the quintic RRMF curves [8] which satisfy (7) with $\deg(\mathcal{A}) = 2$ and $\deg(a,b) = 2$ subject to the vector constraint

$$\mathrm{vect}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*) = \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^*, \tag{8}$$

and the degree 7 RRMF curves [17, 21] that satisfy (7) with $\deg(\mathcal{A}) = 3$ and $\deg(a,b) = 0$ subject to the system of five scalar constraints

$$\begin{aligned} \mathrm{scal}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_1^*) \;&=\; \mathrm{scal}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^*) \;=\; 0,\\ 3\,\mathrm{scal}(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_2^*) + \mathrm{scal}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_3^*) \;&=\; 0,\\ \mathrm{scal}(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_3^*) \;&=\; \mathrm{scal}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_3^*) \;=\; 0. \end{aligned} \tag{9}$$

For brevity, we focus here on the former instance. The latter instance corresponds to the case in which the ERF is itself rotation–minimizing — i.e., the normal–plane rotation (6) is not necessary to define an RMF.

3 Mapping RMFs to rational curves

The construction of RRMF curves that satisfy prescribed boundary conditions (end points and frames) is complicated by the non–linear nature of the constraints on $\mathcal{A}_0,\ldots,\mathcal{A}_m$ that identify RRMF curves, and the end–point displacement equation resulting from integration of the hodograph (2). With the quintic RRMF curves characterized by the vector condition (8), the end–point displacement requires a real degree 6 polynomial to have a positive real root, and this does not hold [13] for all possible input data. For the degree 7 RRMF curves identified by the five scalar constraints (9), a system of four quadratic equations in four real variables (dependent on two free parameters) must be solved numerically [17].

To circumvent these difficulties, we consider here the set of all *rational* PH curves $\tilde{\mathbf{r}}(\xi)$ with the same RMF as a given (polynomial) RRMF curve $\mathbf{r}(\xi)$ defined by the hodograph (2). With these curves, satisfaction of the end–point displacement will impose only *linear* conditions on the available degrees of freedom. Such curves $\tilde{\mathbf{r}}(\xi)$ are evidently of the form

$$\tilde{\mathbf{r}}(\xi) = \int \lambda(\xi)\,\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)\,\mathrm{d}\xi \tag{10}$$

for some rational function $\lambda(\xi)$. However, the integral (10) does not necessarily generate a rational curve for any choice of $\lambda(\xi)$ — in other words, for a prescribed tangent indicatrix $\mathbf{t}(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)/|\mathcal{A}(\xi)|^2$ it is not possible to freely specify the rational parametric speed $\tilde{\sigma}(\xi) = |\tilde{\mathbf{r}}'(\xi)| = |\lambda(\xi)|\,|\mathcal{A}(\xi)|^2$ and ensure that (10) yields a rational curve.

In lieu of (10), we choose a different approach here, wherein $\tilde{\mathbf{r}}(\xi)$ is constructed as the *edge of regression* [28] of a one–parameter family of osculating planes — defined by a family of binormal vectors constructed from $\mathbf{r}'(\xi)$ together with a rational support function $f(\xi)$ that determines their distances from the origin. An explicit form for $\tilde{\mathbf{r}}(\xi)$ has been presented in Proposition 1 of [20], based on specifying the tangent indicatrix through stereographic projection of a plane rational curve. We adopt an alternative approach here, that employs the quaternion product (2) — where $\mathcal{A}(\xi)$ defines an RRMF curve — to specify the tangent indicatrix. Since $\mathbf{r}' = \sigma\,\mathbf{t}$, $\mathbf{r}'' = \sigma'\mathbf{t} + \sigma\,\mathbf{t}'$, and $\mathbf{t}' = \sigma\,\kappa\,\mathbf{n}$, we obtain $\mathbf{r}' \times \mathbf{r}'' = \sigma^3\kappa\,\mathbf{b}$ (where $\mathbf{n}$, $\mathbf{b}$, and κ are the principal normal, binormal, and curvature of $\mathbf{r}(\xi)$). Omitting the factor $\sigma^3\kappa$, we define a (non–unit) vector in the direction of the binormal by

$$\mathbf{u}(\xi) = \mathbf{r}'(\xi) \times \mathbf{r}''(\xi) = (\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)) \times (\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi))'\,. \tag{11}$$

Using standard rules of quaternion algebra, the expression on the right can be written as

$$\sigma(\xi)\,\underbrace{\tfrac{1}{2}(\mathcal{A}'(\xi)\mathcal{A}^*(\xi) - \mathcal{A}(\xi)\mathcal{A}'^*(\xi))}_{\mathbf{w}_1(\xi)} + (\mathcal{A}(\xi)\,\mathbf{i})\,\underbrace{\tfrac{1}{2}(\mathcal{A}'^*(\xi)\mathcal{A}(\xi) - \mathcal{A}^*(\xi)\mathcal{A}'(\xi))}_{\mathbf{w}_2(\xi)}(\mathcal{A}(\xi)\,\mathbf{i})^*,$$

where $\mathbf{w}_1 = \mathrm{vect}(\mathcal{A}'\mathcal{A}^*)$, $\mathbf{w}_2 = \mathrm{vect}(\mathcal{A}'^*\mathcal{A})$ are of equal magnitude (since $\sigma = \mathcal{A}\,\mathcal{A}^*$ we have $\sigma' = \mathcal{A}'\mathcal{A}^* + \mathcal{A}\,\mathcal{A}'^*$, and hence $\mathrm{vect}(\sigma') = \mathrm{vect}(\mathcal{A}'\mathcal{A}^*) + \mathrm{vect}(\mathcal{A}\,\mathcal{A}'^*) = \mathbf{0}$).

Then, for *any* rational function $f(\xi)$, we introduce the family of osculating planes $\Pi(\xi)$ defined in terms of free coordinates $\mathbf{p} = (x, y, z)$ by the equation

$$\Pi(\xi):\quad \mathbf{u}(\xi)\cdot\mathbf{p} = f(\xi). \tag{12}$$

Here $\mathbf{u}(\xi)$ defines the normal to the plane $\Pi(\xi)$, and $f(\xi)/\|\mathbf{u}(\xi)\|$ is its (signed) distance from the origin. The edge of regression of the envelope of this family of osculating planes is a rational PH curve $\tilde{\mathbf{r}}(\xi)$ with the same rational RMF as the polynomial PH curve (4), and *every* rational PH curve with the same rational RMF as the polynomial RRMF curve $\mathbf{r}(\xi)$ can be obtained in this manner.

The explicit form of $\tilde{\mathbf{r}}(\xi)$ in terms of $\mathbf{u}(\xi)$ and $f(\xi)$ has been derived in [20] as

$$\tilde{\mathbf{r}}(\xi) = \frac{f(\xi)\,\mathbf{u}'(\xi)\times\mathbf{u}''(\xi) + f'(\xi)\,\mathbf{u}''(\xi)\times\mathbf{u}(\xi) + f''(\xi)\,\mathbf{u}(\xi)\times\mathbf{u}'(\xi)}{\Delta(\xi)}, \tag{13}$$

where

$$\Delta(\xi) = \mathbf{u}(\xi)\cdot[\,\mathbf{u}'(\xi)\times\mathbf{u}''(\xi)\,] = \det[\,\mathbf{u}(\xi),\mathbf{u}'(\xi),\mathbf{u}''(\xi)\,]. \tag{14}$$

Remark 1 Setting $f(\xi) = c(\xi)/d(\xi)$ for relatively prime polynomials $c(\xi)$ and $d(\xi)$, the homogeneous coordinates $\tilde{W}(\xi),\tilde{X}(\xi),\tilde{Y}(\xi),\tilde{Z}(\xi)$ for the rational curve (13) can be derived and, in the absence of common factors, $\tilde{\mathbf{r}}(\xi)$ has the denominator $\tilde{W}(\xi) = d^3(\xi)\Delta(\xi)$.

Now the vector (11) and its first two derivatives are given by

$$\mathbf{u}(\xi) = \mathbf{r}'(\xi)\times\mathbf{r}''(\xi), \quad \mathbf{u}'(\xi) = \mathbf{r}'(\xi)\times\mathbf{r}'''(\xi), \quad \mathbf{u}''(\xi) = \mathbf{r}''(\xi)\times\mathbf{r}'''(\xi) + \mathbf{r}'(\xi)\times\mathbf{r}''''(\xi),$$

and by using the vector identity

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})\,]\,\mathbf{c} - [\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,]\,\mathbf{d}$$

we can express the cross products in (13) as follows:

$$\begin{aligned} \mathbf{u}(\xi)\times\mathbf{u}'(\xi) &= \det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}'''(\xi)\,]\,\mathbf{r}'(\xi), \\ \mathbf{u}'(\xi)\times\mathbf{u}''(\xi) &= \det[\,\mathbf{r}'(\xi),\mathbf{r}'''(\xi),\mathbf{r}''''(\xi)\,]\,\mathbf{r}'(\xi) + \det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}'''(\xi)\,]\,\mathbf{r}'''(\xi), \\ \mathbf{u}''(\xi)\times\mathbf{u}(\xi) &= -\det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}''''(\xi)\,]\,\mathbf{r}'(\xi) - \det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}'''(\xi)\,]\,\mathbf{r}''(\xi). \end{aligned} \tag{15}$$

Lemma 1 *The polynomial* (14) *occurring in the denominator of the rational curve* (13) *can be expressed in terms of the hodograph* (2) *and its derivatives as*

$$\Delta(\xi) = (\det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}'''(\xi)\,])^2. \tag{16}$$

Proof : This follows directly from substituting the above expressions for $\mathbf{u}(\xi)$, $\mathbf{u}'(\xi)$, $\mathbf{u}''(\xi)$ into (14) and simplifying. ∎

Remark 2 Since (13) is a rational PH curve, it may have real points at infinity. In the case that $f(\xi)$ is a polynomial, they occur only when the denominator $\Delta(\xi)$ has real roots. Since the torsion of $\mathbf{r}(\xi)$ is $\tau(\xi) = \det[\,\mathbf{r}'(\xi),\mathbf{r}''(\xi),\mathbf{r}'''(\xi)\,]/|\mathbf{r}(\xi)\times\mathbf{r}''(\xi)|^2$, Lemma 1 shows that the points at infinity of $\tilde{\mathbf{r}}(\xi)$ correspond precisely to the points of zero torsion on $\mathbf{r}(\xi)$. Consequently, we have the following result.

Lemma 2 *If* $\mathbf{r}(\xi)$ *has no points of zero torsion on a real interval* I*, and* $f(\xi)$ *is polynomial, then* $\tilde{\mathbf{r}}(\xi)$ *has no points at infinity on the interval* I*.*

Remark 3 Every polynomial PH curve satisfies [14] the relation

$$\| \mathbf{r}'(\xi) \times \mathbf{r}''(\xi) \|^2 = 4\, \sigma^2(\xi)\, \rho(\xi), \tag{17}$$

where the polynomial $\rho(\xi)$ can be expressed [14] in terms of the components of (1) and its derivative as

$$\rho = (up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2. \tag{18}$$

Thus the *unit* binormal vector $\mathbf{u}(\xi)/\|\mathbf{u}(\xi)\|$ is rational if and only if $\rho(\xi)$ is a perfect square. Polynomial PH curves that possess this property are known [2, 14, 15] as *double PH curves* — they have rational Frenet frames and curvatures.

Proposition 1 *For the quintic RRMF curves generated from* (2) *by a quadratic quaternion polynomial* $\mathcal{A}_0(1-\xi)^2 + \mathcal{A}_1 2(1-\xi)\xi + \mathcal{A}_2\xi^2$ *with coefficients satisfying* (8)*, the triple product* $\mathbf{r}'(\xi) \cdot (\mathbf{r}''(\xi) \times \mathbf{r}'''(\xi))$ *has the factorization*

$$\mathbf{r}'(\xi) \cdot (\mathbf{r}''(\xi) \times \mathbf{r}'''(\xi)) = \sigma(\xi)\, h(\xi), \tag{19}$$

where $\sigma(\xi) = |\mathcal{A}(\xi)|^2$ *is the quartic parametric speed, and* $h(\xi)$ *is a quadratic polynomial.*

Proof : To simplify the analysis, we consider curves in *canonical form*, with $\mathcal{A}_0 = 1$. This is achieved by multiplying $\mathcal{A}(\xi)$ with $\mathcal{A}_0^{-1} = \mathcal{A}_0^*/|\mathcal{A}_0|^2$, corresponding to the imposition of a scaling/rotation transformation on $\mathbf{r}'(\xi)$, which does not alter its RRMF nature, so that $\mathbf{r}'(0) = (1, 0, 0)$. The condition (8) is then satisfied [8] with coefficients of the form

$$\mathcal{A}_0 = 1, \quad \mathcal{A}_1 = u_1 + v_1\mathbf{i} + p_1\mathbf{j} + q_1\mathbf{k},$$

$$\mathcal{A}_2 = u_1^2 + v_1^2 - p_1^2 - q_1^2 + \gamma\,\mathbf{i} + 2(u_1p_1 - v_1q_1)\,\mathbf{j} + 2(u_1q_1 + v_1p_1)\,\mathbf{k},$$

where $u_1, v_1, p_1, q_1, \gamma$ are free (real) parameters. By forming $\mathbf{r}'(\xi) \cdot (\mathbf{r}''(\xi) \times \mathbf{r}'''(\xi))$ in terms of these coefficients, and factorizing it symbolically in `Maple`, it is seen to be the product of the quartic parametric speed

$$\sigma(\xi) = \sum_{i=0}^{4} \sigma_i \binom{4}{i} (1-\xi)^{4-i}\xi^i$$

with coefficients given [8] by

$$\sigma_0 = 1, \;\; \sigma_1 = u_1, \;\; \sigma_2 = u_1^2 + v_1^2 + \tfrac{1}{3}(p_1^2 + q_1^2), \;\; \sigma_3 = u_1|\mathcal{A}_1|^2 + v_1\gamma, \;\; \sigma_4 = |\mathcal{A}_1|^4 + \gamma^2,$$

and the quadratic polynomial

$$h(\xi) = 48(p_1^2 + q_1^2)\,[\,2\,v_1(1-\xi)^2 + \gamma\,2(1-\xi)\xi + 2(\gamma\,u_1 - |\mathcal{A}_1|^2 v_1)\,\xi^2\,]. \tag{20}$$

To express these results in terms of general coefficients, we note first that

$$\mathbf{r}'(\xi) \cdot (\mathbf{r}''(\xi) \times \mathbf{r}'''(\xi)) = -\,\mathrm{scal}(\mathbf{r}'(\xi)\,\mathbf{r}''(\xi)\,\mathbf{r}'''(\xi)). \tag{21}$$

On multiplying $\mathcal{A}(\xi)$ with $\mathcal{A}_0$ to restore $\mathbf{r}'(\xi)$ to its original configuration, $\mathbf{r}'(\xi), \mathbf{r}''(\xi), \mathbf{r}'''(\xi)$ become $\mathcal{A}_0\,\mathbf{r}'(\xi)\mathcal{A}_0^*$, $\mathcal{A}_0\,\mathbf{r}''(\xi)\mathcal{A}_0^*$, $\mathcal{A}_0\,\mathbf{r}'''(\xi)\mathcal{A}_0^*$. Thus, the quaternion product $\mathbf{r}'(\xi)\,\mathbf{r}''(\xi)\,\mathbf{r}'''(\xi)$ in (21) becomes $|\mathcal{A}_0|^4\mathcal{A}_0(\mathbf{r}'(\xi)\,\mathbf{r}''(\xi)\,\mathbf{r}'''(\xi))\mathcal{A}_0^*$, and since $\mathrm{scal}(\mathcal{A}_0\mathcal{Q}\mathcal{A}_0^*) = |\mathcal{A}_0|^2\,\mathrm{scal}(\mathcal{Q})$ for any quaternion $\mathcal{Q}$, we see that $\mathbf{r}'(\xi)\cdot(\mathbf{r}''(\xi)\times\mathbf{r}'''(\xi))$ is mapped to $|\mathcal{A}_0|^6\,\mathbf{r}'(\xi)\cdot(\mathbf{r}''(\xi)\times\mathbf{r}'''(\xi))$. Hence, since $\sigma(\xi)$ becomes $|\mathcal{A}_0|^2\,\sigma(\xi)$, the polynomial $h(\xi)$ becomes $|\mathcal{A}_0|^4\,h(\xi)$. ∎

Note that the factorization (19) of $\mathbf{r}'(\xi)\cdot(\mathbf{r}''(\xi)\times\mathbf{r}'''(\xi))$ is specific to the quintic RRMF curves satisfying the condition (8) — it does not hold for general quintic PH space curves.

Remark 4 From equations (17)–(18) and Proposition 1, we observe that the torsion of a quintic RRMF curve has the simple form

$$\tau(\xi) = \frac{\mathbf{r}'(\xi)\cdot(\mathbf{r}''(\xi)\times\mathbf{r}'''(\xi))}{|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|^2} = \frac{h(\xi)}{4\,\sigma(\xi)\rho(\xi)},$$

where $h(\xi)$ is quadratic, and $\sigma(\xi)$ and $\rho(\xi)$ are quartic. Furthermore, these curves have the non–constant curvature/torsion ratio

$$\frac{\kappa(\xi)}{\tau(\xi)} = \frac{8\,\rho^{3/2}(\xi)}{\sigma(\xi)\,h(\xi)},$$

and consequently they are not, in general, helical curves [14]. From (20) we observe that the canonical–form RRMF quintic degenerates to a planar curve[2] if either $p_1 = q_1 = 0$ or $v_1 = \gamma = 0$. In both cases, the coefficients $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are linearly dependent.

Example 1 Consider Example 8.2 in [11], for which

$$\mathcal{A}(\xi) = (7\,\xi^2 - 22\,\xi + 10) + (-19\,\xi^2 + 14\,\xi)\,\mathbf{i} + (-26\,\xi^2 + 16\,\xi)\,\mathbf{j} + (-2\,\xi^2 + 12\,\xi)\,\mathbf{k}.$$

The resulting polynomial PH curve

$$\mathbf{r}(\xi) = \begin{pmatrix} -\,54\,\xi^5 + 10\,\xi^4 + 140\,\xi^3 - 220\,\xi^2 + 100\,\xi \\ 192\,\xi^5 - 270\,\xi^4 - 40\,\xi^3 + 120\,\xi^2 \\ 88\,\xi^5 - 470\,\xi^4 + 520\,\xi^3 - 160\,\xi^2 \end{pmatrix}$$

admits an RMF, obtained from the Euler–Rodrigues frame via the normal–plane rotation (6) with $a(\xi) = 27\,\xi^2 - 22\,\xi + 10$ and $b(\xi) = -19\,\xi^2 + 14\,\xi$. The parametric speed of $\mathbf{r}(\xi)$ is

$$\sigma(\xi) = 10\,(109\,\xi^4 - 172\,\xi^3 + 122\,\xi^2 - 44\,\xi + 10).$$

Now the non–unit binormal vector (11) becomes

$$\mathbf{u}(\xi) = 400\begin{pmatrix} 3324\,\xi^6 - 7752\,\xi^5 + 7872\,\xi^4 - 3984\,\xi^3 + 840\,\xi^2 \\ 1225\,\xi^6 - 3030\,\xi^5 + 4230\,\xi^4 - 4640\,\xi^3 + 2790\,\xi^2 - 780\,\xi + 80 \\ -\,633\,\xi^6 + 1854\,\xi^5 - 3804\,\xi^4 + 3288\,\xi^3 - 930\,\xi^2 - 60\,\xi + 60 \end{pmatrix},$$

[2] Every planar PH curve is trivially an RRMF curve.

and expression (16) yields

$$\Delta(\xi) = [\, 960000(2\,\xi^2 - 19\,\xi + 7)(109\,\xi^4 - 172\,\xi^3 + 122\,\xi^2 - 44\,\xi + 10)\,]^2.$$

The polynomial PH curve $\mathbf{r}(\xi)$ corresponds to the support function

$$f(\xi) = 8000\,\xi^3(436\,\xi^6 - 7245\,\xi^5 + 20424\,\xi^4 - 25802\,\xi^3 + 18132\,\xi^2 - 7470\,\xi + 1400).$$

If we replace $f(\xi)$ by the rational function $\tilde{f}(\xi)$ defined as

$$\frac{320000\,\xi^3(11387\,\xi^6 - 290136\,\xi^5 + 893466\,\xi^4 - 1179172\,\xi^3 + 854730\,\xi^2 - 363000\,\xi + 70000)}{(\xi + 10)^3}$$

we obtain from (13) the rational quintic curve

$$\tilde{\mathbf{r}}(\xi) = \frac{800}{(\xi + 10)^5} \begin{pmatrix} -13420\,\xi^5 + 4000\,\xi^4 + 30000\,\xi^3 - 50000\,\xi^2 + 25000\,\xi \\ 46643\,\xi^5 - 67850\,\xi^4 - 7000\,\xi^3 + 30000\,\xi^2 \\ 19776\,\xi^5 - 111200\,\xi^4 + 126000\,\xi^3 - 40000\,\xi^2 \end{pmatrix},$$

which results[3] from cancelling a factor proportional to $\Delta(\xi)$, of degree 12, common to the numerator and denominator of the expression (13). The parametric speed $\tilde{\sigma}(\xi) = \|\tilde{\mathbf{r}}'(\xi)\|$ of $\tilde{\mathbf{r}}(\xi)$ is related to the parametric speed $\sigma(\xi)$ of the RRMF curve $\mathbf{r}(\xi)$ by

$$\tilde{\sigma}(\xi) = \frac{2000000}{(\xi + 10)^6}\,\sigma(\xi).$$

The curves $\mathbf{r}(\xi)$ and $\tilde{\mathbf{r}}(\xi)$ have the same tangent indicatrix and the same rational RMF, as shown in Figure 1. In this example, the polynomial (18) proves to be just a multiple of the parametric speed, $\rho(\xi) = 400\,\sigma(\xi)$, and consequently $\|\mathbf{u}(\xi)\| = 40\,[\,\sigma(\xi)\,]^{3/2}$.

4 Degree reduction of rational curves

When $\deg(\mathcal{A}) = m$, we have $\deg(\mathbf{r}') = 2m$ and hence $\deg(\mathbf{u}) = 4m-2$, $\deg(\mathbf{u}\times\mathbf{u}') = 8m-6$, $\deg(\mathbf{u}'' \times \mathbf{u}) = 8m-7$, $\deg(\mathbf{u}' \times \mathbf{u}'') = 8m-8$, and $\deg(\det[\,\mathbf{r}'(\xi), \mathbf{r}''(\xi), \mathbf{r}'''(\xi)\,]) = 6(m-1)$ due to a cancellation of leading terms. Hence, if we choose a polynomial support function $f(\xi)$ of degree l, the numerator and denominator of the curve (13) are of degree $l+8(m-1)$ and $12(m-1)$ in the absence of any common factors. Since the simplest non–trivial RRMF curves correspond [8] to the case $m = 2$, the minimum degrees of the numerator and the denominator are $l + 8$ and 12, if they are relatively prime. These degrees are even higher if $f(\xi)$ is a rational function with a degree l numerator and a non–constant denominator. However, as the rational quintic constructed in Example 1 shows, curves $\tilde{\mathbf{r}}(\xi)$ of much lower

[3] Note also that $\tilde{\mathbf{r}}(\xi)$ has denominator $\tilde{W}(\xi) = (\xi+10)^5$, rather than $(\xi+10)^9$ as indicated by Remark 1, due to a cancellation of factors common to the numerators and denominators of $f'(\xi)$ and $f''(\xi)$.

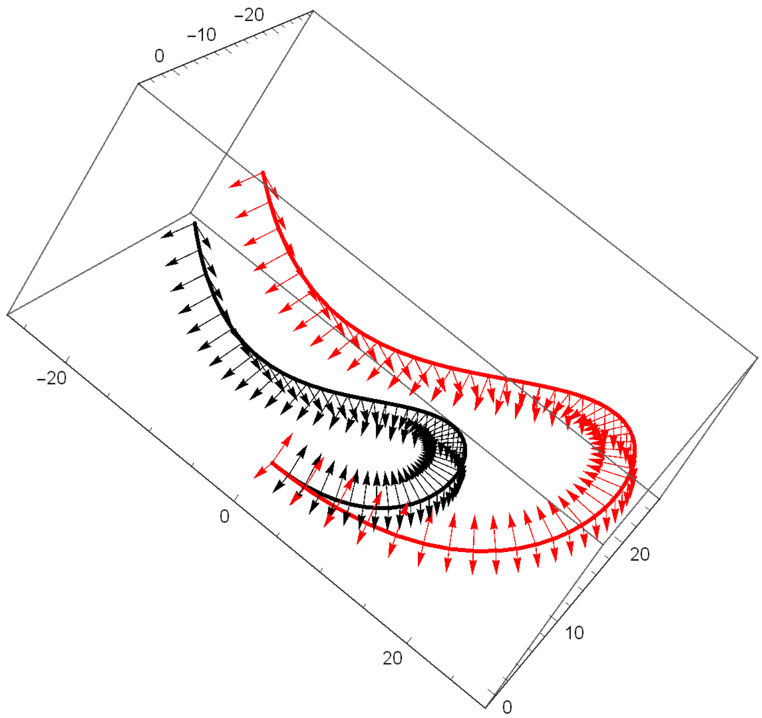

Figure 1: Polynomial PH curve $\mathbf{r}(\xi)$ (black) showing the normal–plane vectors of its rational RMF, and one of the associated rational PH curves $\tilde{\mathbf{r}}(\xi)$ (red) that has the same RMF.

degree are possible, through cancellation of massive factors common to the numerator and denominator in (13). We now investigate in greater detail how this can occur.

Due to the general nature of the function $f(\xi)$, a comprehensive study of when such cancellations occur is a non–trivial task, which we defer to a separate study. At present, we focus on some key observations. First, as noted in Example 1, for a rational function $f(\xi) = c(\xi)/d(\xi)$ the numerators of $f'(\xi)$, $f''(\xi)$ may have factors in common with $d(\xi)$, resulting in cancellations between the numerator of (13) and its denominator $\tilde{W}(\xi) = d^3(\xi)\Delta(\xi)$.

Further cancellations can occur if the numerator of (13) has factors in common with $\Delta(\xi)$. For example, $\Delta(\xi)$ is itself a common factor when $f(\xi) = g(\xi)\Delta^3(\xi)$ for some rational function $g(\xi)$ (moreover, $\tilde{\mathbf{r}}(\xi)$ is a *polynomial* curve if $g(\xi)$ is a polynomial). However, this condition is not necessary for a cancellation to occur, and one might try to identify explicit conditions on the coefficients of $f(\xi)$ such that each coordinate component of the numerator of (13) is divisible by $\Delta(\xi)$. By substituting (15) and (16) into (13), we obtain

$$\tilde{\mathbf{r}} = \frac{\det[\,\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\,](f\mathbf{r}''' - f'\mathbf{r}'' + f''\mathbf{r}') + (f\det[\,\mathbf{r}', \mathbf{r}''', \mathbf{r}''''\,] - f'\det[\,\mathbf{r}', \mathbf{r}'', \mathbf{r}''''\,])\,\mathbf{r}'}{(\det[\,\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\,])^2}, \tag{22}$$

from which it is apparent that $\Delta = (\det[\,\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\,])^2$ cancels out completely between the numerator and denominator of $\tilde{\mathbf{r}}(\xi)$ if $f(\xi) = g(\xi)\,(\det[\,\mathbf{r}'(\xi), \mathbf{r}''(\xi), \mathbf{r}'''(\xi)\,])^3$. This condition is also not necessary for a cancellation, and is not satisfied in Example 1.

Inspecting expression (22) more closely, it transpires that a cancellation of one instance of $\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ between the numerator and denominator of $\tilde{\mathbf{r}}$ occurs when $\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ divides the numerator of $f \det[\mathbf{r}', \mathbf{r}''', \mathbf{r}'''']-f' \det[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''']$. For a rational function $f(\xi) = c(\xi)/d(\xi)$ this implies that

$$c\,d \det[\mathbf{r}', \mathbf{r}''', \mathbf{r}''''] - (c'd - c\,d') \det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''''] \;=\; p \det[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] \tag{23}$$

for some polynomial $p(\xi)$. Now if the denominator $d(\xi)$ is fixed, the condition (23) imposes a system of linear constraints on the unknown coefficients of $c(\xi)$ and $p(\xi)$. Since the left– and right–hand sides of (23) depend linearly on $c(\xi)$ and $p(\xi)$, we may assume that $p(\xi)$ is monic. The constraints are satisfied by both $f(\xi)$ and $\tilde{f}(\xi)$ in Example 1, for which

$$\begin{aligned}
\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''] &= 960000\,(2\,\xi^2 - 19\,\xi + 7)(109\,\xi^4 - 172\,\xi^3 + 122\,\xi^2 - 44\,\xi + 10), \\
\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''''] &= 2880000\,(436\,\xi^5 - 4025\,\xi^4 + 5700\,\xi^3 - 3610\,\xi^2 + 1140\,\xi - 166), \\
\det[\mathbf{r}', \mathbf{r}''', \mathbf{r}''''] &= 11520000\,(327\,\xi^4 - 2415\,\xi^3 + 2573\,\xi^2 - 1159\,\xi + 199).
\end{aligned}$$

For the curve $\mathbf{r}(\xi)$ in Example 1, the choice for $\tilde{f}(\xi)$ actually results in the cancellation of *both* instances of $\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$. In fact, with this curve, other choices for $\tilde{f}(\xi)$ were also found to incur cancellation of both instances. Consider, for example, the choice

$$\tilde{f}(\xi) = \frac{\xi^3\,(a_6\xi^6 + a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0)}{\xi^2 + \xi + 1}. \tag{24}$$

Here the factor ξ^3 ensures that $\tilde{\mathbf{r}}(0) = (0, 0, 0)$, and a simple polynomial without real roots is chosen as the denominator. Hence, since the left–hand side of (23) is of degree 15, and $\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ is degree 6, $p(\xi)$ must be degree 9 to permit satisfaction of (23). This results in a system of 16 linear equations in the unknown coefficients $a_0, \ldots, a_6$ in the numerator of $\tilde{f}(\xi)$ and $p_0, \ldots, p_8$ of $p(\xi)$, which may be solved exactly in rational arithmetic. Using the resulting coefficients $a_0, \ldots, a_6$ in $\tilde{f}(\xi)$ and substituting into (13), we obtain the degree 9 rational curve $\tilde{\mathbf{r}}(\xi)$ defined[4] by the homogeneous coordinates

$$\begin{aligned}
\tilde{W}(\xi) &= (\xi^2 + \xi + 1)^3, \\
\tilde{X}(\xi) &= -7.25\,\xi^9 - 60.31\,\xi^8 - 200.38\,\xi^7 - 215.06\,\xi^6 + 250.85\,\xi^5 \\
&\quad + 375.60\,\xi^4 - 120.20\,\xi^3 + 122.47\,\xi^2 + 446.42\,\xi, \\
\tilde{Y}(\xi) &= 25.78\,\xi^9 + 176.68\,\xi^8 + 430.73\,\xi^7 + 232.28\,\xi^6 - 1289.38\,\xi^5 \\
&\quad - 343.45\,\xi^4 + 1053.10\,\xi^3 + 535.71\,\xi^2, \\
\tilde{Z}(\xi) &= 11.81\,\xi^9 + 25.17\,\xi^8 - 251.99\,\xi^7 - 577.96\,\xi^6 - 1051.13\,\xi^5 \\
&\quad + 1359.91\,\xi^4 + 679.19\,\xi^3 - 714.28\,\xi^2.
\end{aligned} \tag{25}$$

This curve, which has the same rational RMF as the quintic PH curve $\mathbf{r}(\xi)$ introduced in Example 1, is shown in Figure 2. Note that both instances of $\det[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ again cancel between the numerator and denominator of $\tilde{\mathbf{r}}(\xi)$ to yield the curve (25).

[4]The coefficients of $\tilde{\mathbf{r}}(\xi)$ are obtained exactly as rational numbers, but for brevity we present them here as floating–point values with two decimal digits.

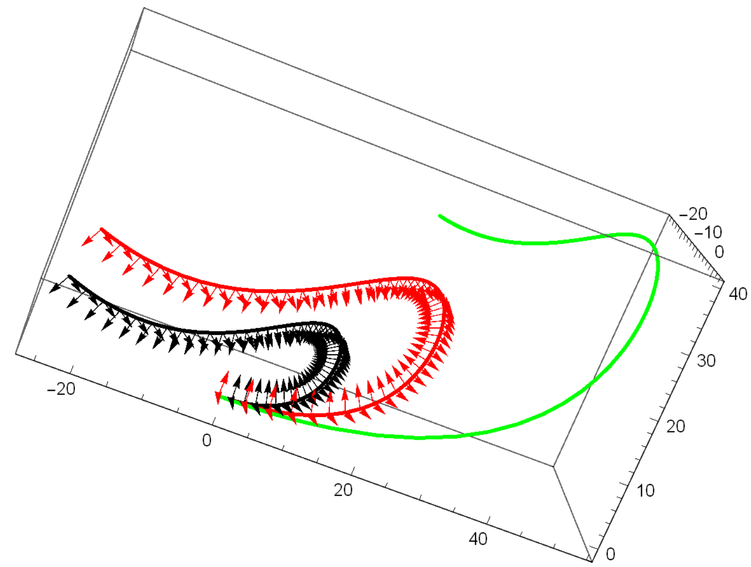

Figure 2: The quintic RRMF curve $\mathbf{r}(\xi)$ employed in Example 1 (black) with its rotation–minimizing frame, and two rational curves that have the same RMF, as defined by $\tilde{f}(\xi)$ as in Example 1 (red) and the alternative form (24) for $\tilde{f}(\xi)$ (green).

5 Interpolation of G^1 Hermite data

The construction of quintic RRMF curves interpolating G^1 Hermite data — i.e., initial/final points $\mathbf{p}_i$ and $\mathbf{p}_f$ and adapted frames $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$ — was addressed in [13]. This problem may be divided into four phases: (1) interpolation of the tangents $\mathbf{t}_i$ and $\mathbf{t}_f$; (2) satisfaction of the constraint identifying the RRMF quintics as a subset of the spatial PH quintics; (3) interpolation of the normal–plane vectors $(\mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{u}_f, \mathbf{v}_f)$; and (4) interpolation of the end–point displacement $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$. The first three admit simple closed–form solutions, and the main difficulty arises in phase (4) — which is algebraically more complicated, and in fact it is known [13] that solutions do not exist for all instances of the given Hermite data. The approach developed herein allows one to circumvent this difficulty, by exploiting the linear dependence of the form (13) on the function $f(\xi)$.

To accomplish this, we construct the hodograph (2) in accordance with the first three phases above, as described in [13]. This hodograph uniquely determines the vector $\mathbf{u}(\xi)$ specified by (11). Without loss of generality we assume $\mathbf{p}_i = (0, 0, 0)$ and $\mathbf{p}_f = (x_f, y_f, z_f)$, and choose the function $f(\xi)$ in (13) such that $\tilde{\mathbf{r}}(0) = \mathbf{p}_i$ and $\tilde{\mathbf{r}}(1) = \mathbf{p}_f$. The simplest $f(\xi)$ satisfying these conditions is a quintic polynomial, which we write in Bernstein form as

$$f(\xi) = \sum_{i=0}^{5} f_i \binom{5}{i} (1-\xi)^{5-i} \xi^i. \tag{26}$$

The choice $f_0 = f_1 = f_2 = 0$ yields $f(0) = f'(0) = f''(0) = 0$, and hence $\tilde{\mathbf{r}}(0) = (0, 0, 0)$. On the other hand, the condition $\tilde{\mathbf{r}}(1) = \mathbf{p}_f$ yields the vector equation

$$\mathbf{u}'(1) \times \mathbf{u}''(1)\, f_5 + \mathbf{u}''(1) \times \mathbf{u}(1)\, 5(f_5 - f_4) + \mathbf{u}(1) \times \mathbf{u}'(1)\, 20(f_5 - 2\, f_4 + f_3) = \Delta(1)\, \mathbf{p}_f\,. \tag{27}$$

Considered as a linear system for f_3, f_4, f_5 it has a non–singular matrix, and thus a unique solution for any choice of $\mathbf{p}_f = (x_f, y_f, z_f)$, when

$$[\,(\mathbf{u}'(1) \times \mathbf{u}''(1)) \times (\mathbf{u}''(1) \times \mathbf{u}(1))\,] \cdot (\mathbf{u}(1) \times \mathbf{u}'(1)) \neq 0.$$

This condition can be simplified to

$$\Delta(1) = \mathbf{u}(1) \cdot (\mathbf{u}'(1) \times \mathbf{u}''(1)) \neq 0,$$

i.e., $\mathbf{u}(1), \mathbf{u}'(1), \mathbf{u}''(1)$ must be linearly independent. However, when $\Delta(1) = 0$, the matrix is singular and there are infinitely–many solutions for f_3, f_4, f_5.

Remark 5 For the polynomial PH curve $\mathbf{r}(\xi)$, the support function $f(\xi)$ is determined by substituting $\mathbf{p} = \mathbf{r}(\xi)$ in (12). Using `Maple`, we observe that the degree of $f(\xi)$ is 9 for a quintic PH curve. Consequently, the family of rational curves constructed above does not incorporate $\mathbf{r}(\xi)$ as a special instance when the support function (26) of degree 5 is used. However, by using a polynomial of degree ≥ 9 for $f(\xi)$, it should be possible to include the quintic PH curve $\mathbf{r}(\xi)$ as a special instance of the family of constructed rational curves.

6 Computed examples

We now illustrate the methodology developed above by some computed examples.

Example 2 In Example 3 of [13], it was observed that no RRMF quintic interpolant $\mathbf{r}(\xi)$ exists for the following data:

$$\mathbf{p}_i = (0, 0, 0)\,, \quad \mathbf{t}_i = \tfrac{1}{2}\,(1, 0, \sqrt{3})\,, \quad \mathbf{u}_i = (0, 1, 0)\,, \quad \mathbf{v}_i = \tfrac{1}{2}\,(-\sqrt{3}, 0, 1)\,,$$

$$\mathbf{p}_f = (1, 0, 0)\,, \quad \mathbf{t}_f = \tfrac{1}{2}\,(1, -\sqrt{2}, 1)\,, \quad \mathbf{u}_f = \tfrac{1}{2}\,(\sqrt{2}, 0, -\sqrt{2})\,, \quad \mathbf{v}_f = \tfrac{1}{2}\,(1, \sqrt{2}, 1)\,.$$

Using the formulation (13), however, one can easily construct interpolants with rational RMFs. With $m = 2$ in (3), matching the end frames determines the coefficients $\mathcal{A}_0$, $\mathcal{A}_2$ as

$$\mathcal{A}_0 = \frac{\ell_0}{2}(\sqrt{3} - \mathbf{j})\,, \quad \mathcal{A}_2 = \frac{\ell_2}{2}(-\sqrt{2} + \mathbf{i} + \mathbf{k})\,,$$

where ℓ_0, ℓ_2 are non–zero free parameters. Moreover, satisfaction of the condition

$$\mathcal{A}_1\, \mathbf{i}\, \mathcal{A}_1^* = \mathrm{vect}(\mathcal{A}_2\, \mathbf{i}\, \mathcal{A}_0^*) \tag{28}$$

that characterizes [8] the quintic RRMF curves determines the coefficient $\mathcal{A}_1$ as

$$\mathcal{A}_1 = \sqrt{a}\, \frac{a\,\mathbf{i} + \mathbf{a}}{\|\, a\,\mathbf{i} + \mathbf{a}\,\|}\, \exp(\phi_1 \mathbf{i}) \,,$$

where we set $\mathbf{a} = \mathrm{vect}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*)$, $a = \|\mathbf{a}\|$, and ϕ_1 is a free parameter. For any non–zero ℓ_0, ℓ_2 and any ϕ_1, the hodograph (2) constructed from the quadratic quaternion polynomial $\mathcal{A}(\xi)$ with Bernstein coefficients $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ satisfies the RRMF condition (28) and interpolates the end frames, but not the end points. Now $\mathbf{r}(0) = \mathbf{p}_i$ is easily achieved by the choice of integration constant, and the main difficulty arises in achieving $\mathbf{r}(1) = \mathbf{p}_f$.

However, using the polynomial $\mathcal{A}(\xi)$ — for any prescribed ℓ_0, ℓ_2, ϕ_1 — to construct the hodograph (2) and corresponding binormal vector field (11), interpolating $\mathbf{p}_i$ and $\mathbf{p}_f$ is easily achieved with (13) by selecting the coefficients of the polynomial (26). For example, taking $\ell_0 = \ell_2 = 1$ and $\phi_1 = 0$, we obtain $\Delta(1) = 1.13857672$ and

$$\begin{aligned}
\mathbf{u}'(1) \times \mathbf{u}''(1) &= (-165.25446779, 278.02677830, -180.38261288)\,, \\
\mathbf{u}''(1) \times \mathbf{u}(1) &= (25.79232761, -\,42.40519904, 27.76134401)\,, \\
\mathbf{u}(1) \times \mathbf{u}'(1) &= (-\,0.53352055, 0.75451200, -\,0.53352055)\,,
\end{aligned}$$

and hence we obtain the solution

$$f_3 = 0.21670731\,, \quad f_4 = -\,0.67509971\,, \quad f_5 = 1.4735748\,.$$

In conjunction with the coefficients $f_0 = f_1 = f_2 = 0$, the quintic polynomial (26) ensures that curve (13) satisfies the end–point interpolation conditions $\tilde{\mathbf{r}}(0) = \mathbf{p}_i$ and $\tilde{\mathbf{r}}(1) = \mathbf{p}_f$.

In the preceding example, arbitrary values were assigned to the free parameters ℓ_0, ℓ_2, ϕ_1. The availability of these free parameters arises from relaxing the condition $\mathbf{r}(1) = \mathbf{p}_f$, and shifting the burden of interpolating this end point to the function $f(\xi)$ in (13). In practice, these free parameters can be exploited to optimize shape properties of the interpolant.

In fact, with $\ell_0 = \ell_2 = 1$ and $\phi_1 = 0$, the interpolant $\tilde{\mathbf{r}}(\xi)$ contains a point at infinity that corresponds (see Lemma 2) to a point of zero torsion on the quintic PH curve $\mathbf{r}(\xi)$. On keeping $\ell_0 = \ell_2 = 1$ and choosing $\phi_1 = 4\pi/5$, however, $\mathbf{r}(\xi)$ has no points of vanishing torsion, and the resulting rational curve $\tilde{\mathbf{r}}(\xi)$ — with the new coefficients $f_3 = -0.125843$, $f_4 = 0.143231$, $f_5 = -2.61809$ — is guaranteed to be finite, as seen in Figure 3.

It is apparent in Figure 3 that the rational curve $\tilde{\mathbf{r}}(\xi)$ has a cusp, corresponding to a parameter value ξ_c such that $\tilde{\mathbf{r}}'(\xi_c) = \mathbf{0}$. Unlike the points at infinity, the rather complicated closed–form expressions (13) and (22) for $\tilde{\mathbf{r}}(\xi)$ make an *a priori* verification that it is free of cusps a non–trivial task. We defer a detailed investigation of this issue to a future study.

Remark 6 Although the form (13) greatly simplifies the end–point interpolation problem, and ensures the existence of interpolants, it should be noted that for any given hodograph $\mathbf{r}'(\xi)$ satisfying the RRMF condition, the rational interpolants are typically of higher degree and do not (in general) have rational arc lengths.

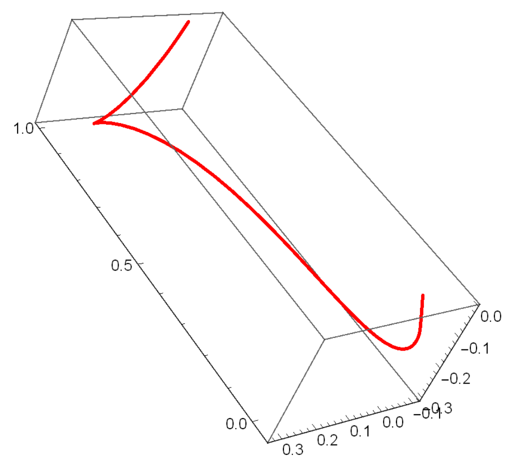

Figure 3: The finite rational interpolant in Example 2, using $\ell_0 = \ell_2 = 1$ and $\phi_1 = 4\pi/5$.

Example 3 Consider the quintic PH curve specified by the quaternion coefficients

$$\mathcal{A}_0 = \mathbf{k}, \quad \mathcal{A}_1 = 1 - \mathbf{j} + \mathbf{k}, \quad \mathcal{A}_2 = 2 + 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$$

in (3), which satisfy the RRMF condition (8). Choosing $\mathbf{r}(0) = (0,0,0)$ the corresponding quadratic polynomial

$$\mathcal{A}(\xi) = 2\,\xi + 2\,\xi^2\,\mathbf{i} + (4\,\xi^2 - 2\,\xi)\,\mathbf{j} + \mathbf{k}$$

generates the polynomial PH curve

$$\mathbf{r}(\xi) = \frac{(-12\,\xi^5 + 20\,\xi^4 - 5\,\xi, 16\,\xi^5 - 10\,\xi^4 + 10\,\xi^2, -20\,\xi^4 + 20\,\xi^3)}{5},$$

with parametric speed

$$\sigma(\xi) = 20\,\xi^4 - 16\,\xi^3 + 8\,\xi^2 + 1.$$

The rational RMF on $\mathbf{r}(\xi)$ is obtained from the ERF through the normal–plane rotation (6) with $a(\xi) = 2\,\xi^2 + 1$ and $b(\xi) = 4\,\xi^2 - 2\,\xi$. For this curve, the non–unit binormal vector (11) becomes

$$\mathbf{u}(\xi) = \begin{pmatrix} 256\,\xi^6 - 384\,\xi^5 + 96\,\xi^4 - 128\,\xi^3 + 48\,\xi^2 \\ 192\,\xi^6 - 288\,\xi^5 + 192\,\xi^4 - 48\,\xi^2 + 24\,\xi \\ 160\,\xi^6 + 144\,\xi^4 - 192\,\xi^3 + 24\,\xi^2 - 4 \end{pmatrix},$$

and the expression (16) yields

$$\Delta(\xi) = [\,\sigma(\xi)h(\xi)\,]^2, \quad h(\xi) = 96\,(2\,\xi^2 + 4\,\xi - 1).$$

The polynomial PH curve $\mathbf{r}(\xi)$ corresponds to the support function

$$f(\xi) = \frac{16}{5}\,\xi^3(20\,\xi^6 + 36\,\xi^5 - 84\,\xi^4 + 84\,\xi^3 - 15\,\xi^2 + 15\,\xi - 5).$$

To satisfy specified end points $\mathbf{p}_i = (0,0,0)$, $\mathbf{p}_f = (1,1,1)$ we replace this with a function $\tilde{f}(\xi)$ of the form (26) with initial coefficients $f_0 = f_1 = f_2 = 0$ and final coefficients f_3, f_4, f_5 satisfying equation (27), which are determined to be

$$f_3 = \frac{232}{5}, \quad f_4 = -\frac{668}{5}, \quad f_5 = 92.$$

The resulting support function is then

$$\tilde{f}(\xi) = 1224\,\xi^5 - 1596\,\xi^4 + 464\,\xi^3.$$

On substituting $\mathbf{u}(\xi)$ and $\tilde{f}(\xi)$ into (13), the numerator and denominator of the resulting curve $\tilde{\mathbf{r}}(\xi)$ are seen to be both of degree 12. In this case, no cancellation of factors common to all of $\tilde{W}, \tilde{X}, \tilde{Y}, \tilde{Z}$ occurs.

Example 4 Consider the following data, which are symmetrical relative to the (y,z) plane (taking into account the inward orientation of the tangent vector at the initial point and outward direction at the final point):

$$\mathbf{p}_i = (13,14,15)\,, \quad \mathbf{t}_i = \tfrac{1}{3}\,(1,2,-2)\,, \quad \mathbf{u}_i = \tfrac{1}{3}(-2,2,1)\,, \quad \mathbf{v}_i = \tfrac{1}{3}\,(2,1,2)\,,$$

$$\mathbf{p}_f = (-13,14,15)\,, \quad \mathbf{t}_f = \tfrac{1}{3}\,(1,-2,2)\,, \quad \mathbf{u}_f = \tfrac{1}{3}\,(2,2,1)\,, \quad \mathbf{v}_f = \tfrac{1}{3}\,(-2,1,2)\,.$$

As in Example 2, with $m = 2$ in (3), matching the end frames yields the coefficients

$$\mathcal{A}_0 = \frac{\ell_0}{\sqrt{3}}(1 + \mathbf{i} + \mathbf{j})\,, \quad \mathcal{A}_2 = \frac{\ell_2}{\sqrt{3}}(1 + \mathbf{i} - \mathbf{j})\,,$$

where ℓ_0, ℓ_2 are non–zero free parameters. For simplicity, we set $\ell_0 = \ell_2 = \sqrt{3}$ and also choose the simple value

$$\mathcal{A}_1 = \sqrt{3}\,\mathbf{i}$$

(corresponding to $\phi_1 = 0$) so the condition (28) for a rational RMF is satisfied. The RMF is obtained from the ERF through the normal–plane rotation (6) with $a(\xi) = 2\,\xi^2 - 2\,\xi + 1$ and $b(\xi) = [\,(2\sqrt{3}-4)\,\xi^2 - (2\sqrt{3}-4)\,\xi + \sqrt{3}\,]/\sqrt{3}$. The symmetry of the data automatically induces symmetry of the hodograph $\mathbf{r}'(\xi)$ and the associated RMF. To ensure symmetry of the rational curve $\tilde{\mathbf{r}}(\xi)$, a symmetrical function $f(\xi)$ must be employed, satisfying $f_0 = f_5$, $f_1 = f_4$ and $f_2 = f_3$ in (26). The linear end–point constraints then yield the values

$$f_0 = f_5 = -68\sqrt{3}, \quad f_1 = f_4 = \frac{4}{5}(-522 + 17\sqrt{3}), \quad f_2 = f_3 = \frac{4}{5}(261 - 292\sqrt{3}).$$

The resulting rational curve, illustrated in Figure 4, is free of points at infinity and cusps, and is symmetrical relative to the (y,z) plane. In this case the numerator and denominator of $\tilde{\mathbf{r}}(\xi)$ are both of degree 10, after cancellation of a common factor $(\xi - \frac{1}{2})^2$ resulting from the symmetry of the input data.

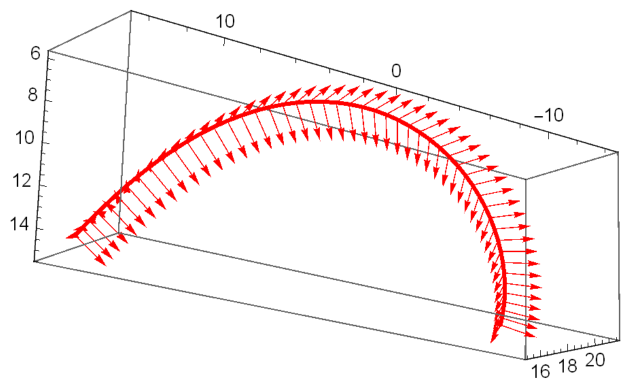

Figure 4: The rational RMF interpolant to the symmetrical data in Example 4.

7 Closure

The simplest non–planar curves that admit rational RMFs comprise subsets of the PH curves of degree 5 and 7, characterized by algebraic constraints on the coefficients of their quaternion pre–image polynomials. However, the construction of such RRMF curves so as to match given end points entails the solution of non–linear equations, incurring questions about the existence of solutions for all possible data. The methodology presented herein circumvents this difficulty, by formulating a means to map the frames of RRMF curves on to rational space curves, whereby the problem of matching end points is linearized.

From a given RRMF curve $\mathbf{r}(\xi)$, a rational space curve $\tilde{\mathbf{r}}(\xi)$ is constructed as the edge of regression of the envelope of a family of osculating planes with normals in the direction of $\mathbf{u}(\xi) = \mathbf{r}'(\xi) \times \mathbf{r}''(\xi)$ and distances from the origin specified in terms of a rational function $f(\xi)$ as $f(\xi)/\|\mathbf{u}(\xi)\|$. A simple closed–form expression for $\tilde{\mathbf{r}}(\xi)$ in terms of $\mathbf{u}(\xi)$ and $f(\xi)$, and their first and second derivatives, was formulated, and $\tilde{\mathbf{r}}(\xi)$ inherits exactly the same rational RMF as $\mathbf{r}(\xi)$ at corresponding points. The methodology was illustrated by computed examples, which also highlight the remarkable simplicity with which the curve $\tilde{\mathbf{r}}(\xi)$ can solve the end–point matching problem, compared to the RRMF curve $\mathbf{r}(\xi)$.

The focus of this study was to elucidate the basic principles underlying the construction of the curves $\tilde{\mathbf{r}}(\xi)$, and to explore their capabilities. Several lines of inquiry deserve further investigation. From a practical viewpoint, it is desirable to more fully develop algorithms to exploit these curves in applications. On the theoretical level, a comprehensive study of the circumstances that result in low–degree curves $\tilde{\mathbf{r}}(\xi)$, by cancellation of factors common to their homogeneous coordinates, will help gain acceptance of them in practical use.

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