# RATIONAL POLYTOPES WITH EHRHART COEFFICIENTS OF ARBITRARY PERIOD 

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#### Abstract

A seminal result of E. Ehrhart states that the number of integer lattice points in the dilation of a rational polytope by a positive integer $k$ is a quasi-polynomial function of $k$ - that is, a "polynomial" in which the coefficients are themselves periodic functions of $k$. Using a result of F. Liu on the Ehrhart polynomials of cyclic polytopes, we construct not-necessarily-convex rational polytopes of arbitrary dimension in which the periods of the coefficient functions appearing in the Ehrhart quasi-polynomial take on arbitrary values.


## 1. Introduction

The Ehrhart function $\operatorname{ehr}_{\mathcal{P}}$ of an $n$-dimensional rational polytope $\mathcal{P} \subseteq \mathbb{R}^{n}$ counts the number of integer lattice points in the $k^{\text {th }}$ dilate of $\mathcal{P}$. That is, $\operatorname{ehr}_{\mathcal{P}}(k)=$ $\left|k \mathcal{P} \cap \mathbb{Z}^{n}\right|$ for integers $k \geq 1$. It is well known that $\operatorname{ehr}_{\mathcal{P}}(k)$ is a degree- $n$ quasipolynomial function of $k$, meaning that

$$
\begin{equation*}
\operatorname{ehr}_{\mathcal{P}}(k)=\sum_{i=0}^{n} c_{i}(k) k^{i}, \quad \text { for } k \in \mathbb{Z}_{\geq 1} \tag{1}
\end{equation*}
$$

where the coefficient functions $c_{i}: \mathbb{Z} \rightarrow \mathbb{Q}$ are periodic functions with finite periods. In other words, writing $\widehat{\mathbb{Q}}$ for the ring of periodic functions $\mathbb{Z} \rightarrow \mathbb{Q}$, each rational polytope $\mathcal{P}$ has an associated Ehrhart quasi-polynomial $\operatorname{ehr}_{\mathcal{P}}(t) \in \widehat{\mathbb{Q}}[t]$. (We refer the reader to [2, 11, 23] for introductions to Ehrhart theory.)

The purpose of this paper is to study the possible periods of the coefficient functions $c_{i}$ appearing in equation (1). Our main result (Theorem 1.2 below) is that these periods may take on arbitrary values. Thus, we offer a contribution to the project of characterizing the Ehrhart quasi-polynomials of all rational polytopes. This latter project has been the subject of a great deal of work for several decades. Many deep constraints on the coefficients $c_{i}$ have been found. See in particular 1, 4, 6-10, 12, 19, 21, 22, 24, 25] and references therein.

It is a remarkable and humbling fact that many of these famous results do not make full use of convexity (cf. [25, Remark 1.12]). That is, the constraints discovered are satisfied by types of rational polytopal balls that are more general than just the convex polytopes, such as the star convex polytopes. Here, by a rational polytopal ball, or a not-necessarily-convex rational polytope, we mean a topological ball in $\mathbb{R}^{n}$ that is a union $\bigcup_{i \in I} \mathcal{P}_{i}$ of a finite family $\left\{\mathcal{P}_{i}: i \in I\right\}$ of convex rational polytopes, all with the same affine span, in which every nonempty intersection

[^0]$\mathcal{P}_{i} \cap \mathcal{P}_{j}, i \neq j$, is a common facet of $\mathcal{P}_{i}$ and $\mathcal{P}_{j}$. Only in dimension $n=2$ do we have a complete characterization of the Ehrhart polynomials of precisely the convex integral polygons 20]. Even here in dimension 2, the case of nonintegral convex polygons remains open [10, 16].

Nonetheless, even in the not-necessarily-convex case, the complete characterization of Ehrhart quasi-polynomials of rational polytope still seems quite far off. Chastened by the difficulty of such a complete characterization, we restrict our attention in this paper and its predecessors ([16, 17]) to the periods of the coefficient functions $c_{i}$. To this end, define the period sequence of $\mathcal{P}$ to be the sequence $\left(p_{0}, p_{1}, \ldots, p_{n}\right)$ in which $p_{i}$ is the (minimum) period of $c_{i}$. That is, $p_{i}$ is the minimum positive integer such that $c_{i}(k)=c_{i}\left(k+p_{i}\right)$ for all $k \in \mathbb{Z}$. Our motivating question is thus: What are the possible period sequences of rational polytopes?

It is well known that, if $\mathcal{P} \subseteq \mathbb{R}^{n}$ is $n$-dimensional, then the leading coefficient of $\operatorname{ehr} \mathcal{P}_{\mathcal{P}}(t)$ is the volume of $\mathcal{P}$. In particular, $c_{n}$ is a constant, so $p_{n}=1$. A result of Beck, Sam, and Woods [3] provides a polytope with period sequence $\left(p_{0}, \ldots, p_{n-1}, 1\right)$, provided that the desired periods $p_{i}$ satisfy the divisibility relations $p_{n-1}\left|p_{n-2}\right| \cdots \mid p_{0}$. In particular, the polytopes constructed in [3] all satisfy $p_{0} \geq p_{1} \geq \cdots \geq p_{n}$. Polytopes can fail to satisfy these inequalities when they exhibit the phenomenon of period collapse [16]. Nonetheless, the construction in [3] gives convex rational polytopes with arbitrary period sequences of the form ( $p, 1, \ldots, 1$ ). (See Theorem 3.3 below.)

A polytope $\mathcal{P} \subseteq \mathbb{R}^{n}$ is integral if all of its vertices lie in the integer lattice $\mathbb{Z}^{n}$. In this case, $\operatorname{ehr}_{\mathcal{P}}(t)$ is simply a polynomial. That is, the period sequence of an integral polytope is $(1, \ldots, 1)$. In [17], we constructed convex rational polytopes with arbitrary period sequences of the form $(1, p, 1, \ldots, 1)$. It is straightforward to glue the constructions in [3] and 17] along a common integral facet to form a convex rational polytope establishing the following.

Theorem 1.1. Let positive integers $p_{0}$ and $p_{1}$ be given. Then there exists a convex $n$-dimensional polytope with period sequence ( $p_{0}, p_{1}, 1, \ldots, 1$ ).

Controlling the periods of higher-degree coefficients proved to be more difficult. In 17], we were able to exploit previously discovered solutions to the system of Diophantine equations known as the ideal Prouhet-Tarry-Escott (PTE) problem [5] to find $n$-dimensional polytopal balls with period sequences of the form $(1, \ldots, 1, p, 1)$, provided that the dimension $n$ satisfied either $3 \leq n \leq 11$ or $n=13$.

The main result of the current paper supersedes the PTE-based construction from [17] by proving the existence of not-necessarily-convex polytopes of arbitrary dimension $n$ with arbitrary period sequences $\left(p_{0}, p_{1}, \ldots, p_{n-1}, 1\right)$.

Theorem 1.2 (Proved in Section 5). Let positive integers $p_{0}, \ldots, p_{n-1}$ be given. Then there exists an n-dimensional polytopal ball $Q_{*}$ such that the period of the coefficient of $t^{i}$ in $\operatorname{ehr}_{Q_{*}}(t)$ is $p_{i}$ for $0 \leq i \leq n-1$.

The proof of Theorem 1.2 depends upon a remarkable property of cyclic polytopes (Theorem 2.1 below). We recall these polytopes and their Ehrhart polynomials in Section 2. In Section 33 we introduce the notation and basic building blocks that we will use in our constructions. In Section 4 we build a rational polytopal ball with a period sequence of the form $(1, \ldots, 1, p, 1, \ldots, 1)$, in which a coefficient function of arbitrary degree has arbitrary period. Finally, in Section 5, we combine
the constructions from Section 4 to build a polytopal ball with an arbitrary period sequence of the form $\left(p_{0}, p_{1}, \ldots, p_{n-1}, 1\right)$.

## 2. CyClic polytopes

Cyclic polytopes are perhaps most famous for their appearance in the Upper Bound Theorem (McMullen [18]): A d-dimensional cyclic polytope attains the maximum number of faces of every dimension among all $d$-dimensional polytopes with the same number of vertices. However, it is the Ehrhart polynomials of cyclic polytopes, rather than their face lattices, that will be of particular interest to us.

We recall the definition of cyclic polytopes. Fix a subset $T \subseteq \mathbb{Z}$ of $n+1$ integers. (The particular subset chosen will not matter for our purposes.) We define a sequence of polytopes $C_{i} \subseteq \mathbb{R}^{i}$, with $0 \leq i \leq n$, as follows. Let $C_{0}:=\{0\} \subseteq \mathbb{R}^{0}$. For $1 \leq i \leq n$, let $\chi_{i}: \mathbb{R} \rightarrow \mathbb{R}^{i}$ be the moment curve $x \mapsto\left(x, x^{2}, \ldots, x^{i}\right)$. Then the cyclic polytope $C_{i} \subseteq \mathbb{R}^{i}$ is the convex hull of the image of $T$ under $\chi_{i}$. That is, $C_{i}:=\operatorname{Conv}\left(\chi_{i}(T)\right)$.

The Ehrhart polynomials of cyclic polytopes are unusual in that all of their coefficients have straightforward geometric interpretations. Such interpretations are always available for the two leading coefficients, $c_{d}$ and $c_{d-1}$, of the Ehrhart polynomial of an arbitrary $d$-dimensional integral polytope $\mathcal{P}$. However, in the general case, no such interpretations exist for the lower-degree coefficients of $\operatorname{ehr}_{\mathcal{P}}(t)$. The cyclic polytopes are a striking exception. In particular, F. Liu 13] proved that the Ehrhart polynomial of $C_{i}$ satisfies a beautiful recursive expression first conjectured by Beck et al. in 1]:

Theorem 2.1 (Liu [13]). The Ehrhart polynomials of the cyclic polytopes are given by

$$
\begin{equation*}
\operatorname{ehr}_{C_{i}}(t)=\operatorname{Vol}\left(C_{i}\right) t^{i}+\operatorname{ehr}_{C_{i-1}}(t), \quad \text { for } 1 \leq i \leq n \tag{2}
\end{equation*}
$$

or, equivalently,

$$
\operatorname{ehr}_{C_{i}}(t)=\sum_{j=0}^{i} \operatorname{Vol}\left(C_{j}\right) t^{j}
$$

where $\operatorname{Vol}\left(C_{j}\right)$ denotes the volume of $C_{j}$ in $\mathbb{R}^{j}$. (By convention, $C_{0}$ has volume 1.)
We remark that the known proofs of this elegant result are far from trivial 13 15].

## 3. Notation and building blocks

In this section, we briefly review notation and results developed in 17, Sections 2 and 3], to which we refer the reader for additional discussion and examples.

Our goal is to build polytopes of arbitrary dimension with arbitrary prescribed period sequences. Since adding a polynomial to a quasi-polynomial does not change the period sequence, we will consider two quasi-polynomials to be equivalent if their difference is a polynomial. Recall that we write $\widehat{\mathbb{Q}}$ for the ring of periodic functions $\mathbb{Z} \rightarrow \mathbb{Q}$.
Definition 3.1. Two quasi-polynomials $q(t), r(t) \in \widehat{\mathbb{Q}}[t]$ are equivalent if $q(t)-$ $r(t) \in \mathbb{Q}[t]$. In this case, we write $q(t) \equiv r(t)$.

The chief convenience of this notation is that, if $\mathcal{Q} \cup \mathcal{R}$ is a union of rational polytopes $\mathcal{Q}$ and $\mathcal{R}$ such that $\mathcal{Q} \cap \mathcal{R}$ is integral, then $\operatorname{ehr}_{\mathcal{Q} \cup \mathcal{R}}(t) \equiv \operatorname{ehr}_{\mathcal{Q}}(t)+\operatorname{ehr}_{\mathcal{R}}(t)$. Since $\mathbb{Q}[t]$ is not an ideal in the ring $\widehat{\mathbb{Q}}[t]$, care must taken when multiplying quasipolynomials. Nonetheless, a limited kind of substitution holds: if $f(t) \in \mathbb{Q}[t]$ and $q(t) \equiv r(t) \in \widehat{\mathbb{Q}}[t]$, then $f(t) q(t) \equiv f(t) r(t)$.

Fix a positive integer $p$. (Typically, $p$ will be the desired period of a coefficient function in the Ehrhart quasi-polynomial of a rational polytope.) Our constructions begin with two fundamental building blocks: the closed line segment $\ell:=\left[-\frac{1}{p}, 0\right] \subseteq$ $\mathbb{R}$, and the convex pentagon $P$ in $\mathbb{R}^{2}$ with vertices $\mathbf{u}^{+}, \mathbf{u}^{-}, \mathbf{v}^{+}, \mathbf{v}^{-}, \mathbf{w}$, where

$$
\begin{equation*}
\mathbf{u}^{ \pm}:= \pm q \mathbf{e}_{1}, \quad \quad \mathbf{v}^{ \pm}:= \pm(q-1) \mathbf{e}_{1}+\mathbf{e}_{2}, \quad \mathbf{w}:=\frac{q}{p} \mathbf{e}_{2} \tag{3}
\end{equation*}
$$

and $q:=p^{2}-p+1$. (Here and below, we write $\mathbf{e}_{i}$ for the $i$ th standard basis vector.)
A key fact, proved in [16], is that the Ehrhart quasi-polynomials of $P$ and $\ell$ are "complements" of each other in the sense that the periodic parts of their coefficients cancel when the quasi-polynomials are added together. That is,

$$
\begin{equation*}
\operatorname{ehr}_{P}(t) \equiv-\operatorname{ehr}_{\ell}(t) \tag{4}
\end{equation*}
$$

Furthermore, this equivalence is respected by the operation of taking $i$-fold pyramids over $P$ and $\ell$. The pyramid $\Delta(\mathcal{Q})$ over a polytope $\mathcal{Q} \subseteq \mathbb{R}^{d}$ is the convex hull of the embedded copy of $\mathcal{Q}$ in $\mathbb{R}^{d+1}$ at height 0 together with the standard basis vector $\mathbf{e}_{d+1}$. That is, $\Delta(\mathcal{Q}):=\operatorname{Conv}\left(\{(\mathbf{x}, 0): \mathbf{x} \in \mathcal{Q}\} \cup\left\{\mathbf{e}_{d+1}\right\}\right)$. This operation may be iterated, yielding the $i$-fold pyramid $\Delta^{i}(\mathcal{Q}):=\Delta\left(\Delta^{i-1}(\mathcal{Q})\right) \subseteq \mathbb{R}^{d+i}$. (Of course, $\Delta^{0}(\mathcal{Q}):=\mathcal{Q}$.)

Proposition 3.2 (17, Proposition 3.1]). Let $P$ and $\ell$ be the pentagon and line segment defined above. Then, for $i \geq 0$,

$$
\begin{equation*}
\operatorname{ehr}_{\Delta^{i}(P)}(t) \equiv-\operatorname{ehr}_{\Delta^{i}(\ell)}(t) \tag{5}
\end{equation*}
$$

Note that the $i$-fold pyramid $\Delta^{i}(\ell) \subseteq \mathbb{R}^{i+1}$ over $\ell$ is the simplex

$$
\Delta^{i}(\ell)=\operatorname{Conv}\left\{\mathbf{0},-\frac{1}{p} \mathbf{e}_{1}, \mathbf{e}_{2}, \ldots, \mathbf{e}_{i+1}\right\}
$$

An important fact for the constructions below is that the period sequence of $\Delta^{i}(\ell)$ is $(p, 1, \ldots, 1)$.

Theorem 3.3 (3, Theorem 2]). Let $\ell:=\left[-\frac{1}{p}, 0\right]$. Then the period sequence of $\Delta^{i}(\ell)$ is the $(i+2)$-tuple $(p, 1, \ldots, 1)$.

## 4. Nonconvex polytopes with period Sequence $(1, \ldots, 1, p, 1, \ldots, 1)$

In this section, we construct an $n$-dimensional nonconvex rational polytope $Q_{i}$ for which all Ehrhart coefficients are constants, except for the coefficient of $t^{i}$, which has period $p$, for arbitrary integers $p \geq 1$ and $0 \leq i \leq n-1$.

In the case where $i=0$, it suffices to set $Q_{0}:=\Delta^{n-1}(\ell)$ by Theorem 3.3. Furthermore, we settled the $n=2$ case in [16]. We thus proceed with the assumption that $i \geq 1$ and $n \geq 3$.

[^1]As in Section 3. let $\ell:=\left[-\frac{1}{p}, 0\right]$ and let $P$ be the pentagon defined in terms of $p$ by equations (3). Consider the $n$-dimensional polytop $\epsilon^{2}$

$$
\begin{equation*}
L_{i}:=\left(C_{i} \times \Delta^{n-i-1}(\ell)\right)-\mathbf{e}_{i+1} \tag{6}
\end{equation*}
$$

and its facet

$$
\begin{equation*}
L_{i}^{\prime}:=\left(C_{i} \times \operatorname{Conv}\left\{\mathbf{0}, \mathbf{e}_{2}, \ldots, \mathbf{e}_{n-i}\right\}\right)-\mathbf{e}_{i+1}, \tag{7}
\end{equation*}
$$

as well as the $n$-dimensional polytope

$$
\begin{equation*}
R_{i}:=\left(C_{i-1} \times \Delta^{n-i-1}(P)\right)+\mathbf{e}_{i+1} \tag{8}
\end{equation*}
$$

and its facet

$$
\begin{equation*}
R_{i}^{\prime}:=\left(C_{i-1} \times \operatorname{Conv}\left\{q \mathbf{e}_{1},-q \mathbf{e}_{1}, \mathbf{e}_{3}, \ldots, \mathbf{e}_{n-i+1}\right\}\right)+\mathbf{e}_{i+1} \tag{9}
\end{equation*}
$$

(Recall from Section 3 that $q:=p^{2}-p+1$.) Observe that $L_{i}^{\prime}$ and $R_{i}^{\prime}$ are $(n-1)$ dimensional integral polytopes in $\mathbb{R}^{n}$, with $L_{i}^{\prime}$ contained in the hyperplane $x_{i+1}=$ -1 and with $R_{i}^{\prime}$ contained in the hyperplane $x_{i+1}=1$. Furthermore, $L_{i}$ is contained in the halfspace $x_{i+1} \leq-1$, and $R_{i}$ is contained in the halfspace $x_{i+1} \geq 1$. Let $M_{i}:=\operatorname{Conv}\left(L_{i}^{\prime} \cup R_{i}^{\prime}\right)$. Then $M_{i}$ is an $n$-dimensional integral polytope lying between the hyperplanes $x_{i+1}=-1$ and $x_{i+1}=1$. In particular, $L_{i}, M_{i}$, and $R_{i}$ have pairwise disjoint interiors and integral intersections.

We are now ready to construct the not-necessarily-convex polytope $Q_{i}$ with period sequence $(1, \ldots, 1, p, 1, \ldots, 1)$. We define

$$
Q_{i}:=L_{i} \cup M_{i} \cup R_{i} .
$$

Theorem 4.1. Fix an arbitrary dimension $n$, degree $i$ with $0 \leq i \leq n-1$, and period $p$. Then the n-dimensional polytopal ball $Q_{i}$ constructed above has an Ehrhart quasi-polynomial $\operatorname{ehr}_{Q_{i}}(t)$ in which all coefficient functions are constants, except for the coefficient of $t^{i}$, which has period $p$.
Proof. As indicated at the beginning of this section, we may assume that $i \geq 1$ and $n \geq 3$. Since $M_{i}$ is an integral polytope meeting $L_{i}$ and $R_{i}$ at integral facets, it follows from the construction above that

$$
\operatorname{ehr}_{Q_{i}}(t) \equiv \operatorname{ehr}_{L_{i}}(t)+\operatorname{ehr}_{R_{i}}(t)
$$

Thus,

$$
\begin{aligned}
\operatorname{ehr}_{Q_{i}}(t) & \equiv \operatorname{ehr}_{C_{i} \times \Delta^{n-i-1}(\ell)}(t)+\operatorname{ehr}_{C_{i-1} \times \Delta^{n-i-1}(P)}(t) \\
& =\operatorname{ehr}_{C_{i}}(t) \operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t)+\operatorname{ehr}_{C_{i-1}}(t) \operatorname{ehr}_{\Delta^{n-i-1}(P)}(t) \\
& \equiv\left(\operatorname{Vol}\left(C_{i}\right) t^{i}+\operatorname{ehr}_{C_{i-1}}(t)\right) \operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t)-\operatorname{ehr}_{C_{i-1}}(t) \operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t) \\
& =\operatorname{Vol}\left(C_{i}\right) t^{i} \operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t)
\end{aligned}
$$

In this sequence of computations, the third line is the crucial step invoking Liu's Theorem [2.1] as well as Proposition 3.2. The second line uses the general fact that $\operatorname{ehr}_{\mathcal{P} \times \mathcal{R}}(t)=\operatorname{ehr} \mathbf{P}_{\mathcal{P}}(t) \operatorname{ehr}_{\mathcal{R}}(t)$.

By Theorem 3.3, all coefficient functions in $\operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t)$ are constants, except for the "constant" coefficient-that is, the coefficient function in the degree-0 term—which has period $p$. Thus, all coefficient functions in $\operatorname{Vol}\left(C_{i}\right) t^{i} \operatorname{ehr}_{\Delta^{n-i-1}(\ell)}(t)$

[^2]are constant functions, except for the coefficient of $t^{i}$, which has period $p$. It follows from the equivalence of quasi-polynomials shown above that the same is true of $\operatorname{ehr}_{Q_{i}}(t)$, as desired.

## 5. Nonconvex polytopes with period sequence $\left(p_{0}, p_{1}, \ldots, p_{n-1}, 1\right)$

In this section, we construct a nonconvex polytope $Q_{*}$ for which the Ehrhart quasi-polynomial has an arbitrary period sequence. Let the desired period sequence be $\left(p_{0}, p_{1}, \ldots, p_{n-1}, 1\right)$, where each $p_{i}$ is a positive integer. The previous section showed how to construct a polytope $Q_{i}$ with period sequence $\left(1, \ldots, 1, p_{i}, 1, \ldots, 1\right)$, where $p_{i}$ is the period of the $i^{\text {th }}$ coefficient. In this section, we will modify that construction so that the resulting "modified $Q_{i}$ " can be glued together to build the polytope $Q_{*}$ with the desired period sequence.

For $0 \leq i \leq n-1$, let $\ell_{i}:=\left[-\frac{1}{p_{i}}, 0\right]$, and let $P_{i}$ be the pentagon defined by equations (3) after replacing $p$ by $p_{i}$ and $q$ by $q_{i}:=p_{i}^{2}-p_{i}+1$.

We first deal with the periods $p_{i}$ with $i \geq 1$. Observe that the construction of $Q_{i}$ in Section 4 goes through if we replace the translations by $\pm \mathbf{e}_{i+1}$ in equations (6)(9) with translations by $\pm k_{i} \mathbf{e}_{i+1}$, where $k_{i}$ is an arbitrary positive integer (to be fixed below), as follows:

$$
\begin{align*}
& L_{i}:=\left(C_{i} \times \Delta^{n-i-1}\left(\ell_{i}\right)\right)-k_{i} \mathbf{e}_{i+1},  \tag{10}\\
& L_{i}^{\prime}:=\left(C_{i} \times \operatorname{Conv}\left\{\mathbf{0}, \mathbf{e}_{2}, \ldots, \mathbf{e}_{n-i}\right\}\right)-k_{i} \mathbf{e}_{i+1},  \tag{11}\\
& R_{i}:=\left(C_{i-1} \times \Delta^{n-i-1}\left(P_{i}\right)\right)+k_{i} \mathbf{e}_{i+1},  \tag{12}\\
& R_{i}^{\prime}:=\left(C_{i-1} \times \operatorname{Conv}\left\{q_{i} \mathbf{e}_{1},-q_{i} \mathbf{e}_{1}, \mathbf{e}_{3}, \ldots, \mathbf{e}_{n-i+1}\right\}\right)+k_{i} \mathbf{e}_{i+1} . \tag{13}
\end{align*}
$$

Likewise, to handle the periodicity $p_{0}$ in the degree- 0 term, we may define a translated version of the polytope $Q_{0}$ from Section 4 as well as one of its facets:

$$
\begin{align*}
& Q_{0}:=\Delta^{n-1}\left(\ell_{0}\right)-k_{0} \mathbf{e}_{1}  \tag{14}\\
& Q_{0}^{\prime}:=\operatorname{Conv}\left\{\mathbf{0}, \mathbf{e}_{2}, \ldots, \mathbf{e}_{n}\right\}-k_{0} \mathbf{e}_{1} . \tag{15}
\end{align*}
$$

As in Section 4 we find that $L_{i}^{\prime}$ and $R_{i}^{\prime}$ are $(n-1)$-dimensional integral polytopes in $\mathbb{R}^{n}$, with $L_{i}^{\prime}$ contained in the hyperplane $x_{i+1}=-k_{i}$ and with $R_{i}^{\prime}$ contained in the hyperplane $x_{i+1}=k_{i}$. Furthermore, $L_{i}$ is contained in the halfspace $x_{i+1} \leq-k_{i}$, and $R_{i}$ is contained in the halfspace $x_{i+1} \geq k_{i}$. Finally, $Q_{0}^{\prime}$ is contained in the hyperplane $x_{1}=-k_{0}$, and $Q_{0}$ is contained in the halfspace $x_{1} \leq-k_{0}$.

We now fix the translation parameters $k_{0}, k_{1}, \ldots, k_{n-1}$ in equations (10)-(15) to be sufficiently large so that the vertices of the facets $Q_{0}^{\prime}, L_{1}^{\prime}, R_{1}^{\prime}, \ldots, L_{n-1}^{\prime}, R_{n-1}^{\prime}$ all lie in convex position. That is, we choose $k_{0}, k_{1}, \ldots, k_{n-1}$ so that each of these facets is a facet of the integral polytope

$$
M:=\operatorname{Conv}\left(Q_{0}^{\prime} \cup L_{1}^{\prime} \cup R_{1}^{\prime} \cup \cdots \cup L_{n-1}^{\prime} \cup R_{n-1}^{\prime}\right)
$$

(The values of $k_{i}$ that are sufficiently large will depend upon the desired periods $p_{0}, p_{1}, \ldots, p_{n-1}$, as well as on the particular $(n+1)$-subset $T \subseteq \mathbb{Z}$ used to construct the cyclic polytopes $C_{i}$ in Section 2.)

Using these translated versions for $L_{i}$ and $R_{i}$, let us redefine the polytope $Q_{i}$ for $1 \leq i \leq n-1$ by setting $Q_{i}:=L_{i} \cup M \cup R_{i}$. As in Section 4, $Q_{i}$ has period sequence $\left(1, \ldots, 1, p_{i}, 1, \ldots, 1\right)$. Finally, we let $Q_{*}:=\bigcup_{i=0}^{n-1} Q_{i}$. As in the proof
of Theorem 4.1. we compute that

$$
\operatorname{ehr}_{Q_{*}}(t) \equiv \sum_{i=0}^{n-1} \operatorname{Vol}\left(C_{i}\right) t^{i} \operatorname{ehr}_{\Delta^{n-i-1}\left(\ell_{i}\right)}(t)
$$

Therefore, the period sequence of $Q_{*}$ is $\left(p_{0}, p_{1}, \ldots, p_{n-1}, 1\right)$, as desired.

## References

[1] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, Integer points in polyhedra - geometry, number theory, algebra, optimization, Contemp. Math., vol. 374, Amer. Math. Soc., Providence, RI, 2005, pp. 15-36, arXiv:math/0402148.
[2] M. Beck and S. Robins, Computing the continuous discretely: Integer-point enumeration in polyhedra, Undergraduate Texts in Mathematics, Springer, New York, 2007.
[3] M. Beck, S. V. Sam, and K. M. Woods, Maximal periods of (Ehrhart) quasi-polynomials, J. Combin. Theory Ser. A 115 (2008), no. 3, 517-525, arXiv:math/0702242.
[4] U. Betke and P. McMullen, Lattice points in lattice polytopes, Monatsh. Math. 99 (1985), no. 4, 253-265.
[5] P. Borwein, Computational excursions in analysis and number theory, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 10, SpringerVerlag, New York, 2002.
[6] B. Braun, Norm bounds for Ehrhart polynomial roots, Discrete Comput. Geom. 39 (2008), no. 1-3, 191-193, arXiv:math/0602464.
[7] F. Breuer, Ehrhart $f^{*}$-coefficients of polytopal complexes are non-negative integers, Electron. J. Combin. 19 (2012), no. 4, Paper 16, 22, arXiv:1202.2652.
[8] C. Haase, B. Nill, and S. Payne, Cayley decompositions of lattice polytopes and upper bounds for $h^{*}$-polynomials, J. Reine Angew. Math. 637 (2009), 207-216, arXiv:0804.3667.
[9] M. Henk and M. Tagami, Lower bounds on the coefficients of Ehrhart polynomials, European J. Combin. 30 (2009), no. 1, 70-83, arXiv:0710.2665.
[10] A. J. Herrmann, Classification of Ehrhart quasi-polynomials of half-integral polygons, Master's thesis, San Francisco State University, August 2010.
[11] T. Hibi, Algebraic Combinatorics on Convex Polytopes, Carslaw Publications, Glebe, N.S.W., Australia, 1992.
[12] , A lower bound theorem for Ehrhart polynomials of convex polytopes, Adv. Math. 105 (1994), no. 2, 162-165.
[13] F. Liu, Ehrhart polynomials of cyclic polytopes, J. Combin. Theory Ser. A 111 (2005), no. 1, 111-127.
[14] _, A note on lattice-face polytopes and their Ehrhart polynomials, Proc. Amer. Math. Soc. 137 (2009), no. 10, 3247-3258.
[15] , Higher integrality conditions, volumes and Ehrhart polynomials, Adv. Math. 226 (2011), no. 4, 3467-3494.
[16] T. B. McAllister and M. Moriarity, Ehrhart quasi-period collapse in rational polygons, J. Combin. Theory Ser. A 150 (2017), 377-385, arXiv:1509.03680.
[17] T. B. McAllister and H. O. Rochais, Periods of Ehrhart coefficients of rational polytopes, Electronic Journal of Combinatorics 25 (2018), no. 1, Paper \#P1.64.
[18] P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika 17 (1970), 179-184.
[19] J. Pfeifle, Gale duality bounds for roots of polynomials with nonnegative coefficients, J. Combin. Theory Ser. A 117 (2010), no. 3, 248-271, arXiv:0707.3010.
[20] P. R. Scott, On convex lattice polygons, Bull. Austral. Math. Soc. 15 (1976), no. 3, 395-399.
[21] R. P. Stanley, Decompositions of rational convex polytopes, Ann. Discrete Math. 6 (1980), 333-342, Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).
[22] _, On the Hilbert function of a graded Cohen-Macaulay domain, J. Pure Appl. Algebra 73 (1991), no. 3, 307-314.
[23] , Enumerative combinatorics. Vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.
[24] A. Stapledon, Inequalities and Ehrhart $\delta$-vectors, Trans. Amer. Math. Soc. 361 (2009), no. 10, 5615-5626, arXiv:0801.0873.
[25] A. Stapledon, Additive number theory and inequalities in Ehrhart theory, Int. Math. Res. Not. IMRN (2016), no. 5, 1497-1540, arXiv:0904.3035.

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[^1]:    ${ }^{1}$ When $p=1, P$ is a triangle.

[^2]:    ${ }^{2}$ We adopt the natural conventions to deal with the extreme cases $i=1$ and $i=n-1$. Hence, $R_{1}:=\Delta^{n-2}(P)+\mathbf{e}_{2}, R_{1}^{\prime}:=\operatorname{Conv}\left\{ \pm q \mathbf{e}_{1}, \mathbf{e}_{3}, \ldots, \mathbf{e}_{n}\right\}+\mathbf{e}_{2}, L_{n-1}^{\prime}:=\left(C_{n-1} \times\{0\}\right)-\mathbf{e}_{n}$, and $R_{n-1}^{\prime}:=\left(C_{n-2} \times \operatorname{Conv}\left\{ \pm q \mathbf{e}_{1}\right\}\right)+\mathbf{e}_{n}$.

