

Critical length: an alternative approach

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Abstract

We provide a numerical method to determine the critical lengths of linear differential operators with constant real coefficients. The need for such a procedure arises when the orders increase. The interest of this article is clearly on the practical side since knowing the critical lengths permits an optimal use of the associated kernels. The efficiency of the procedure is due to its being based on crucial features of Extended Chebyshev spaces on closed bounded intervals.

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1 Introduction

The *critical length* is a crucial notion attached to kernels of linear differential operators with constant real coefficients, which was first introduced in [7]. Such kernels are known to advantageously replace polynomial spaces in many situations. This is due to the fact that, unlike polynomial spaces, they inherently depend on parameters which can be used to modify the solution to classical problems (e.g., interpolation, design, approximation). Nevertheless, to take benefit of these parameters it may be necessary to restrict the length of the interval $[a, b]$ we are working on. For instance, for a given differential operator L , if we are interested in Hermite interpolation, $\mathbb{E}_L := \ker L$ must be an *Extended Chebyshev space* on $[a, b]$. This is ensured if and only if the length $b - a$ is less than a fixed number $\ell_L \in]0, +\infty]$. This number ℓ_L is referred to as the *critical length* of L (or of \mathbb{E}_L). If we want to use \mathbb{E}_L for design [24], we have to require \mathbb{E}_L to contain the constants and the length $b - a$ to be less than the critical length of the space $D\mathbb{E}_L$ obtained by differentiation, which is less than or equal to the critical length of \mathbb{E}_L and is called the *critical length for design* of \mathbb{E}_L .

It is well known that $\ell_L = +\infty$ if and only if the characteristic polynomial of L has only real roots. Therefore, determining the critical lengths concerns only differential operators whose characteristic polynomials have at least one non-real root. The classical approach consists in finding the smallest positive zero of a number of Wronskians attached to L [7, 32]. Unfortunately, this Wronskian approach is generally difficult to carry out in practice, all the more so as the dimension increases. As an example, consider the simplest case of cycloidal spaces (i.e., spaces spanned by polynomials of some degree and the two functions \cos and \sin) commonly used in geometric design, see, e.g., [37, 41, 34, 22, 16, 10, 13, 15]. Their critical lengths were successively investigated in a series of

articles [7, 8, 9]. They were definitely identified in relation to zeros of Bessel functions in [9]. It is worthwhile mentioning a surprising feature attached to this class of spaces: the critical length increases only every two steps of dimension elevation. This property is connected with the fact that, whatever the dimension, only one Wronskian is really involved in the computation of the critical length.

Cycloidal spaces are among the spaces which have also been extensively used during the last decade to build *generalised splines* (which are themselves examples of *Chebyshevian* or *piecewise Chebyshevian splines*) for Isogeometric Analysis purposes [11, 17, 18, 19, 5]. They belong to the larger class of spaces which are closed under reflection, corresponding to either odd or even characteristic polynomials. Such spaces were fully investigated in dimension four in [6], with both the exact computation of all the critical lengths, and the analysis of the shape effects they produce in dimension five, within therefore the critical lengths for design. This study was motivated by the fact that dimension four is the lowest dimension in which hyperbolic and trigonometric functions can be combined within the same space.

Obviously, the critical length should be known prior to any use of a space requiring to work with Extended Chebyshev spaces. To the best of our knowledge, no other computation of critical lengths exist apart from the examples we mentioned, except for trivial cases (e.g., lower dimensions, or spaces resulting from changes of variables). Beyond dimension four an exact computation is not really expectable anyway. This encouraged us to develop an effective numerical procedure instead. An advantage of the proposed algorithm is that it simultaneously provides not only the Bernstein-type bases which can then be used for numerical computations, but also associated generalised derivatives which can serve to develop approximation properties [39].

Our numerical procedure is described and illustrated in Section 4. It is entirely based on the question: *how to connect a number of Extended Chebyshev spaces on consecutive intervals, all of the same dimension, so as to produce a global Extended Chebyshev space of the same dimension?* A numerical answer to this question is briefly presented in Section 3, as an application of a more general numerical test including connection matrices which was developed in [1]. Beforehand, Section 2 provides readers with the necessary background on which the test relies. We more specifically underline the importance of dimension diminishing through generalised derivatives, and its action on Bernstein and Bernstein-like bases, which plays a prominent role in the present work. Section 5 concludes with a few crucial comments drawing readers' attention on why it is both necessary and advantageous to know the critical length. Advantageous: only the knowledge of its critical length (for design) enables us to take the maximum benefit of a given space, in particular in view of constructing splines with pieces taken from different spaces. Necessary: in terms of design, for instance, visual shape preservation is certainly not sufficient, the only safe approach being to work within critical length for design. Not only do these observations show the interest of our numerical approach, but they are all the more crucial as they are somehow in contradiction with rather widely spread practices. Last but not least, we would like to mention that our numerical procedure can also provide useful help to raise conjectures / solve theoretical questions. To cite only one point, it clearly indicates that the every two step increase, or the involvement of only one Wronskian, observed for cycloidal spaces, concerns only a limited subclass of spaces.

2 Background

In this section we gather the basic concepts and properties concerning Extended Chebyshev spaces strongly involved in the present paper. For further acquaintance with the subject, see [14, 39, 36, 20, 25] for instance.

2.1 Extended Chebyshev spaces

Let I be a non-trivial real interval and let $\mathbb{E}_n \subset C^n(I)$ be an $(n+1)$ -dimensional space. Then, \mathbb{E}_n is said to be an *Extended Chebyshev space on I* (for short, EC-space on I), if any non-zero $F \in \mathbb{E}_n$ vanishes at most n times in I , counting multiplicities up to $(n+1)$, or equivalently, if any Hermite interpolation problem in $(n+1)$ data in I is unisolvent in \mathbb{E}_n . It is said to be a *W-space on I* , if the Wronskian of any basis of \mathbb{E}_n never vanishes in I , or equivalently, if any Taylor interpolation problem in $(n+1)$ data in I is unisolvent in \mathbb{E}_n . An $(n+1)$ -dimensional EC-space on I is thus a W-space on I but the converse property is not true, except for $n=0$.

As is well known, the class of all W-spaces on I is closed under integration and multiplication by a sufficiently differentiable function which does not vanish on I , and the same holds true for the class of all EC-spaces on I .

While it is inherent in their definition that EC-spaces are crucial for interpolation, in Theorem 2.3 below we remind the reader why they are crucial for design too. Beforehand let us recall below the definition of bases of the Bernstein-type.

Definition 2.1. Given $a, b \in I$, $a < b$, we say that a sequence (V_0, \dots, V_n) of functions in $C^n(I)$ is a *Bernstein-like basis relative to (a, b)* if, for $i = 0, \dots, n$, the function V_i vanishes exactly i times at a and exactly $(n-i)$ times at b . A *positive Bernstein-like basis relative to (a, b)* is a Bernstein-like basis (V_0, \dots, V_n) relative to (a, b) such that V_i is positive on $]a, b[$ for $i = 0, \dots, n$.

Definition 2.2. Given $a, b \in I$, $a < b$, a Bernstein basis relative to (a, b) is a positive Bernstein-like basis (B_0, \dots, B_n) relative to (a, b) which is normalised, i.e., $\sum_{i=0}^n B_i = \mathbb{I}$, where \mathbb{I} stands for the constant function $\mathbb{I}(x) = 1$ for all $x \in I$.

Most of the time, for the sake of simplicity, it is convenient to include the positivity in the terminology “Bernstein-like basis”, but in the present paper it is essential to state it separately. Indeed, proving the positivity of the various bases is a major concern in the numerical test described in the next section. From now on, D denotes the ordinary differentiation on any interval.

Theorem 2.3. For a given $(n+1)$ -dimensional W-space $\mathbb{E}_n \subset C^n(I)$, supposed to contain the constants, the following properties are equivalent:

- (i) for each $a, b \in I$, $a < b$, \mathbb{E}_n possesses a Bernstein basis relative to (a, b) ;
- (ii) for each $a, b \in I$, $a < b$, $D\mathbb{E}_n$ possesses a Bernstein-like basis relative to (a, b) ;
- (iii) the $(n$ -dimensional) space $D\mathbb{E}_n$ is an EC-space on I ;
- (iv) blossoms exist in the space \mathbb{E}_n .

Furthermore, when (iii) is satisfied, all the classical design algorithms can be developed in \mathbb{E}_n , and for each $a, b \in I$, $a < b$, the Bernstein basis relative to (a, b) is the optimal normalised totally positive basis in \mathbb{E}_n restricted to $[a, b]$.

We mention blossoms only because they are the underlying tool for many of the results involved in the present work. We will not say more on them. Readers interested can refer to [36] and to many articles by the third author. Assuming that (ii) holds true, let (B_0, \dots, B_n) be the Bernstein basis in \mathbb{E}_n , relative to $(a, b) \in I^2$, $a < b$. Its total positivity on $[a, b]$ means that, for any $a \leq x_0 < x_1 < \dots < x_n \leq b$, all minors of the matrix $(B_i(x_j))_{0 \leq i, j \leq n}$ are non-negative. This is known to guarantee shape preserving properties in \mathbb{E}_n , see [12]. The mentioned optimality refers to the fact that we cannot find a better basis regarding this question, see [23, 7]. These comments justify the following definition:

Definition 2.4. For any $n \geq 1$, an $(n+1)$ -dimensional W-space \mathbb{E}_n on I is said to be *good for design* when first, it contains the constants, and second, the space $D\mathbb{E}_n$ is an EC-space on I .

Observe that a W-space \mathbb{E}_n which is good for design is automatically an EC-space on I .

Remark 2.5. The closure of the class of all EC-spaces on I under multiplication by positive functions and integration can be visualised as follows:

$$\begin{aligned}
&\text{step 2:} && \mathbb{E}_{k+1} = (k+2)\text{-dimensional EC-space on } I \\
&&& \text{multiply by any } \uparrow \text{ positive } w \in C^{k+1}(I) \\
&\text{step 1:} && \mathbb{F}_{k+1} = (k+2)\text{-dimensional EC-space good for design on } I, \\
&&& \text{inte-} \uparrow \text{grate} \\
&\text{step 0:} && \mathbb{E}_k = (k+1)\text{-dimensional EC-space on } I
\end{aligned} \tag{1}$$

Select a sequence (w_0, \dots, w_n) of *weight functions on I* , in the sense that, for $i = 0, \dots, n$ w_i is C^{n-i} and positive on I . For $k = 0, \dots, n$, we can then repeatedly apply the dimension increasing scheme (1) starting from the space \mathbb{E}_0 spanned by w_n , where the passage from \mathbb{F}_k to \mathbb{E}_k corresponds to multiplication by w_{n-k} , $k = 0, \dots, n$. The whole process takes place within the class of all EC-spaces on I . If we denote by $EC(w_0, \dots, w_n)$ the final EC-space \mathbb{E}_n , the same notation yields

$$\mathbb{E}_k = EC(w_{n-k}, \dots, w_n), \quad \mathbb{F}_k = EC(\mathbb{1}, w_{n-k+1}, \dots, w_n), \quad k = 0, \dots, n,$$

in which we have added the space \mathbb{F}_0 of all constant functions on I , and the space \mathbb{F}_{n+1} obtained by integration of \mathbb{E}_n . Note that each space \mathbb{F}_k , $k = 1, \dots, n+1$, is good for design on I . Classically, the system (w_0, \dots, w_n) is associated with linear differential operators L_0, \dots, L_n – also named *generalised derivatives* – recursively defined as follows:

$$L_0 F := \frac{F}{w_0}, \quad L_i F := \frac{1}{w_i} D L_{i-1} F, \quad 1 \leq i \leq n. \tag{2}$$

With these notations, $\mathbb{E}_n = EC(w_0, \dots, w_n)$ can be described as the set of all $F \in C^n(I)$ such that $L_n F \in \mathbb{F}_0$.

The previous process relates a classical way to obtain EC-spaces on a given interval, the generalised derivatives enabling the development of important approximation properties modelled on polynomial spaces [39]. In particular, not only does this provide one final EC-space but even a nested sequence of EC-spaces

$$\mathbb{E}_0 \subset \mathbb{E}_1 \subset \dots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n, \quad \text{with } \mathbb{E}_i := EC(w_0, \dots, w_i) \text{ for } i = 0, \dots, n.$$

This is the reason why $\mathbb{E}_n = EC(w_0, \dots, w_n)$ is called the *Extended Complete Chebyshev space associated with (w_0, \dots, w_n)* . The presence of such a nested sequence in \mathbb{E}_n is crucial, for instance, to define associated Chebyshevian divided differences and Newton-type expansions for the solution to any Hermite interpolation problem in \mathbb{E}_n . Conversely, if we start with a given nested sequence $\mathbb{E}_0 \subset \dots \subset \mathbb{E}_i \subset \dots \subset \mathbb{E}_n$, where, for $i = 0, \dots, n$, \mathbb{E}_i is an $(i+1)$ -dimensional W-space on I , then it is well known that it is a nested sequence of EC-spaces on I . More precisely, selecting any sequence (U_0, \dots, U_n) such that $U_i \in \mathbb{E}_i \setminus \mathbb{E}_{i-1}$ for $i = 0, \dots, n$, with $\mathbb{E}_{-1} := \{0\}$, we have [14, 25]

$$\mathbb{E}_i = EC(w_0, \dots, w_i), \quad w_i := \varepsilon_i \frac{W(U_0, \dots, U_{i-2}) W(U_0, \dots, U_i)}{W(U_0, \dots, U_{i-1})^2}, \quad 0 \leq i \leq n. \tag{3}$$

In (3), $W(U_0, \dots, U_i)$ denotes the Wronskian of the sequence (U_0, \dots, U_i) with the convention that $W(\emptyset) = \mathbb{1}$, and $\varepsilon_i = \pm 1$ is chosen so as to ensure the positivity of w_i .

2.2 EC-spaces and dimension diminishing

Is it possible to reverse the process (1) so as to find a system (w_0, \dots, w_n) of weight functions on I such that $\mathbb{E}_n = EC(w_0, \dots, w_n)$? The first step should thus consist in a dimension diminishing procedure as follows:

$$\begin{aligned}
 \text{step 0:} \quad & \mathbb{E}_n \subset C^n(I), \ (n+1)\text{-dimensional} \\
 & \text{division by } a \downarrow \text{positive } w_0 \in \mathbb{E}_n \\
 \text{step 1:} \quad & \mathbb{F}_n = L_0 \mathbb{E}_n \ (n+1)\text{-dimensional containing } \mathbb{I} \\
 & \text{differen-} \downarrow \text{tiation} \\
 \text{step 2:} \quad & \mathbb{E}_{n-1} = DL_0 \mathbb{E}_n \subset C^{n-1}(I) \ n\text{-dimensional}
 \end{aligned} \tag{4}$$

We should therefore first be able to find a positive function in a given EC-space \mathbb{E}_n . Unfortunately, this is not always possible, as proved by the famous counterexample of the space \mathbb{E}_1 spanned on $I = [0, \pi[$ by the two functions \cos, \sin , which is an EC-space on I but which does not contain any non-vanishing function. No equality of the form $\mathbb{E}_1 = EC(w_0, w_1)$ is thus expectable. Nonetheless, such an equality is possible by restriction to any $[0, b]$, with $0 < b < \pi$, as reminded below.

Theorem 2.6. *Let $[a, b]$, $a < b$, be a closed bounded interval, and let $\mathbb{E}_n \subset C^n([a, b])$ be $(n+1)$ -dimensional. Then, the following properties are equivalent:*

- (i) \mathbb{E}_n is an EC-space (resp., an EC-space good for design) on $[a, b]$;
- (ii) there exists a system (w_0, \dots, w_n) (resp., (w_1, \dots, w_n)) of weight functions on $[a, b]$ such that $\mathbb{E}_n = EC(w_0, \dots, w_n)$ (resp., $\mathbb{E}_n = EC(\mathbb{I}, w_1, \dots, w_n)$).

Given an $(n+1)$ -dimensional EC-space \mathbb{E}_n on $[a, b]$, and any positive function $w_0 \in \mathbb{E}_n$, we are certain that the n -dimensional space $DL_0 \mathbb{E}_n$ obtained according to (4) is a W-space on $[a, b]$, but there is no guarantee that it is an EC-space on $[a, b]$. This is clear from the classical example where \mathbb{E}_2 is the three-dimensional EC-space on $[0, 3\pi/2]$ spanned by the functions \mathbb{I}, \cos, \sin , for which the space $D\mathbb{E}_2$, spanned by \cos, \sin is an EC-space on $[0, \pi/2[$ but not on $[0, \pi/2]$. As a matter of fact, one step of dimension diminishing within the class of all EC-spaces on $[a, b]$ can only be done according to the rule specified below [28].

Theorem 2.7. *Let \mathbb{E}_n be an $(n+1)$ -dimensional EC-space on $[a, b]$, and let (V_0, \dots, V_n) denote a positive Bernstein-like basis relative to (a, b) in \mathbb{E}_n . Given a function $w_0 = \sum_{i=0}^n \alpha_i V_i \in \mathbb{E}_n$, the following properties are equivalent:*

- (i) $\alpha_0, \dots, \alpha_n$ are all positive;
- (ii) w_0 is positive on $[a, b]$, and if L_0 stands for the division by w_0 , the $(n+1)$ -dimensional space $L_0 \mathbb{E}$ is an EC-space good for design on $[a, b]$, i.e., the n -dimensional space $DL_0 \mathbb{E}$ is an EC-space on $[a, b]$.

It is worthwhile mentioning the following straightforward but crucial consequence of Theorem 2.7, to be compared with Theorem 2.3, see [28].

Corollary 2.8. *For a given $(n+1)$ -dimensional space $\mathbb{E}_n \subset C^n([a, b])$, $n \geq 1$, known to be an EC-space on $[a, b]$, the following properties are equivalent:*

- (1) \mathbb{E}_n possesses a Bernstein basis relative to (a, b) ;
- (2) \mathbb{E}_n is good for design on $[a, b]$.

Remark 2.9. The effect of generalised differentiation on the bases is well known and we recall it here. Assume that (i) of Theorem 2.7 is satisfied,

- From \mathbb{E}_n to $L_0\mathbb{E}_n$: Division by w_0 yields

$$\mathbb{I} = \sum_{i=0}^n B_i, \quad \text{with } B_i := \frac{\alpha_i V_i}{w_0} \text{ for } i = 0, \dots, n. \quad (5)$$

Clearly, (B_0, \dots, B_n) is the Bernstein basis relative to (a, b) in the space $L_0\mathbb{E}_n$ which is an EC-space good for design on $[a, b]$. Observe that the space $L_0\mathbb{E}_n$ is completely determined by the equivalence class of the sequence $(\alpha_0, \dots, \alpha_n)$ under proportionality. Accordingly, we can build infinitely many different such spaces $L_0\mathbb{E}_n$.

- From $L_0\mathbb{E}_n$ to $DL_0\mathbb{E}_n$: In close relation with the Bernstein basis (B_0, \dots, B_n) , it is convenient to introduce the functions

$$B_i^* := \sum_{k=i}^n B_k = \mathbb{I} - \sum_{k=0}^{i-1} B_k, \quad i = 0, \dots, n. \quad (6)$$

For each $i = 1, \dots, n$, the function B_i^* is characterised by the fact that

B_i^* vanishes exactly i times at a , and $\mathbb{I} - B_i^*$ vanishes exactly $(n - i + 1)$ times at b .

For this reason, these functions are named *transition functions* in the space $L_0\mathbb{E}_n$, see [2]. Let us set

$$\bar{V}_i := DB_{i+1}^* = \sum_{k=i+1}^n DB_k = - \sum_{k=0}^i DB_k, \quad i = 0, \dots, n-1. \quad (7)$$

Clearly, $(\bar{V}_0, \dots, \bar{V}_{n-1})$ is a Bernstein-like basis relative to (a, b) in the space $DL_0\mathbb{E}_n$ [24]. Moreover, expansions in that basis can easily be derived from expansions in the Bernstein basis in $L_0\mathbb{E}_n$, see [24]. From (7) we can also see that each \bar{V}_i is positive close to a . Accordingly, because the space $DL_0\mathbb{E}_n$ is known to be an EC-space on $[a, b]$ (Theorem 2.7), we can conclude that $(\bar{V}_0, \dots, \bar{V}_{n-1})$ is a positive Bernstein-like basis relative to (a, b) which can be used to iterate the process to construct w_1 via Theorem 2.7, and so forth up to one sequence (w_0, \dots, w_n) of weight functions such that $\mathbb{E}_n = EC(w_0, \dots, w_n)$.

Remark 2.10. Starting again with the $(n + 1)$ -dimensional EC-space \mathbb{E}_n on $[a, b]$, consider one sequence of spaces obtained by iteration of Theorem 2.7:

$$\mathbb{E}_n^{\{0\}} := \mathbb{E}_n, \quad \mathbb{E}_n^{\{p\}} := DL_{p-1}\mathbb{E}_n = EC(w_p, \dots, w_n) \text{ for } p = 1, \dots, n,$$

corresponding to one given equality $\mathbb{E}_n = EC(w_0, \dots, w_n)$. For each $p = 0, \dots, n$, the space $L_p\mathbb{E}_n = EC(\mathbb{I}, w_{p+1}, \dots, w_n)$ is an $(n - p + 1)$ -dimensional EC-space good for design on $[a, b]$ in which we denote by $(B_0^{\{p\}}, \dots, B_{n-p}^{\{p\}})$ the Bernstein basis relative to (a, b) . One step of dimension diminishing transforms each Bernstein basis into the next one via (7) and (5). These relations can be read in

the reverse way, which yields, for each $p = n, n-1, \dots, 1$,

$$\begin{aligned}
B_0^{\{p-1\}}(x) &= 1 - \frac{\int_a^x w_p(t) B_0^{\{p\}}(t) dt}{\int_a^b w_p(t) B_0^{\{p\}}(t) dt}, \\
B_i^{\{p-1\}}(x) &= \frac{\int_a^x w_p(t) B_{i-1}^{\{p\}}(t) dt}{\int_a^b w_p(t) B_{i-1}^{\{p\}}(t) dt} - \frac{\int_a^x w_p(t) B_i^{\{p\}}(t) dt}{\int_a^b w_p(t) B_i^{\{p\}}(t) dt}, \quad 1 \leq i \leq n-p, \\
B_{n-p+1}^{\{p-1\}}(x) &= \frac{\int_a^x w_p(t) B_{n-p}^{\{p\}}(t) dt}{\int_a^b w_p(t) B_{n-p}^{\{p\}}(t) dt},
\end{aligned} \tag{8}$$

starting from $B_0^{\{n\}} = \mathbb{I}$. These relations, first obtained in [27] through blossoms, are the analogue of the classical integral recurrence relations for polynomial Bernstein bases. From our comments in Remark 2.9, we know that there are infinitely many essentially different possibilities to go from $B_0^{\{n\}} = \mathbb{I}$ up to the Bernstein basis in $L_0\mathbb{E}_n$ according to (8). It should be observed that, apart from trivial exceptions, none of them can be considered a practical way to calculate the Bernstein basis in $L_0\mathbb{E}_n$, since to the contrary, they are derived from the latter basis.

2.3 Global versus local

We start again with an $(n+1)$ -dimensional EC-space \mathbb{E}_n on $[a, b]$. In \mathbb{E}_n , we consider two positive Bernstein-like bases: the first one relative to (a, b) , say (V_0, \dots, V_n) ; the second relative to (a^*, b^*) , say (V_0^*, \dots, V_n^*) , where $a \leq a^* < b^* \leq b$. For short, we refer to them as global positive Bernstein-like basis / local positive Bernstein-like basis.

Let us expand the global basis in the local one as follows

$$V_i = \sum_{r=0}^n \gamma_{i,r} V_r^*, \quad i = 0, \dots, n. \tag{9}$$

All coefficients of these expansions are known to be positive (see [1] and other references therein), except possibly in accordance with the zero conditions of (V_0, \dots, V_n) at the endpoints, that is, $\gamma_{i,r} = 0$ for $0 \leq r \leq i-1$ if $a^* = a$, and $\gamma_{i,r} = 0$ for $i+1 \leq r \leq n$ if $b^* = b$. We refer to this fact as *the positivity property of local expansions* of the global basis.

Let \mathbb{E}_n^* denote the restriction of \mathbb{E}_n to the interval $[a^*, b^*]$. Take any system (w_0, \dots, w_n) of weight functions on $[a, b]$ such that $\mathbb{E}_n = EC(w_0, \dots, w_n)$. By restriction to $[a^*, b^*]$, it generates a system (w_0^*, \dots, w_n^*) of weight functions on $[a^*, b^*]$ such that $\mathbb{E}_n^* = EC(w_0^*, \dots, w_n^*)$. This means that the successive steps of the corresponding dimension diminishing (4) can be applied simultaneously in \mathbb{E}_n and in \mathbb{E}_n^* , with the two associated sequences of positive Bernstein-like bases $(V_0^{\{p\}}, \dots, V_{n-p}^{\{p\}})$, and $(V_0^{*\{p\}}, \dots, V_{n-p}^{*\{p\}})$, respectively deduced from the initial bases $(V_0^{\{0\}}, \dots, V_n^{\{0\}}) := (V_0, \dots, V_n)$, and $(V_0^{*\{0\}}, \dots, V_n^{*\{0\}}) := (V_0^*, \dots, V_n^*)$. At each step we can expand the global basis $(V_0^{\{p\}}, \dots, V_{n-p}^{\{p\}})$ in the local basis $(V_0^{*\{p\}}, \dots, V_{n-p}^{*\{p\}})$ as

$$V_i^{\{p\}} = \sum_{r=0}^{n-p} \gamma_{i,r}^{\{p\}} V_r^{*\{p\}}, \quad i = 0, \dots, n-p. \tag{10}$$

In this process, for each $p \leq n-1$, the coefficients at level $(p+1)$ can be computed from those of level p . In the special case where the weight functions are taken as

$$w_p := V_0^{\{p\}} + \cdots + V_{n-p}^{\{p\}}, \quad p = 0, \dots, n,$$

then we have [1]:

$$\gamma_{i,r}^{\{p+1\}} = \frac{\sum_{j=i+1}^{n-p} \gamma_{j,r+1}^{\{p\}}}{\sum_{j=0}^{n-p} \gamma_{j,r+1}^{\{p\}}} - \frac{\sum_{j=i+1}^{n-p} \gamma_{j,r}^{\{p\}}}{\sum_{j=0}^{n-p} \gamma_{j,r}^{\{p\}}}, \quad 0 \leq i, r \leq n-p-1. \quad (11)$$

3 Building a global EC space from local EC spaces

Given an $(n+1)$ -dimensional space $\mathbb{E}_n \subset C^n(I)$, how to determine whether or not \mathbb{E}_n is an EC-space on I ? From (ii) of Theorem 2.3, we know that this consists in checking whether all determinants

$$\det \left(\mathbb{U}(x), \dots, \mathbb{U}^{(i-1)}(x), \mathbb{U}(y), \dots, \mathbb{U}^{(j-1)}(y) \right), \quad i, j \geq 0, \quad i+j = n+1,$$

never vanish for $x, y \in I$, $x < y$, where $\mathbb{U} := (U_0, \dots, U_n)^T$ and (U_0, \dots, U_n) is any basis in \mathbb{E}_n . Moreover, in case the space \mathbb{E}_n is known to be a W-space on I , we only have to consider positive integers i, j . Except for small values of n , it is not easy to check this by hand and it is not easy either in the general case to do it numerically.

From now on we consider a closed bounded interval $[a, b]$, $a < b$. From Theorem 2.6, we know that \mathbb{E}_n is an EC-space on $[a, b]$ if and only if,

- we can find a positive function $w_0 \in \mathbb{E}_n$;
- we can find a positive function $w_1 \in DL_0 \mathbb{E}_n$;
-
- we can find a positive function $w_n \in DL_{n-1} \mathbb{E}_n$,

where the notations are according to (2). Now, replacing each “we can find” by “can we find?”, we have at our disposal an easy theoretical test: if at some stage the answer to the “can we find?” question is negative, then the initial space \mathbb{E}_n is not an EC-space on $[a, b]$; if all answers are affirmative, then \mathbb{E}_n is indeed an EC-space on $[a, b]$ and we can even say that (w_0, \dots, w_n) is a system of weight functions associated with \mathbb{E}_n . Observe that, in case \mathbb{E}_n is known to be a W-space on $[a, b]$, we only need affirmative answers until the last but one question.

Clearly, in general this is not a realistic test, since exhibiting such positive functions is more or less like pulling a rabbit out of a hat. Nonetheless, it becomes realistic in the situation addressed subsequently, where the positivity of functions is checked through the positivity of the coefficients of some expansions in appropriate bases.

Throughout the rest of the present subsection, the interval $[a, b]$, $a < b$, is given along with a sequence $\mathbb{T} = (t_1, \dots, t_q)$ of $q \geq 1$ knots interior to $[a, b]$, with

$$t_0 := a < t_1 < \cdots < t_q < t_{q+1} := b,$$

and a given positive integer n . From now on, we change the notations, the index of a space no longer being related to its dimension, but to its interval. For $k = 0, \dots, q$, \mathbb{E}_k is an $(n+1)$ -dimensional EC-space on $[t_k, t_{k+1}]$, in which we select a positive Bernstein-like basis $(V_{k,0}, \dots, V_{k,n})$ relative to (t_k, t_{k+1}) , which we refer to as the k th local positive Bernstein-like basis. Consider the space \mathbb{E} defined by the two conditions:

- $\mathbb{E} \subset C^n([a, b])$;
- for each $k = 0, \dots, q$, the restriction of \mathbb{E} to $[t_k, t_{k+1}]$ is \mathbb{E}_k .

Clearly, the space \mathbb{E} is an $(n+1)$ -dimensional W-space on $[a, b]$, but in general it is not an EC-space on $[a, b]$. A numerical answer to the question “Is \mathbb{E} an EC-space on $[a, b]$ ” can be obtained thanks to the test built in [1] in the larger framework of adjacent EC-spaces tied by connection matrices. Here, it answers the question “*Starting from \mathbb{E} , can we iteratively diminish the dimension as in (4)?*” Subsequently, we briefly recall the main ideas / steps, modelled on the iterative dimension diminishing step described in Theorem 2.7.

• **Numerical test – Step 0:**

In this step 0, we first answer the question: *does the W-space \mathbb{E} possess a Bernstein-like basis relative to (a, b) ?* With the notations already used above, we have to test if

$$\det \left(\mathbb{U}(a), \dots, \mathbb{U}^{(i-1)}(a), \mathbb{U}(b), \dots, \mathbb{U}^{(j-1)}(b) \right) \neq 0 \quad \text{for } i, j \geq 1, i + j = n + 1. \quad (12)$$

If the answer to (12) is negative, we can state that \mathbb{E} is not an EC-space on $[a, b]$. Supposing that it is affirmative, let $(V_0^{\{0\}}, \dots, V_n^{\{0\}})$ be a Bernstein-like basis of \mathbb{E} relative to (a, b) , determined by fixing the first non-zero derivatives at a or b to be in accordance with its possible positivity.

For each $k = 0, \dots, q$, consider the k th local expansion of the previous basis, in the sense of the expansion of its restriction to $[t_k, t_{k+1}]$ in the local positive Bernstein-like basis $(V_{k,0}^{\{0\}}, \dots, V_{k,n}^{\{0\}}) := (V_{k,0}, \dots, V_{k,n})$:

$$V_i^{\{0\}}|_{[t_k, t_{k+1}]} = \sum_{r=0}^n \gamma_{i,k,r}^{\{0\}} V_{k,r}^{\{0\}}, \quad i = 0, \dots, n, \quad k = 0, \dots, q. \quad (13)$$

Answer the question: *do all these local expansions satisfy the positivity property reminded in Subsection 2.3?*

If the answer is affirmative, proceed to Step 1. If not, we know that \mathbb{E} is not an EC-space on $[a, b]$, and the test stops.

• **Numerical test – Step 1:**

Due to the positivity property of all local expansions (13), $(V_0^{\{0\}}, \dots, V_n^{\{0\}})$ is a positive Bernstein-like basis relative to (a, b) . Take

$$w_0 := V_0^{\{0\}} + \dots + V_n^{\{0\}}, \quad w_{0,k} := w_0|_{[t_k, t_{k+1}]}, \quad k = 0, \dots, q. \quad (14)$$

Then, w_0 is positive on $[a, b]$ and each $w_{0,k}$ has positive coordinates in the local positive Bernstein-like basis of \mathbb{E}_k . Accordingly, we can simultaneously diminish the dimension

- globally, via w_0 , within the class of all W-spaces on $[a, b]$, replacing \mathbb{E} by $DL_0\mathbb{E}$;
- for each $k = 0, \dots, q$, locally, via $w_{0,k}$ within the class of all EC-spaces on $[t_k, t_{k+1}]$, replacing \mathbb{E}_k by $DL_0^k\mathbb{E}_k$, where L_0^k is the division by $w_{0,k}$.

Meanwhile, each local positive Bernstein-like basis $(V_{k,0}^{\{0\}}, \dots, V_{k,n}^{\{0\}})$ is transformed into a positive Bernstein-like basis $(V_{k,0}^{\{1\}}, \dots, V_{k,n-1}^{\{1\}})$ of $DL_0^k\mathbb{E}_k$, relative to (t_k, t_{k+1}) , and the global positive Bernstein-like basis $(V_0^{\{0\}}, \dots, V_n^{\{0\}})$ into a global Bernstein-like basis $(V_0^{\{1\}}, \dots, V_{n-1}^{\{1\}})$ of $DL_0\mathbb{E}$, relative to (a, b) . These transformations follow the procedure described in Remark 2.9.

The next step consists in checking if the coefficients of all local expansions of the new global basis satisfy the positivity property, in which case we can continue diminishing the dimension both globally and locally through

$$w_1 := V_0^{\{1\}} + \dots + V_{n-1}^{\{1\}}, \quad w_{1,k} := w_1|_{[t_k, t_{k+1}]}, \quad k = 0, \dots, q.$$

All in all, starting from the positive $\gamma_{i,k,r}^{\{0\}}$, $i = 0, \dots, n$, $k = 0, \dots, q$, Step 1 can be translated into a simple iterative computation of real numbers at level $(p+1)$, $\gamma_{i,k,r}^{\{p+1\}}$, from positive $\gamma_{i,k,r}^{\{p\}}$ at level p , according to the formulæ:

$$\gamma_{i,k,r}^{\{p+1\}} = \frac{\sum_{j=i+1}^{n-p} \gamma_{j,k,r+1}^{\{p\}}}{\sum_{j=0}^{n-p} \gamma_{j,k,r+1}^{\{p\}}} - \frac{\sum_{j=i+1}^{n-p} \gamma_{j,k,r}^{\{p\}}}{\sum_{j=0}^{n-p} \gamma_{j,k,r}^{\{p\}}}, \quad 0 \leq i, r \leq n-p-1, \quad (15)$$

similar to (11). The passage from level p to level $p+1$ is successful when all quantities $\gamma_{i,k,r}^{\{p+1\}}$ are positive (at least in accordance with the positivity property of local expansions). The W-space \mathbb{E} is an EC-space on $[a, b]$ if and only if the test is successful up to level $n-1$. In other words, if, for some integers p, k, i , $p \leq n-1$, some $0 \leq k \leq q$, $0 \leq i \leq n-p$, we obtain a negative $\gamma_{i,k,r}^{\{p\}}$, the step stops, and the W-space \mathbb{E} is not an EC-space.

We have presented above the theoretical test on which the numerical test is based. In its numerical version, “non-zero” or “positive”, etc, is checked up to fixed tolerances, see [1].

Remark 3.1. If the test is successful up to level $(n-1)$, we know that it can be continued successfully until dimension one. In other words, not only can we say that \mathbb{E} is an EC-space on $[a, b]$, but we can even add that

- with $w_{n-1} := V_0^{\{n-1\}} + V_1^{\{n-1\}}$ and $w_n := V_0^{\{n\}}$, we have built one system of weight functions (among infinitely many) associated with \mathbb{E} — i.e., $\mathbb{E} = EC(w_0, \dots, w_n)$ — which depends only on our choice of the initial Bernstein-like basis $(V_0^{\{0\}}, \dots, V_n^{\{0\}})$;
- at each level, the global Bernstein-like basis $(V_0^{\{p\}}, \dots, V_{n-p}^{\{p\}})$ implicitly obtained in the test is a positive Bernstein-like basis relative to (a, b) in the EC-space on $[a, b]$, $DL_{p-1}\mathbb{E} = EC(w_p, \dots, w_n)$;
- at each level, dividing the equality $w_p = V_0^{\{p\}} + \dots + V_{n-p}^{\{p\}}$ by w_p provides us with the Bernstein basis $(B_0^{\{p\}}, \dots, B_{n-p}^{\{p\}})$ relative to (a, b) in the space $L_p\mathbb{E} = EC(\mathbb{I}, w_{p+1}, \dots, w_n)$;
- read in the reverse sense, the relations between the bases at level p and at level $p+1$ on which the test is based, produce one among infinitely many recurrence formulæ of the form (8) [27].

Example 3.2. Here we take $q = 1$. Given $n \geq 1$, let \mathbb{E}_0 be the space spanned on $[t_0, t_1]$ by the functions $1, x, \dots, x^{n-2}, \cos x, \sin x$ (to which we refer here as the *trigonometric space*), and let \mathbb{E}_1 be the space spanned on $[t_1, t_2]$ by the functions $1, x, \dots, x^{n-2}, \cosh x, \sinh x$ (to which we refer here as the *hyperbolic space*). Setting $T := t_1 - t_0$ and $H := t_2 - t_1$, we want to know how to choose the pair (T, H) so that the W-space \mathbb{E} on $[t_0, t_2]$ obtained by the C^n connection of \mathbb{E}_0 and \mathbb{E}_1 is an EC-space on $[t_0, t_2]$. To apply the test, we first have to make sure that the space \mathbb{E}_0 is an EC-space on $[t_0, t_1]$. It is known that this is obtained by requiring $T < \ell_n$, where (see [7, 8, 9] and Subsection 4.3.1 below)

$$\ell_1 = \pi, \quad \ell_2 = \ell_3 = 2\pi, \quad \ell_4 = \ell_5 \approx 8.9868, \quad \ell_6 = \ell_7 \approx 11.5269, \quad \ell_8 = \ell_9 \approx 13.9758, \quad \dots \quad (16)$$

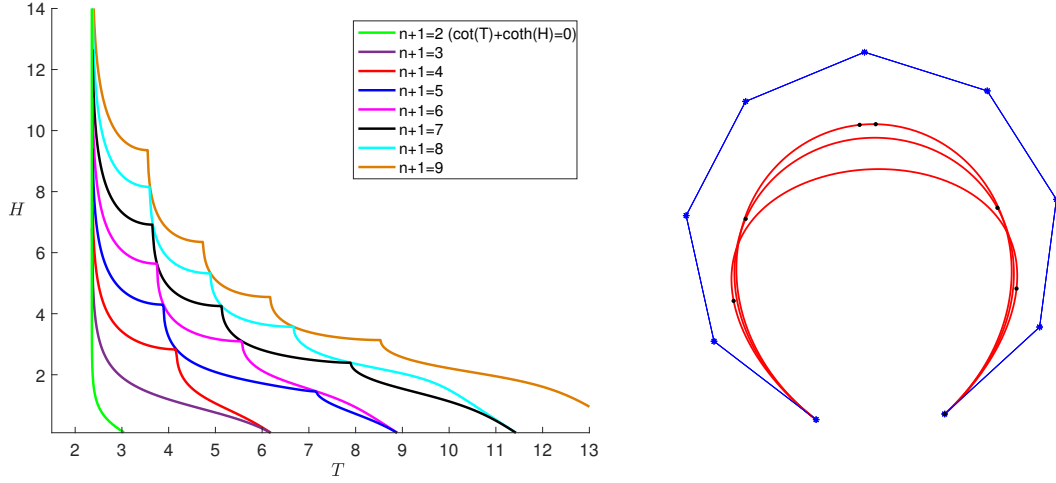


Figure 1: **Left:** In the plane (T, H) , for various values of n , the region producing a global EC-space on $[0, T + H]$ from a trigonometric space on $[0, T]$ and an hyperbolic space on $[T, T + H]$. **Right:** A C^8 HTH-curve on $[0, 2H + T]$ for $H = 2$ and $T = 0.1; 3.14; 6.6545$ (limit curve), see Example 3.2.

On the left picture of Figure 1, the region of the plane (T, H) producing an EC-space on $[t_0, t_2]$ is located below the boundary curve, depending on the dimension. For $n = 1$ and $0 < T < \pi$, this region, limited by the green curve, coincides with the theoretical result obtained in [30], that is,

$$\mathbb{E} \text{ is an EC-space on } [t_0, t_2] \quad \Leftrightarrow \quad \cot T + \coth H > 0.$$

We can see that the boundary curve presents more and more cusps as the dimension increases. The presence of such cusps can be explained by the various determinants (12) appearing in the first part of the test, each of them being expressed as a function of the two variables T, H . The different segments of the boundary curve correspond to the different determinants (12) which effectively vanish at Step 0 in the test. The test could be applied with more sections as well. As an instance, in the right picture, we take three sections, the first and last one are hyperbolic with length H , while the central one is trigonometric with length T . For $n = 8$, we design in the nine-dimensional space \mathbb{E} , within the region of the plane (T, H) ensuring that $D\mathbb{E}$ is an EC-space on $[t_0, t_3] = [0, 2H + T]$. with now t_1 and t_2 as the two interior knots. The curves are obtained with $H = 2$ and, from top to bottom, $T = 0.1; 3.14$; and finally $T = 6.6545$, which corresponds to the limit curve.

4 Numerical Procedure for Critical Length:

In this section we show how to use the numerical test briefly described in the previous one for computing the critical length of a given linear differential operator L of order $(n + 1)$ with constant coefficients, say

$$L := D^{n+1} + \sum_{i=0}^n a_i D^i, \quad \text{with } a_0, \dots, a_n \in \mathbb{R}, \quad (17)$$

where n is a positive integer. Before explaining our approach, it is useful to review some classical facts concerning such operators.

4.1 Preliminaries

We are concerned with $\mathbb{E}_L = \ker(L)$. For a given n , the class of all such kernels coincides with the class of all $(n+1)$ -dimensional W-spaces on \mathbb{R} which are closed under translation, or as well, with the class of all $(n+1)$ -dimensional W-spaces on \mathbb{R} which are closed under differentiation. As is well known, the description of \mathbb{E}_L follows from its characteristic polynomial

$$p_L(x) := x^{n+1} + \sum_{i=0}^n a_i x^i. \quad (18)$$

In particular, \mathbb{E}_L is closed under reflection (i.e., for each $F \in \mathbb{E}_L$, the function $x \mapsto F(-x)$ belongs to \mathbb{E}_L) if and only if the polynomial p_L is either even or odd.

Definition 4.1. The critical length of the L (or, as well, of \mathbb{E}_L) is defined by

$$\ell_L := \sup\{h > 0 \mid \mathbb{E}_L \text{ is an EC-space on } [0, h]\}.$$

Due to \mathbb{E}_L being closed under translation, it is also the supremum of all $h > 0$ such that \mathbb{E}_L is an EC-space on any $[\alpha, \alpha + h]$. Two basic facts are to keep in mind (see, for instance, Proposition 1.15 in [35]).

Proposition 4.2. *The critical length ℓ_L lies in $]0, +\infty]$. More precisely*

- if p_L has only real roots, then $\ell_L = +\infty$;
- otherwise, $\frac{\pi}{M_L} \leq \ell_L < +\infty$, where M_L stands for the maximum imaginary part of all roots of p_L , and \mathbb{E}_L is not an EC-space on $[0, \ell_L]$.

Finally, the classical way to compute the critical length ℓ_L consists in determining the first positive zeros of a number of Wronskians, as reminded below, [7, 32].

Theorem 4.3. *Let S be the unique element of \mathbb{E}_L satisfying $S(0) = S'(0) = \dots = S^{(n-1)}(0) = 0$, $S^{(n)}(0) = 1$. Then, the critical length ℓ_L can be obtained as*

$$\begin{aligned} \ell_L &= \min_{0 \leq k \leq n-1} \inf\{h > 0 \mid W(S, S', \dots, S^{(k)})(h) \neq 0\}, \\ &= \min_{0 \leq k \leq \frac{n-1}{2}} \inf\{|h| > 0 \mid W(S, S', \dots, S^{(k)})(h) \neq 0\}. \end{aligned} \quad (19)$$

When all roots of p_L are real, since $\ell_L = +\infty$, we can thus say that each of the Wronskians $W(S, S', \dots, S^{(k)})$, $k = 0, \dots, n$, keeps the same strict sign on $]0, +\infty[$. When \mathbb{E}_L is invariant under reflection, formulæ (19) can be simplified as follows:

Corollary 4.4. *Suppose that p_L is either odd or even. Then we have*

$$\ell_L = \min_{0 \leq k \leq \frac{n-1}{2}} \inf\{h > 0 \mid W(S, S', \dots, S^{(k)})(h) \neq 0\}. \quad (20)$$

Consider the operator $\widehat{L} := D \circ L$. Given that $\mathbb{E}_L = D\mathbb{E}_{\widehat{L}}$, according to Definition 2.4, we can say that

$$\ell_L := \sup\{h > 0 \mid \mathbb{E}_{\widehat{L}} \text{ is an EC-space good for design on } [0, h]\}.$$

This justifies the following terminology.

Definition 4.5. The critical length ℓ_L is called *the critical length for design of the operator $\widehat{L} := D \circ L$* .

The space $\mathbb{E}_{\widehat{L}}$ being obtained from \mathbb{E}_L by integration, it should be observed that its critical length for design is always less than or equal to its critical length.

4.2 An alternative approach

In this subsection, we assume that p_L has at least one non-real root, and as previously, we denote by M_L the maximum imaginary part of all non-real roots of p_L . We will compute the critical length $\ell_L \in]0, +\infty[$ via the test recalled in the previous section. The computation comprises two successive parts:

4.2.1 Step 1: Rough estimate of ℓ_L

We select a positive number $\ell_0 < \frac{\pi}{M_L}$, close to $\frac{\pi}{M_L}$. From Proposition 4.2, we know that \mathbb{E}_L is an EC-space on $[0, \ell_0]$. Accordingly, we can state that

$$\text{there exists an integer } \mu \geq 1 \text{ such that } \mu\ell_0 < \ell_L \leq (\mu + 1)\ell_0.$$

This integer is equivalently determined by the two properties below

$$\mathbb{E}_L \text{ is an EC-space on } [0, \mu\ell_0] \text{ and } \mathbb{E}_L \text{ is not an EC-space on } [0, (\mu + 1)\ell_0]. \quad (21)$$

Observing that, for each non-negative k , the restriction \mathbb{E}_k of \mathbb{E}_L to $[t_k, t_{k+1}] := [k\ell_0, (k + 1)\ell_0]$ is an EC-space on $[t_k, t_{k+1}]$, we can apply the test on $[t_0, t_{k+1}]$, with the sequence $\mathbb{T} = (t_1, \dots, t_k)$ of interior knots, to determine whether \mathbb{E}_L is an EC-space on $[0, (k + 1)\ell_0]$, successively for $k = 1, 2, \dots$. The integer μ satisfying (21) is the first integer k for which we obtain a negative answer.

4.2.2 Step 2: Search for ℓ_L in $]\mu\ell_0, (\mu + 1)\ell_0]$

The second step consists in localising ℓ_L within the interval $]\mu\ell_0, (\mu + 1)\ell_0]$ by dichotomy. On account of (21), we first want to test if \mathbb{E}_L is an EC-space on $[0, \mu\ell_0 + \frac{\ell_0}{2}]$. Then,

- if the answer is affirmative, test if \mathbb{E}_L is an EC-space on $[0, \mu\ell_0 + \frac{\ell_0}{2} + \frac{\ell_0}{4}]$;
- if the answer is negative, test if \mathbb{E}_L is an EC-space on $[0, \mu\ell_0 + \frac{\ell_0}{2} - \frac{\ell_0}{4}]$;

Continue the same way, that is, at each step increment or decrement the interval length by $\frac{\ell_0}{2^n}$, until $\frac{\ell_0}{2^n}$ is less than a given tolerance.

At each dichotomy step, we apply the test of Section 3. However, in order to avoid numerical problems, we do not apply it using the same first μ intervals of length ℓ_0 and adding an interval which might be of smaller and smaller length, but using only two consecutive intervals both of same variable length. To be more precise, in the first step we take

$$[t_0, t_2] := [0, \ell_0(\mu + \frac{1}{2})], \quad t_1 = \frac{t_2}{2} = \frac{\ell_0}{2}(\mu + \frac{1}{2}), \quad \mathbb{E}_0 := \mathbb{E}_L|_{[t_0, t_1]}, \quad \mathbb{E}_1 := \mathbb{E}_L|_{[t_1, t_2]}.$$

Since $\mu \geq 1$, for $i = 0, 1$, \mathbb{E}_i is an EC-space on $[t_i, t_{i+1}]$, which enables us to apply the test. If the answer is affirmative (resp., negative) we do the same, after incrementing (resp., decrementing) t_2 by $\frac{\ell_0}{4}$, and so forth. At each step of the dichotomic process, each \mathbb{E}_i , $i = 0, 1$, is guaranteed to be an EC-space on $[t_i, t_{i+1}]$.

4.3 Examples

We illustrate the previous procedure with several instances of differential operators, indicated by their characteristic polynomials. We have chosen to illustrate the behaviour of the critical lengths up to dimension nine, since this has proven to be sufficient to formulate theoretical conjectures and, at the same time, it comprises all cases of practical interest for applications (e.g., design). The

computational method of the critical length has no limitations on the dimension of the space, but being a numerical method and working at the limits of the critical lengths it is subject to the machine precision and to the fixed tolerances. The following results are obtained in Matlab with tolerances fixed to 1.0e-30 for the test of Section 3, and 1.0e-10 for the dichotomy procedure.

4.3.1 Basic example: $p_n(x) = x^{n-1}(x^2 + b^2)$, $b > 0$

This is the simplest class of kernels \mathbb{E}_n of differential operators for which the critical lengths are not infinite, spanned on \mathbb{R} by the $(n + 1)$ functions $1, x, \dots, x^{n-2}, \cos(bx), \sin(bx)$ (cycloidal or trigonometric spaces depending on the literature). The corresponding critical lengths $\ell_n(b)$ are shown in Figure 2, top left (under the name ℓ_L), as functions of the variable b , for increasing values of n . Clearly, we have

$$\ell_n(b) = \frac{\ell_n(1)}{b}, \quad b > 0.$$

This explains why each curve resulting from our numerical procedure is a branch of hyperbola. Moreover, the first values of $\ell_n := \ell_n(1)$ are those already given in (16). The critical length ℓ_n was studied in several papers by Carnicer, Mainar, Peña [7, 8, 9], their main results being stated below:

Theorem 4.6. *The successive critical lengths ℓ_n of the spaces \mathbb{E}_n spanned by $1, x, \dots, x^{n-2}, \cos x, \sin x$, $n \geq 1$, satisfy*

$$\ell_{2k} = \ell_{2k+1} = 2j_{k-\frac{1}{2},1} < \ell_{2k+2}, \quad k \geq 1, \quad (22)$$

where, for each positive α , $j_{\alpha,1}$ stands for the first positive zero of the Bessel function of the first kind:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

On the other hand, since p_n is either odd or even, we know that the critical lengths can be computed by (20). It is worthwhile mentioning that, whatever the dimension,

$$\ell_n \text{ is the first positive zero of the Wronskian } W(S_n, \dots, S_n^{(p)}), \text{ where } p := \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where the non-zero function $S_n \in \mathbb{E}_n$ vanishes n times at 0, and where $\lfloor \cdot \rfloor$ stands for the floor function. As shown in [8], none of the other Wronkians involved in (20) vanishes on $]0, +\infty[$.

Our procedure, first ran in this class of spaces, is in perfect accordance with Theorem 4.6. In particular, the two-by-two behaviour can easily be observed in Figure 2.

4.3.2 Other examples of the form $p_n(x) = x^{n-3}p_3(x) = x^{n-3}(x^4 + a_2x^2 + a_0)$, $n \geq 3$

Let us recall that the four-dimensional spaces \mathbb{E}_3 associated with characteristic polynomials $p_3(x) = x^4 + a_2x^2 + a_0$, with at least two non-real roots, were thoroughly investigated in [6]. It is natural to apply the procedure in the class of spaces obtained from them by repeated integration, by comparison with its simplest subclass of cycloidal spaces. In spite of this class being relatively limited and simple, we will already observe different interesting behaviours depending on the polynomial p_3 and on the integer n . This will clearly point out the difficulty of determining the critical length for any given differential operator, and the impossibility to easily foresee what it will be.

- $p_3(x) = (x^2 - a^2)(x^2 + b^2)$, $a, b > 0$:

The space \mathbb{E}_n is spanned by the $(n+1)$ functions $1, x, \dots, x^{n-4}, \cosh(ax), \sinh(ax), \cos(bx), \sin(bx)$. The critical length of \mathbb{E}_n depends on the two parameters a, b , and we denote it by $\ell_n(a, b)$. Now, we clearly have

$$\ell_n(\alpha a, \alpha b) = \frac{1}{\alpha} \ell_n(a, b), \quad \alpha, a, b > 0. \quad (23)$$

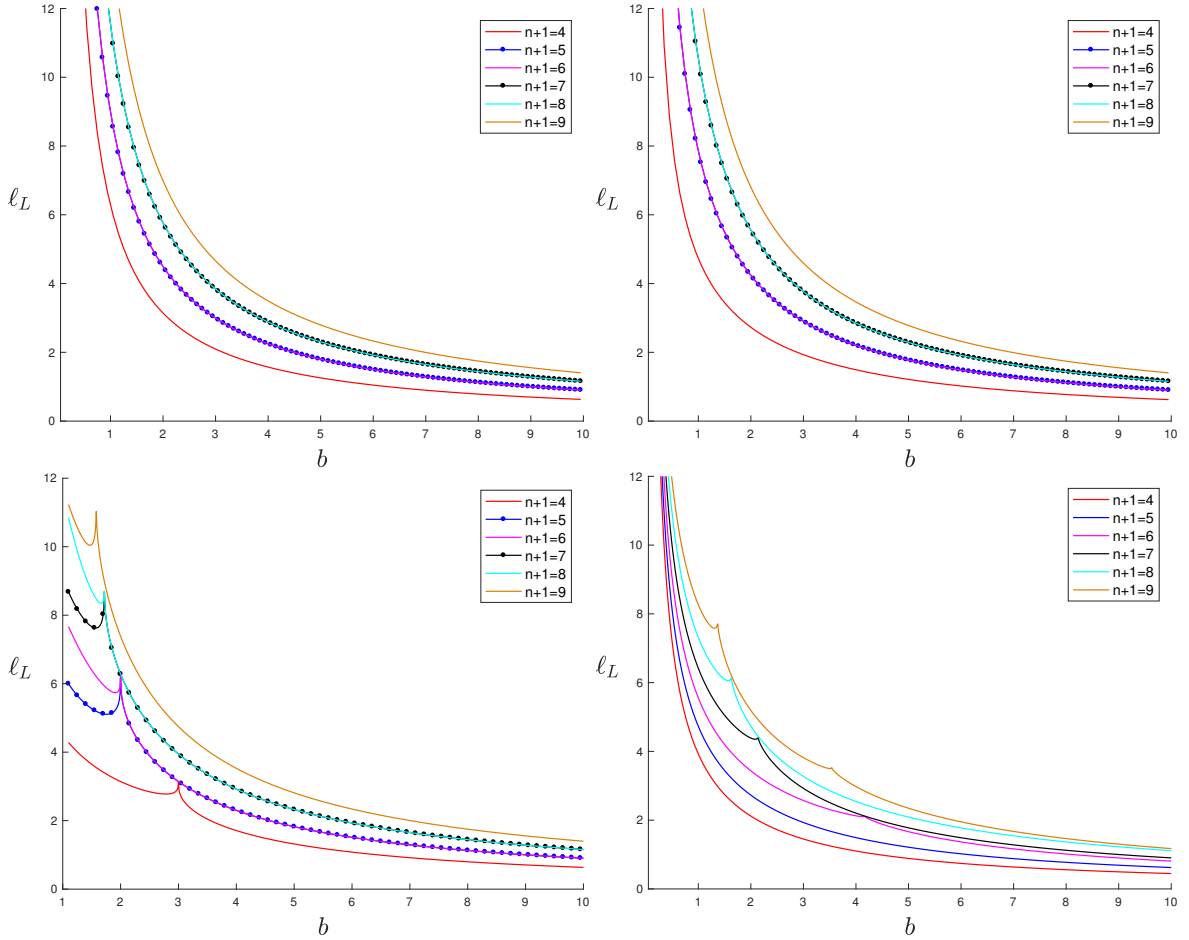


Figure 2: The critical length of the space \mathbb{E}_L as a function of the positive parameter b , where \mathbb{E}_L is spanned by $1, x, \dots, x^{n-2}, \cos(bx), \sin(bx)$ (top left); $1, x, \dots, x^{n-4}, \cosh x, \sinh x, \cos(bx), \sin(bx)$ (top right); $1, x, \dots, x^{n-4}, \cos x, \sin x, \cos(bx), \sin(bx)$, with $b > 1$ (bottom left); $1, x, \dots, x^{n-4}, \cosh x \cos(bx), \cosh x \sin(bx), \sinh x \cos(bx), \sinh x \sin(bx)$ (bottom right).

This makes it sufficient to apply our numerical procedure for the computation of $\ell_L := \ell_n(1, b)$. The corresponding graph in function of the single variable b is shown for increasing values of $n \geq 3$ in Figure 2, top right. For $n = 3$, the procedure confirms the results obtained in [6], namely the fact that the critical length $\ell_3(a, b)$ is the only solution of the equation

$$(b^2 - a^2) \sinh(ax) \sin(bx) = 2ab(1 - \cosh(ax) \cos(bx)), \quad x \in \left] \frac{\pi}{b}, \frac{2\pi}{b} \right[. \quad (24)$$

With the same notation as for the the cycloidal example, it is the first positive zero of the Wronskian $W(S_3, S'_3)$, while the function S_3 has no positive zero. From Figure 2, we conjecture that this class of spaces follows the same two-by-two behaviour as the cycloidal spaces (apart from the connection with Bessel functions) involving only one of the Wronskians in formula (20). Nevertheless, our purpose here is not to solve this conjecture.

- $p_3(x) = (x^2 + a^2)(x^2 + b^2)$, $a, b > 0$:

We first assume that $a \neq b$. Here, the space \mathbb{E}_n is spanned by the $(n+1)$ functions $1, x, \dots, x^{n-4}, \cos(ax), \sin(ax), \cos(bx), \sin(bx)$. Assuming that $a < b$, and taking account of (23), we apply the numerical procedure to compute $\ell_L := \ell_n(1, b)$, for $b > 1$. For each $n \geq 3$, the presence of one cusp in the graph indicates that the two parts of the graph are related to two different Wronskians. Second interesting observation: the right parts of the graphs follow the same two-by-two behaviour as the cycloidal spaces, while the first parts change whenever n increases. The results obtained in [6] for $n = 3$ are confirmed, namely:

- for $a < b \leq 3a$, $\ell_3(a, b)$ is the first zero of S_3 , and it is the only solution of the equation

$$b(\sin(ax) = a \sin(bx)), \quad x \in \left[\frac{\pi}{b} \left\lfloor \frac{b}{a} \right\rfloor, \frac{\pi}{b} \left\lceil \frac{b}{a} \right\rceil \right],$$

where $\lceil \cdot \rceil$ denotes the ceiling function;

- for $b \geq 3a$, $\ell_3(a, b)$ is the first zero of $W(S_3, S'_3)$, and it is the only solution of the equation

$$(b-a) \sin\left(\frac{(b+a)x}{2}\right) + (b+a) \sin\left(\frac{(b-a)x}{2}\right) = 0, \quad x \in \left[\frac{2\pi}{b}, \frac{2\pi}{b-a} \right].$$

When $a = b$, the space \mathbb{E}_n is spanned by the $(n+1)$ functions $1, x, \dots, x^{n-4}, \cos(bx), \sin(bx), x \cos(bx), x \sin(bx)$, and the graphs of the critical lengths $\ell_n(b) = \ell_n(1)/b$ (not shown here) change whenever the dimension increases, with $\ell_n(1)$ in accordance with the left parts of the case $a = 1 < b$ of which it is the limit situation.

- $p_3(x) = x^4 + 2(b^2 - a^2)x^2 + (a^2 + b^2)^2$, $a, b > 0$:

Here, the space \mathbb{E}_n is spanned on \mathbb{R} by the $(n+1)$ functions $1, x, \dots, x^{n-4}, \cosh(ax) \cos(bx), \sinh(ax) \cos(bx), \cosh(ax) \sin(bx), \sinh(ax) \sin(bx)$. In [6], the critical length $\ell_3(a, b)$ was proved to be the only solution of the equation

$$b \tanh(ax) = a \tan(bx), \quad x \in \left[\frac{\pi}{b}, \frac{3\pi}{2b} \right]. \quad (25)$$

It is the first positive zero of S_3 , while the Wronskian $W(S_3, S'_3)$ has no positive zero. As n increases, we can see, in the graph of $\ell_L := \ell_n(1, b)$, the presence of one cusp (for $n = 5, 6, 7$), then two cusps ($n = 8$), indicating that two (resp. three) of the Wronskians appearing in formula (20) are involved in the graph.

4.3.3 More examples

These additional examples are intended to show that our numerical procedure is not limited to classes of spaces obtained by repeated integration.

- More pairs of trigonometric functions:

The critical length of the $(2n)$ -dimensional space \mathbb{E}_{2n-1} spanned by the functions $\cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(nx), \sin(nx)$ is equal to π . This well-known result (see, e.g., [39]) was successfully checked through the numerical procedure for several values of n . But what about other pairs of trigonometric functions? A very partial answer to this question is given for three pairs $\cos(ax), \sin(ax), \cos(bx), \sin(bx), \cos(cx), \sin(cx)$, with $0 < a < b < c$. Without loss of generality, we take $a = 1$, and for several values of $b > 1$, we show in Figure 3 the graph of the critical length as a function of the only variable $c > b$. As a generalisation of what happened with the four-dimensional

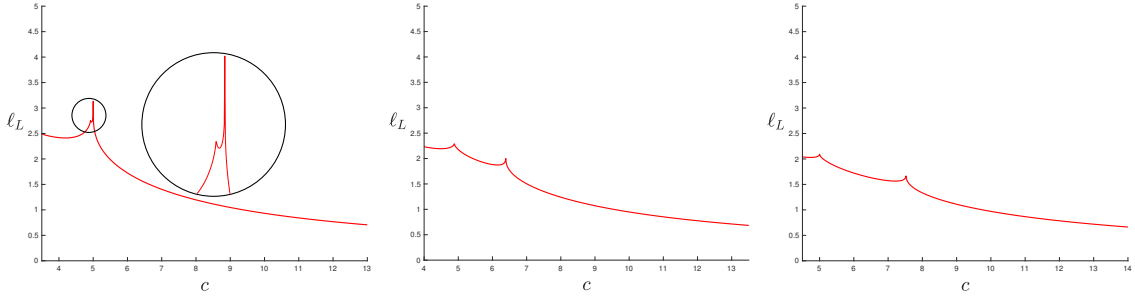


Figure 3: Critical length for the space \mathbb{E}_n spanned by $\cos ax, \sin ax, \cos bx, \sin bx, \cos cx, \sin cx$, with $a = 1$ and, from left to right, $b = 3, 3.5, 4$.

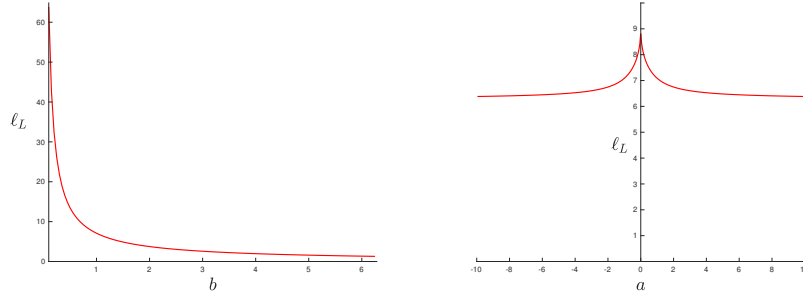


Figure 4: Critical length for the space \mathbb{E}_4 spanned by $1, x, e^{ax}, \cos bx, \sin bx$ with $a = 1$ (left) and $b = 1$ (right).

space spanned by two different pairs (see Figure 2, bottom left) we can see in these examples that the curve presents two cusps (*i.e.*, three active Wronskians) and we can observe how they evolve as b increases. However, these few pictures are certainly not enough to state any possible conjecture.

- A non-symmetric example:

Of course, the procedure is not limited to spaces closed under reflection. This is the reason why we conclude the illustrations with the characteristic polynomial $p_4(x) = x^2(x - a)(x^2 + b^2)$, where a is any non-zero real, and where $b > 0$. In other words, \mathbb{E}_4 is spanned by $1, x, e^{ax}, \cos(bx), \sin(bx)$. In Figure 4, we show the graph of the critical length $\ell_4(a, 1)$ as a function of $a \neq 0$, and the graph of $\ell_4(1, b)$ as a function of $b > 0$. Note that the limit value $\ell_4(0, 1) := \lim_{a \rightarrow 0} \ell_4(a, 1)$ is the value $\ell_4 = 8.9868$ given in (16).

5 Critical length: why?

In this concluding section, we gather crucial remarks which clearly point out the importance of knowing the critical lengths, and thus the interest of our numerical method to compute them.

Due to Theorem 2.6, it may seem tempting to consider systems of weight functions as the starting point of anything concerning EC-spaces on a given closed bounded interval $[a, b]$, all the more so as the classical approximation results generalising the polynomial framework are based on weight functions [14, 39]. If one adopts this angle, the Bernstein bases relative to $[a, b]$ are then naturally provided by the associated integral recurrence relations (8). This approach, thoroughly developed

in [3, 4]¹ was certainly worth attention for theoretical purposes when it was elaborated. It was also commonly used to develop examples, in particular with ordinary (rather than generalised) integration, see for instance [10, 15, 40]. It is important though to draw the reader's attention to its limitations on the practical side.

First of all, if we start with any arbitrary system (w_0, \dots, w_n) of weights functions on $[a, b]$, chances are that the associated space $\mathbb{E}_n := EC(w_0, w_1, \dots, w_n)$ will not present much interest for design or any other application. As a general rule, replacing polynomial spaces by given EC-spaces is motivated by either geometric constraints or likely to happen shape effects, or whatever purpose, but always with the space \mathbb{E}_n of functions which we expect to work in as the starting point. Unless the dimension is really small, neither on which intervals it is an EC-space (possibly good for design) nor possible associated systems of weight functions are easy-to-solve questions. This is why constructing Bernstein bases through (8) may be extremely limiting. With a view to stress this limitation, we take the simplest class of trigonometric spaces spanned by the function $1, x, \dots, x^{n-2}, \cos x, \sin x$, with the following notations:

$$L_n = D^{n+1} + D^{n-1}, \quad \ell_n := \text{critical length of } L_n, \quad n \geq 1.$$

Subsequently, we work with a fixed integer $n \geq 1$, and a fixed interval $[a, b]$, with $0 < b - a < \ell_n$. For each $k \geq 0$, the notations $\mathbb{E}_k, \mathbb{P}_k$, stand for the restrictions to $[a, b]$ of $\ker L_k$, and of the degree k polynomial space, respectively. From the condition $b - a < \ell_n$ and from Theorems 2.6 and 2.7, we know that one can find infinitely many nested sequences

$$\mathbb{E}_0^* \subset \mathbb{E}_1^* \subset \dots \subset \mathbb{E}_{n-1}^* \subset \mathbb{E}_n^* := \mathbb{E}_n, \quad (26)$$

where, for $i = 0, \dots, n-1$, \mathbb{E}_i^* is an $(i+1)$ -dimensional EC-space (or W-space as well) on $[a, b]$, and the search for such nested sequences is equivalent to the search for systems (w_0, \dots, w_n) of weights functions on $[a, b]$ such that $\mathbb{E}_n = EC(w_0, \dots, w_n)$. Now, the space \mathbb{E}_n contains two obvious nested sequences of W-spaces

$$\mathbb{P}_0 \subset \mathbb{P}_1 \subset \dots \subset \mathbb{P}_{n-2} \subset \mathbb{E}_n, \quad \mathbb{E}_1 \subset \mathbb{E}_2 \subset \dots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n. \quad (27)$$

Each of them is “almost” a sequence (26), and we will subsequently discuss when it is “exactly” of the form (26).

- The case $n = 1$. When $n = 1$, both nested sequences (27) reduce to \mathbb{E}_1 , for which the critical length is $\ell_1 = \pi$. The condition $b - a < \pi$ is thus the necessary and sufficient condition for the existence of a one-dimensional W-space \mathbb{E}_0^* on $[a, b]$ contained in \mathbb{E}_1 . We know that there are infinitely many such spaces, corresponding to inclusions of the form

$$\mathbb{E}_0^* = EC(\omega_0) \subset \mathbb{E}_1^* = \mathbb{E}_1 = EC(\omega_0, \omega_1).$$

According to Theorem 2.7 and Remark 2.9 they all are obtained with

$$\omega_0 := \alpha_0 \beta_0 + \alpha_1 \beta_1 \text{ for some positive } \alpha_0, \alpha_1, \quad \omega_1 := \left(\frac{\alpha_1 \beta_1}{\omega_0} \right)', \quad (28)$$

where (β_0, β_1) denotes the Lagrange basis of \mathbb{E}_1 relative to (a, b) , that is

$$\beta_0(x) := \frac{\sin(b-x)}{\sin(b-a)}, \quad \beta_1(x) := \frac{\sin(x-a)}{\sin(b-a)}, \quad x \in [a, b].$$

¹These references address the general framework of integral-positive weight functions, with application to EC-spaces and associated splines. Integral recurrence relations can also be understood through the blossoming approach, as shown in [27], including EC-piecewise spaces or Quasi EC-spaces, and associated splines, see also [29, 30].

- The left nested sequence in (27). Assume that $n \geq 2$. The left nested sequence can be completed into a sequence of the form (26) if and only if we can find an n -dimensional W-space on $[a, b]$, \mathbb{F}_{n-1} such that $\mathbb{P}_{n-2} \subset \mathbb{F}_{n-1} \subset \mathbb{E}_n$. This is known to be possible if and only if the two-dimensional space $D^{n-1}\mathbb{E}_n$ is an EC-space on $[a, b]$, see [26]. Since $D^{n-1}\mathbb{E}_n = \mathbb{E}_1$, this requirement is equivalent to $b - a < \ell_1 = \pi$. Suppose the latter condition to be fulfilled. We then have infinitely many different choices for the subspace \mathbb{F}_{n-1} , namely

$$\mathbb{F}_{n-1} = EC(\underbrace{\mathbb{I}, \mathbb{I}, \dots, \mathbb{I}}_{(n-2) \text{ times}}, \omega_0) \subset \mathbb{E}_n = EC(\underbrace{\mathbb{I}, \mathbb{I}, \dots, \mathbb{I}}_{(n-2) \text{ times}}, \omega_0, \omega_1), \quad (29)$$

where ω_0, ω_1 are defined in (28). The corresponding nested sequence (26) is therefore given by $\mathbb{E}_i^* = EC(w_0, \dots, w_i)$, $i = 0, \dots, n$, with

$$w_i := \mathbb{I} \text{ for } i = 0, \dots, n-2, \quad w_{n-1} := \omega_0, \quad w_n := \omega_1. \quad (30)$$

Since $b - a < \pi = \ell_1$, for each $n \geq 2$, the space \mathbb{E}_n is an EC-space good for design on $[a, b]$. Relative to (a, b) we can thus compute the Bernstein basis in \mathbb{E}_n via the integral recurrence relations formulæ (8) associated with the previous sequence of weight functions.

- The right nested sequence in (27). We now consider the right nested sequence in (27). It can be completed into a sequence of the form (26) if and only if we can find a one-dimensional EC-space $\mathbb{E}_0^* \subset \mathbb{E}_1$, that is, once again if and only if $b - a < \ell_1 = \pi$. Supposing that $b - a < \pi$, through (28), the complete nested sequence $\mathbb{E}_0^* \subset \mathbb{E}_1 \subset \dots \subset \mathbb{E}_n$ provides us with another sequence (w_0, \dots, w_n) of weight functions on $[a, b]$, with $w_0 := \omega_0$, $w_1 := \omega_1$. As for w_2, \dots, w_n they are provided by the nested sequence $DL_1\mathbb{E}_2 \subset \dots \subset DL_1\mathbb{E}_{n-1} \subset DL_1\mathbb{E}_n$ (where the generalised derivative L_1 is defined by ω_0, ω_1), that is, after multiplication by $\omega_0\omega_1$, $\mathbb{P}_0 \subset \mathbb{P}_1 \subset \dots \subset \mathbb{P}_{n-2}$. We can therefore also write $\mathbb{E}_n = EC(w_0, \dots, w_n)$ with

$$w_0 := \omega_0, \quad w_1 := \omega_1, \quad w_2 = 1/(\omega_0\omega_1), \quad w_i := \mathbb{I} \text{ for } i = 3, \dots, n. \quad (31)$$

We thus have at our disposal two obvious ways – (30) and (31) – to obtain the Bernstein basis relative to (a, b) in \mathbb{E}_n via integral recurrence relations (8), and to use it to handle associated curves for design or any other purposes. Unfortunately, this is only valid under the assumption $b - a < \pi$. Most of the time, trigonometric spaces are implicitly studied through (30) for whatever purposes, for e.g., [10, 16]. We can then avoid any reference to weight functions, starting the recurrence formulæ (8) from the Lagrange basis (β_0, β_1) of \mathbb{E}_1 . The same is generally done for the construction for splines [15, 40, 11, 17, 18, 19, 5].

As soon as $n \geq 3$, working under the assumption $b - a < \pi$ is actually a very limited use of trigonometric curves, if only with reference to (16). To emphasise this point, observe that ℓ_n tends to infinity with n due to (22). This limitation is illustrated in Figure 5 for increasing values of $n \geq 4$. A control polygon being given, we show a few corresponding curves in \mathbb{E}_n on $[0, h]$ depending on h . When h increases, they go from the polynomial curve of degree n (visually obtained for $h = 0.1$) to the *critical curve* obtained for $h = \ell_{n-1}^-$ (ℓ_{n-1} being the critical length for design in the space \mathbb{E}_n). In all cases we also show the curve obtained with $h = 3.14$, which is quite far from the critical curve, and can soon hardly be distinguished from the polynomial curve as n increases. This naturally raises the following question: *for $n \geq 4$, given that we simultaneously lose the remarkable simplicity of polynomials, is it really worthwhile replacing the degree n polynomial space on $[a, b]$ by the trigonometric space \mathbb{E}_n when requiring that $b - a < \pi$?*² Under the weaker assumption

²Note that the case $n = 3$ has been considered in many papers: it is well known that, when h increases from 0^+ to $2\pi^-$ the curve evolves from the cubic curve up to the segment joining the extreme control points (critical curve) and $h = \pi$ already yields a curve quite close to the polynomial one. Considering this case along with all pictures in Fig. 5, one can observe that the amplitude of the shape effects permitted by cycloidal spaces weakens in an “oscillating” way as n increases. This corresponds to the fact that the limit curves ignore the (two) central control point(s).

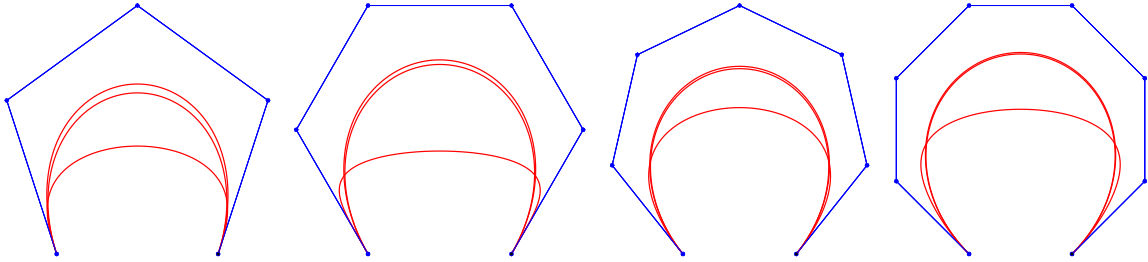


Figure 5: For $n = 4; 5; 6; 7$, Design in the space \mathbb{E}_n spanned by $1, x, \dots, x^{n-2}, \cos x, \sin x$ on $[0, h]$, with, further and further from the control polygon, $h = 0.1$; $h = 3.14$; and $h = \ell_{n-1}^-$ (that is, $h = 6.283$ for $n = 4$; $h = 8.986$ for $n = 5, 6$; $h = 11.526$ for $n = 7$).

$b - a < 2\pi$, we similarly obtain the Bernstein basis relative to (a, b) in the trigonometric space \mathbb{E}_n on $[a, b]$ for any $n > 3$ through formulæ (8) and ordinary integration, starting from the Bernstein basis in \mathbb{E}_3 (for which explicit expressions can be found in [21]). Very soon as n increases, we will be facing the same limitation. This can easily be guessed from the table giving the successive critical lengths in [9].

This simple example clearly illustrates that only the knowledge of the critical length (for design) enables us to take full advantage of the parameter(s) attached to a given space of functions. Not all kernels of differential operators have really appealing features, say for design. When investigating a new space, the most efficient way to learn if it produces remarkable shape effects and / or if it is worthwhile combining it with other spaces to build splines, is to consider its critical curves, that is, the curves obtained close to the critical length for design, which should therefore be determined beforehand, see [6, 31, 33]. Unfortunately, we cannot expect this to be done experimentally from the visual analysis of either the curves or the Bernstein-type basis. Of course, if a parametric curve visibly contradicts the shape of its control polygon, or if one function of the expected Bernstein basis clearly takes negative values, we are certainly beyond the critical length for design. However, it is necessary to stress that we can be beyond it in spite of a visually satisfying behaviour. Moreover, the existence of a Bernstein basis / of a normalised totally positive basis implies that we are within the critical length for design only if, beforehand, we know that we are within the critical length (see Corollary 2.8 and Theorem 4.1 of [7]).

These comments clearly emphasise how important it is to have at our disposal a reliable numerical method for the computation of critical lengths. We would like to mention that our procedure can also be useful on the theoretical side. It is well known that most results of the polynomial framework extend to EC-spaces. Nonetheless, there are a few exceptions. To point out such exceptions, it can be necessary to observe what happens for kernels of linear differential operators with constant coefficients sufficiently close to their critical lengths. This will be illustrated in future work concerning dimension elevation.

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References

- [1] C.V. Beccari, G. Casciola, M-L. Mazure, Piecewise Extended Chebyshev spaces: A numerical test for design, *Applied Math. Comput.*, 296 (2017), 239–256.

- [2] C.V. Beccari, G. Casciola, M.-L. Mazure, Design or not design? A numerical characterisation for piecewise Chebyshevian splines, *Numer. Algor.*, online, <https://doi.org/10.1007/s11075-018-0533-z>
- [3] D. Bister, Ein neuer Zugang für eine verallgemeinerte Klasse von Tschebyscheff-Splines, Dissertation, Karlsruhe, 1996.
- [4] D. Bister and H. Prautzsch, A New Approach to Tchebycheffian B-Splines, in *Curves and Surfaces with Applications in CAGD*, Vanderbilt University Press, 1997, 35–41.
- [5] C. Bracco, T. Lyche, C. Manni, F. Roman, H. Speleers, Generalized spline spaces over T-meshes: Dimension formula and locally refined generalized B-splines, *Applied Math. Comput.*, 272 (2016), 187–198.
- [6] M. Brilleaud, M.-L. Mazure, Mixed hyperbolic/trigonometric spaces for design, *Comp. Math. Applic.*, 64 (2012), 2459–2477.
- [7] J.-M. Carnicer, E. Mainar, J.-M. Peña, Critical Length for Design Purposes and Extended Chebyshev Spaces, *Constr. Approx.*, 20 (2004), 55–71.
- [8] J.-M. Carnicer, E. Mainar, J.-M. Peña, On the Critical Lengths of Cycloidal Spaces, *Constr. Approx.*, 39 (2014), 573–583.
- [9] J.-M. Carnicer, E. Mainar, J.-M. Peña, Critical length of cycloidal spaces are zeros of Bessel functions, *Calcolo*, 54 (2017), 1521–1531.
- [10] Q. Chen, G. Wang, A class of Bézier-like curves, *Comput. Aided Geom. Design*, 20 (2003), 29–39.
- [11] P. Costantini, C. Manni, F. Pelosi, M. L. Sampoli, Quasi-interpolation in isogeometric analysis based on generalized B-splines, *Comput. Aided Geom. Design*, 27 (2010), 656–668.
- [12] T.N.T. Goodman, Shape preserving representations, In *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L.L. Schumaker (eds), Acad. Press, NY, 1989, 333–357.
- [13] M. Hoffmann, Y. Li, G. Wang, Paths of C-Bézier and C-B-spline curves, *Comput. Aided Geom. Design*, 23 (2006), 463–475.
- [14] S.J. Karlin, W.J. Studden, *Tchebycheff Systems: with applications in analysis and statistics*, Wiley Interscience, N.Y., 1966.
- [15] E. Mainar and J.-M. Peña, A general class of Bernstein-like bases, *Comp. & Math. Applic.*, 53 (2007), 1686–1703.
- [16] E. Mainar, J.-M. Peña, J. Sánchez-Reyes, Shape preserving alternatives to the rational Bézier model, *Comput. Aided Geom. Design*, 18 (2001), 37–60.
- [17] C. Manni, F. Pelosi, M.L. Sampoli, Generalized B-splines as a tool in isogeometric analysis, *Comp. Methods Applied Mech. Eng.*, 200 (2011), 867–881.
- [18] C. Manni, F. Pelosi, H. Speleers, Local Hierarchical h-refinements in IgA Based on Generalized B-splines, *Lecture Notes in Computer Science*, 8177 (2014) 341–363.
- [19] C. Manni, A. Reali, H. Speleers, Isogeometric collocation methods with generalized B-splines, *Comp. Math. Appl.*, 70 (2015), 1659–1675
- [20] M.-L. Mazure, Blossoming: a geometrical approach, *Constr. Approx.*, 15 (1999), 33–68.
- [21] M.-L. Mazure, Chebyshev-Bernstein bases, *Comput. Aided Geom. Design*, 16 (1999), 649–669.
- [22] M.-L. Mazure, Chebyshev splines beyond total positivity, *Adv. in Comput. Math.*, 14 (2001), 129–156.
- [23] M.-L. Mazure, Blossoms and optimal bases, *Adv. Comp. Math.*, 20 (2004), 177–203.
- [24] M.-L. Mazure, Chebyshev spaces and Bernstein bases, *Constr. Approx.*, 22 (2005), 347–363.
- [25] M.-L. Mazure, Ready-to-blossom bases in Chebyshev spaces, in *Topics in Multivariate Approximation and Interpolation*, K. Jetter, M. Buhmann, W. Haussmann, R. Schaback, et J. Stoeckler (eds), Elsevier, 12, 2006, 109–148.
- [26] M.-L. Mazure, Bernstein-type operators in Chebyshev spaces, *Numer. Algor.*, 52 (2009), 93–128.
- [27] M.-L. Mazure, On differentiation formulæ for Chebyshevian Bernstein and B-spline bases, *Jaén J. Approx.*, 1 (2009), 111–143.

- [28] M.-L. Mazure, Finding all systems of weight functions associated with a given Extended Chebyshev space, *J. Approx. Theory*, 163 (2011), 363–376.
- [29] M.-L. Mazure, How to build all Chebyshevian spline spaces good for geometric design?, *Numer. Math.*, 119 (2011), 517–556.
- [30] M.-L. Mazure, On a general new class of quasi-Chebyshevian splines, *Num. Algorithms*, 58 (2011), 399–438.
- [31] M.-L. Mazure, On a new criterion to decide whether a spline space can be used for design, *BIT Num. Math.*, 52 (2012), 1009–1034.
- [32] M.-L. Mazure, From Taylor interpolation to Hermite interpolation via duality, *Jaén J. Approx.*, 4 (2012), 15–45.
- [33] M.-L. Mazure, Constructing totally positive piecewise Chebyhevian B-splines, *J. Comput. Applied Math.*, 342 (2018), 550–586.
- [34] M.-L. Mazure, H. Pottmann, Tchebycheff curves, in *Total Positivity and its Applications*, M. Gasca and C.A. Micchelli (eds), Kluwer Ac. Pub., 1996, 187–218.
- [35] K. Melk mi, Orthogonalit  des B-splines de Chebyshev cardinales dans un espace de Sobolev pond r , doctirat de luniversit  Joseph Fourier, Grenoble, 1999.
- [36] H. Pottmann, The geometry of Tchebycheffian splines, *Comput. Aided Geom. Design*, 10 (1993), 181–210.
- [37] H. Pottmann and M.G. Wagner, Helix splines as an example of affine Tchebycheffian splines, *Adv. Comput. Math.* 2 (1994), 123–142.
- [38] A. R th, An OpenGL and C++ based function library for curve and surface modeling in a large class of extended Chebyshev spaces, *Trans. Math. Software*, to appear.
- [39] L.L. Schumaker, *Spline Functions*, Wiley Interscience, N.Y., 1981.
- [40] G. Wang, M. Fang, Unified and extended form of three types of splines, *J. Comp. Applied Math*, 216 (2008), 498–508.
- [41] J. Zhang, C-curves: An extension of cubic curves, *Comput. Aided Geom. Design*, 13 (1996), 199–217.