A table of short-period Tausworthe generators for Markov chain quasi-Monte Carlo

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Abstract

We consider the problem of estimating expectations by using Markov chain Monte Carlo methods and improving the accuracy by replacing IID uniform random points with quasi-Monte Carlo (QMC) points. Recently, it has been shown that Markov chain QMC remains consistent when the driving sequences are completely uniformly distributed (CUD). However, the definition of CUD sequences is not constructive, so an implementation method using short-period Tausworthe generators (i.e., linear feedback shift register generators over the two-element field) that approximate CUD sequences has been proposed. In this paper, we conduct an exhaustive search of shortperiod Tausworthe generators for Markov chain QMC in terms of the t-value, which is a criterion of uniformity widely used in the study of QMC methods. We provide a parameter table of Tausworthe generators and show the effectiveness in numerical examples using Gibbs sampling.

Keywords: Pseudorandom number generation, Quasi-Monte Carlo, Markov chain Monte Carlo, Polynomial lattice point set, Continued fraction expansion

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1. Introduction

We consider the problem of estimating the expectation $E_{\pi}[f(\mathbf{X})]$ by using Markov chain Monte Carlo (MCMC) methods for a target distribution π and

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some function f. For this problem, we want to improve the accuracy by replacing independent and identically distributed (IID) uniform random points with quasi-Monte Carlo (QMC) points. However, typical QMC points (e.g., Sobol', Faure, and Niederreiter-Xing) are not applicable in general. Motivated by a simulation study by Liao [20], Owen and Tribble [24] and Chen et al. [2] proved that Markov chain QMC remains consistent when the driving sequences are completely uniformly distributed (CUD). Here, a sequence $u_0, u_1, u_2, \ldots \in [0, 1)$ is said to be CUD if overlapping s-blocks $(u_i, u_{i+1}, \ldots, u_{i+s-1}), i = 0, 1, 2, \ldots$, are uniformly distributed for every dimension $s \geq 1$.

Levin [19] provided several constructions for CUD sequences, but they are not convenient to implement. Instead, to construct CUD sequences approximately, Tribble and Owen [32] and Tribble [31] proposed an implementation method using short-period linear congruential and Tausworthe generators (i.e., linear feedback shift register generators over the two-element field $\mathbb{F}_2 := \{0, 1\}$) that run for the entire period. Chen et al. [3] implemented short-period Tausworthe generators optimized in terms of the equidistribution property, which is a coarse criterion used in the area of pseudorandom number generation (see [1, §8.1] for the complete parameter table). In the theory of (t, m, s)-nets and (t, s)-sequences, the t-value is a central criterion of uniformity. In fact, typical QMC points (e.g., Sobol', Faure, and Niederreiter-Xing) are optimized in terms of the t-value (see [23, 5]).

The aim of this paper is to conduct an exhaustive search of short-period Tausworthe generators for Markov chain QMC in terms of the *t*-value and to provide a parameter table of Tausworthe generators. It is known that Tausworthe generators can be viewed as polynomial Korobov lattice point sets with a denominator polynomial p(x) and a numerator polynomial q(x)over \mathbb{F}_2 (e.g., see [17, 18]). For dimension s = 2, there is a connection between the *t*-value and continued fraction expansions, that is, the *t*-value is optimal (i.e., the *t*-value is zero) if and only if the partial quotients in the continued fraction of q(x)/p(x) are all of degree one. To satisfy the definition of CUD sequences approximately, we want to search for parameters (p(x), q(x)) whose t-values are optimal for s = 2 and as small as possible for s > 3. As a previous study, in 1993, Tezuka and Fushimi [30] proposed an algorithm to search for such parameters using a polynomial analogue of Fibonacci numbers from the viewpoint of continued fraction expansions. Thus, we refine their algorithm on modern computers, and conduct an exhaustive search again. In addition, we report numerical examples using Gibbs sampling in which the resulting QMC point sets perform better than the existing point sets developed by Chen et al. [3].

One might consider searching for parameters (p(x), q(x)) with *t*-value zero for s = 3. Kajiura et al. [12] proved that there exists no maximal-period Tausworthe generator with this property.

The remainder of this paper is organized as follows: In Section 2, we briefly recall the definition of CUD sequences, Tausworthe generators, and the *t*-value and equidistribution property. Section 3 is devoted to our main results: we describe an exhaustive search algorithm and provide a table of short-period Tausworthe generators for Markov chain QMC. We also compare our new generators with existing generators developed by Chen et al. [3] in terms of the *t*-value and equidistribution property. In Section 4, we present numerical examples using Gibbs sampling. In Section 5, we conclude this paper.

2. Preliminaries

We refer the reader to [23, 5, 17, 16] for general information.

2.1. Discrepancy and completely uniformly distributed sequences

Let $P_s = {\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{N-1}} \subset [0, 1)^s$ be an s-dimensional point set of N elements in the sense of a "multiset". We recall the definition of the discrepancy as a criterion of uniformity of P_s .

Definition 1 (Discrepancy). For a point set $P_s = {\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{N-1}} \subset [0, 1)^s$, the *(star) discrepancy* is defined as

$$D_N^{*s}(P_s) := \sup_J \left| \frac{\nu(J; P_s)}{N} - \operatorname{vol}(J) \right|,$$

where the supremum is taken over every sub-interval $J = [0, t_1) \times \cdots \times [0, t_s) \subset [0, 1)^s$, $\nu(J; P_s)$ is the number of points from P_s that belong to J, and $\operatorname{vol}(J) := t_1 \cdots t_s$ is the volume of J.

If $D_N^{*s}(P_s)$ is close to zero, we regard P_s as highly uniformly distributed.

Next, we define the CUD property for a one-dimensional infinite sequence $\{u_i\}_{i=0}^{\infty} \subset [0, 1).$

Definition 2 (CUD sequences). A one-dimensional infinite sequence $u_0, u_1, u_2, \ldots \in [0, 1)$ is said to be *completely uniformly distributed* (*CUD*) if overlapping *s*-blocks satisfy

$$\lim_{N \to \infty} D_N^{*s} \left((u_0, \dots, u_{s-1}), (u_1, \dots, u_s), \dots, (u_{N-1}, \dots, u_{N+s-2}) \right) = 0$$

for every dimension $s \ge 1$, that is, the sequence of s-blocks $(u_i, \ldots, u_{i+s-1}), i = 0, 1, \ldots$, is uniformly distributed in $[0, 1)^s$ for every dimension $s \ge 1$.

This is one of the definitions of a random sequence from Knuth [13]. From the viewpoint of QMC, it is desirable that D_N^{*s} converges to zero fast if $N \to \infty$; see [7, 6] for details. As a necessary and sufficient condition of Definition 2, Chentsov [4] showed that non-overlapping blocks satisfy

$$\lim_{N \to \infty} D_N^{*s} \left((u_0, \dots, u_{s-1}), (u_s, \dots, u_{2s-1}), \dots, (u_{s(N-1)}, \dots, u_{Ns-1}) \right) = 0$$

for every dimension $s \geq 1$. Thus, we use a sequence $\{u_i\}_{i=0}^{\infty} \subset [0,1)$ for Markov chain QMC in this order.

2.2. Tausworthe generators

We recall some results of Tausworthe generators. Let $\mathbb{F}_2 := \{0, 1\}$ be the two-element field, and perform addition and multiplication over \mathbb{F}_2 (or modulo 2).

Definition 3 (Tausworthe generators [27, 14, 15]). Let $p(x) := x^m - c_1 x^{m-1} - \cdots - c_{m-1} x - c_m \in \mathbb{F}_2[x]$. Consider the linear recurrence

$$a_i \coloneqq c_1 a_{i-1} + \dots + c_m a_{i-m} \in \mathbb{F}_2, \tag{1}$$

whose characteristic polynomial is p(x). Let σ be a step size with $0 < \sigma < 2^m - 1$ and

$$u_i := \sum_{j=0}^{w-1} a_{i\sigma+j} 2^{-j-1} \in [0,1)$$
(2)

be the *output* at step *i*, where *w* is the *word size* of the intended machine. If p(x) is primitive, $(a_0, \ldots, a_{m-1}) \neq (0, \ldots, 0)$, and $gcd(\sigma, 2^m - 1) = 1$, then the sequences (1) and (2) are both purely periodic with maximal period $2^m - 1$. Assume the maximal periodicity and $\sigma \geq w$. A generator in such a class is called a *Tausworthe generator* (or a *linear feedback shift register generator*).

Let $N = 2^m$ and consider a sequence

$$u_0, u_1, \dots, u_{N-2}, u_{N-1} = u_0, \dots \in [0, 1)$$
 (3)

generated from a Tausworthe generator with the period length N-1. We consider s-dimensional overlapping points $\mathbf{u}_i = (u_i, \ldots, u_{i+s-1})$ for $i = 0, 1, \ldots, N-2$, that is, $\mathbf{u}_0 = (u_0, \ldots, u_{s-1})$, $\mathbf{u}_1 = (u_1, \ldots, u_s)$, \ldots , $\mathbf{u}_{N-2} = (u_{N-2}, u_0, \ldots, u_{s-2})$. Adding the origin $\{\mathbf{0}\}$, we regard a point set

$$P_s = \{\mathbf{0}\} \cup \{\mathbf{u}_i\}_{i=0}^{N-2} \subset [0,1)^s \tag{4}$$

as a QMC point set. Note that the cardinality is $|P_s| = 2^m$.

Moreover, Tausworthe generators can be represented as a polynomial analogue of linear congruential generators:

$$q(x) := x^{\sigma} \mod p(x) \tag{5}$$

$$X_i(x) := q(x)X_{i-1}(x) \mod p(x)$$
 (6)

$$X_i(x)/p(x) = a_{i\sigma}x^{-1} + a_{i\sigma+1}x^{-2} + a_{i\sigma+2}x^{-3} + \dots \in \mathbb{F}_2((x^{-1})).$$
(7)

Then, the sequence (2) is expressed as $u_i = \nu_w(X_i(x)/p(x))$, where a map $\nu_w : \mathbb{F}_2((x^{-1})) \to [0, 1)$ is given by $\sum_{j=j_0}^{\infty} k_j x^{-j-1} \mapsto \sum_{j=\max\{0, j_0\}}^{w-1} k_j 2^{-j-1}$, which is obtained by substituting x = 2 into (7) and truncating the value with the word size w. Furthermore, according to [17, § 5.5] and [18], a point set P_s in (4) can also be represented as a *polynomial Korobov lattice* point set:

$$P_s = \left\{ \nu_w \left(\frac{h(x)}{p(x)} (1, q(x), q(x)^2, \dots, q(x)^{s-1}) \right) \mid \deg(h(x)) < m \right\},$$
(8)

where $m = \deg(p(x))$ and the map ν_w is applied component-wise. A pair of polynomials (p(x), q(x)) is a parameter set of P_s . Thus, to construct a point set that approximates CUD sequences in Definition 2, we want to find a pair (p(x), q(x)) with small discrepancies $D_N^{*s}(P_s)$ for each $s \ge 1$.

2.3. Criteria of uniformity

Generally, calculating $D_N^{*s}(P_s)$ is NP-hard [11]. A point set P_s in (4) generated from a Tausworthe generator is a *digital net*, so we can compute the *t*-value closely related to $D_N^{*s}(P_s)$ for $N = 2^m$.

Definition 4 ((t, m, s)-nets). Let $s \ge 1$ and $0 \le t \le m$ be integers. Then, a point set P_s consisting of 2^m points in $[0, 1)^s$ is called a (t, m, s)-net (in base 2) if every subinterval $E = \prod_{j=1}^s [r_j/2^{d_j}, (r_j + 1)/2^{d_j})$ in $[0, 1)^s$ with integers $d_j \ge 0$ and $0 \le r_j < 2^{d_j}$ for $1 \le j \le s$ and of volume 2^{t-m} contains exactly 2^t points of P_s .

For dimension s, the smallest value t for which P_s is a (t, m, s)-net is called the t-value. $D_N^{*s}(P_s) = O(2^t (\log N)^{s-1}/N)$ holds, where the implied constant in the O-notation only depends on s, so a small t-value is desirable. Thus, we want to find Tausworthe generators with pairs of polynomials (p(x), q(x))whose t-values are optimal (i.e., t = 0) for s = 2 and as small as possible for $s \ge 3$. Note that all Tausworthe generators have the t-value zero for s = 1.

Conversely, Chen et al. [3] used the following equidistribution property as a criterion of uniformity:

Definition 5 (s-dimensional equidistribution with *l*-bit accuracy). For $1 \leq s \leq m$ and $1 \leq l \leq m$, a point set P_s consisting of 2^m points in $[0, 1)^s$ is said to be s-dimensionally equidistributed with *l*-bit accuracy if we can partition the s-dimensional unit cube $[0, 1)^s$ into congruent cubic boxes of volume 2^{-sl} by dividing each axis [0, 1) into 2^l intervals, and can obtain an equal number of points from P_s in each box.

For dimension s, the largest value of l for which this definition holds is called the resolution of P_s and denoted by l_s . We have a trivial upper bound $l_s \leq \lfloor m/s \rfloor$. As a criterion of uniformity, a high resolution l_s is desirable. Thus, we define the resolution gap $d_s = \lfloor m/s \rfloor - l_s$ and the sum of resolution gaps $\Delta = \sum_{s=1}^{m} d_s$. If $\Delta = 0$, the generator is said to be fully equidistributed (FE). Note that P_s contains the origin $\{\mathbf{0}\}$ and the output values of a Tausworthe generator for the entire period of $2^m - 1$. Chen et al. [3] implemented FE Tausworthe generators for Markov chain QMC.

3. Main result

3.1. An exhaustive search algorithm using Fibonacci polynomials

To construct a point set that approximates CUD sequences in Definition 2, we search for a pair of polynomials (p(x), q(x)) whose *t*-values are optimal for s = 2 and as small as possible for $s \ge 3$. Thus, we refine the algorithm of Tezuka and Fushimi [30]. For dimension s = 2, there is a connection between the *t*-value of polynomial Korobov lattice point sets (8) and continued fraction expansion of q(x)/p(x). Let

$$\frac{q(x)}{p(x)} = A_0(x) + \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{\ddots} + \frac{1}{A_v(x)}}} =: [A_0(x); A_1(x), A_2(x), \dots, A_v(x)]$$

be the continued fraction expansion of the rational function q(x)/p(x) with a polynomial part $A_0(x) \in \mathbb{F}_2[x]$ and partial quotients $A_k(x) \in \mathbb{F}_2[x]$ satisfying $\deg(A_k(x)) \geq 1$ for $1 \leq k \leq v$.

Theorem 1 ([23, 30]). Let $p(x) \in \mathbb{F}_2[x]$ with $m = \deg(p(x))$ and $q(x) \in \mathbb{F}_2[x]$ with $\deg(q(x)) < m$. Assume $\gcd(p(x), q(x)) = 1$. Then, the twodimensional point set

$$P_2 = \left\{ \nu_w \left(\frac{h(x)}{p(x)} (1, q(x)) \right) \left| \operatorname{deg}(h(x)) < m \right\} \right\}$$

is a (0, m, 2)-net (i.e., the t-value is zero) if and only if the partial quotients in the continued fraction expansion $[0; A_1(x), A_2(x), \ldots, A_v(x)]$ of q(x)/p(x)all have degree one, so v = m.

The next theorem asserts the existence of q(x) with the above property for every irreducible polynomial p(x).

Theorem 2 ([22]). Let p(x) be an irreducible polynomial with $m = \deg(p(x))$ and $q(x) \in \mathbb{F}_2[x]$ with $\deg q((x)) < m$. For each p(x), there are exactly two polynomials q(x) for which the partial quotients of the continued fraction expansion of q(x)/p(x) all have degree one.

In fact, the two polynomials are q(x) and $q^{-1}(x) \mod p(x)$, which mean that we generate Tausworthe generators in normal order and reverse order, respectively. Hence, they yield essentially the same polynomial lattice point set P_s . To obtain (p(x), q(x)) satisfying the above theorems, Tezuka and Fushimi [30] defined a polynomial analogue of Fibonacci numbers as follows:

$$F_k(x) = A_k(x)F_{k-1}(x) + F_{k-2}(x) \qquad (k \ge 2), \tag{9}$$

$$F_0(x) = 1, \quad F_1(x) = A_1(x),$$
(10)

$$A_k(x) = x \text{ or } x + 1 \qquad (k \ge 1).$$
 (11)

They called a pair of polynomials $(F_k(x), F_{k-1}(x))$ a pair of "Fibonacci polynomials" because the partial quotients in the continued fraction of $F_{k-1}(x)/F_k(x)$ are all of degree one. Figure 1 shows the initial part of a tree of Fibonacci polynomials, which was originally illustrated in [28, Figure 4.5]. Note that there are 2^m different pairs $(F_m(x), F_{m-1}(x))$ for Fibonacci polynomials with degree m. From them, we choose a suitable pair (p(x), q(x)) that approximates CUD sequences in Definition 2.



Figure 1: A tree of Fibonacci polynomials.

Now we refine the algorithm of Tezuka and Fushimi [30]. Our exhaustive search algorithm proceeds as follows:

Algorithm 1 An exhaustive search algorithm

- 1: Generate all the pairs $(F_m(x), F_{m-1}(x))$ using the recurrence relation of Fibonacci polynomials (9)–(11).
- 2: Check the primitivity of $F_m(x)$.
- 3: Find σ such that $x^{\sigma} \equiv F_{m-1}(x) \mod F_m(x)$ and $0 < \sigma < 2^m 1$. Check $gcd(\sigma, 2^m 1) = 1$ and $\sigma \geq w$.
- 4: Choose pairs $(F_m(x), F_{m-1}(x))$ whose t-value is equal to or smaller than 3 for s = 3.
- 5: Let $t^{(s)}$ be a *t*-value for dimension *s*. For each $(F_m(x), F_{m-1}(x))$, make a vector $(t^{(4)}, t^{(5)}, t^{(6)}, \ldots, t^{(m)})$ of the *t*-values.
- 6: Sort pairs $(F_m(x), F_{m-1}(x))$ in lexicographic order based on $(t^{(4)}, t^{(5)}, t^{(6)}, \ldots, t^{(m)})$ starting from dimension 4.
- 7: Choose one of the best (or smallest) pairs $(F_m(x), F_{m-1}(x))$ in Step 6.
- 8: Set $(p(x), q(x)) \leftarrow (F_m(x), F_{m-1}(x)).$

In Step 4, this criterion means that the *t*-value is sufficiently small for s = 3; see Remark 2 for details. In Steps 4 and 5, we calculate the *t*-values by using Gaussian elimination [25] instead of solving Diophantine equations in [30, Theorem 1].

Remark 1. In the original paper [30], before Step 2, Tezuka and Fushimi checked the condition

$$F_{m-1}(x)^m + F_{m-1}(x)^n + 1 = 0 \mod F_m(x),$$

where 0 < n < m, to obtain fast Tausworthe generators using trinomial generalized feedback shift register generators. They also restricted the calculation of the *t*-values to only $3 \le s \le 6$. A reason for these conditions might be the difficulty of checking from Steps 2–5 on computers around 1990. As a result, in the range $3 \le m \le 32$, there exist pairs $(F_m(x), F_{m-1}(x))$ only for m = 3, 5, 7, 15, 17, 18, 20, 22, 23, 25, 28, and 31; otherwise, there exists no pair. In the related paper [29], the authors found pairs $(F_m(x), F_{m-1}(x))$ for all $3 \le m \le 21$ under a pentanomial condition. Currently, it is not difficult to remove these conditions when we conduct an exhaustive search on modern computers. In Remark 3, we note a reasonably fast generation method instead of the direct use of Definition 3.

Remark 2. In Step 4, we observed that the smallest *t*-values are 2 or 3 for $10 \le s \le 32$ by exhaustive search. More precisely, there exist pairs

(p(x), q(x)) with t-value two only for $10 \le s \le 14$ and s = 16 and 17, and the number of them are quite few, compared with the number of pairs with t-value three. For example, in the case where s = 17, there exist four pairs with t-value two but 464 pairs with t-value three. Thus, to find a pair (p(x), q(x)) with smaller t-value even for $s \ge 4$, we adopted this criterion.

3.2. Specific parameters

Table 1 lists specific parameters for w = 32,64 and $10 \le m \le 32$. In Table 1, each first and second row shows the coefficients of p(x) and q(x) respectively; for example, 1 1 0 1 means $1+x+x^3$. We also note the step size σ corresponding to q(x). For m = 21 and 28, we obtained the pairs of polynomials (p(x), q(x)) with somewhat large defects $\Delta = 6$ and 4, respectively, so we replaced them by the second-best pairs. Table 2 summarizes the *t*-values and sum of resolution gaps Δ for our new Tausworthe generators (labeled "New") and the existing Tausworthe generators developed by Chen et al. [3] (labeled "Chen") in the range of $2 \le s \le 20$. For $2 \le s \le 5$, our new generators have the t-values equal to or smaller than the existing generators (except for m = 32). It is known that QMC are successful in high-dimensional problems, particularly in the case in which problems are dominated by the first few variables, so we focus on the optimization of leading dimensions. Conversely, from the viewpoint of the FE property, our generators are not FE. We can also optimize both the *t*-values and FE property, but the *t*-values slightly increase. Thus, we prioritized the *t*-values over the FE property for simplicity. The code in C is available at https://github.com/sharase/cud.

Remark 3. We note a reasonably fast generation method for Tausworthe generators. Let $\mathbf{x}_i = (a_{i\sigma}, a_{i\sigma+1}, \ldots, a_{i\sigma+m-1}, a_{i\sigma+m}, \ldots, a_{i\sigma+w-1})^{\mathsf{T}}$ be a *w*-bit state vector at step *i* for $m \leq w$. We can define a state transition $\mathbf{x}_{i+1} = \mathbf{B}\mathbf{x}_i$, where $\mathbf{B} := (\mathbf{b}_0 \ \ldots \ \mathbf{b}_{m-1} \ \mathbf{0} \ \ldots \ \mathbf{0})$ is a $w \times w$ state transition matrix consisting of *w*-bit column vectors $\mathbf{b}_0, \ldots, \mathbf{b}_{m-1}$ and w - m w-bit zero column vectors $\mathbf{0}$. Then, we have the recurrence relation $\mathbf{x}_{i+1} = a_{i\sigma}\mathbf{b}_0 \oplus a_{i\sigma+1}\mathbf{b}_1 \oplus \cdots \oplus a_{i\sigma+m-1}\mathbf{b}_{m-1}$, which can be calculated by adding column vectors \mathbf{b}_j if $a_{i\sigma+j} = 1$ holds for $j = 0, \ldots, m-1$, where the symbol \oplus denotes the bitwise exclusive-or operation. Using this method, we can generate $\{u_i\}_{i=0}^{\infty}$ in (2) with reasonable speed. See [16, §3 and 5.1] for the construction of B.

m = 10	$1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1$
	$0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1 \qquad (\sigma = 70)$
m = 11	$1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1$
	$0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1 \qquad (\sigma = 179)$
m = 12	$1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1$
	$0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1 \qquad (\sigma = 146)$
m = 13	$1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1$
	$1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0$
m = 14	$1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1$
	$1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0$
m = 15	
	$0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \qquad (\sigma = 1028)$
m = 16	
	$1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0$
m = 17	
10	$111101011110111101 (\sigma = 20984)$
m = 18	
10	$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 &$
m = 19	10110111100011001001 00001111100001110101 ($\sigma = 02600$)
	11101010111100001110101 (0 - 92009)
m = 20	$(\sigma - 226826)$
m - 21	
m = 21	$(\sigma = 1127911)$
m = 22	
==	$\begin{array}{c} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0$
m = 23	
	$1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0$
m = 24	1111000110101100010111101
	$1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ (\sigma = 7017398)$
m = 25	1110101100110110010111111
	$0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1$ (\$\sigma = 2947446\$)
m = 26	$1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1$
	$1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \qquad (\sigma = 19101221)$
m = 27	
	$0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ (\sigma = 4397933)$
m = 28	
	$0 0 0 1 1 0 1 0 0 1 1 0 0 0 1 1 1 1 0 0 0 1 0 1 0 1 0 1 1 (\sigma = 167713336)$
m = 29	101000001010101011110001101011
	$\frac{11111010011100001011110000101101}{(\delta = 65169117)}$
m = 30	$(\sigma - 315800840)$
m - 21	
111 - 51	0 0 0 0 1 1 1 1 0 1 0 0 0 1 1 0 1 1 1 1
m = 32	
	0 1 0 0 0 0 1 1 1 0 1 1 0 1 0 1 0 1 0 1

Table 1: Specific parameters of pairs of polynomials (p(x), q(x)) and step sizes σ .

m	dim. s	2	3	4	5	6	- 7	8	9	10	11	12	13	14	15	16	17	18	19	20	Δ
10	New	0	3	3	4	5	5	6	6	6	6	6	6	6	6	6	6	6	6	7	2
	Chen	2	5	5	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	0
11	New	0	3	3	5	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	1
10	Chen	2	5	5	6	6	6	7	7	7	7	7	7	7	7	8	8	8	8	8	0
12	New	0	- ろ - 9	4	5	6 7	6 7	0 7	6 7	6 7	6 7	6 7	87	8	8	8	8	8	8	8	2
12	Now	2	3 2	3	5	6	6	7	7	7	8	8	8	8	8	0	0	0	0	0	0
10	Chen	1	5	5	5	6	8	ģ	9	ģ	9	9	9	9	9	9	9	9	9	9	0
14	New	0	3	4	5	7	7	7	7	8	9	9	9	9	9	9	9	9	9	9	1
11	Chen	1	6	7	7	7	7	8	9	9	9	9	9	9	9	9	9	10	10	10	0
15	New	0	3	4	6	7	8	8	9	9	9	9	10	10	10	10	10	10	10	10	1
	Chen	2	4	5	7	7	7	8	8	9	9	9	9	9	9	9	9	9	10	10	0
16	New	0	3	4	7	7	8	10	10	10	11	11	11	11	11	11	11	11	11	11	1
	Chen	3	4	5	8	8	8	8	8	10	10	10	10	10	10	10	10	10	10	12	0
17	New	0	3	4	7	7	7	8	10	10	10	10	11	11	11	11	11	12	12	12	1
	Chen	2	5	6	10	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11	0
18	New	0	3	5	6	7	9	9	9	10	10	10	10	11	11	11	12	12	13	13	2
10	Chen	3	4	5	7	8	9	9	12	12	12	12	12	12	12	12	12	12	12	12	0
19	New	0	3	5	6	7	12	12	12	12	12	12	12	13	13	13	13	13	13	13	1
20	Chen	2	4	8	8	8	10	10	11	11	12	12	12	12	12	12	12	12	12	12	0
20	Chen	0	3	о о	((10	10	11	11	12	12	13	13	13	13	13	13	13	13	
- 21	Now	0	3	5	8	8	10	10	10	10	13	13	13	13	19	14	14	14	14	14	1
21	Chen	3	6	8	8	8	11	11	11	12	12	12	12	12	12	12	12	13	13	14	
22	New	0	3	5	7	10	10	12	12	12	12	13	13	13	13	15	15	15	15	15	1
	Chen	7	7	7	8	8	14	14	14	14	14	14	14	14	14	14	14	14	14	15	0
23	New	0	3	5	9	9	11	12	13	13	13	13	13	13	13	15	15	15	15	15	1
	Chen	5	5	9	9	9	9	11	15	15	15	15	15	15	15	15	15	15	15	15	0
24	New	0	3	6	8	10	11	12	13	14	14	14	14	15	17	17	17	17	17	17	3
	Chen	5	5	8	8	11	11	11	12	14	14	14	14	14	14	14	15	15	16	16	0
25	New	0	3	6	7	12	12	12	13	13	13	14	14	16	16	16	18	18	18	18	3
	Chen	4	6	8	8	9	10	11	12	12	12	14	16	16	16	16	16	16	16	16	0
26	New	0	3	6	8	12	12	12	13	13	13	14	14	15	15	15	16	16	16	18	2
07	Chen	6	7	7	9	11	11	12	13	13	14	15	15	16	16	16	16	17	17	17	0
27	New	$\begin{bmatrix} 0\\ 2\end{bmatrix}$	3	7	11	11	12	13	13	13	14	14	14	16	16	16 16	16 16	16 16	16	16 17	3
20	Chen	3	0	8	11	12	12	14	14	14	10	15	10	10	10	10	10	10	17	17	0
28	Chop		3 5	(13	9 13	13	13	13	13	15	14 15	10 15	16	16	16	17	17	17	18	18	
20	Now	4	3	13	13	10	13	13	14	10	20	20	20	20	20	20	20	20	20	20	1
23	Chen	5	5	12	12	12	12	14	14	14	20 17	17	20 17	17	17	17	17	20 17	17	20 18	
30	New	0	3	7		12	13	14	14	16	16	16	17	17	17	17	17	17	18	19	1
	Chen	2	7	7	10	13	13	13	14	17	17	17	17	17	17	18	18	18	18	19	0
31	New	0	3	7	9	12	12	15	15	15	16	18	19	19	19	19	19	19	19	20	1
	Chen	2	5	9	10	13	13	15	15	15	15	17	18	18	18	18	18	19	19	19	0
32	New	0	3	7	10	13	14	14	15	15	17	17	17	18	18	20	20	20	20	20	4
	Chen	5	5	9	9	13	13	15	15	15	15	16	16	17	18	18	18	19	19	20	0

Table 2: Comparison of the *t*-values and Δ for our new Tausworthe generators and the existing Tausworthe generators developed by Chen et al. [3].

4. Numerical examples

In this section, we provide numerical examples to confirm the performance of Markov chain QMC.

4.1. Two-dimensional Gaussian Gibbs sampling

Our first example is a systematic Gibbs sampler to generate the twodimensional Gaussian distribution

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

for correlation $\rho \in (-1, 1)$. This can be implemented as

$$X_{i,1} \leftarrow \rho X_{i-1,2} + \sqrt{1 - \rho^2} \Phi^{-1}(u_{2i-2}),$$

$$X_{i,2} \leftarrow \rho X_{i,1} + \sqrt{1 - \rho^2} \Phi^{-1}(u_{2i-1}),$$

where Φ is the cumulative distribution function for the standard normal distribution. For the output values (3) generated from Tausworthe generators, we define two-dimensional non-overlapping points starting from the origin:

$$(0,0), (u_0,u_1), (u_2,u_3), \dots, (u_{N-2},u_0), (u_1,u_2), \dots, (u_{N-3},u_{N-2}),$$
 (12)

where $N = 2^m$. We apply digital shifts, that is, we add (z_1, z_2) to each point in (12) using bitwise exclusive-or \oplus , where z_1 and z_2 are IID samples from U(0, 1).

We estimate $E(X_1)$ and $E(X_2)$ by taking the sample mean. Hence, the true values are zero. We compare the following driving sequences:

- 1. New: our new Tausworthe generators;
- 2. Chen et al. (2012): Tausworthe generators developed by Chen et al. [3]; and
- 3. IID: Mersenne Twister [21].

Figure 4.1 shows a summary of standard deviations (in \log_2 scale) for $\rho = 0, 0.3$ and 0.9 and $12 \le m \le 25$ using 100 digital shifts. Our new generators outperformed Chen's generators for no correlation $\rho = 0$ and weak correlation $\rho = 0.3$. Even for strong correlation $\rho = 0.9$, our new generators were still better than Chen's generators. In Figure 4.1, we generated scatter plots of sampling (X_1, X_2) from our new and Chen's Tausworthe generators for $\rho = 0$

and m = 12. In the scatter plots, Chen's generator has a pattern of wiggly strips of points, which is optimized in terms of 64×64 grids for s = 2, but our generator seems to be highly balanced both for X_1 and X_2 . Therefore, it can be expected that our new generators have better marginal distributions than the existing generators.

In addition, as a test function, we estimated $E(X_1X_2)$, which has the true value ρ . Figure 4.1 shows a summary of standard deviations (in \log_2 scale) for $\rho = 0, 0.3$ and 0.9 and $12 \le m \le 25$ using 100 digital shifts. We obtained the results in which our new generators were superior to Chen's generators especially for $\rho = 0$ and 0.3.

4.2. A hierarchical Bayesian model

Our second example is a hierarchical Bayesian model [9] used in [24, 31, 20]. Following [26, Example 7.12], we explain the problem setting. We consider multiple failures of ten pumps in a nuclear plant, with the data given in Table 3. The modeling is based on the assumption that the number of failures of the *j*th pump follows a Poisson process with parameter λ_j (j = 1, ..., 10). For an observation time t_j , the number of failures X_j is thus a Poisson $\mathcal{P}(\lambda_j t_j)$ random variable. The standard prior distributions are gamma distributions $\mathcal{G}(\alpha, \beta)$ with shape parameter α and rate parameter β , which lead to the hierarchical model

$$X_j \sim \mathcal{P}(\lambda_j t_j), \quad j = 1, \dots, 10,$$

$$\lambda_j \sim \mathcal{G}(\alpha, \beta), \quad j = 1, \dots, 10,$$

$$\beta \sim \mathcal{G}(\gamma, \delta),$$

where the hyperparameter values are $\alpha = 1.802, \gamma = 0.1$, and $\delta = 1$. Our goal is to estimate the posterior means $E[\lambda_j]$ and $E[\beta]$ by taking the sample mean. For this purpose, we use a Gibbs sampler based on the full conditional distributions

$$\lambda_j \mid \beta, t_j, x_j \sim \mathcal{G}(x_j + \alpha, t_j + \beta), \quad j = 1, \dots, 10,$$

$$\beta \mid \lambda_1, \dots, \lambda_{10} \sim \mathcal{G}\left(\gamma + 10\alpha, \delta + \sum_{j=1}^{10} \lambda_j\right).$$

Note that the state vector $(\lambda_1, \ldots, \lambda_{10}, \beta)$ has eleven dimensions. The starting point uses the maximum likelihood estimates x_j/t_j for λ_j together with



Figure 2: Estimation of $E(X_1)$ and $E(X_2)$ for $\rho = 0, 0.3$ and 0.9.



Figure 3: Scatter plots of sampling (X_1, X_2) for $\rho = 0$ and m = 12.



Figure 4: Estimation of $E(X_1X_2)$ for $\rho = 0, 0.3$ and 0.9.

the full conditional mean of β , given the starting λ_j . The Gibbs sampling is driven by inversion of gamma cumulative density functions. Similarly to (12), for the output values (3), we define eleven-dimensional nonoverlapping points $(u_0, \ldots, u_{10}), (u_{11}, \ldots, u_{21}), \ldots, (u_{11(N-2)}, \ldots, u_{11(N-1)-1}),$ starting from the origin $(0, \ldots, 0)$, where $N = 2^m$ and $gcd(2^m - 1, 11) = 1$.

Table 4 shows a summary of sample variances of posterior mean estimates for m = 12, 14, 16, and 18 using 300 digital shifts. Our new Tausworthe generators were comparable to or even better than Chen's Tausworthe generators with a few exceptions (e.g., λ_7, λ_8 , and λ_9 for m = 14). Such exceptions occurred in pumps for short monitoring periods, and this implies that it might be difficult to estimate those parameters with high accuracy from the perspective of Bayesian inference. In any case, our new generators were at least superior to IID uniform random number sequences generated by Mersenne Twister.

Table 3: Number of failures and times of observation of ten pumps in a nuclear plant [8].

						1 1	L		1 L	
Pump j	1	2	3	4	5	6	7	8	9	10
Failures x_j	5	1	5	14	3	19	1	1	4	22
Time t_j	94.32	15.72	62.88	125.76	5.24	31.44	1.05	1.05	2.10	10.48

Remark 4. In our experiments, we set w = 32. In fact, Chen et al. [3] originally defined Tausworthe generators in (2) with *m*-bit precision, that is, $u_i = \sum_{j=0}^{m-1} a_{i\sigma+j} 2^{-j-1} \in [0, 1)$. In this definition, we could not observe clear differences between our new generators and Chen's generators. However, we increased the precision of points and redefined Tausworthe generators with w bits as in (2), and then the differences became clear.

Remark 5. Sequential Monte Carlo (SMC) can be used to perform Bayesian inference when the data are accumulated sequentially rather than being given a priori. Recently, Gerber and Chopin [10] developed a class of algorithms combining SMC and randomized QMC to accelerate convergence.

5. Conclusion

We conducted an exhaustive search of short-period Tausworthe generators for Markov chain QMC in terms of the *t*-value. Our key technique was to use the continued fraction expansion of q(x)/p(x) by refining the algorithm

	m = 12												
	Parame	eter	λ_1		λ_2		λ_3		λ_4	λ_4			
	IID		1.77e-	-07	1.98e-06		4.12e-07		1.96e-07		2.40e-05		
	Cher	1	4.77e-	·11	7.18e-10		8.91e-11		4.69e-11		7.44e-09		
	New	r	8.13e-12		2.41e-10		1.96e-11		9.86e-12		4.11e-09		
Par	ameter		λ_6		λ_7		λ_8		λ_9		λ_{10}		β
	IID	4.1	4e-06	9.7	9e-05	9.0	0e-05	1.()5e-04	4.8	80e-05	2.2	29e-04
(Chen	1.0	9e-09	1.0	3e-07	4.5	3e-08	3.8	81e-08	1.2	23e-08	1.6	8e-07
	New	2.4	4e-10	1.7	8e-07	3.4	9e-08	2.3	8e-08	2.8	81e-09	5.2	1e-08
						m =	: 14						
	Parame	eter	λ_1		λ_2		λ_3		λ_4		λ_5		
	IID		4.33e-	-08	5.44e	-07	9.21e	-08	6.67e-	-08	5.46e-	-06	
	Cher	1	4.86e-	$\cdot 12$	1.07e	-10	1.05e	-11	5.15e	-12	5.64e-	-09	
	New	r	5.96e	-13	$\mathbf{2.48e}$	-11	1.16e	-12	6.13e	-13	1.03e	-09	
Par	ameter		λ_6		λ_7		λ_8		λ_9		λ_{10}		β
	IID	1.1	2e-06	2.2	1e-05	2.3	2e-05	2.4	40e-05	1.2	21e-05	6.2	28e-05
(Chen	9.7	'5e-11	7.0	8e-09	1.3	7e-08	5.6	8e-09	1.4	46e-09	2.4	1e-08
	New	2.1	2e-11	5.6	De-08 2.37e-07			1.18e-08 4.5			61e-10 4.8		6e-09
						m =	: 16						I
	Parame	eter	λ_1		λ_2	<i>m</i> =	16 λ_3		λ_4		λ_5		
	Parame	eter	λ_1 1.08e-	-08	λ_2 1.42e	m =	$\frac{16}{\lambda_3}$ 2.42e	-08	λ_4 1.21e-	-08	λ_5 1.44e-	-06	
	Parame IID Cher	eter 1	λ_1 1.08e- 4.03e-	-08 -13	λ_2 1.42e 8.28e	m = -07 -12	16 λ_3 2.42e 1.07e	-08 -12	λ_4 1.21e- 4.73e-	-08 -13	λ_5 1.44e- 7.05e-	-06 -11	
	Parame IID Cher New	eter n	λ_1 1.08e- 4.03e- 2.78e -	-08 -13 - 14	λ_2 1.42e 8.28e 1.53e	m = -07 -12 -12	16 λ_3 2.42e 1.07e 5.23e	-08 -12 - 14	λ_4 1.21e- 4.73e- 2.40e	-08 -13 -14	λ_5 1.44e- 7.05e- 7.03e -	-06 -11 -11	
Par	Parame IID Cher New rameter	eter n	λ_1 1.08e- 4.03e- 2.78e - λ_6	-08 -13 -14	$\frac{\lambda_2}{1.42e}$ 8.28e 1.53e λ_7	m = -07 -12 -12	= 16 λ_3 2.42e 1.07e 5.23e λ_8	-08 -12 - 14	λ_4 1.21e- 4.73e- 2.40e λ_9	-08 -13 -14	λ_{5} 1.44e- 7.05e- 7.03e - λ_{10}	-06 -11 -11	β
Par	Parame IID Cher New cameter IID	eter	λ_1 1.08e- 4.03e- 2.78e - λ_6 1e-07	-08 -13 -14 5.3	λ_2 1.42e 8.28e 1.53e λ_7 4e-06	m = -07 -12 -12 5.6	$ \begin{array}{r} = 16 \\ \hline \lambda_3 \\ 2.42e \\ 1.07e \\ \textbf{5.23e} \\ \hline \lambda_8 \\ \hline 5e-06 \end{array} $	-08 -12 -14 6.7	λ_4 1.21e- 4.73e- 2.40e λ_9 79e-06	-08 -13 -14 2.6	λ_5 1.44e- 7.05e- 7.03e λ_{10} 51e-06	-06 -11 -11 1.6	β 57e-05
Par	Parame IID Chen New cameter IID Chen	eter 1 3.0 8.7	$\begin{array}{c} \lambda_{1} \\ 1.08e{-} \\ 4.03e{-} \\ 2.78e{-} \\ \lambda_{6} \\ 01e{-}07 \\ 4e{-}12 \end{array}$	-08 -13 -14 5.3 5.1	λ_2 1.42e 8.28e 1.53e λ_7 4e-06 1e-10	m = -07 -12 -12 5.6 5.3		-08 -12 -14 6.7 3.9	$\frac{\lambda_4}{1.21e}$ 4.73e 2.40e $\overline{\lambda_9}$ 79e-06 00e-10	-08 -13 -14 2.6 9.9	λ_5 1.44e- 7.05e- 7.03e - λ_{10} 51e-06 90e-11	-06 -11 -11 1.6 2.2	β 57e-05 20e-09
Par	Parame IID Cher New cameter IID Chen New	eter 1 3.0 8.7 1.5	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ 4e-12 \\ \textbf{2e-12} \end{array}$	-08 -13 -14 5.3 5.1 2.1	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \lambda_7 \\ 4e - 06 \\ 1e - 10 \\ \textbf{6e-10} \end{array}$	m = -07 -12 -12 5.6 5.3 5.2		-08 -12 -14 6.7 3.9 1.7	$\frac{\lambda_4}{1.21e}$ 4.73e 2.40e λ_9 79e-06 90e-10 71e-10	-08 -13 -14 2.6 9.9	λ_5 1.44e- 7.05e- 7.03e λ_{10} 51e-06 90e-11 22e-11	-06 -11 -11 1.6 2.2 1.0	β 57e-05 20e-09 5 e-09
Par	Parame IID Chen New ameter IID Chen New	eter 3.0 8.7 1.5	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ 4e-12 \\ \textbf{2e-12} \end{array}$	-08 -13 -14 5.3 5.1 2.1 0	λ_2 1.42e 8.28e 1.53e λ_7 4e-06 1e-10 6e-10	m = -07 -12 -12 5.6 5.3 5.2 $m =$	$ \frac{16}{\lambda_3} $ 2.42e 1.07e 5.23e λ_8 55e-06 4e-10 3e-10 3e-10	-08 -12 -14 6.7 3.9 1.7	$\begin{array}{c} \lambda_4 \\ 1.21e \\ 4.73e \\ 2.40e \\ \overline{\lambda_9} \\ 79e - 06 \\ 00e - 10 \\ \mathbf{71e - 10} \end{array}$	-08 -13 -14 2.6 9.9 2.2	$\frac{\lambda_5}{1.44e}$ 7.05e- 7.03e - $\overline{\lambda_{10}}$ 51e-06 90e-11 22e-11	-06 -11 -11 1.6 2.2 1.0	β 57e-05 20e-09 5e-09
Par	Parame IID Chen New Chen IID Chen New Parame	eter 3.0 8.7 1.5	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ 4e-12 \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \lambda_1 \end{array}$	-08 -13 -14 5.3 5.1 2.1	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \hline \lambda_7 \\ 4e-06 \\ 1e-10 \\ \textbf{6e-10} \\ \hline \lambda_2 \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 $m =$	$ \begin{array}{c} 16 \\ \hline \lambda_3 \\ 2.42e \\ 1.07e \\ \hline 5.23e \\$	-08 -12 -14 6.7 3.9 1.7	$\frac{\lambda_4}{1.21e}$ 4.73e 2.40e $\overline{\lambda_9}$ 79e-06 00e-10 '1e-10 λ_4	-08 -13 -14 2.6 9.9 2.2	λ_5 1.44e- 7.05e- 7.03e - λ_{10} δ_{1e} -06 ∂_{0e} -11 22e-11 λ_5	-06 -11 -11 1.6 2.2 1.0	β 57e-05 20e-09 5e-09
Par	Parame IID Chen New Chen New Parame IID	eter 3.0 8.7 1.5 eter	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ 74e-12 \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \lambda_1 \\ 2.48e- \end{array}$	-08 -13 -14 5.3 5.1 2.10	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \textbf{1.53e} \\ \lambda_7 \\ 4e-06 \\ 1e-10 \\ \textbf{6e-10} \\ \textbf{6e-10} \\ \hline \lambda_2 \\ 3.21e \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 $m =$ -08	$ = 16 $ $ \lambda_3 $ $ 2.42e $ $ 1.07e $ $ 5.23e $ $ \delta_8 $ $ 55e-06 $ $ 64e-10 $ $ 3e-10 $ $ = 18 $ $ \lambda_3 $ $ 7.49e $	-08 -12 -14 6.7 3.9 1.7	$\begin{array}{c} \lambda_4 \\ 1.21e \\ 4.73e \\ \textbf{2.40e} \\ \hline \lambda_9 \\ \hline 79e - 06 \\ 00e - 10 \\ \hline \textbf{'1e-10} \\ \hline \lambda_4 \\ \hline 3.47e \\ \end{array}$	-08 -13 -14 2.6 9.9 2.2	$\begin{array}{c} \lambda_5 \\ 1.44e \\ 7.05e \\ \textbf{7.03e} \\ \textbf{7.03e} \\ \lambda_{10} \\ 51e \\ 00e \\ 11 \\ \textbf{22e-11} \\ \lambda_5 \\ 3.88e \end{array}$	-06 -11 -11 1.6 2.2 1.0	β 57e-05 20e-09 5 e-09
Par	Parame IID Chen New Chen IID Chen New Parame IID Chen IID Chen	eter 3.0 8.7 1.5 eter	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ \textbf{'4e-12} \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \hline \lambda_1 \\ 2.48e-\\ 2.50e- \end{array}$	-08 -13 -14 5.3 5.1 2.1 -09 -14	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \textbf{1.53e} \\ \lambda_7 \\ 4e-06 \\ 1e-10 \\ \textbf{6e-10} \\ \textbf{6e-10} \\ \hline \lambda_2 \\ 3.21e \\ 1.05e \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 $m =$ -08 -12	$ \begin{array}{c} 16 \\ \hline \lambda_3 \\ 2.42e \\ 1.07e \\ \textbf{5.23e} \\ \textbf{5.23e} \\ \hline \lambda_8 \\ 55e-06 \\ \textbf{64e-10} \\ \textbf{3e-10} \\ \textbf{3e-10} \\ \textbf{5.12e} \\ \hline \lambda_3 \\ 7.49e \\ \textbf{5.12e} \end{array} $	-08 -12 -14 6.7 3.9 1.7 -09 -14	$\begin{array}{r} \lambda_4 \\ 1.21e \\ 4.73e \\ 2.40e \\ \hline \lambda_9 \\ \hline 79e-06 \\ 00e-10 \\ \hline 1e-10 \\ \hline \lambda_4 \\ 3.47e \\ 2.58e \end{array}$	-08 -13 -14 2.6 9.9 2.2 -09 -14	$\frac{\lambda_5}{1.44e}$ 7.05e- 7.03e - $\overline{\lambda_{10}}$ 51e-06 90e-11 22e-11 $\overline{\lambda_5}$ 3.88e- 1.86e-	-06 -11 -11 1.6 2.2 1.0 -07 -11	β 57e-05 20e-09 5e-09
Par	Parame IID Chen New Chen New Parame IID Chen IID Chen New	eter 3.0 8.7 1.5 eter	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \textbf{2.78e-} \\ \lambda_6 \\ 11e-07 \\ 4e-12 \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \textbf{2.48e-} \\ 2.50e-\\ \textbf{2.41e-} \end{array}$	-08 -13 -14 5.3 5.1 2.10 -09 -14 -15	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ 1.53e \\ \hline \lambda_7 \\ 4e-06 \\ 1e-10 \\ 6e-10 \\ \hline \hline \lambda_2 \\ 3.21e \\ 1.05e \\ 7.97e \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 m = -08 -12 -14		-08 -12 -14 6.7 3.9 1.7 -09 -14 -15	$\begin{array}{r} \lambda_4 \\ 1.21e \\ 4.73e \\ 2.40e \\ \hline \lambda_9 \\ \hline 79e-06 \\ 00e-10 \\ \hline 1e-10 \\ \hline \lambda_4 \\ \hline 3.47e \\ 2.58e \\ 1.86e \end{array}$	-08 -13 -14 2.6 9.8 2.2 2.2 -09 -14 -15	$\begin{array}{c} \lambda_5 \\ 1.44e \\ 7.05e \\ \hline 7.03e \\ \hline \lambda_{10} \\ \hline \delta 1e - 06 \\ 00e - 11 \\ \hline 22e - 11 \\ \hline \lambda_5 \\ \hline 3.88e \\ 1.86e \\ 1.72e \end{array}$	-06 -11 -11 1.6 2.2 1.0 -07 -11 -12	β 57e-05 20e-09 5e-09
Par	Parame IID Chen New Chen New Parame IID Chen New Chen New	eter 3.0 8.7 1.5 eter	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \lambda_6 \\ 1e-07 \\ 4e-12 \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \hline \lambda_1 \\ 2.48e-\\ 2.50e-\\ \textbf{2.41e-} \\ \hline \lambda_6 \end{array}$	-08 -13 -14 5.3 5.1 2.1 -09 -14 -15	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \textbf{1.53e} \\ \lambda_7 \\ 4e - 06 \\ 1e - 10 \\ \textbf{6e-10} \\ \textbf{6e-10} \\ \hline \textbf{6e-10} \\ \hline \textbf{3.21e} \\ 1.05e \\ \textbf{7.97e} \\ \hline \lambda_7 \\ \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 m = -08 -12 -14	$ \begin{array}{c} 16 \\ \hline \lambda_3 \\ 2.42e \\ 1.07e \\ \textbf{5.23e} \\ \textbf{5.23e} \\ \hline \lambda_8 \\ \hline 5e-06 \\ \textbf{64e-10} \\ \textbf{3e-10} \\ \textbf{3e-10} \\ \hline \textbf{3e-10} \\ \hline \textbf{3e-10} \\ \hline \textbf{5.12e} \\ \textbf{5.12e} \\ \textbf{5.60e} \\ \hline \lambda_8 \\ \end{array} $	-08 -12 -14 6.7 3.9 1.7 -09 -14 -15	$\begin{array}{c} \lambda_4 \\ 1.21e \\ 4.73e \\ 2.40e \\ \overline{\lambda_9} \\ 79e - 06 \\ 00e - 10 \\ \hline 1e - 10 \\ \hline 1e - 10 \\ \hline \lambda_4 \\ 3.47e \\ 2.58e \\ 1.86e \\ \overline{\lambda_9} \end{array}$	-08 -13 -14 2.(9.(2.2 2.2 -09 -14 -15	$\begin{array}{r} \lambda_5 \\ 1.44e \\ 7.05e \\ \textbf{7.03e} \\ \textbf{7.03e} \\ \lambda_{10} \\ \hline \textbf{51e-06} \\ 00e-11 \\ \textbf{22e-11} \\ \hline \textbf{22e-11} \\ \hline \lambda_5 \\ 3.88e \\ 1.86e \\ \textbf{1.72e} \\ \hline \lambda_{10} \\ \end{array}$	-06 -11 -11 1.6 2.2 1.0 -07 -11 -12	β 57e-05 20e-09 5e-09 β
Par	Parame IID Chen New Chen New Parame IID Chen New Chen New Chen IID Chen IID	eter 3.0 8.7 1.5 eter	$\begin{array}{c} \lambda_1 \\ 1.08e-\\ 4.03e-\\ \textbf{2.78e-} \\ \textbf{2.78e-} \\ \lambda_6 \\ \textbf{11e-07} \\ \textbf{4e-12} \\ \textbf{2e-12} \\ \textbf{2e-12} \\ \hline \lambda_1 \\ \textbf{2.48e-} \\ \textbf{2.50e-} \\ \textbf{2.41e-} \\ \lambda_6 \\ \textbf{30e-08} \end{array}$	-08 -13 -14 5.3 5.1 2.10 -09 -14 -15 -13	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ 1.53e \\ \hline 1.53e \\ 4e-06 \\ 1e-10 \\ \hline 6e-10 \\ \hline \lambda_2 \\ 3.21e \\ 1.05e \\ \hline 7.97e \\ \hline \lambda_7 \\ 4e-06 \end{array}$	m = -07 -12 -12 -12 5.6 5.3 5.2 m = -08 -12 -14 1.6		-08 -12 -14 6.7 3.9 1.7 -09 -14 -15	$\begin{array}{c} \lambda_4 \\ 1.21e \\ 4.73e \\ 2.40e \\ \hline \lambda_9 \\ \hline 79e-06 \\ 00e-10 \\ \hline 1e-10 \\ \hline \lambda_4 \\ 3.47e \\ 2.58e \\ 1.86e \\ \hline \lambda_9 \\ \hline 55e-06 \end{array}$	-08 -13 -14 2.6 9.5 2.2 -09 -14 -15 6.6	$\begin{array}{c} \lambda_5 \\ 1.44e \\ 7.05e \\ \hline 7.03e \\ \hline \hline 7.03e \\ \hline \hline 31e - 06 \\ 00e - 11 \\ \hline 22e - 11 \\ \hline \hline 22e - 11 \\ \hline \hline \lambda_5 \\ 3.88e \\ 1.86e \\ \hline 1.72e \\ \hline \lambda_{10} \\ \hline \hline 35e - 07 \\ \end{array}$	-06 -11 -11 1.6 2.2 1.0 -07 -11 -12 4.2	$\frac{\beta}{57e-05}$ 20e-09 5e-09 $\frac{\beta}{24e-06}$
Par	Parame IID Chen New Chen New Parame IID Chen IID Chen Sameter IID Chen	eter 3.0 8.7 1.5 eter 1.3 1.3	$\begin{array}{c} \lambda_1 \\ 1.08e- \\ 4.03e- \\ 2.78e- \\ 2.78e- \\ \lambda_6 \\ 11e-07 \\ 4e-12 \\ 2e-12 \\ \hline 2e-12 \\ \hline \lambda_1 \\ 2.48e- \\ 2.50e- \\ 2.41e- \\ \lambda_6 \\ \hline 30e-08 \\ 39e-12 \\ \end{array}$	-08 -13 -14 5.3 5.1 2.10 -09 -14 -15 1.3 7.5	$\begin{array}{c} \lambda_2 \\ 1.42e \\ 8.28e \\ \textbf{1.53e} \\ \textbf{1.53e} \\ \lambda_7 \\ 4e-06 \\ 1e-10 \\ \textbf{6e-10} \\ \textbf{6e-10} \\ \hline \textbf{6e-10} \\ \hline \textbf{3.21e} \\ 1.05e \\ \textbf{7.97e} \\ \hline \lambda_7 \\ 4e-06 \\ 2e-11 \end{array}$	m = -07 -12 -12 5.6 5.3 5.2 $m =$ -08 -12 -14 1.6 1.8		-08 -12 -14 6.7 3.9 1.7 -09 -14 -15 1.6 9.8	$\begin{array}{r} \lambda_{4} \\ 1.21e \\ 4.73e \\ 2.40e \\ \hline \\ 2.9e \\ \hline \\ 2.9e \\ -06 \\ \hline \\ 00e \\ -10 \\ \hline \\ 1e \\ -10 \\ \hline \\ \hline \\ 3.47e \\ 2.58e \\ \hline \\ 1.86e \\ \hline \\ \hline \\ \hline \\ \hline \\ 35e \\ -06 \\ \hline \\ 33e \\ -11 \\ \hline \end{array}$	-08 -13 -14 2.6 9.8 2.2 2.2 -09 -14 -15 6.6 1.6	$\begin{array}{c} \lambda_5 \\ 1.44e \\ 7.05e \\ \textbf{7.03e} \\ \textbf{7.03e} \\ \textbf{7.03e} \\ \textbf{5.03e} \\ \textbf{7.03e} \\ \textbf{5.05e} \\ \textbf{7.03e} \\ \textbf{5.05e} \\ \textbf{7.03e} \\ \textbf{5.05e} \\ \textbf{7.03e} \\ \textbf{5.05e} \\ \textbf{7.05e} \\ \textbf{5.05e} \\ \textbf{7.05e} \\ \textbf{5.05e} \\ \textbf{7.05e} \\ 7.$	-06 -11 -11 1.6 2.2 1.0 -07 -11 -12 4.2 9.0	β 57e-05 20e-09 5e-09 5e-09 β 24e-06 01e-10

Table 4: Variance of posterior mean estimates for pump failure data.

of Tezuka and Fushimi [30] on modern computers. As a result, we obtained the point sets with *t*-values optimal for s = 2 and small for $s \ge 3$. We also reported numerical examples using Gibbs sampling in which our new generators performed better than the existing generators of Chen et al. [3]. The code in C is available at https://github.com/sharase/cud.

As a future work, we will attempt more realistic numerical examples as in [1, 32, 31]. For this purpose, we believe that the next task is to embed our new and existing generators into several programming languages for statistical computing; for example, R, Stan, and Python. Thus, we are now planning a software implementation of Markov chain QMC.

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