Positivity Preserving Truncated Euler–Maruyama Method for Stochastic Lotka–Volterra Competition Model *

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Abstract

The well-known stochastic Lotka–Volterra model for interacting multi-species in ecology has some typical features: highly nonlinear, positive solution and multi-dimensional. The known numerical methods including the tamed/truncated Euler-Maruyama (EM) applied to it do not preserve its positivity. The aim of this paper is to modify the truncated EM to establish a new positive preserving truncated EM (PPTEM). To simplify the proof as well as to make our theory more understandable, we will first develop a nonnegative preserving truncated EM (NPTEM) and then establish the PPTEM. Of course, we should point out that the NPTEM has its own right as many SDE models in applications have their nonnegative solutions.

Key words: Stochastic differential equation, Lotka–Volterra competition model, positivity preserving truncated Euler-Maruyama method, strong convergence.

1 Introduction

Numerical methods for stochastic differential equations (SDEs) have become one of most popular research areas in the study of SDEs. Up to 2002, most of the existing strong convergence theory in this area requires the coefficients of the SDEs to be globally Lipschitz continuous (see, e.g., [8, 11, 17]). Higham, Mao and Stuart in 2002 published a very influential paper [4] (Google citation 606) which opened a new chapter—to study the strong convergence question for numerical approximations under the local Lipschitz condition. Given that the classical Euler–Maruyama (EM) method may fail to work for SDEs under the local Lipschitz condition but without the linear growth condition (i.e., highly nonlinear SDEs) (see, e.g., [5, 6]), implicit methods have therefore naturally been used to study the numerical solutions to highly nonlinear SDEs (see, e.g., [15, 20, 21]).

*This work is entirely theoretical and the results can be reproduced using the methods described in this paper.

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Nevertheless, the explicit EM method has its simple algebraic structure, cheap computational cost and acceptable convergence rate under the global Lipschitz condition. Influenced by [4], several modified EM methods have recently been developed for the highly nonlinear SDEs. These include the tamed EM method [7, 18, 19], the tamed Milstein method [2], the stopped EM method [10], the truncated EM method [12, 13].

On the other hand, many SDE models in applications have their special properties. For example, the square root process and mean-reverting square root process in finance have nonnegative solutions (see, e.g., [9, 11]). The stochastic Lotka–Volterra model for interacting multi-species in ecology has positive solutions (see, e.g., [1, 14, 11]). The SDE SIS model in epidemiology has positive solutions (see, e.g., [3]). These SDE models are all highly nonlinear. If we apply the modified EM methods mentioned above to these SDEs, they fail to preserve the nonnegativity or positivity. Although there are some implicit numerical methods which can preserve these properties (see, e.g., [20]), explicit methods would be more desired as explained above.

Therefore there is a need to develop explicit numerical schemes which can preserve the nonnegativity or positivity for highly nonlinear SDEs. The aim of this paper is to modify the truncated EM method to create a new positivity preserving truncated EM (PPTEM) for the well-known stochastic Lotka–Volterra model for interacting multi-species in ecology. The reason why we will concentrate on this model is because it has typical features: highly nonlinear, positive solution and multi-dimensional. Consequently, the methods developed in this paper can be applicable to other SDE models, e.g., the 1-dimensional SDE SIS model.

Our approach is to establish a new nonnegative preserving truncated EM (NPTEM) and then the more desired PPTEM. The reader may wonder if it is enough to study the PPTEM only but not the NPTEM given that the solution of the underlying stochastic Lotka–Volterra model is positive. The reasons why we study both NPTEM and PPTEM are: (a) Mathematically speaking, we need to show the convergence of the NPTEM solutions to the true solution first, from which we can then show the convergence of the PPTEM more easily. (b) The NPTEM has its own right as many SDE models in applications have their solutions taking nonnegative values, for example, the well-known square root process and mean-reverting square root process in finance (see, e.g., [9, 11]).

2 Preliminary

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be a scalar Brownian motion defined on the complete probability space. If $\Omega_1 \subset \Omega$, denote by Ω_1^c its complement, namely $\Omega_1^c = \Omega - \Omega_1$. Denote by I_{Ω_1} the indicator function of Ω_1, namely $I_{\Omega_1}(\omega) = 1$ if $\omega \in \Omega_1$ and 0 otherwise.

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathbb{R}^{d\times d}$ the space of real-valued $d \times d$ matrices. If A is a vector or matrix, its transpose is denoted by A^T. Let $\mathbb{R}_+^d = \{(x_1, \cdots, x_d)^T \in \mathbb{R}^d : x_i > 0,\ 1 \leq i \leq d\}$ and $\bar{\mathbb{R}}_+^d = \{(x_1, \cdots, x_d)^T \in \mathbb{R}^d : x_i \geq 0,\ 1 \leq i \leq d\}$. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

Consider the d-dimensional stochastic Lotka–Volterra model (see, e.g., [1, 11])

$$dx(t) = \text{diag}(x_1(t), x_2(t), ..., x_d(t))[(b - Ax(t))dt + \sigma dB(t)], \tag{2.1}$$

where $x(t) = (x_1(t), \cdots, x_d(t))^T$ is the state of the d interacting species and the system parameters $b = (b_1, \cdots, b_d)^T \in \mathbb{R}^d$, $\sigma = (\sigma_1, \cdots, \sigma_d)^T \in \mathbb{R}^d$, $A = (a_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$. It is worth noting that the scalar Brownian motion $B(t)$ in this paper can be generalised into a multi-dimensional one without any difficulty but we leave the details to the reader. We impose the following assumption as a standing hypothesis, which is the only one for this paper.

Assumption 2.1 *All elements of A are nonnegative, namely $a_{ij} \geq 0$ for all $1 \leq i, j \leq d$.*

From the ecological point of view, this assumption means that the d interacting species are competitive. The SDE (2.1) has been studied intensively by many authors. For example, it is known (see, e.g., [11, Theorem 2.1 on p.381]) that under Assumption 2.1, for any initial value $x(0) \in \mathbb{R}^d_+$, the SDE (2.1) has a unique global solution $x(t)$ on $t \geq 0$ and the solution will remain to be in $\mathbb{R}^d_+$ with probability one (namely, $x(t) \in \mathbb{R}^d_+$ a.s. for all $t \geq 0$).

Throughout this paper, we set

$$\bar{b} = \max_{1\leq i\leq d} |b_i|, \quad \bar{\sigma} = \max_{1\leq i\leq d} |\sigma_i|, \quad \bar{a} = \max_{1\leq i,j\leq d} a_{ij}. \tag{2.2}$$

From now on, we will fix the initial value $x(0) \in \mathbb{R}^d_+$ arbitrarily and, of course, $x(t)$ is the corresponding solution. We will also fix two real numbers $T > 0$ and $p \geq 2$ arbitrarily. We will further use C to stand for generic positive real constants dependent on $x(0), T, b, A, \sigma, p$ but *independent of the step size* Δ which we will use in the next section. Please note that the values of C may change between occurrences. Let us present two lemmas which will play their useful role in this paper.

Lemma 2.2 *Under Assumption 2.1,*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x(t)|^p\Big) \leq C. \tag{2.3}$$

Proof. Recalling that $x(t) \in \mathbb{R}^d_+$ and applying the Itô formula and Assumption 2.1, we can easily show from (2.1) that

$$d(x_i(t))^p \leq p[\bar{b} + 0.5(p-1)\bar{\sigma}^2](x_i(t))^p dt + p\sigma_i(x_i(t))^p dB(t),$$

for $t \geq 0$ and every $i = 1, \cdots, d$. By the Burkholder–Davis–Gundy inequality (see, e.g., [16, p.76]), it is straightforward to show that

$$\mathbb{E}\Big(\sup_{0\leq u\leq t} (x_i(u))^p\Big) \leq C + C\int_0^t \mathbb{E}\Big(\sup_{0\leq u\leq s} (x_i(u))^p\Big) ds, \quad \forall t \in [0, T].$$

An application of the well-known Gronwall inequality gives

$$\mathbb{E}\Big(\sup_{0\leq u\leq T} (x_i(u))^p\Big) \leq C.$$

This implies the required assertion (2.3). $\Box$

Lemma 2.3 *Under Assumption 2.1,*

$$\mathbb{E}\Big(\sup_{0\le t\le T}[x_i(t)-1-\log(x_i(t))]\Big)\le C,\quad 1\le i\le d. \tag{2.4}$$

Proof. For each i, by the Itô formula, we have

$$\begin{aligned}&d[x_i(t)-1-\log(x_i(t))]\\ \le&\Big(-b_i+0.5\sigma_i^2+b_ix_i(t)+\sum_{j=1}^d a_{ij}x_j(t)\Big)dt+\sigma_i(x_i(t)-1)dB(t).\end{aligned}$$

By Lemma 2.2, the first and second moments of the solution is bounded (by C) for $t\in[0,T]$. Applying the Burkholder–Davis–Gundy inequality (see, e.g., [16, p.76]), we can then derive that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le t\le T}[x_i(t)-1-\log(x_i(t))]\Big)\\ \le&C+\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t\sigma_i(x_i(s)-1)dB(s)\Big)\\ \le&C+3\mathbb{E}\Big(\int_0^T|\sigma_i(x_i(s)-1)|^2ds\Big)^{1/2}\\ \le&C+3\bar\sigma\Big(\int_0^T 2(\mathbb{E}|x_i(s)|^2+1)ds\Big)^{1/2}\\ \le&C+3\bar\sigma\sqrt{2T(C+1)},\end{aligned}$$

which is the desired assertion (2.4). □

3 Definitions of New Numerical Schemes

In this section, we will develop two numerical schemes. The first one will be called the NPTEM scheme, while the second one the PPTEM scheme. We have explained in Section 1 why we do not only study the PPTEM but also the NPTEM in this paper, although the solution of the underlying SDE (2.1) is positive with probability one.

3.1 Nonnegativity preserving truncated EM method

To define the NPTEM scheme, it would be convenient to treat the SDE (2.1) in $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$. For this purpose, we need to extend the definition of the coefficients of the SDE from $\mathbb{R}^d_+$ to $\mathbb{R}^d$. We denote the coefficients by

$$F_1(x)=(b_1x_1,\cdots,b_dx_d)^T,\quad F_2(x)=-\text{diag}(x_1,\cdots,x_d)Ax,\quad G(x)=(\sigma_1x_1,\cdots,\sigma_dx_d)^T$$

for $x\in\bar{\mathbb{R}}^d_+$. Define a mapping $\hat\pi_0:\mathbb{R}^d\to\bar{\mathbb{R}}^d_+$ by

$$\hat\pi_0(x)=(x_1\vee 0,\cdots,x_d\vee 0)^T\quad\text{for } x\in\mathbb{R}^d.$$

Define $f_1,f_2,g:\mathbb{R}^d\to\mathbb{R}^d$ by

$$f_1(x)=F_1(\hat\pi_0(x)),\quad f_2(x)=F_2(\hat\pi_0(x)),\quad g(x)=G(\hat\pi_0(x))\quad\text{for } x\in\mathbb{R}^d.$$

Obviously, $f_1(x) = F_1(x)$ etc. if $x \in \bar{\mathbb{R}}^d_+$. In other words, f_1, f_2, g are the extended functions of F_1, F_2, G, respectively. Recalling that the solution of the SDE (2.1) has the property that $x(t) \in \mathbb{R}^d_+$ a.s. for all $t \geq 0$, we can therefore write the SDE (2.1) as the following equation

$$dx(t) = [f_1(x(t)) + f_2(x(t))]dt + g(x(t))dB(t) \tag{3.1}$$

in $\mathbb{R}^d$. We observe that f_1 and g are linearly bounded, namely

$$|f_1(x)| \leq \bar{b}|x|, \quad |g(x)| \leq \bar{\sigma}|x|, \quad \forall x \in \mathbb{R}^d, \tag{3.2}$$

but f_2 is not. The classical EM method is therefore not applicable to the SDE (see, e.g., [5, 7]). The truncated EM method established by [12, 13] may be applied but it cannot preserve nonnegativity, not mentioning positivity.

The aim of this subsection is to modify the truncated EM method in order to create a new NPTEM method. For this purpose, we first choose a strictly increasing continuous function $\mu : [1, \infty) \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{x \in \mathbb{R}^d,\, |x| \leq u} |f_2(x)| = \sup_{x \in \bar{\mathbb{R}}^d_+,\, |x| \leq u} |F_2(x)| \leq \mu(u), \quad \forall u \geq 1. \tag{3.3}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(1), \infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h} \geq 1 \vee \mu(1) \vee |x(0)|$ and a strictly decreasing function $h : (0, 1] \to [\mu(1), \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1]. \tag{3.4}$$

Note that for $x \in \bar{\mathbb{R}}^d_+$,

$$|F_2(x)|^2 = \sum_{i=1}^d x_i^2 \Big(\sum_{j=1}^d a_{ij} x_j \Big)^2 \leq \sum_{i=1}^d x_i^2 \Big(\sum_{j=1}^d a_{ij}^2 \Big) |x|^2 \leq |A|^2 |x|^4.$$

We can hence let $\mu(u) = |A|u^2$, while let $h(\Delta) = \hat{h}\Delta^{-\theta}$ for some $\theta \in (0, 1/4]$. In other words, there are lots of choices for $\mu(\cdot)$ and $h(\cdot)$.

For a given step size $\Delta \in (0, 1]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \leq \mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = \big(|x| \wedge \mu^{-1}(h(\Delta)) \big) \frac{x}{|x|},$$

where we set $x/|x| = 0$ when $x = 0$. That is, π_Δ maps x to itself or $\mu^{-1}(h(\Delta))x/|x|$ depending on $|x| \leq \mu^{-1}(h(\Delta))$ or not. It is useful to see that for all $x \in \mathbb{R}^d$,

$$f_2(\hat{\pi}_0(\pi_\Delta(x))) = f_2(\pi_\Delta(x)) = F_2(\hat{\pi}_0(\pi_\Delta(x))). \tag{3.5}$$

Hence

$$|f_2(\hat{\pi}_0(\pi_\Delta(x)))| = |f_2(\pi_\Delta(x))| \leq \mu(\mu^{-1}(h(\Delta))) = h(\Delta). \tag{3.6}$$

Moreover, noting $\hat{\pi}_0(\pi_\Delta(x)) = (|x| \wedge \mu^{-1}(h(\Delta)))\hat{\pi}_0(x)/|x|$, we also have

$$x^T f_2(\hat{\pi}_0(\pi_\Delta(x))) = x^T f_2(\pi_\Delta(x)) = (\hat{\pi}_0(x))^T F_2(\hat{\pi}_0(\pi_\Delta(x))) \leq 0. \tag{3.7}$$

We can now form the discrete-time NPTEM solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ by setting $\bar{X}_\Delta(0) = X_\Delta(0) = x(0)$ and computing

$$\bar{X}_\Delta(t_{k+1}) = \bar{X}_\Delta(t_k) + [f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k))]\Delta + g(\bar{X}_\Delta(t_k))\Delta B_k, \tag{3.8}$$

$$X_\Delta(t_{k+1}) = \hat{\pi}_0(\pi_\Delta(\bar{X}_\Delta(t_{k+1}))), \tag{3.9}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Please note that $\bar{X}_\Delta(t_{k+1})$ is an intermediate step in order to get the NPTEM solution $X_\Delta(t_{k+1})$. We extend the definitions of both $\bar{X}_\Delta(\cdot)$ and $X_\Delta(\cdot)$ from the grid points t_k to the whole $t \geq 0$ by defining

$$\bar{X}_\Delta(t) = \sum_{k=0}^{\infty} \bar{X}_\Delta(t_k) I_{[t_k, t_{k+1})}(t) \tag{3.10}$$

and

$$X_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t) \tag{3.11}$$

for $t \geq 0$. Clearly, $X_\Delta(t) = \hat{\pi}_0(\pi_\Delta(\bar{X}_\Delta(t)))$ so it preserves the nonnegativity although $\bar{X}_\Delta(t)$ does not.

3.2 Positivity preserving truncated EM method

For each step size $\Delta \in (0, 1]$, define one more truncation mapping $\hat{\pi}_\Delta : \mathbb{R}^d \to \mathbb{R}^d_+$ by

$$\hat{\pi}_\Delta(x) = (\Delta \vee x_1, \cdots, \Delta \vee x_d)^T, \quad x \in \mathbb{R}^d.$$

Please note that $\hat{\pi}_\Delta$ maps $\mathbb{R}^d$ to $\mathbb{R}^d_+$ while $\hat{\pi}_0$ to $\bar{\mathbb{R}}^d_+$ so they are different. The PPTEM solution is defined by

$$X^+_\Delta(t) = \hat{\pi}_\Delta(\pi_\Delta(\bar{X}_\Delta(t))), \quad t \geq 0, \tag{3.12}$$

where $\bar{X}_\Delta(t)$ has already been defined by (3.10).

The reader may wonder why we do not define the PPTEM in a similar fashion as the NPTEM, namely by replacing $\hat{\pi}_0$ in (3.9) with $\hat{\pi}_\Delta$ while keeping everything else unchanged. This is certainly possible but the mathematics will become slightly more complicated because $\hat{\pi}_\Delta$ does not preserve the nice property that π_Δ has while $\hat{\pi}_0$ does. More precisely, π_Δ maps $\mathbb{R}^d$ into the ball in $\mathbb{R}^d$ with center 0 and radius $\mu^{-1}(h(\Delta))$ but $\hat{\pi}_\Delta$ may map some x in the ball outside the ball. In terms of mathematics, we have

$$|\pi_\Delta(x)| \leq \mu^{-1}(h(\Delta)), \quad \forall x \in \mathbb{R}^d,$$

but we may have

$$|\hat{\pi}_\Delta(\pi_\Delta(x))| > \mu^{-1}(h(\Delta))$$

for some $x \in \mathbb{R}^d$ with $|x| \leq \mu^{-1}(h(\Delta))$. For example, if $x = (\mu^{-1}(h(\Delta)), 0, \cdots, 0)^T$, then $\hat{\pi}_\Delta(\pi_\Delta(x)) = (\mu^{-1}(h(\Delta)), \Delta, \cdots, \Delta)^T$ and

$$|\hat{\pi}_\Delta(\pi_\Delta(x))| = \sqrt{(\mu^{-1}(h(\Delta)))^2 + (d-1)\Delta^2} > \mu^{-1}(h(\Delta)).$$

4 Main Results

4.1 Statement of main results

Our aim of this paper is to show that both NPTEM solution $X_\Delta(t)$ and PPTEM solution $X^+_\Delta(t)$ converge to the true solution $x(t)$ in L^p for any $p \ge 2$ as described in the following main theorems of this paper.

Theorem 4.1 *Under Assumption 2.1,*

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\le t\le T} |X_\Delta(t) - x(t)|^p \Big) = 0. \tag{4.1}$$

Theorem 4.2 *Under Assumption 2.1,*

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\le t\le T} |X^+_\Delta(t) - x(t)|^p \Big) = 0. \tag{4.2}$$

The proof of these theorems are highly technical. To make it more understandable, we break it into a number of lemmas in the next subsection and prove the theorems afterward.

4.2 Lemmas

For the mathematical analysis, we need to define a new process

$$x_\Delta(t) = x(0) + \int_0^t [f_1(\bar{X}_\Delta(s)) + f_2(X_\Delta(s))]ds + \int_0^t g(\bar{X}_\Delta(s))dB(s) \tag{4.3}$$

for $t \ge 0$. We observe that $x_\Delta(t_k) = \bar{X}_\Delta(t_k)$ for all $k \ge 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = [f_1(\bar{X}_\Delta(t)) + f_2(X_\Delta(t))]dt + g(\bar{X}_\Delta(t))dB(t). \tag{4.4}$$

We also denote the ith component of $x_\Delta(t)$, $X_\Delta(t)$ or $\bar{X}_\Delta(t)$ by $x_{\Delta,i}(t)$, $X_{\Delta,i}(t)$ or $\bar{X}_{\Delta,i}(t)$, respectively.

By (3.2) and (3.6), it is easy to see from (3.8) that for any $q \ge 2$, $\mathbb{E}|\bar{X}_\Delta(t_1)|^q < \infty$ and then, by induction, $\mathbb{E}|\bar{X}_\Delta(t_k)|^q < \infty$ for all $k \ge 1$. By (4.3) we can then further see that $\mathbb{E}|x_\Delta(t)|^q < \infty$ for all $t \ge 0$. But we will show a better result (see Lemma 4.4).

We start with the following lemma, which shows that $x_\Delta(t)$ and $\bar{X}_\Delta(t)$ are close to each other in the sense of L^p.

Lemma 4.3 *For any* $\Delta \in (0, 1]$, *we have*

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \le C\Delta^{p/2}(h(\Delta))^p, \quad \forall t \in [0, T]. \tag{4.5}$$

Consequently

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = 0, \quad \forall t \in [0, T]. \tag{4.6}$$

Proof. By (3.6),

$$|f_2(X_\Delta(t))| = |f_2(\hat{\pi}_0(\pi_\Delta(\bar{X}_\Delta(t))))| \le h(\Delta). \tag{4.7}$$

Using this and (3.2), we can easily show from (4.3) that

$$\mathbb{E}|x_\Delta(t)|^p \le C(h(\Delta))^p + C\int_0^t \mathbb{E}|\bar{X}_\Delta(s)|^p ds$$

for $t \in [0,T]$. This implies

$$\begin{aligned}\sup_{0\le u\le t} \mathbb{E}|x_\Delta(u)|^p &\le C(h(\Delta))^p + C\int_0^t \mathbb{E}|\bar{X}_\Delta(s)|^p ds \\ &\le C(h(\Delta))^p + C\int_0^t \Big(\sup_{0\le u\le s} \mathbb{E}|x_\Delta(u)|^p\Big) ds.\end{aligned}$$

The well-known Gronwall inequality shows

$$\sup_{0\le u\le T} \mathbb{E}|x_\Delta(u)|^p \le C(h(\Delta))^p. \tag{4.8}$$

Now, for any $t \in [0,T]$, there is a unique $k \ge 0$ such that $t \in [t_k, t_{k+1})$ and hence $\bar{X}_\Delta(t) = \bar{X}_\Delta(t_k) = x_\Delta(t_k)$. It then follows from (4.3) that

$$\begin{aligned}&\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = \mathbb{E}|x_\Delta(t) - x_\Delta(t_k)|^p \\ \le& C\Delta^{p-1}\mathbb{E}\int_{t_k}^t [|f_1(\bar{X}_\Delta(s)|^p + |f_2(X_\Delta(s))|^p] ds + C\Delta^{(p-2)/2}\int_{t_k}^t |g(\bar{X}_\Delta(s)|^p ds.\end{aligned}$$

This, along with (3.2), (3.6) and (4.8), implies

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \le C\Delta^p(h(\Delta))^p + C\Delta^{p/2}(h(\Delta))^p \le C\Delta^{p/2}(h(\Delta))^p$$

which is the first assertion. Noting from (3.4) that $\Delta^{p/2}(h(\Delta))^p \le \Delta^{p/4}$, we obtain the second assertion from the first one immediately. □

The following lemma shows a much better result than (4.8).

Lemma 4.4 *Let Assumption 2.1 hold. Then*

$$\sup_{0<\Delta\le 1} \mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t)|^p\Big) \le C. \tag{4.9}$$

Proof. Fix any $\Delta \in (0,1]$. By the Itô formula and the Burkholder-Davis-Gundy inequality etc., it is not difficult (see, e.g., [11, pp.59-63]) to show that

$$\mathbb{E}\Big(\sup_{0\le u\le t} |x_\Delta(u)|^p\Big) \le C + C\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s} |x_\Delta(u)|^p\Big) ds + J_1(t) \tag{4.10}$$

for $t \in [0,T]$, where

$$J_1(t) = \mathbb{E}\Big(\sup_{0\le u\le t} \int_0^u p|x_\Delta(s)|^{p-2} x_\Delta^T(s) f_2(X_\Delta(s)) ds\Big).$$

By (3.6) and (3.7), we have

$$\begin{aligned}x_\Delta^T(s) f_2(X_\Delta(s)) &= \big([x_\Delta(s) - \bar{X}_\Delta(s)]^T + \bar{X}_\Delta^T(s)\big) f_2(\hat{\pi}_0(\pi_\Delta(\bar{X}_\Delta(s)))) \\ &\le h(\Delta)|x_\Delta(s) - \bar{X}_\Delta(s)|.\end{aligned}$$

Hence

$$J_1(t) \le \mathbb{E}\int_0^t p|x_\Delta(s)|^{p-2}h(\Delta)|x_\Delta(s) - \bar{X}_\Delta(s)|ds.$$

Using the Young inequality

$$pa^{p-2}b \le (p-2)a^p + 2b^{p/2}, \quad \forall a, b \ge 0,$$

as well as Lemma 4.3, we can then derive that

$$\begin{aligned} J_1(t) &\le \mathbb{E}\int_0^t \big[(p-2)|x_\Delta(s)|^p + 2(h(\Delta))^{p/2}|x_\Delta(s) - \bar{X}_\Delta(s)|^{p/2}\big]ds \\ &\le (p-2)\int_0^t \mathbb{E}|x_\Delta(s)|^p ds + 2(h(\Delta))^{p/2}\int_0^T (\mathbb{E}|x_\Delta(s) - \bar{X}_\Delta(s)|^p)^{1/2}ds \\ &\le (p-2)\int_0^t \mathbb{E}|x_\Delta(s)|^p ds + C\Delta^{p/4}(h(\Delta))^p \\ &\le (p-2)\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s}|x_\Delta(u)|^p\Big)ds + C, \end{aligned}$$

where we have used (3.4) in the last step. Substituting this into (4.10) yields

$$\mathbb{E}\Big(\sup_{0\le u\le t}|x_\Delta(u)|^p\Big) \le C + C\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s}|x_\Delta(u)|^p\Big)ds.$$

An application of the Gronwall inequality gives

$$\mathbb{E}\Big(\sup_{0\le u\le T}|x_\Delta(u)|^p\Big) \le C.$$

As this holds for any $\Delta \in (0,1]$ while C is independent of Δ, we see the required assertion (4.9). $\Box$

The following lemma improves the second assertion in Lemma 4.3.

Lemma 4.5 *Let Assumption 2.1 hold. Then*

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) = 0. \tag{4.11}$$

Proof. Let m be the integer part of T/Δ. Then, by (3.2) and (4.7) as well as Lemma 4.4, we derive that

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \\ \le&\mathbb{E}\Big(\max_{0\le k\le m}\sup_{t_k\le t\le t_{k+1}}\big|[f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k))](t-t_k) + g(\bar{X}_\Delta(t_k))(B(t) - B(t_k))\big|^p\Big) \\ \le&C\mathbb{E}\Big(\max_{0\le k\le m}[|\bar{X}_\Delta(t_k)|^p + (h(\Delta))^p]\Delta^p\Big) + J_2 \\ \le&C\Delta^p\mathbb{E}\Big(\max_{0\le k\le m}|x_\Delta(t_k)|^p + (h(\Delta))^p\Big) + J_2 \\ \le&C\Delta^p[C + (h(\Delta))^p] + J_2 \le C\Delta^p(h(\Delta))^p + J_2, \end{aligned} \tag{4.12}$$

where

$$J_2 = C\mathbb{E}\Big(\max_{0\le k\le m}\Big[|\bar{X}_\Delta(t_k))|^p \sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^p\Big]\Big).$$

Now, choose a sufficiently large integer $n \ge 3 \vee p$, dependent on p and T, for which

$$\Big(\frac{2n}{2n-1}\Big)^p (T+1)^{p/2n} \le 2. \tag{4.13}$$

But, by the Hölder inequality,

$$\begin{aligned} J_2 &\le C\Big\{\mathbb{E}\Big(\max_{0\le k\le m}\Big[|\bar{X}_\Delta(t_k))|^{2n} \sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)\Big\}^{p/2n} \\ &\le C\Big(\sum_{k=0}^m \mathbb{E}\Big[|\bar{X}_\Delta(t_k))|^{2n} \sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)^{p/2n}. \end{aligned}$$

But, by Lemma 4.4 (replacing p there by $2n$ though n here depends on p), $\mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}$ is bounded by C for every t_k. Note also that for each k, $\bar{X}_\Delta(t_k)$ is independent of $\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}$. We hence have

$$\begin{aligned} J_2 &\le C\Big(\sum_{k=0}^m \mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}\,\mathbb{E}\Big[\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)^{p/2n} \\ &\le C\Big(\sum_{k=0}^m \mathbb{E}\Big[\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)^{p/2n}. \end{aligned}$$

By the Doob martingale inequality (see, e.g., [11, Theorem 3.8 on p.14]), we further derive that

$$\begin{aligned} J_2 &\le C\Big(\sum_{k=0}^m \Big[\frac{2n}{2n-1}\Big]^{2n}\mathbb{E}|B(t_{k+1})-B(t_k)|^{2n}\Big)^{p/2n} \\ &\le C\Big(\sum_{k=0}^m \Big[\frac{2n}{2n-1}\Big]^{2n}(2n-1)!!\Delta^n\Big)^{p/2n} \\ &\le C\Big(\Big[\frac{2n}{2n-1}\Big]^{2n}(T+1)(2n-1)!!\Delta^{n-1}\Big)^{p/2n}, \end{aligned}$$

where $(2n-1)!! = (2n-1)\times(2n-3)\times\cdots\times 3\times 1$. (Please note that C above should depend on n but as n depends on p and T, we can still use the generic positive real constants.) However,

$$[(2n-1)!!]^{1/n} \le \frac{1}{n}\sum_{i=1}^n (2i-1) = n.$$

Thus

$$J_2 \le Cn^{p/2}\Big(\frac{2n}{2n-1}\Big)^p (T+1)^{p/2n}\Delta^{p(n-1)/2n}.$$

Using (4.13) while noting $(n-1)/2n \ge 1/3$ as we choose $n\ge 3$, we obtain

$$J_2 \le C\Delta^{p/3}.$$

Substituting this into (4.12) gives

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{X}_\Delta(t)|^p\Big) \le C(h(\Delta))^p\Delta^p + C\Delta^{p/3} \le C(h(\Delta))^p\Delta^{p/3}.$$

But, by (3.4),

$$(h(\Delta))^p \Delta^{p/3} = \Delta^{p/12}(\Delta^{1/4}h(\Delta))^p \le C\Delta^{p/12}.$$

We hence obtain

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \le C\Delta^{p/12}.$$

This implies the required assertion (4.11). $\Box$

In the remaining of this section, we need a couple of new notations. For each $r > |x_0|$, define the stopping times

$$\tau_r = \inf\{t \ge 0 : |x(t)| \ge r\}$$

and

$$\rho_{\Delta,r} = \inf\{t \ge 0 : |x_\Delta(t)| \ge r\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Moreover, we set

$$\theta_{\Delta,r} = \tau_r \wedge \rho_{\Delta,r} \tag{4.14}$$

and define the closed ball

$$S_r = \{x \in \mathbb{R}^d : |x| \le r\}.$$

The following lemma shows both $x(t \wedge \theta_{\Delta,r})$ and $x_\Delta(t \wedge \theta_{\Delta,r})$ are close to each other.

Lemma 4.6 *Let Assumption 2.1 hold. Then for each $r > |x_0|$, there is a $\Delta_1 = \Delta_1(r) \in (0, 1]$ such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x(t\wedge\theta_{\Delta,r}) - x_\Delta(t\wedge\theta_{\Delta,r})|^p\Big) \le C_r\Delta^{p/2}, \quad \forall \Delta \in (0, \Delta_1], \tag{4.15}$$

where C_r is a positive constant dependent on r, T etc. but independent of Δ.

Proof. Define

$$f_{2,r}(x) = f_2\big((|x| \wedge r)x/|x|\big) \quad \text{for } x \in \mathbb{R}^d.$$

Obviously, $f_{2,r}(\cdot)$ is bounded and globally Lipschitz continuous in $\mathbb{R}^d$ but its Lipschitz constant depends on r. Consider the SDE

$$dy(t) = [f_1(y(t)) + f_{2,r}(y(t))]dt + g(y(t))dB(t) \tag{4.16}$$

on $t \ge 0$ with the initial value $y(0) = x(0)$. It has a unique global solution $y(t)$ on $t \ge 0$. For each step size $\Delta \in (0, 1]$, we can apply the EM method to the SDE (4.16). That is, we form the EM solutions $Y_\Delta(t_k) \approx y(t_k)$ for $t_k = k\Delta$ by setting $Y_\Delta(0) = x(0)$ and computing

$$Y_\Delta(t_{k+1}) = Y_\Delta(t_k) + [f_1(Y_\Delta(t_k)) + f_{2,r}(Y_\Delta(t_k))]\Delta + g(Y_\Delta(t_k))\Delta B_k, \tag{4.17}$$

for $k = 0, 1, \cdots$. Extend the definitions of $Y_\Delta(\cdot)$ from the grid points t_k to the whole $t \ge 0$ by setting

$$Y_\Delta(t) = \sum_{k=0}^{\infty} Y_\Delta(t_k) I_{[t_k,t_{k+1})}(t), \tag{4.18}$$

and then define the Itô process

$$y_\Delta(t) = x(0) + \int_0^t [f_1(Y_\Delta(s)) + f_{2,r}(Y_\Delta(s))]ds + \int_0^t g(Y_\Delta(s))dB(s) \tag{4.19}$$

for $t \geq 0$. It is well known (see, e.g., [8, 11]) that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|y(t)-y_\Delta(t)|^p\Big)\leq C_r\Delta^{p/2}. \tag{4.20}$$

Let us relate $y(t)$ and $y_\Delta(t)$ to $x(t)$ and $x_\Delta(t)$, respectively. It is straightforward to see that

$$x(t\wedge\tau_r)=y(t\wedge\tau_r)\ \text{ a.s for all } t\in[0,T]. \tag{4.21}$$

We now choose $\Delta_1 \in (0,1]$ sufficiently small for $\mu^{-1}(h(\Delta_1)) \geq r$. Obviously, for all $\Delta \in (0,\Delta_1]$,

$$f_2(\pi_\Delta(x))=f_{2,r}(x),\quad \forall x\in S_r.$$

This, together with (3.5), yields

$$f_2(\hat{\pi}_0(\pi_\Delta(x)))=f_{2,r}(x),\quad \forall x\in S_r.$$

Comparing (3.8), (4.3) with (4.17),(4.19), we then see that

$$x_\Delta(t\wedge\rho_{\Delta,r})=y_\Delta(t\wedge\rho_{\Delta,r})\ \text{ a.s for all } t\in[0,T] \tag{4.22}$$

provided $\Delta \in (0,\Delta_1]$. Combining (4.20) - (4.22), we obtain the desired assertion (4.15) immediately. □

4.3 Proof of Theorem 4.1

We are finally in a position to prove our main theorems. We prove Theorem 4.1 first in this subsection and then Theorem 4.2 next. Obviously,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|X_\Delta(t)-x(t)|^p\Big)\leq 3^{p-1}(J_3(\Delta)+J_4(\Delta)+J_5(\Delta)), \tag{4.23}$$

where

$$\begin{aligned}
J_3(\Delta)&=\mathbb{E}\Big(\sup_{0\leq t\leq T}|X_\Delta(t)-\bar{X}_\Delta(t)|^p\Big),\\
J_4(\Delta)&=\mathbb{E}\Big(\sup_{0\leq t\leq T}|\bar{X}_\Delta(t)-x_\Delta(t)|^p\Big),\\
J_5(\Delta)&=\mathbb{E}\Big(\sup_{0\leq t\leq T}|x_\Delta(t)-x(t)|^p\Big).
\end{aligned}$$

By Lemma 4.5, we already have that $J_4(\Delta) \to 0$ as $\Delta \to 0$. To complete the proof, we hence only need to show both $J_3(\Delta)$ and $J_5(\Delta)$ tend to 0.

Let us first show $J_5(\Delta) \to 0$. Let $\varepsilon \in (0,1)$ be arbitrary. Recalling the definition (4.14) of $\theta_{r,\Delta}$ and using Lemmas 2.2 and 4.4, we can derive that

$$\begin{aligned}
&\mathbb{P}(\theta_{r,\Delta}\leq T)\leq\mathbb{P}(\tau_r\leq T)+\mathbb{P}(\rho_{r,\Delta}\leq T)\\
=&\frac{1}{r^p}\Big[\mathbb{E}\big(|x(\tau_r)|^pI_{\{\tau_r\leq T\}}\big)+\mathbb{E}\big(|x_\Delta(\rho_{r,\Delta})|^pI_{\{\rho_{r,\Delta}\leq T\}}\big)\Big]\\
\leq&\frac{1}{r^p}\Big[\mathbb{E}\Big(\sup_{0\leq t\leq T}|x(t)|^p\Big)+\mathbb{E}\Big(\sup_{0\leq t\leq T}|x_\Delta(t)|^p\Big)\Big]\\
\leq&\frac{C}{r^p}.
\end{aligned}$$

We can hence choose a real number $r = r(\varepsilon)$ so large that

$$\mathbb{P}(\theta_{r,\Delta} \le T) \le \varepsilon^2.$$

For this r, by Lemma 4.6, we have

$$\mathbb{E}\Big(\sup_{0\le t\le T} |x(t\wedge\theta_{\Delta,r}) - x_\Delta(t\wedge\theta_{\Delta,r})|^p\Big) \le C_r \Delta^{p/2}, \quad \forall \Delta \in (0,\Delta_1],$$

where Δ_1 now depends on ε (as r dependent on ε). Thus, for $\Delta \in (0,\Delta_1]$, we derive

$$\begin{aligned} J_5(\Delta) =& \mathbb{E}\Big(I_{\{\theta_{r,\Delta}\le T\}} \sup_{0\le t\le T} |x_\Delta(t) - x(t)|^p\Big) + \mathbb{E}\Big(I_{\{\theta_{r,\Delta}>T\}} \sup_{0\le t\le T} |x_\Delta(t) - x(t)|^p\Big) \\ \le& \big[\mathbb{P}(\theta_{r,\Delta}\le T)\big]^{1/2} \Big[\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - x(t)|^{2p}\Big)\Big]^{1/2} \\ &+ \mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t\wedge\theta_{r,\Delta}) - x(t\wedge\theta_{r,\Delta})|^p\Big) \\ \le& C\varepsilon + C_r\Delta^{p/2}. \end{aligned}$$

But, by Lemma 4.4 (recalling p is arbitrary once again),

$$\begin{aligned} &\Big[\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - x(t)|^{2p}\Big)\Big]^{1/2} \\ \le& 2^{(2p-1)/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t)|^{2p}\Big) + \mathbb{E}\Big(\sup_{0\le t\le T} |x(t)|^{2p}\Big)\Big]^{1/2} \le C. \end{aligned}$$

We then have

$$J_5(\Delta) \le C\varepsilon + C_r\Delta^{p/2}, \quad \forall \Delta \in (0,\Delta_1].$$

This implies

$$\limsup_{\Delta\to 0} J_5(\Delta) \le C\varepsilon.$$

As ε is arbitrary, we must have that $J_5(\Delta) \to 0$ as $\Delta \to 0$.

Let us finally show $J_3(\Delta) \to 0$ to complete our proof of Theorem 4.1. By Lemmas 2.2 and 4.4, we can find a positive number $r = r(\varepsilon)$ so large that

$$\mathbb{P}(\Omega_1) \ge 1 - \varepsilon/3, \tag{4.24}$$

where

$$\Omega_1 = \{|x(t)| \vee |x_\Delta(t)| < r \text{ for all } 0 \le t \le T\}.$$

For a sufficiently small $\delta \in (0,1)$, define

$$\zeta_{\delta,i} = \inf\{t \ge 0 : x_i(t) \le \delta\}, \quad 1 \le i \le d.$$

By Lemma 2.3,

$$\begin{aligned} &\mathbb{P}(\zeta_{\delta,i} \le T) = \mathbb{E}\Big(I_{\{\zeta_{\delta,i}\le T\}} \frac{x_i(\zeta_{\delta,i}) - 1 - \log(x_i(\zeta_{\delta,i}))}{\delta - 1 - \log(\delta)}\Big) \\ \le& \frac{1}{\delta - 1 - \log(\delta)} \mathbb{E}\Big(\sup_{0\le t\le T} [x_i(t) - 1 - \log(x_i(t))]\Big) \le \frac{C}{\delta - 1 - \log(\delta)}. \end{aligned}$$

Noting that $\delta - 1 - \log(\delta) \to \infty$ as $\delta \to 0$, we can find a $\delta = \delta(\varepsilon)$ so small that

$$\mathbb{P}(\zeta_{\delta,i} \le T) \le \frac{\varepsilon}{3d}, \quad 1 \le i \le d.$$

Set $\zeta_\delta = \min_{1\le i\le d} \zeta_{\delta,i}$. Then

$$\mathbb{P}(\zeta_\delta \le T) \le \mathbb{P}\Big(\cup_{i=1}^d \{\zeta_{\delta,i} \le T\}\Big) \le \sum_{i=1}^d \mathbb{P}(\zeta_{\delta,i} \le T) \le \varepsilon/3.$$

So $\mathbb{P}(\zeta_\delta > T) \ge 1 - \varepsilon/3$. This implies

$$\mathbb{P}(\Omega_2) \ge 1 - \varepsilon/3, \tag{4.25}$$

where

$$\Omega_2 = \Big\{ \min_{1\le i\le d} \inf_{0\le t\le T} x_i(t) > \delta \Big\}.$$

On the other hand, for the pair of chosen r and δ, define

$$\Omega_\Delta = \Big\{ \sup_{0\le t\le T} |x(t\wedge\theta_{\Delta,r}) - x_\Delta(t\wedge\theta_{\Delta,r})| < \delta/2 \Big\}.$$

By Lemma 4.6 and the well-known Chebyshev inequality, we see that there is a $\Delta_1 = \Delta_1(\varepsilon)$ (as $r = r(\varepsilon)$) such that $\mu^{-1}(h(\Delta_1)) \ge r$ and

$$\mathbb{P}(\Omega_\Delta^c) = \mathbb{P}\Big(\sup_{0\le t\le T} |x(t\wedge\theta_{\Delta,r}) - x_\Delta(t\wedge\theta_{\Delta,r})| \ge \delta/2 \Big) \le \frac{C_r \Delta^{p/2}}{(\delta/2)^p}, \quad \forall \Delta \in (0, \Delta_1].$$

Consequently, there is a $\Delta_2 = \Delta_2(\varepsilon) \in (0, \Delta_1]$ such that

$$\mathbb{P}(\Omega_\Delta) \ge 1 - \varepsilon/3, \quad \forall \Delta \in (0, \Delta_2]. \tag{4.26}$$

Set $\Omega_{3,\Delta} = \Omega_1 \cap \Omega_2 \cap \Omega_\Delta$. Combining (4.24) - (4.26) gives

$$\mathbb{P}(\Omega_{3,\Delta}) \ge 1 - \varepsilon, \quad \forall \Delta \in (0, \Delta_2]. \tag{4.27}$$

From now on, we consider any step size $\Delta \in (0, \Delta_2]$. Note that for every $\omega \in \Omega_{3,\Delta}$, $\theta_{\Delta,r} > T$,

$$\sup_{0\le t\le T} |\bar{X}_\Delta(t)| \le \sup_{0\le t\le T} |x_\Delta(t)| \le r \le \mu^{-1}(h(\Delta_1)) \le \mu^{-1}(h(\Delta)), \tag{4.28}$$

and

$$\begin{aligned} \inf_{0\le t\le T} \bar{X}_{\Delta,i}(t) &\ge \inf_{0\le t\le T} x_{\Delta,i}(t) \ge \inf_{0\le t\le T} x_i(t) - \sup_{0\le t\le T} |x_i(t) - x_{\Delta,i}(t)| \\ &> \delta - \sup_{0\le t\le T} |x(t) - x_\Delta(t)| > \delta - \delta/2 = \delta/2. \end{aligned} \tag{4.29}$$

In other words, for every $\omega \in \Omega_{3,\Delta}$, $\bar{X}_\Delta(t) \in \mathbb{R}_+^d$ with $|\bar{X}_\Delta(t)| \le \mu^{-1}(h(\Delta))$, whence $X_\Delta(t) = \hat{\pi}_0(\pi_\Delta(\bar{X}_\Delta(t))) = \bar{X}_\Delta(t)$ for all $t \in [0, T]$. Consequently,

$$\begin{aligned} J_3(\Delta) &= \mathbb{E}\Big(I_{\Omega_{3,\Delta}^c} \sup_{0\le t\le T} |X_\Delta(t) - \bar{X}_\Delta(t)|^p \Big) \\ &\le \big[\mathbb{P}(\Omega_{3,\Delta}^c)\big]^{1/2} \Big[\mathbb{E}\Big(\sup_{0\le t\le T} |X_\Delta(t) - \bar{X}_\Delta(t)|^{2p} \Big)\Big]^{1/2} \\ &\le 2^p \sqrt{\varepsilon} \Big[\mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t)|^{2p} \Big)\Big]^{1/2} \\ &\le C\sqrt{\varepsilon} \end{aligned}$$

provided $\Delta \in (0, \Delta_2]$, where Lemma 4.4 has been used once again. As ε is arbitrary, we must have that $J_3(\Delta) \to 0$ as $\Delta \to 0$. This completes our proof of Theorem 4.1.

4.4 Proof of Theorem 4.2

Once again, it is obvious that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-x(t)|^p\Big)\le 3^{p-1}(J_4(\Delta)+J_5(\Delta)+J_6(\Delta)), \tag{4.30}$$

where $J_4(\Delta), J_5(\Delta)$ have been defined before and

$$J_6(\Delta)=\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar X_\Delta(t)|^p\Big).$$

Clearly, all we need to do is to show that $J_6(\Delta)\to 0$ as $\Delta\to 0$. Let $\Delta\in(0,\Delta_2\wedge(\delta/2)]$ be arbitrary. We see from (4.28) and (4.29) that for every $\omega\in\Omega_{3,\Delta}$, $\bar X_\Delta(t)\in\mathbb{R}_+^d$ with $|\bar X_\Delta(t)|\le\mu^{-1}(h(\Delta))$ and $\inf_{0\le t\le T}\bar X_{\Delta,i}(t)>\delta/2$, whence $X_\Delta^+(t)=\hat\pi_\Delta(\pi_\Delta(\bar X_\Delta(t)))=\bar X_\Delta(t)$ for all $t\in[0,T]$. Consequently,

$$\begin{aligned}J_6(\Delta)&=\mathbb{E}\Big(I_{\Omega_{3,\Delta}^c}\sup_{0\le t\le T}|X_\Delta^+(t)-\bar X_\Delta(t)|^p\Big)\\&\le\big[\mathbb{P}(\Omega_{3,\Delta}^c)\big]^{1/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar X_\Delta(t)|^{2p}\Big)\Big]^{1/2}\\&\le 2^p\sqrt{\varepsilon}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big)+\mathbb{E}\Big(\sup_{0\le t\le T}|\bar X_\Delta(t)|^{2p}\Big)\Big]^{1/2}.\end{aligned}$$

But, by Lemma 4.4,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\bar X_\Delta(t)|^{2p}\Big)\le C.$$

On the other hand, for any $x\in\mathbb{R}^d$,

$$\begin{aligned}|\hat\pi_\Delta(x)|^{2p}&=\Big(\sum_{i=1}^d(\Delta\vee x_i)^2\Big)^p\le\Big(\sum_{i=1}^d(\Delta^2+|x_i|^2)\Big)^p\\&\le(d+|x|^2)^p\le 2^{p-1}(d^p+|x|^{2p}).\end{aligned}$$

So

$$\begin{aligned}|X_\Delta^+(t)|^{2p}&=|\hat\pi_\Delta(\pi_\Delta(\bar X_\Delta(t)))|^{2p}\le 2^{p-1}(d^p+|\pi_\Delta(\bar X_\Delta(t))|^{2p})\\&\le 2^{p-1}(d^p+|\bar X_\Delta(t)|^{2p}).\end{aligned}$$

Consequently

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big)\le C.$$

In other words, we have showed that

$$J_6(\Delta)\le C\sqrt{\varepsilon}$$

provided $\Delta\in(0,\Delta_2\wedge(\delta/2)]$. As ε is arbitrary, we must have that $J_6(\Delta)\to 0$ as $\Delta\to 0$. This completes our proof of Theorem 4.2.

5 Examples with Simulations

In this section, we will discuss two examples for illustration.

Example 5.1 To illustrate as well as to verify our new PPTEM scheme, we consider the scalar stochastic Lotka–Volterra competitive model

$$dx(t) = x(t)[(b - ax(t))dt + \sigma dB(t)] \tag{5.1}$$

for a single species, where individuals within the species are competitive, $x(t) \in (0, \infty)$, b, a, σ are all positive numbers. The main reason we discuss this model is because it has an explicit solution so that we can compare it with the NPTEM numerical solution in order to verify the NPTEM scheme.

We write the Lotka–Volterra model (5.1) as the SDE (3.1) in $\mathbb{R}$ by defining

$$\begin{cases} f_1(x) = bx, \quad f_2(x) = -ax^2, \quad g(x) = \sigma x \quad \text{for } x \geq 0, \\ f_1(x) = f_2(x) = g(x) = 0 \quad \text{for } x < 0. \end{cases} \tag{5.2}$$

Define $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mu(u) = au^2$ for $u \geq 1$. Its inverse function of $\mu^{-1} : [a, \infty) \to \mathbb{R}_+$ has the form $\mu^{-1}(u) = \sqrt{u/a}$ for $u \geq a$. Let $\hat{h} = 1 \vee a \wedge x(0)$ and define the strictly decreasing function $h : (0, 1] \to [a, \infty)$ by $h(\Delta) = \hat{h}\Delta^{-\theta}$ for some $\theta \in (0, 1/4]$. Hence $\mu^{-1}(h(\Delta)) = \sqrt{\hat{h}/a\Delta^\theta}$. The mapping $\hat{\pi}_\Delta(\pi_\Delta(\cdot)) : \mathbb{R} \to \left[\Delta, \sqrt{\hat{h}/a\Delta^\theta}\right]$ has the form

$$\hat{\pi}_\Delta(\pi_\Delta(x)) = (\Delta \vee x) \wedge \sqrt{\hat{h}/a\Delta^\theta}, \quad \text{for } x \in \mathbb{R}.$$

We first apply the NPTEM to the Lotka–Volterra model (5.1) (namely the SDE (3.1) with f_1, f_2 and g being defined by (5.2)). That is, set $\bar{X}_\Delta(0) = X_\Delta(0) = x(0)$ and compute

$$\bar{X}_\Delta(t_{k+1}) = \bar{X}_\Delta(t_k) + [f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k))]\Delta + g(\bar{X}_\Delta(t_k))\Delta B_k, \tag{5.3}$$

$$X_\Delta(t_{k+1}) = (0 \vee \bar{X}_\Delta(t_{k+1})) \wedge \sqrt{\hat{h}/a\Delta^\theta} \tag{5.4}$$

for $k = 0, 1, \cdots$, and then extend the definitions of $X_\Delta(\cdot)$ from the grid points t_k to the whole $t \geq 0$ by (3.11). The PPTEM solution is then defined by

$$X^+_\Delta(t) = (\Delta \vee X_\Delta(t)) \wedge \sqrt{\hat{h}/a\Delta^\theta}, \quad t \geq 0.$$

By Theorem 4.2, we can conclude that $X^+_\Delta(T)$ converges to $x(t)$ defined by (5.6) in the sense that

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |X^+_\Delta(t) - x(t)|^p\Big) = 0. \tag{5.5}$$

Given an initial value $x(0) > 0$, the solution $x(t)$ remains to be positive. Let $z(t) = 1/x(t)$. By the Itô formula,

$$dz(t) = [a + (\sigma^2 - b)z(t)]dt - \sigma z(t)dB(t).$$

By the variation-of-constants formula (see, e.g., [11, Theorem 3.1 on p.96]),

$$z(t) = \exp\big(-[b - 0.5\sigma^2]t - \sigma B(t)\big)\Big(z(0) + a\int_0^t \exp\big([b - 0.5\sigma^2]s + \sigma B(s)\big)ds\Big).$$

This gives the explicit solution of (5.1):

$$x(t) = \exp\big([b - 0.5\sigma^2]t + \sigma B(t)\big) \left(\frac{1}{x(0)} + a \int_0^t \exp\big([b - 0.5\sigma^2]s + \sigma B(s)\big) ds \right)^{-1}. \quad (5.6)$$

Although the integration in this formula cannot be calculated analytically, it can be approximated numerically by the Riemann sum. More precisely, define

$$\phi(t) = \exp\big([b - 0.5\sigma^2]t + \sigma B(t)\big), \quad 0 \le t \le T. \quad (5.7)$$

In the remaining of this example, we set $\Delta = T/N$ for an integer $N > T$ and $t_k = k\Delta$ for $0 \le k \le N$. We approximate $\int_0^{t_k} \phi(s)ds$ by

$$\Psi_\Delta(t_k) = \sum_{i=0}^{k-1} 0.5\Delta[\phi(t_i) + \phi(t_{i+1})], \quad 0 \le k \le N \quad (5.8)$$

and of course set $\Psi_\Delta(t_0) = 0$. We then form the discrete-time Riemann approximate solutions $Y_\Delta(t_k) \approx x(t_k)$ by

$$Y_\Delta(t_k) = \phi(t_k)/(1/x(0) + a\Psi_\Delta(t_k)), \quad 0 \le k \le N. \quad (5.9)$$

We will show in appendix A that

$$\lim_{N\to\infty} \mathbb{E}\Big(\sup_{0\le k\le N} |Y_\Delta(t_k) - x(t_k)|^2 \Big) = 0. \quad (5.10)$$

Although it is sufficient to compare our new PPTEM solutions $X_\Delta^+(t_k)$ with $Y_\Delta(t_k)$, we will do better by comparing it with the well-known backward Euler-Maruyama (BEM) scheme (see, e.g., [4]) as well. To be more precise, the BEM applied to the Lotka–Volterra model is to form the discrete-time BEM solutions $Z_\Delta(t_k) \approx x(t_k)$ by setting $Z_\Delta(0) = x(0)$ and computing

$$Z_\Delta(t_{k+1}) = Z_\Delta(t_k) + [f_1(Z_\Delta(t_k)) + f_2(Z_\Delta(t_{k+1}))]\Delta + g(Z_\Delta(t_k))\Delta B_k$$

for $k \ge 0$. It is known that

$$\lim_{N\to\infty} \mathbb{E}\Big(\sup_{0\le k\le N} |Z_\Delta(t_k) - x(t_k)|^2 \Big) = 0.$$

For numerical simulations, we let $b = 10$, $a = 1$, $\sigma = 0.5$, $x(0) = 6$ and choose $\theta = 1/4$, $\hat{h} = 1000$, whence $\mu^{-1}(h(\Delta)) = \sqrt{\hat{h}/\Delta^\theta}$. The simulations in Figure 5.1 show the sample paths of the solution for $t \in [0, 10]$ by three schemes of the PPTEM, Riemann and BEM. The simulations in the left graph use $\Delta = 10^{-3}$ while in the right $\Delta = 10^{-4}$.

They show that three sample paths generated by the three schemes are very close to each other. More precisely, the simulations are designed to produce the squares of the max differences between PPTEM and Riemann as well as BEM and Riemann:

$$\max_{0\le k\le 10^4} |X_\Delta^+(t_k) - Y_\Delta(t_k)|^2 = 0.002809 \quad \text{and} \quad \sup_{0\le k\le 10^4} |Z_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.005086$$

when $\Delta = 10^{-3}$; while

$$\max_{0\le k\le 10^5} |X_\Delta^+(t_k) - Y_\Delta(t_k)|^2 = 0.0002235 \quad \text{and} \quad \sup_{0\le k\le 10^5} |Z_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.0002527$$

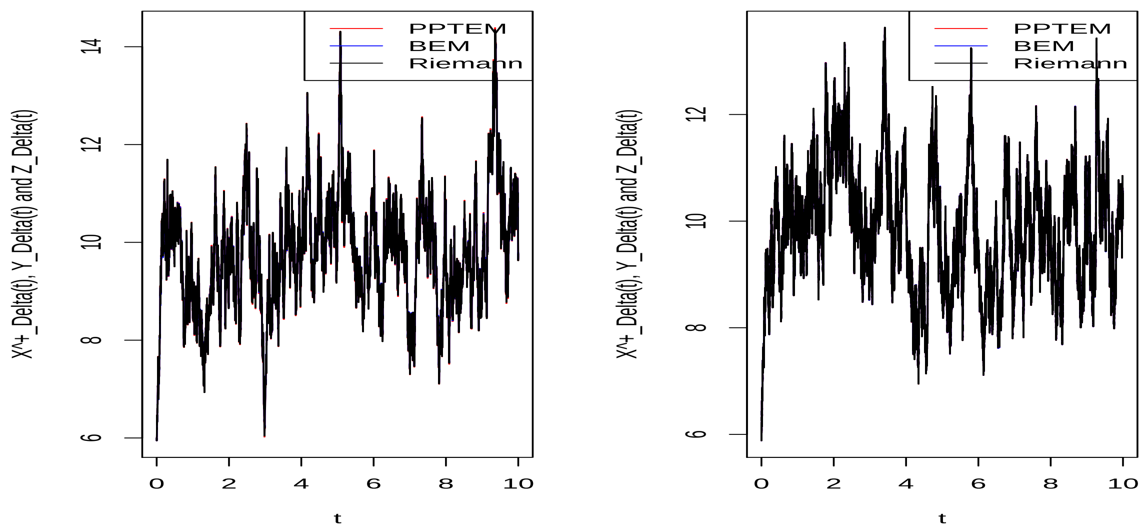

Figure 5.1: The computer simulations of the sample paths of the solution to equation (5.1) by PPTEM, Riemann and BEM. Left: $\Delta = 10^{-3}$. Right: $\Delta = 10^{-4}$.

when $\Delta = 10^{-4}$. These seem to indicate that PPTEM is closer to Riemann than BEM. To confirm this, we repeat the above simulations 100 times (namely, simulate 100 sample paths for each of the three scheme) and produce the mean squares (MS) of the max differences:

$$\frac{1}{100}\sum_{j=1}^{100}\Big(\sup_{0\le k\le N}|X_\Delta^{+,j}(t_k) - Y_\Delta^j(t_k)|^2\Big) \quad \text{and} \quad \frac{1}{100}\sum_{j=1}^{100}\Big(\sup_{0\le k\le N}|Z_\Delta^j(t_k) - Y_\Delta^j(t_k)|^2\Big),$$

where j stands for the jth sample paths. To reduce the time of simulations without losing any necessary illustration, we only simulate the paths for $t \in [0,1]$ but we make comparisons for $\Delta = 10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5}. The outcomes of the simulations are shown in Figure 5.2. They show that our new PPTEM solutions are closer to Riemann solutions than BEM slightly. They also indicate that our new PPTEM solutions converge to the true solution with the rate of order 0.5, though we have not proved this in theory yet but we will tackle it in the future.

Example 5.2 Let us now discuss a 3-dimensional Lotka–Volterra SDE model

$$dx(t) = \mathrm{diag}(x_1(t), x_2(t), x_3(t))[(b - Ax(t))dt + \sigma dB(t)], \tag{5.11}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state of the 3 interacting species and the system parameters

$$b = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}, \ A = \begin{bmatrix} 0 & -0.3 & -0.2 \\ -0.3 & 0 & -0.1 \\ -0.1 & -0.2 & 0 \end{bmatrix}, \ \sigma = \begin{bmatrix} 3.8 \\ 4.5 \\ 4.2 \end{bmatrix}.$$

Unlike Example 5.1, there is so far no explicit solution to a multi-dimensional Lotka–Volterra SDE model. We hence choose the SDE model (5.11), where the 3 species will all become extinct with probability 1 (see, e.g., [1, 11]). Applying our PPTEM to the model

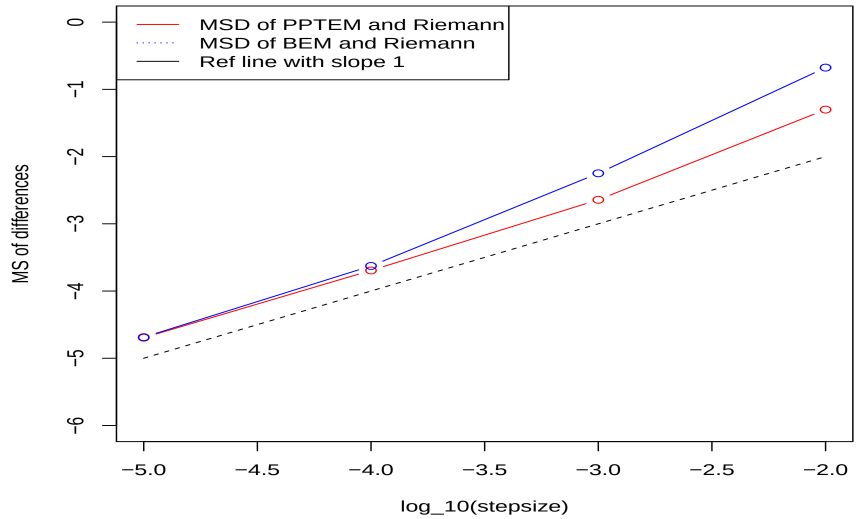

Figure 5.2: Mean squares of differences for $\Delta = 10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5} with sample size of 100.

will enable us to test if the scheme will not only preserve the positivity of the solution but also reproduce the extinction. For numerical simulations, we let $x(0) = (3, 2, 1)^T$ and use $\Delta = 10^{-5}$. The simulations in Figure 5.3 show the sample paths of the solution for $t \in [0, 10]$ by the PPTEM. They support the theoretical results.

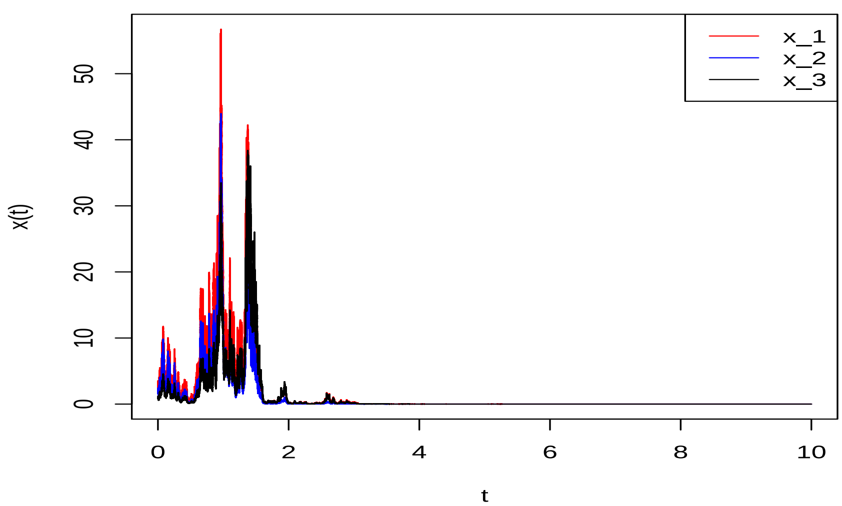

Figure 5.3: The computer simulations of the sample paths of the solution to equation (5.11) by PPTEM.

6 Conclusion

In this paper we developed two new numerical methods, NPTEM and PPTEM, for the highly nonlinear multi-dimensional stochastic Lotka–Volterra model. Although PPTEM should be more appropriate for the Lotka–Volterra model as its solution is positive, we discuss NPTEM first and then PPTEM in order to simplify the proofs as well as to make our theory more understandable. However, we emphasise once again that the NPTEM has its own right as many SDE models in applications have their nonnegative solutions.

A Riemann approximate solutions

In this appendix we will prove (5.10), namely that the Riemann approximate solutions (5.9) converge to the true solution of the SDE (5.1) in L^2. Note that $\phi(t)$ defined by (5.7) is the solution to the following linear scalar SDE

$$d\phi(t) = b\phi(t)dt + \sigma\phi(t)dB(t) \tag{A.1}$$

on $t \in [0, T]$ with the initial value $\phi(0) = 1$. It is known (see, e.g., [11, p.304]) that for any positive integer $n \geq 2$,

$$\mathbb{E}|\phi(t)|^n = e^{nt[b+0.5\sigma^2(n-1)]} \leq e^{nT[b+0.5\sigma^2(n-1)]} =: K_n, \quad 0 \leq t \leq T, \tag{A.2}$$

where =: stands for 'denoted by'. Applying the Hölder inequality and the property of the Itô integral (see, e.g., [11, p.39]) we can then derive that, for $0 \leq s < t \leq T$ with $t - s \leq 1$,

$$\begin{aligned}\mathbb{E}|\phi(t) - \phi(s)|^n \leq& 2^{n-1}(t-s)^{n-1}\int_s^t \mathbb{E}|b\phi(u)|^n du \\ &+ 2^{n-1}(0.5n(n-1))^{n/2}(t-s)^{(n-2)/2}\int_s^t \mathbb{E}|\sigma\phi(u)|^n du \\ &\leq \bar{K}_n(t-s)^{n/2},\end{aligned} \tag{A.3}$$

where $\bar{K}_n = 2^{n-1}K_n\big(b^n + \sigma^n(0.5n(n-1))^{n/2}\big)$. Note that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq k\leq N}|Y_\Delta(t_k) - x(t_k)|^2\Big) &\leq \Big[\mathbb{E}\Big(\sup_{0\leq k\leq N}|Y_\Delta(t_k) - x(t_k)|^{2n}\Big)\Big]^{1/n} \\ &\leq \Big[\sum_{k=1}^N \mathbb{E}|Y_\Delta(t_k) - x(t_k)|^{2n}\Big]^{1/n},\end{aligned} \tag{A.4}$$

recalling $Y_\Delta(0) = x(0)$. Set $I(t_k) = \int_0^{t_k}\phi(s)ds$. It follows from (5.6) and (5.9) that

$$\begin{aligned}|x(t_k) - Y_\Delta(t_k)| &= \Big|\frac{\phi(t_k)}{1/x(0) + aI(t_k)} - \frac{\phi(t_k)}{1/x(0) + a\Psi_\Delta(t_k)}\Big| \\ =&\frac{a\phi(t_k)|I(t_k) - \Psi_\Delta(t_k)|}{(1/x(0) + I(t_k))(1/x(0) + a\Psi_\Delta(t_k))} \leq a|x(0)|^2\phi(t_k)|I(t_k) - \Psi_\Delta(t_k)|.\end{aligned}$$

Hence, by the Hölder inequality,

$$\begin{aligned}\mathbb{E}|x(t_k) - Y_\Delta(t_k)|^{2n} &\leq a^{2n}|x(0)|^{4n}(\mathbb{E}|\phi(t_k)|^{4n})^{1/2}(\mathbb{E}|I(t_k) - \Psi_\Delta(t_k)|^{4n})^{1/2} \\ &\leq a^{2n}|x(0)|^{4n}\sqrt{K_{4n}}(\mathbb{E}|I(t_k) - \Psi_\Delta(t_k)|^{4n})^{1/2}\end{aligned} \tag{A.5}$$

for any integer $n \geq 2$. But

$$\mathbb{E}|I(t_k) - \Psi_\Delta(t_k)|^{4n} \leq k^{4n-1} \sum_{i=0}^{k-1} \mathbb{E}\Big| \int_{t_{i-1}}^{t_i} \phi(s)ds - 0.5\Delta(\phi(t_{i-1}) + \phi(t_i))\Big|^{4n}$$

$$\begin{aligned}
=&k^{4n-1} \sum_{i=0}^{k-1} \mathbb{E}\Big| \int_{t_{i-1}}^{t_i} 0.5(\phi(s) - \phi(t_{i-1}))ds + \int_{t_{i-1}}^{t_i} 0.5(\phi(s) - \phi(t_i))ds\Big|^{4n} \\
\leq&0.5(2k\Delta)^{4n-1} \sum_{i=0}^{k-1} \Big(\int_{t_{i-1}}^{t_i} \mathbb{E}|\phi(s) - \phi(t_{i-1})|^{4n} ds + \int_{t_{i-1}}^{t_i} \mathbb{E}|\phi(s) - \phi(t_i)|^{4n} ds\Big) \\
\leq&(2k\Delta)^{4n} \bar{K}_{4n} \Delta^{2n} \leq (2T)^{4n} \bar{K}_{4n} \Delta^{2n}, \qquad\qquad (A.6)
\end{aligned}$$

where (A.3) has been used. Substituting (A.6) into (A.5) yields

$$\mathbb{E}|x(t_k) - Y_\Delta(t_k)|^{2n} \leq (2aT)^{2n} |x(0)|^{4n} \sqrt{K_{4n} \bar{K}_{4n}}\, \Delta^n. \tag{A.7}$$

Substituting this into (A.4) implies

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0 \leq k \leq N} |Y_\Delta(t_k) - x(t_k)|^2\Big) \leq& \Big[(2aT)^{2n} |x(0)|^{4n} \sqrt{K_{4n} \bar{K}_{4n}}\, \Delta^n N\Big]^{1/n} \\
=& 4a^2 |x(0)|^4 \big(\sqrt{K_{4n} \bar{K}_{4n}} T^{2n+1} \Delta^{n-1}\big)^{1/n}. \qquad (A.8)
\end{aligned}$$

In particular, choosing $n = 16$, we get

$$\mathbb{E}\Big(\sup_{0 \leq k \leq N} |Y_\Delta(t_k) - x(t_k)|^2\Big) \leq 4a^2 |x(0)|^4 (K_{64} \bar{K}_{64})^{1/32} T^{33/16} \Delta^{15/16}. \tag{A.9}$$

This implies (5.10) immediately.

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