Accepted Manuscript

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PII: S0377-0427(16)30443-5
DOI: http://dx.doi.org/10.1016/j.cam.2016.09.023
Reference: CAM 10811

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 20 June 2016

Lie symmetries and equivalence transformations for the Barenblatt-Gilman model

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Abstract

In this paper we have considered the Barenblatt-Gilman equation which models the nonequilibrium countercurrent capillary impregnation. The equation of this model is a third-order equation and the unknown function concerns to the effective water saturation.
We have applied the classical method to get the Lie group classification with respect to unknown function and we have constructed the equivalence transformations. We have also obtained the invariant solutions for some forms of the equation, including travelling wave solutions based on the Jacobi elliptic sine function.

Keywords: Barenblatt-Gilman equation, Lie group analysis, equivalence transformations, travelling wave solutions

1. Introduction

Naturally, the study of partial differential equations plays a vital role in the physical sciences. These equations are often non-linear and solving them requires unique and creative methods. Most well-known techniques have a common feature: they exploit symmetries.

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Symmetry analysis have continuously been in focus of research, it is applicable to both linear and nonlinear differential equations, so it is a powerful and fundamental tool. Nowadays several well-known authors are researching in this field [1], [2], [3], [4], [5], [6], even generalizing the method to study systems of the first order ODEs [7], fractional differential equations [8], [9] and obtaining conservation laws [10], [11]. Along the same line we have applied this method, its theory and detailed description can be found in [12], [13], [14], [15] and so on, to the Barenblatt-Gilman equation.

In broad strokes we can find the Lie algebra of admitted operators for different forms of unknown coefficient function and use certain subalgebras to construct invariant solutions.

To solve the classification problem completely, we have also obtained the equivalence transformations which transform the given equation in another one of the same class, preserving the differential structure. A practical guide for calculation of invariants for families of linear and nonlinear differential equations with special emphasis on the use of infinite equivalence Lie algebras can be find in [16]. Equivalence transformations are playing an important role in equations or systems with arbitrary functions, allowing to select suitable forms of the arbitrary functions. In [17], [18], [19], [20] it is possible to find a description and application of the equivalence transformations, including examples for systems of differential equations and fractional differential equations.

Equation considered in the present paper is based on theory of counterflow capillary impregnation of a porous medium. It has been studied extensively due to its applications in various fields such as soil science, petroleum, crystal growth and flip chip underfilling [21], [22], [23] and [24].

In [25] the physical model of the non-equilibrium effects in a simultaneous flow of two immiscible fluids in porous media is presented. The Barenblatt-Gilman equation is as follows

$$u_t = \alpha \triangle \Phi(u) + \alpha\lambda \left(\triangle\Phi(u)\right)_t \tag{1}$$

where the function Φ is the effective water saturation.

2. Equivalence generators

An equivalence transformation of the equation (1) is a change of variables $(t, x, u) \longrightarrow (\bar{t}, \bar{x}, \bar{u})$ carrying the equation into an equation of the same form. That way, the resultant equation and the original are said to be equivalent and where the function $\bar{\Phi}$ may, in general, be different from the original function Φ.

There are two main methods for calculating the equivalence transformations: the direct search for equivalence transformations and the infinitesimal method suggested by Ovsyannikov [26]. In this paper, we obtained the equivalence transformations by means of the second method.

As before, now we have differentiated two cases. The first with $\alpha \neq 0$ and $\lambda = 0$ and another with α and λ non zero.

Let us consider first the case with $\alpha \neq 0$ and $\lambda = 0$. To start, we have extended the space of variables, adding Φ as a new variable, and then we looked for Y the generator of the continuous group of equivalence transformations that have the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu \frac{\partial}{\partial \Phi} \tag{2}$$

where $\xi^1,\ \xi^2$ and η depend on $x,\ t,\ u$ and μ depends on $x,\ t,\ u,\ \Phi$.

Next, we have written Barenblatt-Gilman equation (1), considering $\lambda = 0$, in the following extended form

$$\begin{cases} u_t = \alpha \Phi_{uu} u_x^2 + \alpha \Phi_u u_{xx} \\ \Phi_t = 0,\ \Phi_x = 0 \end{cases} \tag{3}$$

The prolongation of the operator (2) to all variables involved in equation (3) is given by the usual prolongation procedure and has the form

$$\tilde{Y} = Y + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \omega_1 \frac{\partial}{\partial \Phi_t} + \omega_2 \frac{\partial}{\partial \Phi_x} + \omega_3 \frac{\partial}{\partial \Phi_u} + \omega_{33} \frac{\partial}{\partial \Phi_{uu}}$$

where $\zeta_1,\ \zeta_2$ and ζ_{22} are given by the usual prolongation formula and the other four coefficients $\omega_1,\ \omega_2,\ \omega_3$ and ω_{33} are obtained by applying the secondary prolongation procedure:

$$
\begin{aligned}
\zeta_1\big|_{(3)} &= \eta_t + \eta_u u_t - \xi^1_t u_t - \xi^1_u u_t^2 - \xi^2_t u_x - \xi^2_u u_t u_x;\\
\zeta_2\big|_{(3)} &= \eta_x + \left(\eta_u - \xi^2_x\right) u_x - \xi^1_u u_t u_x - \xi^2_u u_x^2 - \xi^1_x u_t;\\
\zeta_{22}\big|_{(3)} &= \eta_{xx} + \left(2\eta_{xu} - \xi^2_{xx}\right) u_x + \left(\eta_u - 2\xi^2_x\right) u_{xx} + \left(\eta_{uu} - 2\xi^2_{xu}\right) u_x^2\\
&\quad -\xi^1_{xx} u_t - 2\xi^2_x u_{tx} - 2\xi^1_{xu} u_x u_t - \xi^1_u u_{xx} u_t - 2\xi^1_u u_x u_{tx} - \xi^1_u u u_x^2 u_t\\
&\quad -\xi^2_{uu} u_x^2 - 3\xi^2_u u_x u_{xx};\\
\omega_1\big|_{(3)} &= \mu_t - \Phi_u \eta_t;\\
\omega_2\big|_{(3)} &= \mu_x - \Phi_u \eta_x;\\
\omega_3\big|_{(3)} &= \mu_u + \Phi_u \mu_\Phi - \Phi_u \eta_u;\\
\omega_{33}\big|_{(3)} &= \mu_{uu} + \Phi_{uu}\left(\mu_\Phi - \eta_u\right) + \Phi_u\left(2\mu_{\Phi u} - \eta_{uu}\right) + \Phi_u^2 \mu_{\Phi\Phi};
\end{aligned}
$$

Then, taking into account that the infinitesimal invariance test for the system (3) has the form

$$
\begin{cases}
\left(-\zeta_1 + 2\alpha\Phi_{uu}\zeta_2 u_x + \alpha\Phi_u\zeta_{22} + \alpha\omega_3 u_{xx} + \alpha\omega_{33} u_x^2\right)\Big|_{(3)} = 0\\
\omega_1\big|_{(3)} = 0,\ \omega_2\big|_{(3)} = 0
\end{cases}
$$

and substituting here the expressions for $\zeta_1,\ \zeta_2,\ \zeta_{22},\ \omega_1,\ \omega_2,\ \omega_3$ and ω_{33} and splitting the resultant equations we have determined

$$
\begin{aligned}
&\xi^1(t) = C_1 t + C_4\\
&\xi^2(x) = C_2 x + C_5\\
&\eta(u) = \left(C_1 - 2C_2 + C_3\right) u + C_6\\
&\mu(\Phi) = C_3\Phi + C_7
\end{aligned}
$$

where $C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6$ and C_7 are constants.

Finally, based on the above results, the equivalence algebra for the equation (3) is a seven-dimensional Lie algebra spanned by

$$
\begin{aligned}
&Y_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}\\
&Y_2 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}\\
&Y_3 = u\frac{\partial}{\partial u} + \Phi\frac{\partial}{\partial \Phi}\\
&Y_4 = \frac{\partial}{\partial t},\ Y_5 = \frac{\partial}{\partial x},\ Y_6 = \frac{\partial}{\partial u},\ Y_7 = \frac{\partial}{\partial \Phi}
\end{aligned}
$$

On the other hand, let us consider now the case with $\alpha \neq 0$ and $\lambda \neq 0$. The generator of the continuous group of equivalence transformations have the same form (2) as before and the extended form of the Barenblatt-Gilman equation is

$$\left\{\begin{array}{ll} u_t = & \lambda\alpha\Phi_{uuu}u_x^2u_t + 2\lambda\alpha\Phi_{uu}u_xu_{xt} + \lambda\alpha\Phi_{uu}u_tu_{xx} + \lambda\alpha\Phi_u u_{xxt} \\ & \alpha\Phi_{uu}u_x^2 + \alpha\Phi_u u_{xx}, \\ \Phi_t = & 0, \\ \Phi_x = & 0. \end{array}\right. \tag{4}$$

Following the process, we have prolonged the operator (2) to all variables involved in equations (4) and it has the form

$$\begin{aligned} \tilde{Y} = \ & Y + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{21}\frac{\partial}{\partial u_{xt}} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{221}\frac{\partial}{\partial u_{xxt}} + \omega_1\frac{\partial}{\partial \Phi_t} \\ & +\omega_2\frac{\partial}{\partial \Phi_x} + \omega_3\frac{\partial}{\partial \Phi_u} + \omega_{33}\frac{\partial}{\partial \Phi_{uu}} + \omega_{333}\frac{\partial}{\partial \Phi_{uuu}} \end{aligned}$$

whose coefficients were calculated as before, $\zeta_1,\ \zeta_2,\ \zeta_{21},\ \zeta_{22}$ and ζ_{221} are given by the usual prolongation formula and $\omega_1,\ \omega_2,\ \omega_3,\ \omega_{33}$ and ω_{333} are obtained by applying the secondary prolongation procedure.

In such a way that the infinitesimal invariance test for the system (4) has the form

$$\left\{\begin{array}{l} \left.\left(\begin{array}{l} -\zeta_1 + \lambda\alpha\omega_{333}u_x^2u_t + \lambda\alpha\Phi_{uuu}2\zeta_2u_xu_t + \lambda\alpha\Phi_{uuu}\zeta_1u_x^2 \\ +2\lambda\alpha\omega_{33}u_xu_{xt} + 2\lambda\alpha\Phi_{uu}\zeta_2u_{xt} + 2\lambda\alpha\Phi_{uu}\zeta_{21}u_x \\ +\lambda\alpha\omega_{33}u_tu_{xx} + \lambda\alpha\Phi_{uu}\zeta_1u_{xx} + \lambda\alpha\Phi_{uu}\zeta_{22}u_t + \lambda\alpha\omega_3u_{xxt} \\ +\lambda\alpha\Phi_u\zeta_{221} + 2\alpha\Phi_{uu}\zeta_2u_x + \alpha\Phi_u\zeta_{22} + \alpha\omega_3u_{xx} + \alpha\omega_{33}u_x^2 \end{array}\right)\right|_{(3)} = 0 \\ \omega_1\big|_{(3)} = 0,\ \omega_2\big|_{(3)} = 0 \end{array}\right.$$

At the end, substituting the expressions for $\zeta_1,\ \zeta_2,\ \zeta_{21},\ \zeta_{22},\ \zeta_{221},\ \omega_1,\ \omega_2,\ \omega_3,$ ω_{33} and ω_{333} and splitting the equations, we have obtained the equivalence algebra for the equation (4)

$$\begin{aligned} Y_1 &= x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 3\Phi\frac{\partial}{\partial \Phi} \\ Y_2 &= \frac{\partial}{\partial t},\ Y_3 = \frac{\partial}{\partial x},\ Y_4 = \frac{\partial}{\partial u},\ Y_5 = \frac{\partial}{\partial \Phi} \end{aligned}$$

3. Lie symmetry analysis

Lie classical method is specially useful in the study of Barenblatt-Gilman equation (1) due to its arbitrary function because while we are searching for symmetries it will provide a set of special forms for the unknown function Φ where it is possible to choose.

We have started applying it to (1)

$$F(x,t,u,u_t,u_x,\ldots)=0$$

and considering the invariance of the equation under the Lie group transformation with infinitesimal generator of the form

$$V=\xi(x,t,u)\partial x+\varphi(x,t,u)\partial t+\eta(x,t,u)\partial u \tag{5}$$

By Criterion of Invariance we have required that

$$\tilde{V}F=0 \text{ when } F=0$$

where $\tilde{V}=\mathrm{pr}^{(3)}V$ is the third prolongation of the vector field (5). This yields to an overdetermined linear system of 31 equations for the infinitesimals $\xi(x,t,u,v)$, $\varphi(x,t,u,v)$ and $\eta(x,t,u,v)$. The solutions of this system depend on the function Φ and α, λ parameters. Emphasise that with respect to equivalence transformations from section before, only non-equivalent $\Phi(u)$ functions are listed.

We have obtained the following classification of (1) in two ways:

1. For $\alpha,\ \lambda\neq 0$:
   - Case 1: Φ arbitrary function.

     $$V_1=\frac{\partial}{\partial x},\quad V_2=\frac{\partial}{\partial t}$$

   - Case 2: $\Phi=\mathrm{e}^u$

     Infinitesimal generators are V_1,V_2 and

     $$V_3^1=x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$$

- Case 3: $\Phi = \ln u$

  Infinitesimal generators are V_1, V_2 and

$$V_3^2 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$$

- Case 4: $\Phi = u^{\gamma+1}$

  Infinitesimal generators are V_1, V_2 and

$$V_3^3 = x\frac{\partial}{\partial x} - \frac{2}{\gamma}u\frac{\partial}{\partial u}$$

- Case 5: $\Phi = u^{-1/3}$

  Infinitesimal generators are V_1, V_2 and

$$V_3^4 = x\frac{\partial}{\partial x} - \frac{3}{2}u\frac{\partial}{\partial u}, \quad V_4 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$$

- Case 6: $\Phi = u$ Infinitesimal generators are V_1, V_2 and

$$V_3^5 = u\frac{\partial}{\partial u}, \quad V_\chi = \chi(x,t)\frac{\partial}{\partial u},$$

  where $\chi(x,t)$ is an arbitrary solution of the linear equation

$$\alpha\chi_{xx} + (\alpha\lambda\chi_{xx} - \chi)_t = 0.$$

2. For $\lambda = 0$ the equation takes the form $u_t = \alpha\Delta\Phi(u)$

- The cases from 1 to 5 admits the additional infinitesimal generator

$$V_5 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$$

- The case 6, with linear Φ (standard linear diffusion equation), have 3 more admitted operators than with $\lambda \neq 0$:

$$V_4 = xu\frac{\partial}{\partial u} - 2\alpha t\frac{\partial}{\partial x}, \quad V_5 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$$

$$V_6 = (ux^2 + 2\alpha t)\frac{\partial}{\partial u} - 4\alpha tx\frac{\partial}{\partial x} - 4\alpha t^2\frac{\partial}{\partial t}.$$

4. Reductions and exact solutions

In the first section we have obtained the vector fields of the Barenblatt-Gilman equation. In this section, we have investigated the symmetry reductions and exact solutions of the equation (1) considering $\alpha \neq 0$ and $\lambda \neq 0$.

- Case 1: If Φ is an arbitrary function from generator $\omega \mathbf{V}_1 + \mu \mathbf{V}_2$ we have obtained travelling wave reductions

  $$z = \mu x - \omega t, \quad u(x,t) = h(z),$$

  where $h(z)$ satisfies

  $$\begin{aligned} \lambda\alpha \left(h'\right)^3 \mu^2 \omega \Phi_{hhh} + (3\lambda\alpha h' h'' \mu^2 \omega - \alpha \left(h'\right)^2 \mu^2)\Phi_{hh} + \\ (\lambda\alpha h''' \mu^2 \omega - \alpha h'' \mu^2)\Phi_h - h'\omega = 0 \end{aligned} \tag{6}$$

  Let us assume that equation (6) has solution of the form $h = H(z)$, where $H(z)$ is a solution of Jacobi equation

  $$(H')^2 = r + pH^2 + qH^4,$$

  with r, p and q constants.

  Substituting $h = H(z)$ into equation (6) we can obtain an equation in the form

  $$\alpha_1 \Phi_{hhh} + \alpha_2 \Phi_{hh} + \alpha_3 \Phi_h + \alpha_4 = 0$$

  where $\alpha_i = \alpha_i(h)$ with $i = 1, \ldots, 4$, which can be resolved for Φ.

  As a continuation we studied the procedure for $h = \operatorname{sn}(z,k)$ and we have obtained the following results: If

  $$h(z) = \operatorname{sn}(z,k) \tag{7}$$

  is the Jacobi elliptic sine function, by substituting (7) into (6) we have obtained

  $$\begin{aligned} (3\lambda\alpha J_2 J_3 \left(-{J_3}^2 J_1 - {J_2}^2 k^2 J_1\right) \mu^2 \omega - \alpha {J_2}^2 {J_3}^2 \mu^2) F_{hh} \\ + \left(\lambda\alpha \left(4 J_3 {J_1}^2 k^2 J_2 - {J_3}^3 J_2 - {J_2}^3 k^2 J_3\right) \mu^2 \omega \right. \\ \left. - \alpha \left(-{J_3}^2 J_1 - {J_2}^2 k^2 J_1\right) \mu^2\right) F_h + \lambda\alpha {J_2}^3 {J_3}^3 \mu^2 \omega F_{hhh} - J_2 J_3 \omega = 0, \end{aligned}$$

where $J_1 = \mathrm{sn}(z,m)$, $J_2 = \mathrm{cn}(z,m)$ and $J_3 = \mathrm{dn}(z,m)$. Taking into account that $\mathrm{cn}^2(z,k) = 1 - \mathrm{sn}^2(z,k) = 1 - (h)^2$ and $\mathrm{dn}^2(z,k) = 1 - m\mathrm{sn}^2(z,k) = 1 - k\,(h)^2$ (see, e.g. [27]), $F(h)$ must satisfy

$$\alpha_1 F_{hhh} + \alpha_2 F_{hh} + \alpha_3 F_h - \alpha_4 = 0, \tag{8}$$

where

$$\begin{aligned}
\alpha_1 &= \lambda\alpha\left(1-kh^2\right)^3\mu^2\omega, && (9)\\
\alpha_2 &= 3\lambda\alpha\left(1-kh^2\right)\left(-\left(1-kh^2\right)h-\left(1-kh^2\right)k^2h\right)\mu^2\omega && (10)\\
&\quad -\alpha\left(1-kh^2\right)^2\mu^2, &&\\
\alpha_3 &= \lambda\alpha\left(4\left(1-kh^2\right)h^2k^2-\left(1-kh^2\right)^2-\left(1-kh^2\right)^2k^2\right)\mu^2\omega && (11)\\
&\quad -\alpha\left(-\left(1-kh^2\right)h-\left(1-kh^2\right)k^2h\right)\mu^2, &&\\
\alpha_4 &= \left(1-kh^2\right)\omega. && (12)
\end{aligned}$$

Solving (8) with α_i given in (9)-(12) we have obtained the function $F(h)$ for which (7) is solution of equation (6). Consequently, an exact solution of equation (1) is

$$u(x,t) = a\,\mathrm{sn}^b(\mu x - \lambda t, m).$$

As an example, for $k = 0$, equation (8) is

$$\lambda\alpha\mu^2\omega F_{hhh} + (-3\lambda\alpha\mu^2\omega h - \alpha\mu^2)F_{hh} + \left(-\lambda\alpha\mu^2\omega + \alpha\mu^2\right)F_h h - \omega = 0$$

and setting $H = F'$, we have obtained

$$\lambda\alpha\mu^2\omega H_{hh} + (-3\lambda\alpha\mu^2\omega h - \alpha\mu^2)H_h + (-\lambda\alpha\mu^2\omega + \alpha\mu^2)Hh - \omega = 0 \tag{13}$$

The solutions H of the equation (13) are the Kummer functions: $\mathrm{KummerM}(\gamma,\nu,z)$ and $\mathrm{KummerU}(\gamma,\nu,z)$, for more information about them see, e.g. [27].

Taking into account that $\mathrm{sn}(z,0) = \sin(z)$, we have concluded that

$$u(x,t) = \sin(\mu x - \omega t)$$

is a solution of equation (1).

In similar way, it is possible to get solutions for subcases (ii) and (iii).

- Case 2: If $\Phi = \mathrm{e}^u$, besides $\mathbf{V}_1$ and $\mathbf{V}_2$, we have found the infinitesimal generator $\mathbf{V}_3^1$.

  For $\mathbf{V}_3^1$ we have obtained the symmetry reduction

$$z = t, \qquad u = 2\ln(x) + h(z),$$

  where $h(z)$ satisfies

$$-2\,\lambda\,\alpha\,e^h\,h' + h' - 2\,\alpha\,e^h = 0.$$

- Case 3: If $\Phi(u) = \ln(u)$, furthermore $\mathbf{V}_1$ and $\mathbf{V}_2$ there is another infinitesimal generator $\mathbf{v}_3^2$.

  For $\mathbf{V}_3^2$ the similarity variables and similarity solutions are:

$$z = t, \qquad u(x,t) = \tfrac{1}{x^2}h(z),$$

  where $h\,(t) = 2\,\alpha\,t + c_0$.

- Case 4: If $\Phi(u) = u^{\gamma+1}$, in addition to $\mathbf{V}_1$ and $\mathbf{V}_2$, the equation has an extra symmetry $\mathbf{V}_3^3$.

  For $\mathbf{V}_3^3$ the similarity variables and similarity solutions are:

$$z = t, \qquad u(x,t) = x^{\frac{2}{\gamma}}\,h(z), \tag{14}$$

  where $h(z)$ satisfies

$$2\lambda\alpha h^\gamma h_z\gamma^2 - h_z\gamma^2 + 2\alpha h^{\gamma+1}\gamma + 6\lambda\alpha h^\gamma h_z\gamma + 4\alpha h^{\gamma+1} + 4\lambda\alpha h^\gamma h_z = 0. \tag{15}$$

  If $\gamma \neq 0$ the solution in implicit form is

$$-\frac{2\lambda\alpha h^\gamma \log h\gamma^2 + (6\lambda\alpha h^\gamma \log h + 1)\,\gamma + 4\lambda\alpha h^\gamma \log h}{2\alpha h^\gamma\gamma + 4\alpha h^\gamma} = z + c_0$$

- Case 5: If $\Phi(u) = u^{-1/3}$, as well as $\mathbf{V}_1$ and $\mathbf{V}_2$ the equation has an extra symmetry, namely,

$$\mathbf{V}_3^4 = x\frac{\partial}{\partial x} - \frac{3}{2}u\frac{\partial}{\partial u}.$$

For $\mathbf{V}_3^4$ the similarity variables and similarity solutions are:

$$z = t, \qquad u(x,t) = \frac{1}{x^{\frac{3}{2}}} h(z),$$

where $h(z)$ satisfies

$$4\, h^{\frac{4}{3}}\, h' + \lambda\, \alpha\, h' - 3\, \alpha\, h = 0.$$

For $\mathbf{V}_4$ the similarity variables and similarity solutions are:

$$z = t, \qquad u(x,t) = \frac{1}{x^3} h(z),$$

where h is a constant.

5. Conclusions

In this paper, we have considered the basic equation for the effective water saturation reduces due to the incompressibility of both fluids. In sec. 2 we have looked for the equivalence generators considering two different cases, the first one for $\alpha \neq 0,\ \lambda = 0$ and the second one for $\alpha \neq 0,\ \lambda \neq 0$. And we have got a seven-dimensional equivalence algebra for the first case and a five-dimensional algebra for the last one.

We have studied the Lie symmetries of Barenblatt-Gilman equation in section 3 and we have found the classification of functions Φ for which we have obtained the Lie group of point transformations admitted by the (1) equation and its Lie algebra.

Finally, we have presented the symmetry reductions, similarity variables and the reduced ODEs, in sec. 4, for all the different Φ functions. Without forgetting that we have also obtained travelling wave solutions.

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