Tamed EM schemes for neutral stochastic differential delay equations with superlinear diffusion coefficients

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Abstract

In this article, we propose two types of explicit tamed Euler-Maruyama (EM) schemes for neutral stochastic differential delay equations with superlinearly growing drift and diffusion coefficients. The first type is convergent in the $\mathcal{L}^q$ sense under the local Lipschitz plus Khasminskii-type conditions. The second type is of order half in the mean-square sense under the Khasminskii-type, global monotonicity and polynomial growth conditions. Moreover, it is proved that the partially tamed EM scheme has the property of mean-square exponential stability. Numerical examples are provided to illustrate the theoretical findings.

Keywords: Neutral stochastic differential delay equations, tamed EM scheme, super-linear growth, strong convergence, mean-square stability.

1. Introduction

As an important type of stochastic differential equations (SDEs), neutral stochastic differential delay equations (NSDDEs) play a significant part in many application fields, such as automatic control, biology, power system and finance [1, 2]. In general, such SDEs with the neutral term do not have any explicit solutions, and we must be content ourselves with an approximation via a numerical approach. Due to the simple algebraic structure, easy implementation and acceptable convergence rate, Euler-type schemes have been introduced to approximate the exact solutions of NSDDEs. Li and Cao [3] presented a two-step Euler-Maruyama (EM) scheme for NSDDEs and studied the mean-square stability of the scheme under the linear growth condition. Mo et al. [4] proposed a split-step theta-method for NSDDEs with Poisson jumps, they also discussed the exponential stability of the method. Ji and Yuan [5] analyzed the convergence rate of tamed EM for NSDDEs with diffusion coefficients of linear growth. Influenced by the work of Mao [6], Lan and Xia [7] developed a modified truncated EM method for SDEs, they extended this method to the case of NSDDEs and obtained the exponential stability of the scheme in [8].

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Other related work on this topic can be found in [9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and the references therein.

So far as we know, most of the convergence rate results on the numerical schemes for NSDDEs require the conditions that the drift coefficients satisfy the linear or one-sided linear growth condition and the diffusion coefficients satisfy the linear growth condition, see e.g., [5, 10, 11, 19]. There are limited results on the convergence rate under weaker conditions than these. Zhou and Jin [9] discussed the strong convergence of the backward EM scheme for highly nonlinear NSDDEs, where the diffusion coefficient is polynomially growing with respect to delay term while for non-delay part the coefficient is linearly growing. As a result, our effort is devoted to investigating the strong convergence of explicit numerical approximations, whose convergence order can arrive at one half, for NSDDEs with superlinearly growing drift and diffusion coefficients. It is known that tamed (balanced) or truncation techniques can be used to cope with the superlinearly growing parts appearing in the drift and diffusion coefficients when SDEs are considered. For strong schemes for such SDEs, several types of methods have been introduced: tamed EM schemes, originally proposed by Hutzenthaler et al. in [20], where the coefficients are approximated by the function of the form $F(x)/(1+\Delta^{\alpha}|F(x)|)$ $(0<\alpha\leq 1)$ to control their superlinear growth, see e.g., [21, 22, 23, 24, 24, 25, 26, 27, 28]; truncated EM schemes, originally proposed by Mao in [6], where coefficients of superlinear growth can be bounded by the truncated function, see e.g., [29, 30, 31, 32, 33, 34, 35]. Moreover, another method, called Semi-Discrete (SD) method, originally proposed by Halidias [36], also attracts researchers' attention. A major advantage of the SD method is the domain preservation of the solution process, a property that the EM method in general do not preserve [37]. There is an ongoing research of the method and its properties, see for instance [38] for an application in a delay model with jumps, and the recent [39] for the convergence order.

Inspired by the taming idea in Sabanis [21] together with the truncation techniques from Mao [29], we propose a class of tamed EM scheme for NSDDEs with coefficients of superlinear growth. According to the scheme we derive some crucial properties P1-P3, which mean that the modified coefficients f_Δ and g_Δ conserve the Khasminskii-type condition and behave linearly for a fixed step size, see (3.11), (3.12) and (3.13). Based on these properties, a uniform moment bound for the numerical solutions is established and then the tamed EM method can be shown to converge strongly and conserve the stability in the mean-square sense.

The main contribution of this paper is to develop two types of explicit tamed EM schemes for NSDDEs, in which both drift and diffusion coefficients can be growing superlinearly, and investigate the strong convergence, mean-square stability of the schemes for NSDDE (2.1). We extend the tamed EM scheme presented in Sabanis [21] to the case of NSDDEs. Furthermore, when the neutral term D is absent in NSDDE (2.1), compared with the convergence results of the truncated EM schemes for stochastic delay differential equations (SDDEs) from Fei et al. [32, Theorem 3.6], our results obtain a better convergence order under almost the same conditions, see Theorem 4.1.

The rest of the paper is organized as follows. In the next section, we present some preliminaries and assumptions on the NSDDEs. The tamed EM scheme is proposed in Section 3. The discussion of the strong convergence of the tamed EM is given in Section 4. In Section 5, we show the reproduction of mean-square stability of numerical solutions for the exact solution. Numerical examples are presented in Section 6. In the final section, we close the paper by our conclusion.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\tau > 0$ be a constant and denote by $C([-\tau, 0]; \mathbb{R}^d)$ the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Let $B(t)$ be an m-dimensional Brownian motion. If A is a vector or matrix, its transpose is denoted by A^T. If $X \in \mathbb{R}^d$, then $|X|$ is the Euclidean norm. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{(A^T A)}$. For two real numbers a and b, $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. For a set G, its indicator function is denoted by $\mathbb{I}_G$. The scalar product of two vectors $X, Y \in \mathbb{R}^d$ is denoted by $\langle X, Y\rangle$ or $X^T Y$. Denote by $\lfloor a \rfloor$ the largest integer which is less or equal to a.

Consider a neutral stochastic differential delay equation of the form

$$d[x(t) - D(x(t-\tau))] = f(x(t)), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t), \; t \geq 0, \tag{2.1}$$

with the initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^d)$, where $D : \mathbb{R}^d \to \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are Borel-measurable functions. Unless specified otherwise, we assume that the initial data ξ satisfies the following condition: there is a pair of constants $K_1 > 0$ and $\varrho \in (0, 1]$ such that

$$|\xi(t) - \xi(s)| \leq K_1|t - s|^{\varrho}, \; \forall s, t \in [-\tau, 0]. \tag{2.2}$$

Moreover, we assume that $D(0) = 0$ and there exists a constant $\nu \in (0, 1)$ such that

$$|D(x) - D(y)| \leq \nu|x - y|, \; \forall x, y \in \mathbb{R}^d. \tag{2.3}$$

Consider the following assumptions:

Assumption 2.1. ***(Local Lipschitz condition)*** *For any $R > 0$, there exists a constant L_R depending on R such that*

$$|f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \leq L_R(|x - \bar{x}| + |y - \bar{y}|),$$

$\forall x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ *with* $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq R$.

Assumption 2.2. ***(Khasminskii-type condition)*** *There exist positive constants K_2 and $p_0 > 2$ such that*

$$(x - D(y))^T f(x, y) + \frac{p_0 - 1}{2}|g(x, y)|^2 \leq K_2(1 + |x|^2 + |y|^2), \quad \forall x, y \in \mathbb{R}^d.$$

Under Assumptions 2.1 and 2.2, NSDDE (2.1) has a unique global solution $x(t)$ on $t \in [-\tau, \infty)$. In addition, we have the following result regarding the moments of $x(t)$, the proof is similar to that of Mao [40, p.213, Theorem 4.5] and is therefore omitted.

Lemma 2.3. *Suppose that Assumption 2.1 and 2.2 hold. Then*

$$\sup_{-\tau\leq t\leq T} \mathbb{E}|x(t)|^{p_0} < \infty, \; \forall T > 0. \tag{2.4}$$

Denote by $\mathcal{U}$ the family of continuous function $U : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ such that for any $R > 0$, there exists a positive constant $\bar{L}_R$ for which

$$U(x, \bar{x}) \le \bar{L}_R |x - \bar{x}|^2, \ \forall x, \bar{x} \in \mathbb{R}^d \text{ with } |x| \vee |\bar{x}| \le R.$$

Assumption 2.4. ***(Global monotonicity with U function and polynomial growth conditions)*** *There exist constants $p_1 > 2$, $l \ge 0$ and $K_3 > 0, K_4 > 0$ as well as a function $U \in \mathcal{U}$ such that*

$$\begin{aligned}&(x - \bar{x} - D(y) + D(\bar{y}))^T (f(x, y) - f(\bar{x}, \bar{y})) + \frac{p_1 - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ &\le K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2) - U(x, \bar{x}) + U(y, \bar{y}), \ \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{2.5}$$

and

$$|f(x, y) - f(\bar{x}, \bar{y})| \le K_4(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x - \bar{x}| + |y - \bar{y}|), \ \forall x, y \in \mathbb{R}^d, \tag{2.6}$$

as well as

$$|g(x, y) - g(\bar{x}, \bar{y})|^2 \le K_4(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x - \bar{x}|^2 + |y - \bar{y}|^2), \ \forall x, y \in \mathbb{R}^d. \tag{2.7}$$

From (2.6) and (2.7), we have the following growth condition

$$|f(x, y)| \le K_5(1 + |x|^{l+1} + |y|^{l+1}) \quad \text{and} \quad |g(x, y)|^2 \le K_5(1 + |x|^{l+2} + |y|^{l+2}), \ \forall x, y \in \mathbb{R}^d, \tag{2.8}$$

where $K_5 = 6K_4 \vee (4K_4 + 2|g(0, 0)|^2 + |f(0, 0)|^2)$.

Remark 2.5. *If the neutral term D vanishes, the global monotonicity condition with U function (2.5) reduces to Fei et al. [32, Assumption 2.3] and Guo et al. [34, Assumption 5.1]. In view of Fei et al. [32, Example 6.2], the presence of U function in global monotonicity condition will make the choice of the drift and diffusion coefficients for SDDEs more flexible.*

3. Tamed EM scheme for NSDDEs

Assume that the step size Δ is a fraction of τ. Define $\Delta = \tau/m \in (0, 1]$ for some positive integer m and $\kappa(t) := \lfloor t/\Delta \rfloor \Delta$, for any $t \ge -\tau$. The discrete-time tamed EM scheme for NSDDE (2.1) is defined as follows:

$$\begin{aligned}&y_\Delta^{k+1} = D(y_\Delta^{k+1-m}) + y_\Delta^k - D(y_\Delta^{k-m}) + f_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta + g_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta B_k, \ k = 0, 1, 2, \cdots, \\ &y_\Delta^k = \xi(k\Delta), \ k = -m, -m + 1, \cdots, 0, \end{aligned} \tag{3.1}$$

where $\Delta B_k = B((k+1)\Delta - k\Delta)$, the modified coefficients $f_\Delta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g_\Delta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are Borel-measurable and satisfy some conditions given below. Define a continuous-time step process $\bar{Y}_\Delta(t)$ on $t \in [-\tau, \infty)$ by

$$\bar{Y}_\Delta(t) = \sum_{k=-m}^{\infty} y_\Delta^k \mathbb{I}_{[k\Delta, (k+1)\Delta)}(t),$$

where $\mathbb{I}$ is the indicator function. Then $\bar{Y}_\Delta(t-\tau) = y_\Delta^{k-m}$, for any $t \in [k\Delta, (k+1)\Delta)$ with $k \geq 0$. Define a new continuous-time process $Y_\Delta(t)$ on $t \in [-\tau, \infty)$ by

$$\begin{aligned} Y_\Delta(t) &= D(Y_\Delta(t-\tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^t f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))ds \\ &\quad + \int_0^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s), \ \ t \geq 0, \\ Y_\Delta(t) &= \xi(t), \ \ -\tau \leq t \leq 0. \end{aligned} \tag{3.2}$$

Obviously, $Y_\Delta(t)$ is well defined on $[0, \tau]$. Once the process $Y_\Delta(t)$ on this interval is known, we can proceed this argument on $[\tau, 2\tau]$, $[2\tau, 3\tau]$ etc. and hence obtain the process $Y_\Delta(t)$ on the entire interval $[-\tau, \infty)$. Moreover, $Y_\Delta(t)$ is an Itô process on $[0, \infty)$ with Itô differential

$$d[Y_\Delta(t) - D(Y_\Delta(t-\tau))] = f_\Delta(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))dt + g_\Delta(\bar{Y}_\Delta(t), \bar{Y}_\Delta(t-\tau))dB(t). \tag{3.3}$$

From (3.1) and (3.2), we conclude that

$$Y_\Delta(\Delta) = D(\xi(\Delta-\tau)) + \xi(0) - D(\xi(-\tau)) + f_\Delta(\xi(0), \xi(-\tau))\Delta + g_\Delta(\xi(0), \xi(-\tau))\Delta B_0 = y_\Delta^1.$$

Similarly, we can show that for any $t \in [k\Delta, (k+1)\Delta)$ with $k \geq 0$,

$$Y_\Delta(k\Delta) = y_\Delta^k = \bar{Y}_\Delta(t), \tag{3.4}$$

that is, the discrete and continuous tamed EM solutions coincide at the grid points. Thus, it is useful to know that for any $t \in [k\Delta, (k+1)\Delta)$ with $k \geq 0$,

$$Y_\Delta(k\Delta) - D(Y_\Delta(k\Delta - \tau)) = \bar{Y}_\Delta(t) - D(\bar{Y}_\Delta(t-\tau)) = y_\Delta^k - D(y_\Delta^{k-m}) \tag{3.5}$$

and

$$\begin{aligned} &Y_\Delta(t) - D(Y_\Delta(t-\tau)) - \bar{Y}_\Delta(t) + D(\bar{Y}_\Delta(t-\tau)) \\ &= \int_{k\Delta}^t f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))ds + \int_{k\Delta}^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s). \end{aligned} \tag{3.6}$$

Remark 3.1. *In most of the existing work on the numerical methods for NSDDEs, e.g., Ji and Yuan [5], Lan [8], Tan and Yuan [19], $D(\bar{Y}_\Delta(t-\tau))$, rather than $D(Y_\Delta(t-\tau))$, as an approximation to $D(x(t-\tau))$, appears in the equation* (3.2)*, which determines another form of continuous-time process $Y_\Delta^*(t)$ defined by*

$$\begin{aligned} Y_\Delta^*(t) &= D(\bar{Y}_\Delta(t-\tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^t f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))ds \\ &\quad + \int_0^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s), \ \ t \geq 0. \end{aligned}$$

We then have the following form of difference between the exact solution $x(t)$ and the tamed EM solution $Y_\Delta^*(t)$,

$$\begin{aligned} &x(t) - Y_\Delta^*(t) - D(x(t-\tau)) + D(\bar{Y}_\Delta(t-\tau)) \\ &= \int_0^t \Big(f(x(s), x(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big)ds \\ &\quad + \int_0^t \Big(g(x(s), x(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big)dB(s), \ \ t \geq 0. \end{aligned} \tag{3.7}$$

If we apply the Itô formula to (3.7) *and use the global monotonicity condition with U function* (2.5)*, then there will be an extra term expressed by the following we have to address,*

$$\mathbb{E}\int_0^t \Big(-U(x(s), Y_\Delta^*(s)) + U(x(s-\tau), Y_\Delta^*(\kappa(s)-\tau))\Big)ds, \ t \ge 0. \tag{3.8}$$

Note that the two time variables of the second U function in (3.8) *are asynchronous, it is difficult to treat* (3.8) *as an appropriate form. However, if we use* $Y_\Delta(t)$ *given by* (3.2)*,* (3.8) *becomes*

$$\mathbb{E}\int_0^t \Big(-U(x(s), Y_\Delta(s)) + U(x(s-\tau), Y_\Delta(s-\tau))\Big)ds, \ t \ge 0. \tag{3.9}$$

We observe from (4.35) *that* (3.9) *can be addressed well. From the practical point of view, using* $D(Y_\Delta(t-\tau))$ *to approximate* $D(x(t-\tau))$ *in* (3.7) *avoids the presence of asynchronous time. Moreover,* $Y_\Delta(t)$ *preserves the useful property* (3.4) *that* $Y_\Delta^*(t)$ *has.*

Let us make some conditions on the coefficients of the scheme (3.1). Suppose that there is a constant $\alpha \in (0, 1/2]$ such that the following conditions hold:
P1. For any $R > 0$, there is a positive constant N_R depending on R such that for any $x, y \in \mathbb{R}^d$,

$$\sup_{|x|\vee|y|\le R} |f(x,y) - f_\Delta(x,y)| \vee |g(x,y) - g_\Delta(x,y)| \le N_R\Delta^\alpha. \tag{3.10}$$

P2. There is a positive constant $\hat{K}_1$ such that for any $\Delta \in (0, 1]$,

$$|f_\Delta(x,y)| \le \hat{K}_1\Delta^{-\alpha}(1+|x|+|y|) \wedge |f(x,y)|, \ \forall x, y \in \mathbb{R}^d, \tag{3.11}$$

and

$$|g_\Delta(x,y)|^2 \le \hat{K}_1\Delta^{-\alpha}(1+|x|^2+|y|^2) \wedge |g(x,y)|^2, \ \forall x, y \in \mathbb{R}^d. \tag{3.12}$$

P3. There is a positive constant $\hat{K}_2$ such that

$$(x - D(y))^T f_\Delta(x,y) + \frac{p_0 - 1}{2}|g_\Delta(x,y)|^2 \le \hat{K}_2(1+|x|^2+|y|^2), \ \forall x, y \in \mathbb{R}^d. \tag{3.13}$$

Property P3 means that modified coefficients f_Δ and g_Δ preserve the Khasminskii-type condition 2.2. While property P2 implies that for any $\Delta \in (0, 1]$, f_Δ and g_Δ satisfy the linear growth condition which guarantees the existence of a unique solution to (3.3).

Now, we propose two types of modified coefficients f_Δ and g_Δ in (3.1). Let $\alpha \in (0, 1/2]$, define

$$f_\Delta(x,y) := \pi_\Delta(x,y)f(x,y) \quad \text{and} \quad g_\Delta(x,y) := \pi_\Delta(x,y)g(x,y), \tag{3.14}$$

where $\pi_\Delta : \mathbb{R}^d \times \mathbb{R}^d \to (0, 1)$ is defined by

$$\textbf{Type I:} \quad \pi_\Delta(x,y) = \frac{1}{1+\Delta^\alpha(|f(x,y)| + |g(x,y)|^2)}, \ \forall x, y \in \mathbb{R}^d, \ \Delta \in (0, 1], \tag{3.15}$$

or

$$\textbf{Type II:} \quad \pi_\Delta(x,y) = \frac{1}{1+\Delta^\alpha(|x|^l + |y|^l)}, \ \forall x, y \in \mathbb{R}^d, \ \Delta \in (0, 1]. \tag{3.16}$$

Remark 3.2. *Under Assumptions 2.1 and 2.2, we can show that the modified coefficients f_Δ and g_Δ with Type I given by* (3.15) *satisfy conditions P1-P3. If Assumptions 2.1 is replaced by Assumption 2.4, the modified coefficients Type II also satisfy P1-P3. This type of tamed EM scheme allows us to produce the optimal rate of convergence. But if we are only interested in the strong convergence (without order) of the numerical scheme, then using the tamed EM Type I may suffice.*

4. Strong convergence at time $T > 0$

4.1. Order of strong convergence of tamed EM (Type II) under monotonicity condition

Theorem 4.1. *Suppose that Assumptions 2.2 and 2.4 hold with $p_0 \geq 4 \vee 2(1 + 2l)$ and $\alpha \in (0, 1/2]$ is arbitrary. Then the tamed EM solution $Y_\Delta(t)$ or $\bar{Y}_\Delta(t)$ with modified coefficients Type II converges to the exact solution $x(t)$ of NSDDE* (2.1) *with order $\alpha \wedge \varrho$ in the mean-square sense, i.e.,*

$$\sup_{0\leq t\leq T} \mathbb{E}|x(t) - Y_\Delta(t)|^2 \leq C\Delta^{2(\alpha\wedge\varrho)} \quad \text{and} \quad \sup_{0\leq t\leq T} \mathbb{E}|x(t) - \bar{Y}_\Delta(t)|^2 \leq C\Delta^{2(\alpha\wedge\varrho)}, \ \forall \Delta \in (0, 1], \tag{4.1}$$

where the positive constant $C := C(T, v, \|\xi\|, p_0, p_1, K_1, K_2, K_3, K_4, l)$. In particular, letting $\alpha = 1/2$ yields that

$$\Big[\mathbb{E}|x(T) - Y_\Delta(T)|^2\Big]^{1/2} \leq C\Delta^{0.5\wedge\varrho} \quad \text{and} \quad \Big[\mathbb{E}|x(T) - \bar{Y}_\Delta(T)|^2\Big]^{1/2} \leq C\Delta^{0.5\wedge\varrho}, \forall \Delta \in (0, 1]. \tag{4.2}$$

From now on, C denotes a genetic positive real constant dependent on T, v, $\|\xi\|$ etc. but independent of Δ.

Remark 4.2. *It should be pointed out that if the neutral term D vanishes in NSDDE* (2.1)*, Theorem 4.1 reduces to the convergence result of tamed EM scheme for SDDEs. In this case, compared with the convergence result of the truncated EM scheme in Fei et al. [32, Corollary 3.8], which has an order $(0.5 - \varepsilon) \wedge \varrho$ in the mean-square sense, where $\varepsilon \in (0, 1/4]$ is arbitraty, we observe from* (4.2) *that our scheme has a better convergence order under almost the same conditions as [32, Corollary 3.8].*

Lemma 4.3. *Suppose that P2 and P3 hold with $p_0 \geq 4$ and $\alpha \in (0, 1/2]$ is arbitrary. Then the tamed EM solution $Y_\Delta(t)$ defined by* (3.2) *satisfies*

$$\sup_{0<\Delta\leq 1} \sup_{0\leq t\leq T} \mathbb{E}|Y_\Delta(t)|^{p_0} \leq C,$$

where the positive constant $C := C(p_0, T, K_2, \|\xi\|, v)$.

Proof. Let P2 and P3 hold with $p_0 \geq 4$ and $\alpha \in (0, 1/2]$. Then for any $t \in [0, T]$, applying the Itô formula to (3.2) and P3, i.e., (3.13), we derive that

$$
\begin{aligned}
&\mathbb{E}|Y_\Delta(t) - D(Y_\Delta(t-\tau))|^{p_0} \\
&\leq |\xi(0) - D(\xi(-\tau))|^{p_0} + p_0\mathbb{E}\int_0^t \Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} \\
&\quad \times \Big([Y_\Delta(s) - D(Y_\Delta(s-\tau))]^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) + \frac{p_0-1}{2}|g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2\Big)\Big]ds \\
&= |\xi(0) - D(\xi(-\tau))|^{p_0} + p_0\mathbb{E}\int_0^t \Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} \\
&\quad \times \Big([\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))]^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) + \frac{p_0-1}{2}|g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2\Big)\Big]ds \\
&\quad + p_0\mathbb{E}\int_0^t \Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} \\
&\quad \times [Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \bar{Y}_\Delta(s) + D(\bar{Y}_\Delta(s-\tau))]^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&\leq |\xi(0) - D(\xi(-\tau))|^{p_0} + C\mathbb{E}\int_0^t \Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2}(1 + |\bar{Y}_\Delta(s)|^2 + |\bar{Y}_\Delta(s-\tau)|^2)\Big]ds + p_0 J(t) \\
&\leq C + C\mathbb{E}\int_0^t \Big[(|Y_\Delta(s)|^{p_0-2} + |D(Y_\Delta(s-\tau))|^{p_0-2})(1 + |\bar{Y}_\Delta(s)|^2 + |\bar{Y}_\Delta(s-\tau)|^2)\Big]ds + p_0 J(t) \\
&\leq C + C\mathbb{E}\int_0^t \Big[1 + |Y_\Delta(s)|^{p_0} + |Y_\Delta(s-\tau)|^{p_0} + |\bar{Y}_\Delta(s)|^{p_0} + |\bar{Y}_\Delta(s-\tau)|^{p_0}\Big]ds + p_0 J(t) \\
&\leq C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|Y_\Delta(u)|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0}\Big]ds + p_0 J(t),
\end{aligned}
$$

where

$$
\begin{aligned}
J(t) := \mathbb{E}\int_0^t &\Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} \\
&\times (Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \bar{Y}_\Delta(s) + D(\bar{Y}_\Delta(s-\tau)))^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds.
\end{aligned} \tag{4.3}
$$

From (3.4), we have the following useful estimate:

$$
\sup_{0\leq u\leq s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} \leq \sup_{0\leq u\leq s}\mathbb{E}|Y_\Delta(u)|^{p_0}. \tag{4.4}
$$

Consequently,

$$
\mathbb{E}|Y_\Delta(t) - D(Y_\Delta(t-\tau))|^{p_0} \leq C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds + p_0 J(t). \tag{4.5}
$$

We observe that

$$
\begin{aligned}
J(t) = \mathbb{E}\int_0^t &\Big[|\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2} \\
&\times (Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \bar{Y}_\Delta(s) + D(\bar{Y}_\Delta(s-\tau)))^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&+ \mathbb{E}\int_0^t \Big[\Big(|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} - |\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times (Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \bar{Y}_\Delta(s) + D(\bar{Y}_\Delta(s-\tau)))^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&=: J_1(t) + J_2(t).
\end{aligned} \tag{4.6}
$$

Using (3.6) and P2, we have the following estimate

$$
\begin{aligned}
J_1(t) &= \mathbb{E}\int_0^t \Big[|\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}\Big(\int_{\kappa(s)}^s f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))du\Big)^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&\quad + \mathbb{E}\int_0^t \Big[|\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}\Big(\int_{\kappa(s)}^s g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))dB(u)\Big)^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&\le \mathbb{E}\int_0^t \Big[|\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}|f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 \int_{\kappa(s)}^s du\Big]ds \\
&\le C\Delta^{1-2\alpha}\mathbb{E}\int_0^t |\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}(1 + |\bar{Y}_\Delta(s)|^2 + |\bar{Y}_\Delta(s-\tau)|^2)ds \\
&\le C\int_0^t \Big[1 + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u) - D(\bar{Y}_\Delta(u-\tau))|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u-\tau)|^{p_0}\Big]ds \\
&\le C\int_0^t \Big[1 + \|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u) - D(\bar{Y}_\Delta(u-\tau))|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0}\Big]ds \\
&\le C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds,
\end{aligned} \tag{4.7}
$$

where (4.4) has been used in the derivation of the last inequality.

For some $p_0 \ge 4$, applying the Itô formula to $|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2}$, we conclude from (3.2) or (3.6) that

$$
\begin{aligned}
J_2(t) &= \mathbb{E}\int_0^t \Big[\Big(|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2} - |\bar{Y}_\Delta(s) - D(\bar{Y}_\Delta(s-\tau))|^{p_0-2}\Big) \\
&\quad \times \Big(Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \bar{Y}_\Delta(s) + D(\bar{Y}_\Delta(s-\tau))\Big)^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&\le \mathbb{E}\int_0^t \Big[\Big\{(p_0-2)\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u) - D(Y_\Delta(u-\tau))^T f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))du \\
&\quad + \frac{(p_0-2)(p_0-3)}{2}\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-4}|g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|^2du \\
&\quad + (p_0-2)\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u) - D(Y_\Delta(u-\tau))^T g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))dB(u)\Big\} \\
&\quad \times \Big(\int_{\kappa(s)}^s f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))du + \int_{\kappa(s)}^s g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))dB(u)\Big)^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
&=: \sum_{i=1}^6 J_{2i}(t),
\end{aligned}
$$

where

$$
J_{21}(t) = C\mathbb{E}\int_0^t \Big[\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u) - D(Y_\Delta(u-\tau))^T f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))du
$$

$$
\times\Big(\int_{\kappa(s)}^{s} f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))du\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds
$$

$$
J_{22}(t)=C\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u)-D(Y_\Delta(u-\tau))^T f_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))du
$$

$$
\times\Big(\int_{\kappa(s)}^{s} g_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))dB(u)\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds
$$

$$
J_{23}(t)=C\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-4}|g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))|^2du
$$

$$
\times\Big(\int_{\kappa(s)}^{s} f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))du\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds
$$

$$
J_{24}(t)=C\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-4}|g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))|^2du
$$

$$
\times\Big(\int_{\kappa(s)}^{s} g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))dB(u)\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds
$$

$$
J_{25}(t)=C\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u)-D(Y_\Delta(u-\tau))^T g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))dB(u)
$$

$$
\times\Big(\int_{\kappa(s)}^{s} f_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))du\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds
$$

$$
J_{26}(t)=C\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-4}(Y_\Delta(u)-D(Y_\Delta(u-\tau))^T g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))dB(u)
$$

$$
\times\Big(\int_{\kappa(s)}^{s} g_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))dB(u)\Big)^T f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds. \tag{4.8}
$$

By P2 and (4.4), we have

$$
\begin{aligned}
J_{21}(t)&\le C\mathbb{E}\int_0^t\Big[|f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|^2\Delta\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-3}|f_\Delta(\bar Y_\Delta(u),\bar Y_\Delta(u-\tau))|du\Big]ds\\
&\le C\Delta\mathbb{E}\int_0^t\Big[|f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|^3\int_{\kappa(s)}^{s}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-3}du\Big]ds\\
&\le C\Delta^{1-3\alpha}\mathbb{E}\int_0^t\Big[\int_{\kappa(s)}^{s}(1+|\bar Y(\kappa(s))|^3+|\bar Y(\kappa(s)-\tau)|^3)|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0-3}du\Big]ds\\
&\le C\Delta^{2-3\alpha}\int_0^t\Big[1+\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0}+\sup_{0\le u\le s}\mathbb{E}|\bar Y_\Delta(u)|^{p_0}+\sup_{0\le u\le s}\mathbb{E}|\bar Y_\Delta(u-\tau)|^{p_0}\Big]ds\\
&\le C+C\int_0^t\Big[\|\xi\|^{p_0}+\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds.
\end{aligned}\tag{4.9}
$$

Recall the Young inequality: for $r_1^{-1}+r_2^{-1}=1$, $r_1,r_2>1$,

$$
ab\le\frac{a^{r_1}}{r_1}+\frac{b^{r_2}}{r_2},\ \forall a,b>0.
$$

Letting $r_1 = p_0$, $r_2 = \frac{p_0}{p_0-1}$ in the above inequality, applying the Hölder and Burkholder-Davis-Gundy inequalities, we have

$$
\begin{aligned}
J_{22}(t) \le{}& C\mathbb{E}\int_0^t \Big[\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-3}|f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|du \\
&\times \Big|\int_{\kappa(s)}^s g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))dB(u)\Big|\Big|f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big|\Big]ds \\
\le{}& C\mathbb{E}\int_0^t \Big[\Big(|f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-3}|f_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|du\Big)^{p_0/(p_0-1)} \\
&+ \Big|\int_{\kappa(s)}^s g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))dB(u)\Big|^{p_0}\Big]ds \\
\le{}& C\mathbb{E}\int_0^t \Big[\Big(\Delta^{-2\alpha}\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0-3}(1 + |\bar{Y}_\Delta(s)|^2 + |\bar{Y}_\Delta(s-\tau)|^2)du\Big)^{p_0/(p_0-1)}\Big]ds \\
&+ C\int_0^t \Big[\mathbb{E}\Big(\int_{\kappa(s)}^s |g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|^2 du\Big)^{p_0/2}\Big]ds \\
\le{}& C\Delta^{(1-2\alpha)p_0/(p_0-1)}\int_0^t \Big[1 + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} \\
&+ \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u-\tau)|^{p_0}\Big]ds + C\Delta^{(1-\alpha)p_0/2}\int_0^t \Big[1 + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u-\tau)|^{p_0}\Big]ds \\
\le{}& C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds. \qquad (4.10)
\end{aligned}
$$

Again using P2 and noting that $\alpha \in (0, 1/2]$, we have

$$
\begin{aligned}
J_{26}(t) ={}& C\mathbb{E}\int_0^t \Big[\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau)|^{p_0-4}(Y_\Delta(u) - D(Y_\Delta(u-\tau))^T|g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|^2du \\
&\times f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds \\
\le{}& \mathbb{E}\int_0^t \Big[\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau)|^{p_0-3}|g_\Delta(\bar{Y}_\Delta(u), \bar{Y}_\Delta(u-\tau))|^2du|f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|\Big]ds \\
\le{}& C\Delta^{-2\alpha}\mathbb{E}\int_0^t \Big[\int_{\kappa(s)}^s |Y_\Delta(u) - D(Y_\Delta(u-\tau)|^{p_0-3}|(1 + |\bar{Y}_\Delta(u)|^2 + |\bar{Y}(u-\tau)|^2)du \\
&\times (1 + |\bar{Y}_\Delta(s)| + |\bar{Y}(s-\tau)|)\Big]ds \\
\le{}& C\Delta^{1-2\alpha}\int_0^t \Big[1 + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|\bar{Y}_\Delta(u-\tau)|^{p_0}\Big]ds \\
\le{}& C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds. \qquad (4.11)
\end{aligned}
$$

Similarly, we can derive that

$$
J_{23}(t) + J_{24}(t) + J_{25}(t) \le C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}\Big]ds. \qquad (4.12)
$$

From (4.9)-(4.12), we have

$$J_2(t) \le C + C\int_0^t \Big[\|\xi\|^{p_0} + \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^{p_0}\Big] ds. \tag{4.13}$$

Inserting (4.7) and (4.13) into (4.6), we derive from (4.5) that

$$\sup_{0\le u\le t} \mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0} \le C + C\int_0^t \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^{p_0} ds. \tag{4.14}$$

Recall the following inequality: for $p > 1$, $\varepsilon > 0$ and $a, b \in \mathbb{R}$,

$$|a+b|^p \le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}\left(\frac{|a|^p}{\varepsilon} + |b|^p\right), \tag{4.15}$$

see [40, Lemma 4.1, p.211]. Consequently,

$$\begin{aligned}
|Y_\Delta(u)|^{p_0} &= |Y_\Delta(u) - D(Y_\Delta(u-\tau)) + D(Y_\Delta(u-\tau))|^{p_0} \\
&\le (1+\varepsilon^{\frac{1}{p_0-1}})^{p_0-1}\left(\frac{|D(Y_\Delta(u-\tau))|^{p_0}}{\varepsilon} + |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0}\right) \\
&\le (1+\varepsilon^{\frac{1}{p_0-1}})^{p_0-1}\left(\frac{\nu^{p_0}|(Y_\Delta(u-\tau))|^{p_0}}{\varepsilon} + |Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0}\right).
\end{aligned}$$

Letting $\varepsilon = \left(\frac{\nu}{1-\nu}\right)^{p_0-1}$ and taking expectations, we have

$$\mathbb{E}|Y_\Delta(u)|^{p_0} \le \nu\mathbb{E}|Y_\Delta(u-\tau)|^{p_0} + \frac{1}{(1-\nu)^{p_0-1}}\mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0}, \ \forall u \ge 0. \tag{4.16}$$

Therefore,

$$\begin{aligned}
\sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^{p_0} &\le \nu \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u-\tau)|^{p_0} + \frac{1}{(1-\nu)^{p_0-1}} \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0} \\
&\le \nu\|\xi\|^{p_0} + \nu \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^{p_0} + \frac{1}{(1-\nu)^{p_0-1}} \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0}, \ \forall s \ge 0.
\end{aligned}$$

Rearranging this gives

$$\sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^{p_0} \le \frac{\nu}{1-\nu}\|\xi\|^{p_0} + \frac{1}{(1-\nu)^{p_0}} \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u) - D(Y_\Delta(u-\tau))|^{p_0}, \ \forall s \ge 0. \tag{4.17}$$

Plugging (4.14) into (4.17) and applying the Gronwall inequality complete the proof. □

Lemma 4.4. *Suppose that Assumptions 2.2 and 2.4 hold with $p_0 \ge 4 \vee 2(1+2l)$ and $\alpha \in (0, 1/2]$ is arbitrary. Then for any $p \in \left[2, \frac{p_0}{1+2l}\right]$, the tamed EM solution $Y_\Delta(t)$ defined by (3.2) with modified coefficients Type II has the property that*

$$\mathbb{E}\int_0^T |f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p ds \le C\Delta^{\alpha p}, \ \forall \Delta \in (0,1], \tag{4.18}$$

and

$$\mathbb{E}\int_0^T |g(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p ds \le C\Delta^{\alpha p}, \ \forall \Delta \in (0,1]. \tag{4.19}$$

Proof. Let $\alpha \in (0, 1/2]$ and $p \in \left[2, \dfrac{p_0}{1+2l}\right]$. Consider the tamed EM scheme Type II. In view of Lemma 4.3, we have

$$
\begin{aligned}
&\mathbb{E}\int_0^T |f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p ds \\
&\le \Delta^{\alpha p}\mathbb{E}\int_0^T \left[\frac{(|\bar{Y}_\Delta(s)|^l + |\bar{Y}_\Delta(s-\tau)|^l)^p |f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p}{(1+\Delta^\alpha(|\bar{Y}_\Delta(s)|^l + |\bar{Y}_\Delta(s-\tau)|^l))^p}\right] ds \\
&\le C\Delta^{\alpha p}\mathbb{E}\int_0^T \Big[(|\bar{Y}_\Delta(s)|^{lp} + |\bar{Y}_\Delta(s-\tau)|^{lp})(1 + |\bar{Y}_\Delta(s)|^{l+1} + |\bar{Y}_\Delta(s-\tau)|^{l+1})^p\Big] ds \\
&\le C\Delta^{\alpha p}\int_0^T \mathbb{E}\Big(1 + |\bar{Y}_\Delta(s)|^{p(1+2l)} + |\bar{Y}_\Delta(s-\tau)|^{p(1+2l)}\Big) ds \\
&\le C\Delta^{\alpha p},
\end{aligned}
$$

which yields (4.18). Applying the same techniques gives (4.19). □

The following lemma provides the closeness between the two continuous versions of the tamed EM solutions in the sense of $\mathcal{L}^p$.

Lemma 4.5. *Suppose that Assumptions 2.2 and 2.4 hold with $p_0 \ge 4 \vee (2+l)$ and $\alpha \in (0, 1/2]$ is arbitrary. Then for any $p \in \left[2, \dfrac{p_0}{1+l/2}\right]$, the tamed EM solution $Y_\Delta(t)$ defined by (3.2) with modified coefficients Type II has the property that*

$$\sup_{0\le t\le T} \mathbb{E}|Y_\Delta(t) - \bar{Y}_\Delta(t)|^p \le C\Delta^{(\varrho\wedge 0.5)p}, \ \forall \Delta \in (0, 1]. \tag{4.20}$$

Proof. Let $p \in \left[2, \dfrac{p_0}{1+l/2}\right]$. Consider the tamed EM scheme Type II. Recall (3.6) that

$$Y_\Delta(t) - \bar{Y}_\Delta(t) = D(Y_\Delta(t-\tau)) - D(\bar{Y}_\Delta(t-\tau)) + \varphi_\Delta(t), \ \forall t \in [0, T], \tag{4.21}$$

where

$$\varphi_\Delta(t) := \int_{\kappa(t)}^t f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))ds + \int_{\kappa(t)}^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s).$$

We first show that there is a positive constant $\bar{c}_p$ dependent of p such that

$$\mathbb{E}|\varphi_\Delta(t)|^p \le \bar{c}_p\Delta^{0.5p}, \ \forall t \in [0, T]. \tag{4.22}$$

By the elementary inequality, we have that for any $t \in [0, T]$

$$
\begin{aligned}
\mathbb{E}|\varphi_\Delta(t)|^p &= \mathbb{E}\Big|\int_{\kappa(t)}^t f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))ds + \int_{\kappa(t)}^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s)\Big|^p \\
&\le 2^{p-1}\Delta^{p-1}\mathbb{E}\int_{\kappa(t)}^t |f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^p ds + 2^{p-1}\mathbb{E}\Big|\int_{\kappa(t)}^t g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))dB(s)\Big|^p.
\end{aligned}
\tag{4.23}
$$

On the basis of the Hölder inequality, Lemma 4.3 and P2, we have

$$
\begin{aligned}
2^{p-1}\Delta^{p-1}\mathbb{E}\int_{\kappa(t)}^{t}|f_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^p ds &\le C\Delta^{p-1}\mathbb{E}\int_{\kappa(t)}^{t}\Big[\Delta^{-p\alpha}(1+|\bar{Y}_\Delta(s)|^p+|\bar{Y}(s-\tau)|^p)\Big]ds\\
&\le C\Delta^{p(1-\alpha)}. \qquad (4.24)
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality, P2 and (2.8), we have

$$
\begin{aligned}
\mathbb{E}\Big|\int_{\kappa(t)}^{t} g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))dB(s)\Big|^p &\le C\mathbb{E}\Big|\int_{\kappa(t)}^{t}|g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^2 ds\Big|^{p/2}\\
&\le C\mathbb{E}\Big(\int_{\kappa(t)}^{t}(1+|\bar{Y}_\Delta(s)|^{l+2}+|\bar{Y}_\Delta(s-\tau)|^{l+2})ds\Big)^{p/2}\\
&\le C\mathbb{E}\Big((1+|Y_\Delta(\kappa(t))|^{l+2}+|Y_\Delta(\kappa(t)-\tau)|^{l+2})\Delta\Big)^{p/2}\\
&\le C\Delta^{p/2}\Big(1+\sup_{0\le s\le t}\mathbb{E}|Y_\Delta(s)|^{p(l/2+1)}\Big)\le C\Delta^{p/2}. \qquad (4.25)
\end{aligned}
$$

Noting that $\alpha\in(0,1/2]$, from (4.23)-(4.25), we get (4.22). On the other hand, by (4.15) and (2.3), we see from (4.21) that

$$
\begin{aligned}
|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p &= |D(Y_\Delta(t-\tau))-D(\bar{Y}_\Delta(t-\tau))+\varphi_\Delta(t)|^p\\
&\le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}\left(\frac{|D(Y_\Delta(t-\tau))-D(\bar{Y}_\Delta(t-\tau))|^p}{\varepsilon}+|\varphi_\Delta(t)|^p\right)\\
&\le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}\left(\frac{\nu^p|Y_\Delta(t-\tau)-\bar{Y}_\Delta(t-\tau)|^p}{\varepsilon}+|\varphi_\Delta(t)|^p\right),\ \forall t\in[0,T].
\end{aligned}
$$

Letting $\varepsilon=\left(\frac{\nu}{1-\nu}\right)^{p-1}$ gives that

$$
|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p\le \nu|Y_\Delta(t-\tau)-\bar{Y}_\Delta(t-\tau)|^p+\frac{1}{(1-\nu)^{p-1}}|\varphi_\Delta(t)|^p,\ \forall t\in[0,T]. \qquad (4.26)
$$

Taking expectations on the both sides of (4.26) and using (4.22), (2.2), we have

$$
\begin{aligned}
\mathbb{E}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p &\le \nu\mathbb{E}|Y_\Delta(t-\tau)-\bar{Y}_\Delta(t-\tau)|^p+\frac{1}{(1-\nu)^{p-1}}\mathbb{E}|\varphi_\Delta(t)|^p\\
&\le \nu\sup_{-\tau\le t\le T}\mathbb{E}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p+\frac{\bar{c}_p}{(1-\nu)^{p-1}}\Delta^{0.5p}\\
&\le \nu\sup_{0\le t\le T}\mathbb{E}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p+\nu K_1^p\Delta^{\varrho p}+\frac{\bar{c}_p}{(1-\nu)^{p-1}}\Delta^{0.5p}. \qquad (4.27)
\end{aligned}
$$

As this holds for any $t\in[0,T]$, thus

$$
\sup_{0\le t\le T}\mathbb{E}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p\le \nu\sup_{0\le t\le T}\mathbb{E}|Y_\Delta(t)-\bar{Y}_\Delta(t)|^p+\left(\nu K_1^p+\frac{\bar{c}_p}{(1-\nu)^{p-1}}\right)\Delta^{(\varrho\wedge 0.5)p}. \qquad (4.28)
$$

Rearranging this implies the desired assertion. □

Proof of Theorem 4.1. Consider the tamed EM solution $Y_\Delta(t)$ with modified coefficients Type II. Denote

$$e_\Delta(t) := x(t) - Y_\Delta(t) - D(x(t-\tau)) + D(Y_\Delta(t-\tau)), \ \forall t \in [0,T]. \tag{4.29}$$

Thus, by (2.3) and the elementary inequality, we have that for any $t \in [0,T]$,

$$\begin{aligned}|x(t) - Y_\Delta(t)|^2 &\le (1+\varepsilon)|D(x(t-\tau)) - D(Y_\Delta(t-\tau))|^2 + (1+\varepsilon^{-1})|e_\Delta(t)|^2 \\ &\le (1+\varepsilon)\nu^2|x(t-\tau) - Y_\Delta(t-\tau)|^2 + (1+\varepsilon^{-1})|e_\Delta(t)|^2.\end{aligned} \tag{4.30}$$

Note that $x(t) = Y_\Delta(t)$ for any $t \in [-\tau, 0]$. Then letting $\varepsilon = \dfrac{1-\nu}{\nu}$ and taking expectations on the both sides of (4.30), we have

$$\begin{aligned}\sup_{0\le u\le t}\mathbb{E}|x(u) - Y_\Delta(u)|^2 &\le \nu\sup_{0\le u\le t}\mathbb{E}|x(u-\tau) - Y_\Delta(u-\tau)|^2 + \frac{1}{1-\nu}\sup_{0\le u\le t}\mathbb{E}|e_\Delta(u)|^2 \\ &\le \nu\sup_{0\le u\le t}\mathbb{E}|x(u) - Y_\Delta(u)|^2 + \frac{1}{1-\nu}\sup_{0\le u\le t}\mathbb{E}|e_\Delta(u)|^2, \ \forall t\in[0,T].\end{aligned}$$

Rearranging this gives

$$\sup_{0\le u\le t}\mathbb{E}|x(u) - Y_\Delta(u)|^2 \le \frac{1}{(1-\nu)^2}\sup_{0\le u\le t}\mathbb{E}|e_\Delta(u)|^2, \ \forall t\in[0,T]. \tag{4.31}$$

By the Itô formula and the elementary inequality, we have that for any $t \in [0,T]$

$$\begin{aligned}\mathbb{E}|e_\Delta(t)|^2 = \mathbb{E}\int_0^t &\Big[2e_\Delta^T(s)[f(x(s),x(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))] \\ &+ |g(x(s),x(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^2\Big]ds \\ \le \mathbb{E}\int_0^t &\Big[2e_\Delta^T(s)[f(x(s),x(s-\tau)) - f(Y_\Delta(s),Y_\Delta(s-\tau))] \\ &+ (p_1-1)|g(x(s),x(s-\tau)) - g(Y_\Delta(s),Y_\Delta(s-\tau))|^2\Big]ds \\ &+ \mathbb{E}\int_0^t 2e_\Delta^T(s)[f(Y_\Delta(s),Y_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))]ds \\ &+ \frac{p_1-1}{p_1-2}\mathbb{E}\int_0^t |g(Y_\Delta(s),Y_\Delta(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^2ds.\end{aligned} \tag{4.32}$$

By Assumption 2.4, we have

$$\mathbb{E}|e_\Delta(t)|^2 \le I_1(t) + I_2(t), \tag{4.33}$$

where

$$\begin{aligned}I_1(t) := \mathbb{E}\int_0^t &\Big[|e_\Delta(s)|^2 + 2K_3(|x(s) - Y_\Delta(s)|^2 + |x(s-\tau) - Y_\Delta(s-\tau)|^2) \\ &- 2U(x(s),Y_\Delta(s)) + 2U(x(s-\tau),Y_\Delta(s-\tau))\Big]ds\end{aligned}$$

and

$$\begin{aligned} I_2(t) : = \mathbb{E} \int_0^t \Big[& |f(Y_\Delta(s), Y_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 \\ & + \frac{p_1 - 1}{p_1 - 2} |g(Y_\Delta(s), Y_\Delta(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 \Big] ds. \end{aligned}$$

Recalling that for any $s \in [-\tau, 0]$, $x(s) = Y_\Delta(s)$ and $U(x(s), Y_\Delta(s)) = 0$, we have

$$\int_0^t |x(s-\tau) - Y_\Delta(s-\tau)|^2 ds \leq \int_0^t |x(s) - Y_\Delta(s)|^2 ds, \tag{4.34}$$

and

$$\int_0^t U(x(s-\tau), Y_\Delta(s-\tau)) ds \leq \int_{-\tau}^t U(x(s), Y_\Delta(s)) ds = \int_0^t U(x(s), Y_\Delta(s)) ds, \tag{4.35}$$

as well as

$$|e_\Delta(s)|^2 \leq 2|x(s) - Y_\Delta(s)|^2 + 2\nu^2 |x(s-\tau) - Y_\Delta(s-\tau)|^2, \ \forall s \in [0, t]. \tag{4.36}$$

Inserting (4.36) into $I_1(t)$ and using (4.34), (4.35), we have

$$I_1(t) \leq (4K_3 + 2 + 2\nu^2) \int_0^t \mathbb{E}|x(s) - Y_\Delta(s)|^2 ds. \tag{4.37}$$

To estimate $I_2(t)$, we observe that

$$I_2(t) \leq I_{21}(t) + I_{22}(t),$$

where

$$\begin{aligned} I_{21}(t) :=& 2 \int_0^T \mathbb{E}|f(Y_\Delta(s), Y_\Delta(s-\tau)) - f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 ds \\ & + \frac{2(p_1 - 1)}{p_1 - 2} \int_0^T \mathbb{E}|g(Y_\Delta(s), Y_\Delta(s-\tau)) - g(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 ds, \end{aligned}$$

and

$$\begin{aligned} I_{22}(t) :=& 2 \int_0^T \mathbb{E}|f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 ds \\ & + \frac{2(p_1 - 1)}{p_1 - 2} \int_0^T \mathbb{E}|g(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 ds. \end{aligned}$$

By the condition $p_0 \geq 4 \vee 2(1 + 2l)$, we have

$$\frac{2p_0}{p_0 - 2l} \leq \frac{2p_0}{2 + 2l} < \frac{p_0}{1 + l/2}.$$

Thus, according to Lemma 4.4, we have

$$I_{22}(t) \leq C\Delta^{2\alpha}, \tag{4.38}$$

and applying the Hölder inequality, Lemmas 4.3 and 4.5, as well as Assumption 2.4 we have that for any $s \in [0, T]$

$$
\begin{aligned}
&\mathbb{E}|f(Y_\Delta(s), Y_\Delta(s-\tau)) - f(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 \\
&\leq C\mathbb{E}\Big[(1 + |Y_\Delta(s)|^{2l} + |Y_\Delta(s-\tau)|^{2l} + |\bar{Y}_\Delta(s)|^{2l} + |\bar{Y}_\Delta(s-\tau)|^{2l}) \\
&\quad \times (|Y_\Delta(s) - \bar{Y}_\Delta(s)|^2 + |Y_\Delta(s-\tau) - \bar{Y}_\Delta(s-\tau)|^2)\Big] \\
&\leq C\Big(1 + \mathbb{E}|Y_\Delta(s)|^{p_0} + \mathbb{E}|Y_\Delta(s-\tau)|^{p_0} + \mathbb{E}|\bar{Y}_\Delta(s)|^{p_0} + \mathbb{E}|\bar{Y}_\Delta(s-\tau)|^{p_0}\Big)^{2l/p_0} \\
&\quad \times \Big(\mathbb{E}|Y_\Delta(s) - \bar{Y}_\Delta(s)|^{2p_0/(p_0-2l)} + \mathbb{E}|Y_\Delta(s-\tau) - \bar{Y}_\Delta(s-\tau)|^{2p_0/(p_0-2l)}\Big)^{(p_0-2l)/p_0} \\
&\leq C\Delta^{2\varrho\wedge 1}. \qquad (4.39)
\end{aligned}
$$

Similarly, we can deduce that

$$\mathbb{E}|g(Y_\Delta(s) - Y_\Delta(s-\tau)) - g(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^2 \leq C\Delta^{2\varrho\wedge 1}. \qquad (4.40)$$

Noting that $\alpha \in (0, 1/2]$, combining (4.37), (4.38), (4.39) and (4.40) together, we observe from (4.33) that

$$\mathbb{E}|e_\Delta(t)|^2 \leq C\int_0^t \mathbb{E}|x(s) - Y_\Delta(s)|^2 ds + C\Delta^{2(\alpha\wedge\varrho)}.$$

As this holds for any $t \in [0, T]$, thus

$$\sup_{0\leq u\leq t} \mathbb{E}|e_\Delta(u)|^2 \leq C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|x(u) - Y_\Delta(u)|^2 ds + C\Delta^{2(\alpha\wedge\varrho)}.$$

Substituting this into (4.31) gets

$$\sup_{0\leq u\leq t} \mathbb{E}|x(u) - Y_\Delta(u)|^2 \leq C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|x(u) - Y_\Delta(u)|^2 ds + C\Delta^{2(\alpha\wedge\varrho)}.$$

Applying the Gronwall inequality give the first assertion in (4.1). Combining this with Lemma 4.5 yields the second in (4.1). Thus, the proof is complete. □

4.2. Strong convergence (without order) of tamed EM (Type I) under local Lipschitz condition

In this section, we mainly discuss the convergence issue of tamed EM scheme when global monotonicity and polynomial growth conditions are replaced by the local Lipschitz condition. Note that Lemma 4.3 requires the condition $p_0 \geq 4$, however, if we are only concerned with the convergence (without order) of the tamed EM scheme, then this condition can be substituted by a more relaxed $p_0 > 2$. The price we pay for this is a narrow scope of the parameter α, which will lead to a decrease in the convergence rate in view of Theorem 4.1. Borrowing the method of the proof of moments boundedness in Mao [6, Lemma 3.2], we have the following lemma.

Lemma 4.6. *Suppose that Assumptions 2.1 and 2.2 hold with $p_0 > 2$ and $\alpha \in (0, 1/3]$ is arbitrary. Then the tamed EM solution $Y_\Delta(t)$ given by (3.2) with modified coefficients Type I satisfies*

$$\sup_{0<\Delta\leq 1}\sup_{0\leq t\leq T} \mathbb{E}|Y_\Delta(t)|^{p_0} \leq C,$$

where the positive constant $C := C(p_0, T, K_2, \|\xi\|, \nu)$.

Proof. Let Assumptions 2.1 and 2.2 hold with $p_0 > 2$. Consider the tamed EM scheme Type I. Let us begin with an assertion that for any $\hat{p} > 0$,

$$\begin{aligned}&\mathbb{E}\left[\left|Y_\Delta(t) - \bar{Y}_\Delta(t) - D(Y_\Delta(t-\tau)) + D(\bar{Y}_\Delta(t-\tau))\right|^{\hat{p}}\middle|\mathcal{F}_{\kappa(t)}\right]\\&\leq C\Delta^{\hat{p}(1-\alpha)/2}(1 + |\bar{Y}_\Delta(t)|^{\hat{p}} + |\bar{Y}_\Delta(t-\tau)|^{\hat{p}}),\ \forall t \geq 0,\end{aligned} \tag{4.41}$$

where C is a positive constant independent of Δ. Recall that

$$Y_\Delta(t) - \bar{Y}_\Delta(t) - D(Y_\Delta(t-\tau)) + D(\bar{Y}_\Delta(t-\tau)) = \varphi_\Delta(t),\ \forall t \geq 0,$$

where $\varphi_\Delta(t)$ is defined in (4.21), namely

$$\varphi_\Delta(t) = f_\Delta(Y_\Delta(\kappa(t)), Y_\Delta(\kappa(t)-\tau))(t-\kappa(t)) + g_\Delta(Y_\Delta(\kappa(t)), Y_\Delta(\kappa(t)-\tau))(B(t) - B(\kappa(t))). \tag{4.42}$$

Then for any $\hat{p} \geq 2$, using P2 gives

$$\begin{aligned}\mathbb{E}\left[\left|\varphi_\Delta(t)\right|^{\hat{p}}\middle|\mathcal{F}_{\kappa(t)}\right] &\leq C\mathbb{E}\left[\left|f_\Delta(Y_\Delta(\kappa(t)), Y_\Delta(\kappa(t)-\tau))(t-\kappa(t))\right|^{\hat{p}}\middle|\mathcal{F}_{\kappa(t)}\right]\\&\quad + C\mathbb{E}\left[\left|g_\Delta(Y_\Delta(\kappa(t)), Y_\Delta(\kappa(t)-\tau))(B(t)-B(\kappa(t)))\right|^{\hat{p}}\middle|\mathcal{F}_{\kappa(t)}\right]\\&\leq C\Delta^{\hat{p}(1-\alpha)}(1 + |Y_\Delta(\kappa(t))|^{\hat{p}} + |Y_\Delta(\kappa(t)-\tau)|^{\hat{p}})\\&\quad + C\Delta^{\hat{p}(1-\alpha)/2}(1 + |Y_\Delta(\kappa(t))|^{\hat{p}} + |Y_\Delta(\kappa(t)-\tau)|^{\hat{p}})\\&\leq C\Delta^{\hat{p}(1-\alpha)/2}(1 + |\bar{Y}_\Delta(t)|^{\hat{p}} + |\bar{Y}_\Delta(t-\tau)|^{\hat{p}}),\end{aligned} \tag{4.43}$$

this also holds for any $0 < \hat{p} < 2$ due to the Hölder inequality. Thus, we get the assertion (4.41). Now recall (4.5) that

$$\mathbb{E}|Y_\Delta(t) - D(Y_\Delta(t-\tau))|^{p_0} \leq C + C\int_0^t \left[\|\xi\|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|Y_\Delta(u)|^{p_0}\right]ds + p_0 J(t),\ t \geq 0, \tag{4.44}$$

with $J(t)$ defined in (4.3), i.e.,

$$\begin{aligned}J(t) = \mathbb{E}\int_0^t &\Big[|Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0-2}\\&\times(Y_\Delta(s) - \bar{Y}_\Delta(s) - D(Y_\Delta(s-\tau)) + D(\bar{Y}_\Delta(s-\tau)))^T f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))\Big]ds.\end{aligned}$$

By the Young inequality, we have

$$J(t) \leq \frac{p_0-2}{p_0}\mathbb{E}\int_0^t |Y_\Delta(s) - D(Y_\Delta(s-\tau))|^{p_0}ds + \frac{2}{p_0}\Pi(t), \tag{4.45}$$

where

$$\Pi(t) = \mathbb{E}\int_0^t |Y_\Delta(s) - \bar{Y}_\Delta(s) - D(Y_\Delta(s)) + D(\bar{Y}_\Delta(s-\tau))|^{p_0/2}|f_\Delta(\bar{Y}_\Delta(s), \bar{Y}_\Delta(s-\tau))|^{p_0/2}ds.$$

Let $\alpha \in (0, 1/3]$, using (4.41) and P2 as well as (4.4) yields

$$\begin{aligned}\Pi(t) &\leq C\Delta^{(1-\alpha)p_0/4-\alpha p_0/2}\mathbb{E}\int_0^t \left(1 + |\bar{Y}_\Delta(s))|^{p_0/2} + |\bar{Y}_\Delta(s-\tau))|^{p_0/2}\right)^2 ds\\&\leq C\Delta^{(1-3\alpha)p_0/4}\int_0^t \left[1 + \sup_{0\leq u\leq s}\mathbb{E}|\bar{Y}_\Delta(u)|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|\bar{Y}_\Delta(u-\tau)|^{p_0}\right]ds\\&\leq C\int_0^t \left[1 + \|\xi\|^{p_0} + \sup_{0\leq u\leq s}\mathbb{E}|Y_\Delta(u)|^{p_0}\right]ds.\end{aligned} \tag{4.46}$$

We observe from (4.44) and (4.45) as well as (4.46) that

$$\sup_{0\le u\le t}\mathbb{E}|Y_\Delta(u)-D(Y_\Delta(u-\tau))|^{p_0}\le C+C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^{p_0}ds.$$

Inserting this into (4.17) and using the Gronwall inequality give the desired assertion. □

Lemma 4.7. *Suppose that Assumptions 2.1 and 2.2 hold with $p_0>2$ and $\alpha\in(0,1/3]$ is arbitrary. Consider the tamed EM solution $Y_\Delta(t)$ defined by (3.2) with modified coefficients Type I. For any real number $R>\|\xi\|$ and $\Delta\in(0,1]$, define the stoping time $\bar\rho_{\Delta,R}=\inf\{t\ge 0:|Y_\Delta(t)|\ge R\}$. Then*

$$\mathbb{P}(\bar\rho_{\Delta,R}\le T)\le\frac{C}{R^2},$$

where C is a positive constant independent of Δ and R.

Proof. Write $\bar\rho_{\Delta,R}=\bar\rho$ for short. By the Itô formula and (3.13), we have

$$\begin{aligned}
&\mathbb{E}|Y_\Delta(T\wedge\bar\rho)-D(Y_\Delta(T\wedge\bar\rho-\tau))|^2-|\xi(0)-D(\xi(-\tau))|^2\\
&=\mathbb{E}\int_0^{T\wedge\bar\rho}\Big[2(Y_\Delta(s)-D(Y_\Delta(s-\tau)))^Tf_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))+|g_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|^2\Big]ds\\
&=\mathbb{E}\int_0^{T\wedge\bar\rho}\Big[2(\bar Y_\Delta(s)-D(\bar Y_\Delta(s-\tau)))^Tf_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))+|g_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|^2\Big]ds+2I^\star(T)\\
&\le 2\hat K_2\mathbb{E}\int_0^{T\wedge\bar\rho}(1+|\bar Y_\Delta(s)|^2+|\bar Y_\Delta(s-\tau)|^2)ds+2I^\star(T)\\
&\le 2\hat K_2\int_0^T\mathbb{E}(1+|\bar Y_\Delta(s)|^2+|\bar Y_\Delta(s-\tau)|^2)ds+2I^\star(T)
\end{aligned}\tag{4.47}$$

where

$$I^\star(T)=\mathbb{E}\int_0^{T\wedge\bar\rho}\Big[(Y_\Delta(s)-\bar Y_\Delta(s)-D(Y_\Delta(s-\tau))+D(\bar Y_\Delta(s-\tau)))^Tf_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))\Big]ds.$$

By (3.11) and (4.41) as well as the condition that $\alpha\in(0,1/3]$, we have

$$\begin{aligned}
I^\star(T)&\le\int_0^T\mathbb{E}\Big[|Y_\Delta(s)-\bar Y_\Delta(s)-D(Y_\Delta(s-\tau))+D(\bar Y_\Delta(s-\tau))||f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|\Big]ds\\
&=\int_0^T\mathbb{E}\Big(|f_\Delta(\bar Y_\Delta(s),\bar Y_\Delta(s-\tau))|\mathbb{E}\Big[|Y_\Delta(s)-\bar Y_\Delta(s)-D(Y_\Delta(s-\tau))+D(\bar Y_\Delta(s-\tau))|\Big|\mathcal{F}_{\kappa(s)}\Big]\Big)ds\\
&\le C\Delta^{(1-3\alpha)/2}\int_0^T\mathbb{E}(1+|\bar Y_\Delta(s)|^2+|\bar Y_\Delta(s-\tau)|^2)ds\\
&\le C\int_0^T\mathbb{E}(1+|\bar Y_\Delta(s)|^2+|\bar Y_\Delta(s-\tau)|^2)ds.
\end{aligned}\tag{4.48}$$

Inserting this into (4.47) and using Lemma 4.6 give

$$\mathbb{E}|Y_\Delta(T\wedge\bar\rho)-D(Y_\Delta(T\wedge\bar\rho-\tau))|^2\le C.\tag{4.49}$$

Note that

$$
\begin{aligned}
\mathbb{I}_{\{\bar{\rho}\le T\}}|Y_\Delta(\bar{\rho}) - D(Y_\Delta(\bar{\rho}-\tau))| &\ge \mathbb{I}_{\{\bar{\rho}\le T\}}\Big(|Y_\Delta(\bar{\rho})| - |D(Y_\Delta(\bar{\rho}-\tau))|\Big) \\
&\ge R - \nu\mathbb{I}_{\{\bar{\rho}\le T\}}|Y_\Delta(\bar{\rho}-\tau)| \ge R - \nu R = (1-\nu)R. \qquad (4.50)
\end{aligned}
$$

Thus, from (4.49) and (4.50), we get

$$
\begin{aligned}
\mathbb{P}\left(\bar{\rho}\le T\right) &\le \frac{\mathbb{E}\left[\mathbb{I}_{\{\bar{\rho}\le T\}}|Y_\Delta(\bar{\rho}) - D(Y_\Delta(\bar{\rho}-\tau))|^2\right]}{(1-\nu)^2R^2} \\
&\le \frac{\mathbb{E}\left[|Y_\Delta(T\wedge\bar{\rho}) - D(Y_\Delta(T\wedge\bar{\rho}-\tau))|^2\right]}{(1-\nu)^2R^2} \\
&\le \frac{C}{(1-\nu)^2R^2}, \qquad (4.51)
\end{aligned}
$$

which gives the desired assertion. □

Similarly, we can show the following lemma.

Lemma 4.8. *Suppose that Assumptions 2.1 and 2.2 hold with $p_0 > 2$. For any real number $R > \|\xi\|$, define the stoping time $\rho_R = \inf\{t \ge 0 : |x(t)| \ge R\}$. Then*

$$
\mathbb{P}\left(\rho_R \le T\right) \le \frac{C}{R^2},
$$

where C is a positive constant independent of R.

Theorem 4.9. *Suppose that Assumptions 2.1 and 2.2 hold with $p_0 > 2$ and $\alpha \in (0, 1/3]$ is arbitrary. Consider the tamed EM scheme Type I. Then for any $q \in [2, p_0)$,*

$$
\lim_{\Delta\to 0}\mathbb{E}|x(T) - Y_\Delta(T)|^q = 0 \quad \text{and} \quad \lim_{\Delta\to 0}\mathbb{E}|x(T) - \bar{Y}_\Delta(T)|^q = 0. \qquad (4.52)
$$

Proof. For any $R > \|\xi\|$, denote $\theta_{\Delta,R} := \rho_R \wedge \bar{\rho}_{\Delta,R}$ and $e^\star_\Delta(t) := x(t) - Y_\Delta(t)$, recall $e_\Delta(t) = x(t) - D(x(t-\tau)) - Y_\Delta(t) + D(Y_\Delta(t-\tau))$. Then for any $q \in [2, p_0)$ and $\eta > 0$, the Young inequality gives

$$
\begin{aligned}
\mathbb{E}|e^\star_\Delta(T)|^q &= \mathbb{E}\left[|e^\star_\Delta(T)|^q\mathbb{I}_{\{\theta_{\Delta,R}>T\}}\right] + \mathbb{E}\left[|e^\star_\Delta(T)|^q\mathbb{I}_{\{\theta_{\Delta,R}\le T\}}\right] \\
&\le \mathbb{E}\left[|e^\star_\Delta(T)|^q\mathbb{I}_{\{\theta_{\Delta,R}>T\}}\right] + \frac{q\eta}{p_0}\mathbb{E}|e^\star_\Delta(T)|^q + \frac{p_0-q}{p_0\eta^{q/(p_0-q)}}\mathbb{P}\left(\theta_{\Delta,R}\le T\right). \qquad (4.53)
\end{aligned}
$$

In this subsection, C_R denotes a positive constant depending on R, its value may be different for different appearance. By Lemmas 2.3 and 4.6, we have

$$
\mathbb{E}|e^\star_\Delta(T)|^q \le 2^{q-1}\left(\mathbb{E}|x(T)|^q + \mathbb{E}|Y_\Delta(T)|^q\right) \le C. \qquad (4.54)
$$

While by Lemmas 4.7 and 4.8

$$
\mathbb{P}\left(\theta_{\Delta,R}\le T\right) \le \mathbb{P}\left(\rho_R\le T\right) + \mathbb{P}\left(\bar{\rho}_{\Delta,R}\le T\right) \le \frac{C}{R^2}. \qquad (4.55)
$$

Plugging (4.54) and (4.55) into (4.53), we get

$$\mathbb{E}|e_\Delta^\star(T)|^q \le \mathbb{E}\left[|e_\Delta^\star(T)|^q \mathbb{I}_{\{\theta_{\Delta,R}>T\}}\right] + \frac{Cq\eta}{p_0} + \frac{C(p_0-q)}{p_0 R^2 \eta^{q/(p_0-q)}}. \tag{4.56}$$

Next, we shall prove that for any $R>0$, there exist a positive constant C_R such that

$$\mathbb{E}|e_\Delta^\star(T\wedge\theta_{\Delta,R})|^q \le C_R \Delta^{(\alpha\wedge\varrho)q}. \tag{4.57}$$

In the same way as Lemma 4.5 was proved, applying Assumption 2.1 and P2, we can show that

$$\sup_{0\le t\le T} \mathbb{E}|Y(t)-\bar{Y}(t)|^q \le C_R \Delta^{q/2}. \tag{4.58}$$

In the similar way as the (4.31) was obtained, we have

$$\sup_{0\le u\le T} \mathbb{E}|e_\Delta^\star(u\wedge\theta_{\Delta,R})|^q \le \frac{1}{(1-\nu)^q} \sup_{0\le u\le T} \mathbb{E}|e_\Delta(u\wedge\theta_{\Delta,R})|^q. \tag{4.59}$$

By the stochastic inequality and Assumption 2.1, we have that for any $t\in[0,T]$,

$$\begin{aligned}
\mathbb{E}|e_\Delta(t\wedge\theta_R)|^q &\le C\mathbb{E}\int_0^{t\wedge\theta_R} |f(x(s),x(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\theta_R} |g(x(s),x(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&\le C_R\mathbb{E}\int_0^{t\wedge\theta_R} \Big[|x(s)-Y_\Delta(s)|^q + |x(s-\tau)-Y_\Delta(s-\tau)|^q\Big] ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\theta_R} |f(Y_\Delta(s),Y_\Delta(s-\tau)) - f(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\theta_R} |f(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau)) - f_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\theta_R} |g(Y_\Delta(s),Y_\Delta(s-\tau)) - g(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\theta_R} |g(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau)) - g_\Delta(\bar{Y}_\Delta(s),\bar{Y}_\Delta(s-\tau))|^q ds \\
&=: \sum_{j=1}^{5} \Pi_j(t).
\end{aligned} \tag{4.60}$$

By (2.2), we get

$$\Pi_1(t) \le C_R \int_0^T \sup_{0\le u\le s} \mathbb{E}|e_\Delta^\star(u\wedge\theta_{\Delta,R})|^q ds + C_R\Delta^{\varrho q}. \tag{4.61}$$

By Assumption 2.1, (2.2) and (4.58), we have

$$\Pi_2(t)+\Pi_4(t) \le C_R\int_0^T \mathbb{E}|Y_\Delta(s)-\bar{Y}_\Delta(s)|^q ds + C_R\int_0^T |\xi(s)-\xi(\kappa(s))|^q ds \le C_R\Delta^{(0.5\wedge\varrho)q}. \tag{4.62}$$

According to P1, we obtain

$$\begin{aligned}\Pi_3(t) &\leq C\int_0^T \mathbb{E}|f(\bar{Y}_\Delta(s\wedge\theta_{\Delta,R}),\bar{Y}_\Delta(s\wedge\theta_{\Delta,R}-\tau))-f_\Delta(\bar{Y}_\Delta(s\wedge\theta_{\Delta,R}),\bar{Y}_\Delta(s\wedge\theta_{\Delta,R}-\tau))|^q ds\\ &\leq CN_R^q\Delta^{\alpha q}=C_R\Delta^{\alpha q}. \end{aligned}\tag{4.63}$$

Similarly, we can show

$$\Pi_5(t)\leq C_R\Delta^{\alpha q}. \tag{4.64}$$

From (4.61)-(4.64), we derive from (4.60) that

$$\sup_{0\leq u\leq T}\mathbb{E}|e_\Delta(u\wedge\theta_{\Delta,R})|^q\leq C_R\int_0^T\sup_{0\leq u\leq s}\mathbb{E}|e^\star_\Delta(u\wedge\theta_{\Delta,R})|^q ds+C_R\Delta^{(\alpha\wedge\varrho)q}. \tag{4.65}$$

In the light of (4.59) and (4.65), we have

$$\sup_{0\leq u\leq T}\mathbb{E}|e^\star_\Delta(u\wedge\theta_{\Delta,R})|^q\leq C_R\int_0^T\sup_{0\leq u\leq s}\mathbb{E}|e^\star_\Delta(u\wedge\theta_{\Delta,R})|^q ds+C_R\Delta^{(\alpha\wedge\varrho)q}. \tag{4.66}$$

Now, using the Gronwall inequality gives the assertion (4.57). Inserting (4.57) into (4.56) gives

$$\mathbb{E}|e^\star_\Delta(T)|^q\leq\frac{Cq\eta}{p_0}+\frac{C(p_0-q)}{p_0R^2\eta^{q/(p_0-q)}}+C_R\Delta^{(\alpha\wedge\varrho)q}. \tag{4.67}$$

Then for any $\varepsilon>0$ we can choose η such that

$$\frac{Cq\eta}{p_0}<\frac{\varepsilon}{3},$$

and then take R such that

$$\frac{C(p_0-q)}{p_0R^2\eta^{q/(p_0-q)}}<\frac{\varepsilon}{3},$$

finally for such R choose Δ sufficiently small for

$$C_R\Delta^{(\alpha\wedge\varrho)q}<\frac{\varepsilon}{3},$$

so that, in (4.67),

$$\mathbb{E}|e^\star_\Delta(T)|^q<\varepsilon,$$

as required. □

5. Mean-square stability

Let us concentrate on the mean-square stability of the tamed EM scheme for NSDDE (2.1) in this section. We assume that f and g can be decomposed as $f(x,y)=F_1(x,y)+F(x,y)$ and $g(x,y)=G_1(x,y)+G(x,y)$, where $F_1,F:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ and $G_1,G:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^{d\times n}$. Moreover,

$$F_1(0,0)=F(0,0)=G_1(0,0)=G(0,0)=0, \tag{5.1}$$

the coefficients F_1,F,G_1,G satisfy the following conditions.

Assumption 5.1. *For any $R > 0$, there exists constants $\hat{L}$ and $\tilde{L}_R$ depending on R such that*

$$|F_1(x,y) - F_1(\bar{x},\bar{y})| \vee |G_1(x,y) - G_1(\bar{x},\bar{y})| \leq \hat{L}(|x-\bar{x}| + |y-\bar{y}|),$$

for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and

$$|F(x,y) - F(\bar{x},\bar{y})| \vee |G(x,y) - G(\bar{x},\bar{y})| \leq \tilde{L}_R(|x-\bar{x}| + |y-\bar{y}|),$$

for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

Assumption 5.2. *There exist nonnegative constants ϑ, λ_1, λ_2, λ_3 and λ_4 satisfying $\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$ such that*

$$\begin{aligned} &2\langle x - D(y), F_1(x,y)\rangle + (1+\vartheta)|G_1(x,y)|^2 \leq -\lambda_1|x|^2 + \lambda_2|y|^2, \\ &2\langle x - D(y), F(x,y)\rangle + (1+\vartheta^{-1})|G(x,y)|^2 \leq \lambda_3|x|^2 + \lambda_4|y|^2, \end{aligned} \tag{5.2}$$

for any $x, y \in \mathbb{R}^d$.

When $\vartheta = 0$, we set $\vartheta^{-1}|G(x,y)|^2 = 0$, when $\vartheta = \infty$, we set $\vartheta|G_1(x,y)|^2 = 0$. Moreover, we see from Assumption 5.2 that

$$2\langle x - D(y), f(x,y)\rangle + |g(x,y)|^2 \leq -(\lambda_1 - \lambda_3)|x|^2 + (\lambda_2 + \lambda_4)|y|^2, \ \forall x, y \in \mathbb{R}^d. \tag{5.3}$$

Thus, the solution to NSDDEs (2.1) is stable exponentially in mean-square sense, see e.g., Zong and Wu [10, Theorem 3.1]. We state this result as a lemma.

Lemma 5.3. *Suppose that Assumptions 5.1 and 5.2 hold. Then for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^d)$, the solution $x(t;\xi)$ to the NSDDE (2.1) has the property that*

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t;\xi)|^2}{t} \leq -\left(\gamma^\star \wedge \frac{2}{\tau}\log\frac{1}{\nu}\right), \tag{5.4}$$

where $\gamma^\star$ is the unique root of the following equation

$$\gamma^\star(1+\nu) - (\lambda_1 - \lambda_3) + e^{\gamma^\star\tau}\Big(\gamma^\star\nu(\nu+1) + \lambda_2 + \lambda_4\Big) = 0. \tag{5.5}$$

The following lemma shows that the partially tamed coefficients f_Δ and g_Δ conserve the stable condition 5.3.

Lemma 5.4. *Suppose that Assumptions 5.1 and 5.2 hold and $\alpha \in (0, 1/3]$ is arbitrary. Define the following partially modified coefficients f_Δ and g_Δ by*

$$f_\Delta(x,y) = F_1(x,y) + F_\Delta(x,y) \quad \textit{and} \quad g_\Delta(x,y) = G_1(x,y) + G_\Delta(x,y), \tag{5.6}$$

where

$$F_\Delta(x,y) := \pi_\Delta(x,y)F(x,y) \quad \textit{and} \quad G_\Delta(x,y) := \pi_\Delta(x,y)G(x,y)$$

with

$$\pi_\Delta(x,y) = \frac{1}{1 + \Delta^\alpha(|F(x,y)| + |G(x,y)|^2)}, \ \forall x, y \in \mathbb{R}^d, \ \Delta \in (0, 1].$$

Then

$$2(x - D(y))^T f_\Delta(x, y) + |g_\Delta(x, y)|^2 \le -(\lambda_1 - \lambda_3)|x|^2 + (\lambda_2 + \lambda_4)|y|^2, \ \forall x, y \in \mathbb{R}^d, \tag{5.7}$$

and

$$|f_\Delta(x, y)|^2 \Delta \le \epsilon_\Delta(|x|^2 + |y|^2), \ \forall x, y \in \mathbb{R}^d, \tag{5.8}$$

where $\epsilon_\Delta = 4(\hat{L} + \tilde{L}_1)^2\Delta + 4\Delta^{1-2\alpha}$.

Proof. Write $\pi_\Delta(x, y) = \pi_\Delta$ for short. Note that $0 < \pi_\Delta < 1$. By Assumptions 5.1 and 5.2 as well as (5.6), we have

$$\begin{aligned} 2(x - D(y))^T F_\Delta(x, y) + (1 + \vartheta^{-1})|G_\Delta(x, y)|^2 &= 2(x - D(y))^T \pi_\Delta F(x, y) + (1 + \vartheta^{-1})|\pi_\Delta G(x, y)|^2 \\ &\le 2\pi_\Delta(x - D(y))^T F(x, y) + \pi_\Delta(1 + \vartheta^{-1})|G(x, y)|^2 \\ &\le \pi_\Delta(\lambda_3|x|^2 + \lambda_4|y|^2) \\ &\le \lambda_3|x|^2 + \lambda_4|y|^2, \ \forall x, y \in \mathbb{R}^d. \end{aligned} \tag{5.9}$$

Consequently,

$$\begin{aligned} 2(x - D(y))^T f_\Delta(x, y) + |g_\Delta(x, y)|^2 &\le 2(x - D(y))^T F_1(x, y) + (1 + \vartheta)|G_1(x, y)|^2 \\ &\quad + 2(x - D(y))^2 F_\Delta(x, y) + (1 + \vartheta^{-1})|G_\Delta(x, y)|^2 \\ &\le -(\lambda_1 - \lambda_3)|x|^2 + (\lambda_2 + \lambda_4)|y|^2, \ \forall x, y \in \mathbb{R}^d. \end{aligned} \tag{5.10}$$

Now let us estimate (5.8). By Assumption 5.1 and condition (5.1), we have

$$|F_1(x, y)| \le \hat{L}(|x| + |y|), \ \forall x, y \in \mathbb{R}^d. \tag{5.11}$$

For any $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \le 1$, by Assumption 5.1 and condition (5.1), we have

$$|F_\Delta(x, y)| = |\pi_\Delta F(x, y)| \le |F(x, y)| \le \tilde{L}_1(|x| + |y|).$$

While for any $x, y \in \mathbb{R}^d$ with $|x| \vee |y| > 1$, by (5.6), we have

$$|F_\Delta(x, y)| = \frac{|F(x, y)|}{1 + \Delta^\alpha(|F(x, y)| + |G(x, y)|^2)} \le \Delta^{-\alpha} \le \Delta^{-\alpha}(|x| + |y|).$$

Thus,

$$|F_\Delta(x, y)| \le (\tilde{L}_1 + \Delta^{-\alpha})(|x| + |y|), \ \forall x, y \in \mathbb{R}^d. \tag{5.12}$$

Consequently, by (5.11) and (5.12), we obtain

$$\begin{aligned} |f_\Delta(x, y)|^2\Delta &\le (\hat{L} + \tilde{L}_1 + \Delta^{-\alpha})^2(|x| + |y|)^2\Delta \\ &\le 4(\hat{L} + \tilde{L}_1)^2\Delta + 4\Delta^{1-2\alpha}(|x|^2 + |y|^2), \ \forall x, y \in \mathbb{R}^d. \end{aligned}$$

Thus, the proof is complete. □

The following theorem shows that the tamed EM solution can share the mean-square stability of the exact solution.

Theorem 5.5. *Suppose that Assumptions 5.1 and 5.2 hold and $\alpha \in (0, 1/3]$ is arbitrary. Choose $\Delta^\star \in (0, 1]$ such that $\epsilon_{\Delta^\star} \le (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)/2$, where ϵ_Δ is defined in* (5.8)*. Then for any $\Delta \in (0, \Delta^\star]$ and any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^d)$, the tamed EM approximation y_Δ^k defined by* (3.1) *with modified coefficients* (5.6) *has the property that*

$$\limsup_{k\to\infty} \frac{\log(\mathbb{E}|y_\Delta^k|^2)}{k\Delta} \le -\left(\gamma_\Delta^\star \wedge \frac{2}{\tau}\log\frac{1}{\nu}\right), \tag{5.13}$$

where $\gamma_\Delta^\star$ is the unique root of the following equation

$$\frac{1 - e^{-\gamma_\Delta^\star \Delta}}{\Delta}(1+\nu) - (\lambda_1 - \lambda_3 - \epsilon_\Delta) + e^{\gamma_\Delta^\star \tau}\left(\frac{1 - e^{-\gamma_\Delta^\star \Delta}}{\Delta}\nu(1+\nu) + (\lambda_2 + \lambda_4 + \epsilon_\Delta)\right) = 0. \tag{5.14}$$

Moreover,

$$\lim_{\Delta\to 0} \gamma_\Delta^\star = \gamma^\star. \tag{5.15}$$

Proof. Let Assumptions 5.1 and 5.2 hold. Consider the tamed EM scheme (3.1) with f_Δ and g_Δ given by (5.6). Then

$$\begin{aligned} |y_\Delta^{k+1} - D(y_\Delta^{k+1-m})|^2 = {} & |y_\Delta^k - D(y_\Delta^{k-m})|^2 + 2(y_\Delta^k - D(y_\Delta^{k-m}))^T f_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta \\ & + |g_\Delta(y_\Delta^k, y_\Delta^{k-m})|^2\Delta + |f_\Delta(y_\Delta^k, y_\Delta^{k-m})|^2\Delta^2 + M_k, \quad k = 0, 1, 2, \cdots, \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} M_k = {} & 2(y_\Delta^k - D(y_\Delta^{k-m}))^T g_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta B_k + 2(f_\Delta(y_\Delta^k, y_\Delta^{k-m}))^T [g_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta B_k]\Delta \\ & + |g_\Delta(y_\Delta^k, y_\Delta^{k-m})\Delta B_k|^2 - |g_\Delta(y_\Delta^k, y_\Delta^{k-m})|^2\Delta. \end{aligned}$$

Obviously, $\mathbb{E}M_k = 0$. Denote $z_\Delta^k := y_\Delta^k - D(y_\Delta^{k-m})$. Now choose $\Delta^\star \in (0, 1]$ such that $\epsilon_{\Delta^\star} \le (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)/2$, where $\epsilon_{\Delta^\star}$ is defined in Lemma 5.4. Then for any $\Delta \in (0, \Delta^\star]$, taking expectations on both sides of (5.16) and applying Lemma 5.4, we have

$$\begin{aligned} \mathbb{E}|z_\Delta^{k+1}|^2 & \le \mathbb{E}|z_\Delta^k|^2 - (\lambda_1 - \lambda_3)\mathbb{E}|y_\Delta^k|^2\Delta + (\lambda_2 + \lambda_4)\mathbb{E}|y_\Delta^{k-m}|^2\Delta + \epsilon_\Delta(\mathbb{E}|y_\Delta^k|^2 + \mathbb{E}|y_\Delta^{k-m}|^2)\Delta \\ & = \mathbb{E}|z_\Delta^k|^2 - (\lambda_1 - \lambda_3 - \epsilon_\Delta)\mathbb{E}|y_\Delta^k|^2\Delta + (\lambda_2 + \lambda_4 + \epsilon_\Delta)\mathbb{E}|y_\Delta^{k-m}|^2\Delta. \end{aligned} \tag{5.17}$$

For any $r > 1$, we have

$$\begin{aligned} r^{(k+1)\Delta}\mathbb{E}|z_\Delta^{k+1}|^2 - r^{k\Delta}\mathbb{E}|z_\Delta^k|^2 \le {} & -(\lambda_1 - \lambda_3 - \epsilon_\Delta)r^{(k+1)\Delta}\mathbb{E}|y_\Delta^k|^2\Delta + (\lambda_2 + \lambda_4 + \epsilon_\Delta)r^{(k+1)\Delta}\mathbb{E}|y_\Delta^{k-m}|^2\Delta \\ & + (r^{(k+1)\Delta} - r^{k\Delta})\mathbb{E}|z_\Delta^k|^2. \end{aligned} \tag{5.18}$$

By the elementary inequality, we have

$$|x - D(y)|^2 \le (1+\nu)|x|^2 + (1+\nu^{-1})\nu^2|y|^2 = (1+\nu)|x|^2 + (\nu^2+\nu)|y|^2, \ \forall x, y \in \mathbb{R}^d. \tag{5.19}$$

Then we see from (5.18) that

$$
\begin{aligned}
r^{k\Delta}\mathbb{E}|z_\Delta^k|^2 \le{}& \mathbb{E}|z_\Delta^0|^2 - (\lambda_1-\lambda_3-\epsilon_\Delta)\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^j|^2\Delta + (\lambda_2+\lambda_4+\epsilon_\Delta)\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^{j-m}|^2\Delta \\
&+ \sum_{j=0}^{k-1}(r^{(j+1)\Delta}-r^{j\Delta})\Big((1+\nu)\mathbb{E}|y_\Delta^j|^2 + (\nu^2+\nu)\mathbb{E}|y_\Delta^{j-m}|^2\Big) \\
={}& |\xi(0)-D(\xi(-\tau))|^2 + \Big[-(\lambda_1-\lambda_3-\epsilon_\Delta)\Delta + (1-r^{-\Delta})(1+\nu)\Big]\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^j|^2 \\
&+ \Big[(\lambda_2+\lambda_4+\epsilon_\Delta)\Delta + (1-r^{-\Delta})(\nu^2+\nu)\Big]\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^{j-m}|^2,\ k=1,2,\cdots. \qquad (5.20)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^{j-m}|^2 &= \sum_{j=-m}^{-1} r^{(j+1+m)\Delta}\mathbb{E}|y_\Delta^j|^2 + \sum_{j=0}^{k-m-1} r^{(j+1+m)\Delta}\mathbb{E}|y_\Delta^j|^2 \\
&\le \frac{r^\tau}{1-r^{-\Delta}}\|\xi\|^2 + r^\tau\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^j|^2,\ k=1,2,\cdots. \qquad (5.21)
\end{aligned}
$$

Substituting this into (5.20), we obtain

$$
r^{k\Delta}\mathbb{E}|z_\Delta^k|^2 \le H_\Delta(r) - \bar{H}_\Delta(r)\sum_{j=0}^{k-1} r^{(j+1)\Delta}\mathbb{E}|y_\Delta^j|^2\Delta,\ k=1,2,\cdots, \qquad (5.22)
$$

where

$$
H_\Delta(r) := \left[\frac{r^\tau}{1-r^{-\Delta}}\Big[(\lambda_2+\lambda_4+\epsilon_\Delta)\Delta + (1-r^{-\Delta})(\nu^2+\nu)\Big] + (1+\nu)^2\right]\|\xi\|^2, \qquad (5.23)
$$

$$
\bar{H}_\Delta(r) := (\lambda_1-\lambda_3-\epsilon_\Delta) - \frac{1-r^{-\Delta}}{\Delta}(1+\nu) - \left[(\lambda_2+\lambda_4+\epsilon_\Delta) + \frac{1-r^{-\Delta}}{\Delta}(\nu^2+\nu)\right]r^\tau. \qquad (5.24)
$$

For any $\Delta \in (0, \Delta^\star]$, we have

$$
\bar{H}_\Delta(1) = \lambda_1-\lambda_2-\lambda_3-\lambda_4-2\epsilon_\Delta > 0, \qquad (5.25)
$$

and

$$
\bar{H}_\Delta(\bar{r}) < 0,\ \text{with } \bar{r} = \left(\frac{\lambda_1-\lambda_3-\epsilon_\Delta}{\lambda_2+\lambda_4+\epsilon_\Delta}\right)^{1/\tau} > 1, \qquad (5.26)
$$

as well as

$$
\frac{d\bar{H}_\Delta(r)}{dr} < 0. \qquad (5.27)
$$

From (5.25)-(5.27), there is a positive constant $r^\star_\Delta \in (1, \bar{r})$ such that $\bar{H}_\Delta(r^\star_\Delta) = 0$. If $1 < r < r^\star_\Delta$ and $1 < r < \nu^{-\frac{2}{\tau}}$, which means $1 - r^\tau \nu^2 > 0$, by the elementary inequality and the connection between y^k_Δ and z^k_Δ, we see from (5.22) that

$$\begin{aligned} r^{k\Delta}\mathbb{E}|y^k_\Delta|^2 &\le r^{k\Delta}(1+\eta)\mathbb{E}|z^k_\Delta|^2 + r^{k\Delta}(1+\eta^{-1})\nu^2\mathbb{E}|y^{k-m}_\Delta|^2 \\ &\le (1+\eta)H_\Delta(r^\star_\Delta) + \Big(r^\tau(1+\eta^{-1})\nu^2\Big)\, r^{(k-m)\Delta}\mathbb{E}|y^{k-m}_\Delta|^2, \ k = 0, 1, 2, \cdots, \end{aligned} \tag{5.28}$$

where inequality $r^{k\Delta}\mathbb{E}|z^k_\Delta|^2 < (r^\star_\Delta)^{k\Delta}\mathbb{E}|z^k_\Delta|^2 \le H_\Delta(r^\star_\Delta)$ has been used and η is a positive constant to be determined. Denote $a_k := r^{k\Delta}\mathbb{E}|y^k_\Delta|^2$, then (5.28) becomes

$$a_k \le (1+\eta)H_\Delta(r^\star_\Delta) + \Big(r^\tau(1+\eta^{-1})\nu^2\Big)\, a_{k-m}, \ k = 0, 1, 2, \cdots. \tag{5.29}$$

Hence,

$$\begin{aligned} &a_i \le (1+\eta)H_\Delta(r^\star_\Delta) + \Big(r^\tau(1+\eta^{-1})\nu^2\Big) \sup_{-m\le i\le k} a_i, \ 0 \le i \le k, \\ &a_i \le \|\xi\|^2 < H_\Delta(r^\star_\Delta), \ -m \le i \le 0. \end{aligned}$$

Thus,

$$\sup_{-m\le i\le k} a_i \le (1+\eta)H_\Delta(r^\star_\Delta) + \Big(r^\tau(1+\eta^{-1})\nu^2\Big) \sup_{-m\le i\le k} a_i, \ k = 0, 1, 2, \cdots. \tag{5.30}$$

Now, taking $\eta > \dfrac{r^\tau\nu^2}{1 - r^\tau\nu^2}$, i.e., $r^\tau(1+\eta^{-1})\nu^2 < 1$, we obtain from (5.30) that

$$\sup_{-m\le i\le k} r^{i\Delta}\mathbb{E}|y^i_\Delta|^2 \le \frac{(1+\eta)H_\Delta(r^\star_\Delta)}{1 - r^\tau(1+\eta^{-1})\nu^2} < \infty, \ k = 0, 1, 2, \cdots. \tag{5.31}$$

Therefore,

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|y^k_\Delta|^2}{k\Delta} \le -\log r. \tag{5.32}$$

Similarly, if $1 < r^\star_\Delta \le r$ and $1 < r < \nu^{-\frac{2}{\tau}}$, replacing r by $r^\star_\Delta$ in the above procedure between (5.28) and (5.31), we also have

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|y^k_\Delta|^2}{k\Delta} \le -\log r^\star_\Delta. \tag{5.33}$$

Combining (5.32) and (5.33) gives

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|y^k_\Delta|^2}{k\Delta} \le -\log(r^\star_\Delta \wedge r), \ \text{with } 1 < r < \nu^{-\frac{2}{\tau}}, \tag{5.34}$$

where $r^\star_\Delta$ is the unique root of (5.24). Taking $r = \mathrm{e}^\gamma \to \nu^{-\frac{2}{\tau}}$ and $r^\star_\Delta = \mathrm{e}^{\gamma^\star_\Delta}$, then (5.34) becomes (5.13). Finally, notice that $\epsilon_\Delta \to 0$ and $\dfrac{1 - \mathrm{e}^{-\gamma^\star_\Delta\Delta}}{\Delta} \to \gamma^\star_\Delta$ as $\Delta \to 0$. Comparing (5.5) with (5.14), we obtain the desired assertion (5.15). Thus, the proof is finished. □

6. Numerical examples

In this section, we carry out some numerical experiments to support the findings derived. In the following two examples, the diffusion coefficients of NSDDEs are superlinearly growing and therefore one can not apply the results of Li and Cao [3], Ji and Yuan [5], Zong and Wu [10] and Tan [11].

Example 6.1. *Consider the following one-dimensional NSDDE:*

$$\begin{aligned} d\Big[x(t)+\frac{1}{2}x(t-\tau)\Big] &= \Big[6x(t)+\frac{5}{2}x(t-\tau)-5\Big(x(t)+\frac{1}{2}x(t-\tau)\Big)\Big|x(t)+\frac{1}{2}x(t-\tau)\Big|\Big]dt \\ &\quad+\Big[x(t)+\sin x(t-\tau)+\Big|x(t)+\frac{1}{2}x(t-\tau)\Big|^{3/2}\Big]dB(t),\ \ t\geq 0, \\ x(t) &= 1,\ \ -\tau\leq t\leq 0, \end{aligned} \tag{6.1}$$

where $B(t)$ is a scalar Wiener process and $\tau = 1/8$. We first verify the Assumptions 2.1, 2.2 and 2.4. Obviously, Assumptions 2.1 is satisfied. For any $x, y \in \mathbb{R}$, set $f(x,y) = f_1(x,y) + f_2(x,y)$, $g(x,y) = g_1(x,y) + g_2(x,y)$, where $f_1(x,y) = a(x+0.5y) - a(x+0.5y)|x+0.5y|$, $f_2(x,y) = x$, $g_1(x,y) = |x+0.5y|^{3/2}$, $g_2(x,y) = x + \sin y$, and $a = 5$. Donote $D(y) = -0.5y$, $X := x + 0.5y = x - D(y)$, $Y := \bar{x} + 0.5\bar{y} = \bar{x} - D(\bar{y})$, $F(X) := aX(1-|X|)$, $G(X) := |X|^{3/2}$. Then $f_1(x,y) = F(X)$, $g_1(x,y) = G(X)$, $f_1(\bar{x},\bar{y}) = F(Y)$, $g_1(\bar{x},\bar{y}) = G(Y)$. If $4 \leq p_0 \leq a+1 = 6$, then

$$\begin{aligned} &2\langle x-D(y), f(x,y)\rangle + (p_0-1)|g(x,y)|^2 \\ &= 2(x+0.5y)\Big(6x+\frac{5}{2}y-a(x+0.5y)|x+0.5y|\Big)+(p_0-1)\Big(x+\sin y+|x+0.5y|^{3/2}\Big)^2 \\ &\leq 2(x+0.5y)\Big(6x+\frac{5}{2}y\Big)+2(p_0-1)(x+\sin y)^2+2(p_0-1-a)|x+0.5y|^3 \\ &\leq 2(x+0.5y)\Big(6x+\frac{5}{2}y\Big)+2(p_0-1)(x+\sin y)^2 \\ &\leq K_2(1+|x|^2+|y|^2), \end{aligned} \tag{6.2}$$

which means that Assumption 2.2 is satisfied. Moreover, we can derive that

$$\begin{aligned} &\langle x-D(y)-\bar{x}+D(\bar{y}), f_1(x,y)-f_1(\bar{x},\bar{y})\rangle = \langle X-Y, a(X-Y)-a(X|X|-Y|Y|)\rangle \\ &= a|X-Y|^2 - a(X-Y)(X|X|-Y|Y|) \leq a|X-Y|^2 - a(|X|+|Y|)(|X|-|Y|)^2, \end{aligned} \tag{6.3}$$

where we have used the following estimates

$$-\langle X-Y, X|X|-Y|Y|\rangle \leq -(|X|+|Y|)(|X|-|Y|)^2,\ \ \forall X, Y\in\mathbb{R}^d,$$

and

$$|g_1(x,y)-g_1(\bar{x},\bar{y})|^2 = |G(X)-G(Y)|^2 = (|X|^{3/2}-|Y|^{3/2})^2 \leq 2(|X|+|Y|)(|X|-|Y|)^2, \tag{6.4}$$

see [21, Appendix, p.2104]. If $p_1 \leq a/2+1 = 3.5$, then we conclude from (6.3) *and* (6.4) *that*

$$\begin{aligned} &2\langle x-D(y)-\bar{x}+D(\bar{y}), f_1(x,y)-f_1(\bar{x},\bar{y})\rangle + 2(p_1-1)|g_1(x,y)-g_1(\bar{x},\bar{y})|^2 \\ &\leq 2a|X-Y|^2 - 2a(|X|+|Y|)(|X|-|Y|)^2 + 4(p_1-1)(|X|+|Y|)(|X|-|Y|)^2 \\ &= 2a|X-Y|^2 + [4(p_1-1)-2a](|X|+|Y|)(|X|-|Y|)^2 \\ &\leq 2a|X-Y|^2 = 2a|x-\bar{x}-D(y)+D(\bar{y})|^2 \leq 4a(|x-\bar{x}|^2+|y-\bar{y}|^2). \end{aligned} \tag{6.5}$$

Table 1: ϵ_Δ and $\gamma^\star_\Delta$ with different step sizes for solving (5.14) in example (6.2): $\lambda_1 = 5/2$, $\lambda_2 = 13/8$, $\lambda_3 = 0$, $\lambda_4 = 1/4$, $\hat{L} = 2$, $\tilde{L}_1 = 3$, $\tau = 1$, $\nu = 1/2$, $\alpha = 1/4$, $\Delta^\star = 0.0015$, $\frac{2}{\tau}\log\frac{1}{\nu} = 1.3863$, $\gamma^\star = 0.1427$

Δ	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
ϵ_Δ	0.2265	0.0500	0.0136	0.0041	0.0013	0.0004
$\gamma^\star_\Delta$	0.0389	0.1196	0.1364	0.1408	0.1421	0.1425

Thus,

$$
\begin{aligned}
&2\langle x - D(y) - \bar{x} + D(\bar{y}), f(x,y) - f(\bar{x},\bar{y})\rangle + (p_1-1)|g(x,y) - g(\bar{x},\bar{y})|^2 \\
&\le 2\langle x - D(y) - \bar{x} + D(\bar{y}), f_1(x,y) - f_1(\bar{x},\bar{y})\rangle + 2(p_1-1)|g_1(x,y) - g_1(\bar{x},\bar{y})|^2 \\
&\quad +2\langle x - D(y) - \bar{x} + D(\bar{y}), f_2(x,y) - f_2(\bar{x},\bar{y})\rangle + 2(p_1-1)|g_2(x,y) - g_2(\bar{x},\bar{y})|^2 \\
&\le 4a(|x-\bar{x}|^2 + |y-\bar{y}|^2) + 2\langle x + 0.5y - \bar{x} - 0.5\bar{y}, x - \bar{x}\rangle + 2(p_1-1)|x - \bar{x} + \sin y - \sin\bar{y}|^2 \\
&\le 4a(|x-\bar{x}|^2 + |y-\bar{y}|^2) + 2|x-\bar{x}|^2 + |y-\bar{y}||x-\bar{x}| + 4(p_1-1)(|x-\bar{x}|^2 + |y-\bar{y}|^2) \\
&\le 4(a+p_1)(|x-\bar{x}|^2 + |y-\bar{y}|^2),
\end{aligned}
\tag{6.6}
$$

which impies that the global monotonicity condition is satisfied. In addition,

$$
\begin{aligned}
|f_1(x,y) - f_1(\bar{x},\bar{y})| &= a|(X - Y) - (X|X| - Y|Y|)| \\
&\le a(1 + |X| + |Y|)|X - Y| \\
&\le a\Big(1 + |x| + |y| + |\bar{x}| + |\bar{y}|\Big)\Big(|x-\bar{x}| + |y-\bar{y}|\Big).
\end{aligned}
\tag{6.7}
$$

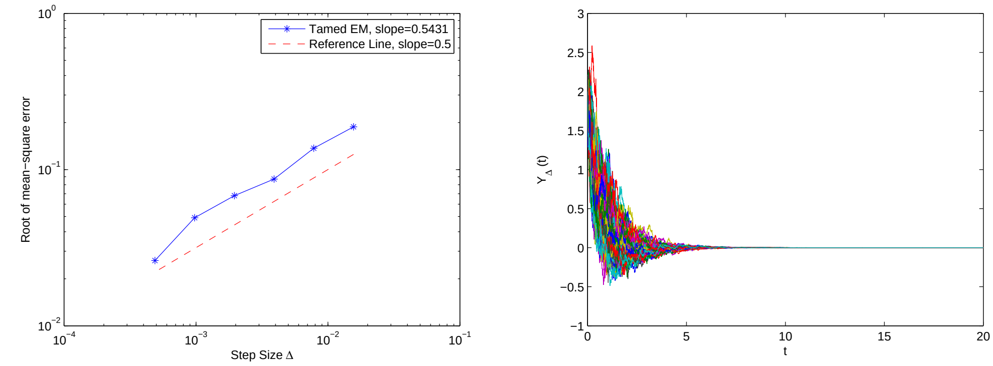
Thus, we derive from the definitions of f and g that polynomial growth condition is satisfied. Taking $l = 1$, $p_0 = 6$, $p_1 = 3.5$, $p = 2$ and $\varrho = 0.5$, we conclude from Theorem 4.1 that the tamed EM solution $Y_\Delta(t)$ with modified coefficients Type II is convergent to the exact solution $x(t)$ with order one half in the sense of mean-square. Now define the root of mean-square error

$$
\hat{e}_\Delta(T) := \Big(\mathbb{E}|x(T) - Y_\Delta(T)|^2\Big)^{1/2}.
$$

Set $\alpha = 1/2$, we apply the tamed EM scheme (3.1) with modified coefficients Type II to approximate the exact solution $x(t)$ of NSDDE (6.1). Tamed EM solution $Y_\Delta(t)$ with step size $\Delta = 2^{-14}$ is taken as the replacement of the exact solution $x(t)$. Fig.1(a) shows the root of mean-square errors $\hat{e}_\Delta(T)$ between the exact solution $x(T)$ and the tamed EM solution $Y_\Delta(T)$ with different step sizes $2^{-6}, 2^{-7}, \cdots 2^{-11}$ at time $T = 1$ for 500 simulations. A least square fit of errors $\hat{e}_\Delta$ produces the strong convergence order 0.5431 and is thus close to the theoretical value 0.5.

Example 6.2. *Consider the following one-dimensional NSDDE:*

$$
\begin{aligned}
d\left[x(t) - \frac{1}{2}\sin x(t-\tau)\right] &= \left[-2x(t) - x^3(t) + \frac{1}{2}\sin x(t-\tau)\right]dt + \left[\frac{1}{2}x^2(t) + \frac{1}{4}x(t-\tau)\right]dB(t), \ t \ge 0, \\
x(t) &= 2, \ -\tau \le t \le 0,
\end{aligned}
\tag{6.8}
$$



(a) Root of mean-square errors $\hat{e}_\Delta(T)$ for (6.1) **(b)** 200 sample paths of $Y_\Delta(t)$ for (6.8)

Fig. 1. Numerical simulations for (6.1) and (6.8)

where $B(t)$ is a scalar Wiener process and $\tau = 1$. Set $D(y) = \dfrac{1}{2}\sin y$ with $\nu = 1/2$, $f(x,y) = F_1(x,y) + F(x,y)$, $g(x,y) = G_1(x,y) + G(x,y)$, where $F_1(x,y) = -2x + \frac{1}{2}\sin y$, $G_1(x,y) = \frac{1}{4}y$, $F(x,y) = -x^3$ and $G(x,y) = \frac{1}{2}x^2$. We compute

$$|F_1(x,y) - F_1(\bar{x},\bar{y})| \vee |G_1(x,y) - G_1(\bar{x},\bar{y})| \le 2(|x-\bar{x}| + |y-\bar{y}|), \tag{6.9}$$

for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ and

$$|F(x,y) - F(\bar{x},\bar{y})| \vee |G(x,y) - G(\bar{x},\bar{y})| \le (3R^2 \vee R)(|x-\bar{x}| + |y-\bar{y}|), \tag{6.10}$$

for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le R$. Thus, Assumption 5.1 is satisfied. By the elementary inequality, we have the following estimates

$$\begin{aligned} 2\langle x - D(y), F_1(x,y)\rangle + 2|G_1(x,y)|^2 &= \left\langle 2x - \sin y, -2x + \frac{1}{2}\sin y \right\rangle + 2\left(\frac{1}{4}y\right)^2 \\ &= -4x^2 + 3x\sin y - \frac{1}{2}\sin^2 y + \frac{1}{8}y^2 \le -4x^2 + \frac{3}{2}(x^2+y^2) + \frac{1}{8}y^2 = -\frac{5}{2}x^2 + \frac{13}{8}y^2, \end{aligned}$$

and

$$\begin{aligned} &2\langle x - D(y), F(x,y)\rangle + 2|G(x,y)|^2 = \left\langle 2x - \sin y, -x^3 \right\rangle + 2\left(\frac{1}{2}x^2\right)^2 \\ &= -2x^4 + x^3\sin y + \frac{1}{2}x^4 = -\frac{3}{2}x^4 + x^3\sin y \le -\frac{3}{2}x^4 + \frac{3}{4}x^4 + \frac{1}{4}\sin^4 y \\ &\le -\frac{3}{2}x^4 + \frac{3}{4}x^4 + \frac{1}{4}y^2 \le \frac{1}{4}y^2, \end{aligned}$$

where the Young inequality and the inequality that $\sin^2 y \le y^2$ have been used. Thus Assumptions 5.2 is satisfied with $\lambda_1 = \frac{5}{2}$, $\lambda_2 = \frac{13}{8}$, $\lambda_3 = 0$, $\lambda_4 = \frac{1}{4}$ and $\vartheta = 1$. From (6.9) and (6.10), we have $\hat{L} = 2$ and $\tilde{L}_1 = 3$. Let $\alpha = 1/4$, solving $\epsilon_{\Delta^\star} = 4(\hat{L} + \tilde{L}_1)^2\Delta^\star + 4(\Delta^\star)^{1-2\alpha} = (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)/2$

gives $\Delta^\star = 0.0015$. Computational results such as ϵ_Δ and $\gamma^\star_\Delta$ as well as $\gamma^\star$ are shown in Table 1. According to Lemma 5.3, the exact solution $x(t)$ of NSDDE (6.8) is mean-square exponentially stable with exponent $-\left(\gamma^\star \wedge \frac{2}{\tau}\log\frac{1}{\nu}\right)$. On the other hand, based on Theorem 5.5 the tamed EM solution $Y_\Delta(t)$ with modified coefficients given by (5.6) is also mean-square exponentially stable with exponent $-\left(\gamma^\star_\Delta \wedge \frac{2}{\tau}\log\frac{1}{\nu}\right)$ for any $\Delta \in (0, \Delta^\star]$. Fig.1(b) plots the sample paths of tamed EM solutions $Y_\Delta(t)$ applied to NSDDE (6.8) for 200 simulations with step size $\Delta = 0.001$ and $\alpha = 1/4$. We see from Fig.1(b) and Table 1 that the numerical solution is stable and $\gamma^\star_\Delta$ tends to $\gamma^\star$ as Δ goes to zero. Our experiments confirm the conclusion from Theorem 5.5.

7. Conclusion

In this work, we mainly examine the strong convergence and stability of tamed EM scheme for NSDDEs, where the drift and diffusion coefficients may be allowed to grow superlinearly. By virtue of tamed technology, uniform boundedness of numerical solutions is obtained and then strong convergence results are established. The results show that the tamed EM approximation $Y(t)$ can arrive at an order one half of strong convergence. Meanwhile, it is proved that the tamed EM solution has the property of reproduction of mean-square stability for the exact solution. Numerical experiments are provided to show the agreement with the theoretical results.

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