

Numerical analysis of a viscoplastic contact problem with normal compliance, unilateral constraint, memory term and friction

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ABSTRACT

In this paper, we consider a model which describes the quasistatic contact between a viscoplastic body and a foundation. The material's behavior is modeled with a rate-type viscoplastic constitutive law with an internal state variable. The contact is modeled with normal compliance, unilateral constraint, memory term, and friction which is under a total slip-dependent version of Coulomb's law. For the weak formulation of the problem, which is in the form of a system coupling two nonlinear integral equations with a history-dependent variational–hemivariational inequality, we introduce a fully discrete scheme and derive an error estimate. Under appropriate regularity assumptions, we obtain an optimal-order error estimate in finite element spaces. Finally, numerical results are reported to show the performance of the numerical method.

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1. Introduction

In this paper, we discuss the numerical approximation of a system coupling two nonlinear integral equations with a history-dependent variational–hemivariational inequality, which models a quasistatic viscoplastic contact.

First, we mention the notation of hemivariational inequality, which is the main tool in our paper. Hemivariational inequality was introduced by Panagiotopoulos in the 1980s [1,2] and is based on the properties of the generalized gradient of a locally Lipschitz function [3]. Compared with variational inequality, which is based on arguments of monotonicity and convexity, hemivariational inequality is concerned with problems with nonconvex and nonsmooth functionals.

Then we briefly recall the contact problem which was studied in [4]. Suppose that a viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a Lipschitz continuous boundary Γ . The boundary is divided into three mutually disjoint measurable parts Γ_1 , Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The body forces of density f_0 act on Ω and surface tractions of density f_2 act on Γ_2 . The viscoplastic body is in contact with a foundation over the surface Γ_3 . Assume that the contact process is quasistatic and we study it on the finite time interval $[0, T]$. Use \mathbb{S}^d to denote the linear space of second-order symmetric tensors on \mathbb{R}^d . The classical formulation of the contact problem can be written as follows.

Problem P Find a displacement field $u : \Omega \times (0, T) \rightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0, T) \rightarrow \mathbb{S}^d$ and an internal state variable $\kappa : \Omega \times (0, T) \rightarrow \mathbb{R}^m$ such that

$$\dot{\sigma}(t) = \mathcal{E}(\dot{u}(t)) + \mathcal{G}(\sigma(t), \varepsilon(u(t)), \kappa(t)) \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$\dot{\kappa}(t) = G(\sigma(t), \varepsilon(u(t)), \kappa(t)) \quad \text{in } \Omega \times (0, T), \quad (1.2)$$

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$$\operatorname{Div} \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega \times (0, T), \quad (1.3)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, T), \quad (1.4)$$

$$\sigma(t)v = f_2(t) \quad \text{on } \Gamma_2 \times (0, T), \quad (1.5)$$

$$-\sigma_\tau(t) \in \partial_{\text{CJ}} j_\tau \left(\int_0^t \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_\tau(t) \right) \quad \text{on } \Gamma_3 \times (0, T), \quad (1.6)$$

there exists $\xi : \Gamma_3 \times (0, T) \rightarrow \mathbb{R}$ which satisfies

$$\left. \begin{aligned} u_v(t) &\leq g, \quad \sigma_v(t) + p(u_v(t)) + \xi(t) \leq 0, \\ (u_v(t) - g)(\sigma_v(t) + p(u_v(t)) + \xi(t)) &= 0, \\ 0 &\leq \xi(t) \leq \int_0^t b(t-s)u_v^+(s)ds, \\ \xi(t) &= 0 \quad \text{if } u_v(t) < 0, \\ \xi(t) &= \int_0^t b(t-s)u_v^+(s)ds \quad \text{if } u_v(t) > 0, \end{aligned} \right\} \quad \text{on } \Gamma_3 \times (0, T), \quad (1.7)$$

and moreover

$$u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad \kappa(0) = \kappa_0 \quad \text{in } \Omega. \quad (1.8)$$

Here $\varepsilon(u)$ denotes the (small) linearized strain tensor. Its components are given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, where $u_{i,j} = \partial u_i / \partial x_j$. The dot above a variable represents the derivative with respect to the time variable t . $\operatorname{Div} \sigma = (\sigma_{ij,j})$ stands for the divergence of a stress tensor σ , and the convention of summation over repeated indices is used in this paper. Denote $u^+ = \max\{0, u\}$. We do not indicate the dependence on the spatial variable x in order to simplify the notation.

Eqs. (1.1) and (1.2) show the rate-type viscoplastic constitutive law with internal state variable κ . Operator \mathcal{E} is linear and describes the elastic properties of the material whereas \mathcal{G} is a nonlinear constitutive function which describes the viscoplastic behavior. The internal state variable κ is a vector-valued function and its evolution is governed by differential Eq. (1.2). G is a nonlinear constitutive function with values in \mathbb{R}^m and m is a positive integer. Viscoplastic models can be used to describe the behavior of real materials like rubbers, metals, rocks and so on. For the comprehensive studies, we refer to Refs. [5–7].

Eq. (1.3) is the normalized equilibrium equation. Eqs. (1.4) and (1.5) are the displacement and the traction boundary conditions, respectively. The multivalued friction law (1.6) is a total slip-dependent version of Coulomb's law with σ_τ being the tangential stress. It is modeled by the Clarke subdifferential of a function which is locally Lipschitz and, in general, nonconvex in its second variable. Function j_τ depends on the accumulated slip over the whole time interval $[0, t]$ since the current roughness of the contact surface Γ_3 could depend on the accumulation of history sliding. Concrete examples of this Coulomb's law are provided in Section 6 of [8]. Condition (1.7) models the contact with normal compliance, unilateral constraint and memory term. We use σ_v and u_v to represent the normal stress and normal displacement, respectively. Function p is a Lipschitz continuous increasing function which vanishes for a negative argument, function b is positive and $g > 0$. The condition shows that the normal stress vanishes when there is a separation between the viscoplastic body and the foundation. When there is a penetration, the contact follows normal compliance condition with memory term ξ and once the penetration limit g is reached, the contact follows the Signorini unilateral condition. The initial conditions are given in (1.8).

Problem P has been investigated in [4]. The weak formulation is derived and the existence of a unique weak solution is proved as the main result. Our paper is a continuation of [4]. We propose a fully discrete scheme for the weak formulation and present an error estimate. Under appropriate regularity assumptions, the error estimate is of optimal-order. Theoretical results are also illustrated numerically.

Compared with the viscoelastic material [9–13], the viscoplastic material has a different form of constitutive law. As Eqs. (1.1) shows, it has an implicit expression of stress field σ . Consequently, the weak formulation of the viscoplastic contact model cannot be decoupled and we have to consider both u and σ , bringing difficulties to unique solvability and numerical analysis. On the other hand, the weak formulation of such models naturally includes a history-dependent term.

Publications devoted to hemivariational inequalities with viscoplastic materials are limited and we list them in the following. The unique solvability of viscoplastic contact models with subdifferential boundary conditions can be found in [4, 14–17]. [14] concerns a dynamic contact problem. In [15], two history-dependent hemivariational inequalities, one of which can model a quasistatic viscoplastic contact, are proved to have a unique weak solution. [16] considers a dynamic contact problem with damage. A quasistatic contact with memory term is studied in [4], and moreover the contact with memory and damage terms is studied in [17]. To our knowledge, there are few papers devoted to numerical analysis. The only one found is [18], which presents a numerical approximation of a quasistatic frictionless contact. The novelty of the present paper compared with [18] lies in two aspects. The model considered here also takes into account: (i) internal state κ , (ii) memory term on Γ_3 , (iii) friction which is under a total slip-dependent version of Coulomb's law. Moreover, we present a more general numerical example to illustrate the theoretical results.

The paper is structured as follows. In Section 2, we review some preliminary material. Recall from [4], we list assumptions on the data, present the weak formulation of Problem P and show the existence and uniqueness result.

In Section 3, we introduce a fully discrete scheme for the weak formulation, derive an error bound and show that the error estimate is of optimal-order under suitable regularity assumptions. Finally, in Section 4, we provide a numerical example.

2. Preliminaries

In this section, we recall notation and a result on unique solvability.

Let X be a Banach space. Let $\varphi : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. From [3], the generalized directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

The generalized gradient of φ at x , denoted by $\partial_{CI}\varphi(x)$, is a subset of a dual space X^* given by $\partial_{CI}\varphi(x) = \{\zeta \in X^* | \varphi^0(x; v) \leq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}$. We also present two useful properties [19]:

$$\varphi^0(x; v) = \max\{\langle \zeta, v \rangle | \zeta \in \partial_{CI}\varphi(x)\}, \quad (2.1)$$

$$\varphi^0(x; v_1 + v_2) \leq \varphi^0(x; v_1) + \varphi^0(x; v_2). \quad (2.2)$$

The inner products and the corresponding norms on \mathbb{R}^d and \mathbb{S}^d are given by

$$u \cdot v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \quad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

Since Γ is the Lipschitz continuous boundary of Ω , the unit outward normal vector exists a.e. on Γ and is denoted by $\nu = (\nu_i) \in \mathbb{R}^d$. For vector field $v \in \Gamma$, the normal and tangential components of v are $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. Similarly, for tensor field $\sigma : \Omega \rightarrow \mathbb{S}^d$, the normal and tangential components are $\sigma_\nu = (\sigma \nu) \cdot \nu$ and $\sigma_\tau = \sigma \nu - \sigma_\nu \nu$. Note again that the boundary Γ is divided into three mutually disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$.

Introduce the following Hilbert spaces with their inner products

$$V = \{v = (v_i) \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_1\}, \quad (u, v)_V = \int_{\Omega} \varepsilon(u) : \varepsilon(v) dx,$$

$$Q = \{\tau = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) : \tau_{ij} = \tau_{ji}\}, \quad (\tau, \sigma)_Q = \int_{\Omega} \tau : \sigma dx.$$

The associated norms in V and Q are denoted respectively by $\|\cdot\|_V$ and $\|\cdot\|_Q$. Completeness of the space $(V, \|\cdot\|_V)$ follows from the use of Korn's inequality, which is allowed under the assumption $\text{meas}(\Gamma_1) > 0$. We also denote by γ the trace operator from V to $L^2(\Gamma_3; \mathbb{R}^d)$ with the norm $\|\gamma\| = \|\gamma\|_{\mathcal{L}(V, L^2(\Gamma_3; \mathbb{R}^d))}$.

Denote the space of fourth-order tensor fields by

$$Q^\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), 1 \leq i, j, k, l \leq d\}.$$

It is a real Banach space with the norm $\|\mathcal{E}\|_{Q^\infty} = \max_{1 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$. Then

$$\|\mathcal{E}\tau\|_Q \leq d \|\mathcal{E}\|_{Q^\infty} \|\tau\|_Q, \quad \text{for all } \mathcal{E} \in Q^\infty, \tau \in Q.$$

We present the following assumptions on the data of Problem P.

- $\left\{ \begin{array}{l} \text{H}(\mathcal{E}): \text{The elasticity tensor } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ satisfies} \\ \text{(i) } \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), 1 \leq i, j, k, l \leq d; \\ \text{(ii) There exists } m_{\mathcal{E}} > 0 \text{ such that } \mathcal{E}\tau : \tau \geq m_{\mathcal{E}} \|\tau\|_{\mathbb{S}^d}^2 \text{ for all } \tau \in \mathbb{S}^d \text{ and a.e. in } \Omega. \end{array} \right.$
- $\left\{ \begin{array}{l} \text{H}(\mathcal{G}): \text{The constitutive function } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \rightarrow \mathbb{S}^d \text{ satisfies} \\ \text{(i) There exists } L_{\mathcal{G}} > 0 \text{ such that } \|\mathcal{G}(x, \sigma_1, \varepsilon_1, \kappa_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2, \kappa_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{G}}(\|\sigma_1 - \sigma_2\|_{\mathbb{S}^d} + \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \\ \quad + \|\kappa_1 - \kappa_2\|_{\mathbb{R}^m}) \text{ for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ and } \kappa_1, \kappa_2 \in \mathbb{R}^m, \text{ a.e. } x \in \Omega; \\ \text{(ii) The mapping } x \mapsto \mathcal{G}(x, \sigma, \varepsilon, \kappa) \text{ is measurable on } \Omega \text{ for all } \sigma, \varepsilon \in \mathbb{S}^d \text{ and } \kappa \in \mathbb{R}^m; \\ \text{(iii) The mapping } x \mapsto \mathcal{G}(x, 0, 0, 0) \text{ belongs to } Q. \end{array} \right.$
- $\left\{ \begin{array}{l} \text{H}(G): \text{The constitutive function } G : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^m \text{ satisfies} \\ \text{(i) There exists } L_G > 0 \text{ such that } \|G(x, \sigma_1, \varepsilon_1, \kappa_1) - G(x, \sigma_2, \varepsilon_2, \kappa_2)\|_{\mathbb{R}^m} \leq L_G(\|\sigma_1 - \sigma_2\|_{\mathbb{S}^d} + \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} \\ \quad + \|\kappa_1 - \kappa_2\|_{\mathbb{R}^m}) \text{ for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ and } \kappa_1, \kappa_2 \in \mathbb{R}^m, \text{ a.e. } x \in \Omega; \\ \text{(ii) The mapping } x \mapsto G(x, \sigma, \varepsilon, \kappa) \text{ is measurable on } \Omega \text{ for all } \sigma, \varepsilon \in \mathbb{S}^d \text{ and } \kappa \in \mathbb{R}^m; \\ \text{(iii) The mapping } x \mapsto G(x, 0, 0, 0) \text{ belongs to } L^2(\Omega; \mathbb{R}^m). \end{array} \right.$

$H(f)$: The densities of body forces and surface tractions satisfy

$$f_0 \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad f_2 \in C(0, T; L^2(\Gamma_2; \mathbb{R}^d)).$$

$$\left\{ \begin{array}{l} H(p): \text{ The normal compliance function } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+ \text{ satisfies} \\ \text{(i) There exists } L_p > 0 \text{ such that } |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(ii) } (p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0 \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(iii) The mapping } x \mapsto p(x, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}; \\ \text{(iv) } p(x, r) = 0 \text{ for all } r < 0, \text{ a.e. } x \in \Gamma_3. \end{array} \right.$$

$H(b)$: The surface memory function $b \in C(\mathbb{R}_+; L^\infty(\Gamma_3))$ is positive for all $t \in \mathbb{R}_+$ and a.e. $x \in \Gamma_3$.

(H_0) : The initial conditions are $u_0 \in U$, $\sigma_0 \in Q$, $\kappa_0 \in L^2(\Omega, \mathbb{R}^m)$, where U is the set of admissible displacements, i.e.

$$U = \{v \in V : v_\nu \leq g \text{ on } \Gamma_3\}.$$

$$\left\{ \begin{array}{l} H(j_\tau): \text{ The tangential superpotential } j_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ satisfies} \\ \text{(i) } j_\tau(\cdot, r, \xi) \text{ is measurable on } \Gamma_3 \text{ for all } (r, \xi) \in \mathbb{R} \times \mathbb{R}^d \text{ and there exists } e \in L^2(\Gamma_3; \mathbb{R}^d) \text{ such that} \\ \quad j_\tau(\cdot, r, e(\cdot)) \in L^1(\Gamma_3); \\ \text{(ii) } j_\tau(x, r, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } x \in \Gamma_3 \text{ and all } r \in \mathbb{R}; \\ \text{(iii) } \|\partial j_\tau(x, r, \xi)\|_{\mathbb{R}^d} \leq \tilde{c}_0 + \tilde{c}_1 |r| + \tilde{c}_2 \|\xi\|_{\mathbb{R}^d} \text{ for all } (r, \xi) \in \mathbb{R} \times \mathbb{R}^d \text{ and a.e. } x \in \Gamma_3; \\ \text{(iv) There exist constants } \tilde{m}_r > 0 \text{ and } m_j \geq 0 \text{ such that } j_\tau^0(x, r_1, \xi_1; \xi_2 - \xi_1) + j_\tau^0(x, r_2, \xi_2; \xi_1 - \xi_2) \leq \\ \quad \tilde{m}_r |r_1 - r_2| \cdot \|\xi_1 - \xi_2\|_{\mathbb{R}^d} + m_j \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2, \text{ for all } r_1, r_2 \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^d \text{ and a.e. } x \in \Gamma_3. \end{array} \right.$$

Exploiting standard argument, we obtain the weak formulation of Problem P.

Problem P_V . Find a displacement field $u : (0, T) \rightarrow U$, a stress field $\sigma : (0, T) \rightarrow Q$ and an internal state variable $\kappa : (0, T) \rightarrow L^2(\Omega; \mathbb{R}^m)$ such that $u(0) = u_0$, $\sigma(0) = \sigma_0$, $\kappa(0) = \kappa_0$ and

$$\sigma(t) = \int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds + \sigma_0 - \mathcal{E}\varepsilon(u_0) + \mathcal{E}\varepsilon(u(t)), \quad (2.3)$$

$$\kappa(t) = \int_0^t G(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds + \kappa_0, \quad (2.4)$$

$$\begin{aligned} & (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + \left(\int_0^t b(t-s) u_v^+(s) ds, v_v^+ - u_v^+(t) \right)_{L^2(\Gamma_3)} + (p(u_v(t)), v_v - u_v(t))_{L^2(\Gamma_3)} \\ & + \int_{\Gamma_3} j_\tau^0 \left(\int_0^t \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_\tau(t); v_\tau - u_\tau(t) \right) d\Gamma \geq (f_0(t), v - u(t))_{L^2(\Omega; \mathbb{R}^d)} + (f_2(t), v - u(t))_{L^2(\Gamma_2; \mathbb{R}^d)} \end{aligned} \quad (2.5)$$

hold for all $v \in U$ and every $t \in (0, T)$.

By Riesz representation theorem, we define the function $f : (0, T) \rightarrow V$ by

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v d\Gamma \quad \text{for all } v \in V, t \in (0, T).$$

Note that condition $H(j_\tau)(iv)$ is equivalent to the relaxed monotonicity condition of subdifferential

$$(\xi_1^* - \xi_2^*) \cdot (\xi_1 - \xi_2) \geq -\tilde{m}_r |r_1 - r_2| \cdot \|\xi_1 - \xi_2\|_{\mathbb{R}^d} - m_j \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2,$$

for all $r_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^d$, $\xi_i^* \in \partial_{CJ} j_\tau(x, r_i, \xi_i)$, $i = 1, 2$ and a.e. $x \in \Gamma_3$. From [4], we have the existence and uniqueness result for Problem P_V .

Theorem 1. Assume $H(\mathcal{E})$, $H(\mathcal{G})$, $H(G)$, $H(f)$, $H(p)$, $H(b)$, (H_0) , $H(j_\tau)$ and

$$m_\mathcal{E} > \max\{\sqrt{3}\tilde{c}_1, m_j\} \|\gamma\|^2. \quad (2.6)$$

Then Problem P_V has a unique solution with the following regularity

$$u \in C(0, T; U), \quad \sigma \in C(0, T; Q), \quad \kappa \in C(0, T; L^2(\Omega; \mathbb{R}^m)). \quad (2.7)$$

3. A fully discrete scheme and error estimate

In this section, we introduce a fully discrete scheme for Problem P_V and provide a result on error estimate. We recall the discrete Gronwall inequality [20] first.

Lemma 2. For a fixed T , let $0 = t_0 < t_1 < \dots < t_N = T$ and $k_n = t_n - t_{n-1}$ for $n = 1, 2, \dots, N$. Assume $\{g_n\}_{n=0}^N$ and $\{e_n\}_{n=0}^N$ are two sequences of non-negative numbers satisfying

$$e_n \leq c g_n + c \sum_{i=1}^n k_i e_{i-1}, \quad n = 1, \dots, N$$

for a constant $c > 0$. Then, for another constant $c > 0$ independent of N ,

$$\max_{0 \leq n \leq N} e_n \leq c \max_{0 \leq n \leq N} g_n.$$

For simplicity, denote $Y = L^2(\Omega; \mathbb{R}^m)$. Let $V^h \subset V$, $Q^h \subset Q$, $Y^h \subset Y$ be finite-dimensional spaces approximating spaces V , Q and Y . We use $U^h := V^h \cap U$ to approximate the convex set U . Here $h > 0$ denotes the spatial discretization parameter. We assume that

$$\varepsilon(V^h) \subset Q^h. \quad (3.1)$$

This assumption is very natural and is valid as long as the polynomial degree for the space V^h is at most one higher than that for the space Q^h .

We use a possibly non-uniform partition of the time interval $[0, T] : 0 = t_0 < t_1 < t_2 < \dots < t_N = T$. We denote the time step size $k_n = t_n - t_{n-1}$ and the maximal time step size $k = \max_n k_n$ for $n = 1, \dots, N$. For a continuous function $g = g(t)$, we write $g_n = g(t_n)$. Everywhere in the sequel, c denotes a positive constant, whose value may change from line to line and is independent of the discretization parameters h and k .

Let $\mathcal{P}_{Q^h} : Q \rightarrow Q^h$ and $\mathcal{P}_{Y^h} : Y \rightarrow Y^h$ be the orthogonal projections defined through relations

$$\begin{aligned} (\mathcal{P}_{Q^h} q, q^h)_Q &= (q, q^h)_Q & \forall q \in Q, \quad q^h \in Q^h, \\ (\mathcal{P}_{Y^h} \beta, \beta^h)_Y &= (\beta, \beta^h)_Y & \forall \beta \in Y, \quad \beta^h \in Y^h. \end{aligned}$$

The orthogonal projections have useful non-expansive property:

$$\|\mathcal{P}_{Q^h} q\|_Q \leq \|q\|_Q \quad \text{for all } q \in Q, \quad \|\mathcal{P}_{Y^h} \beta\|_Y \leq \|\beta\|_Y \quad \text{for all } \beta \in Y.$$

Choose $u_0^h \in U^h$, $\sigma_0^h \in Q^h$, $\kappa_0^h \in Y^h$ to be approximations of initial values $u_0 \in U$, $\sigma_0 \in Q$, $\kappa_0 \in Y$. We construct the following fully discrete approximation scheme for Problem P_V .

Problem P_V^{hk} Find $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset U^h$, $\sigma^{hk} = \{\sigma_n^{hk}\}_{n=0}^N \subset Q^h$ and $\kappa^{hk} = \{\kappa_n^{hk}\}_{n=0}^N$ such that $u_0^{hk} = u_0^h$, $\sigma_0^{hk} = \sigma_0^h$, $\kappa_0^{hk} = \kappa_0^h$ and for $n = 1, 2, \dots, N$,

$$\sigma_n^{hk} = \sum_{j=1}^n k_j \mathcal{P}_{Q^h} G(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}) + \sigma_0^{hk} - \mathcal{P}_{Q^h} \varepsilon \varepsilon(u_0^{hk}) + \mathcal{P}_{Q^h} \varepsilon \varepsilon(u_n^{hk}), \quad (3.2)$$

$$\kappa_n^{hk} = \sum_{j=1}^n k_j \mathcal{P}_{Y^h} G(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}) + \kappa_0^{hk}, \quad (3.3)$$

$$(\sigma_n^{hk}, \varepsilon(v^h - u_n^{hk}))_Q + \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_v^{h,+} - u_{nv}^{hk,+} \right)_{L^2(\Gamma_3)} + (p(u_{nv}^{hk}), v_v^h - u_{nv}^{hk})_{L^2(\Gamma_3)} \quad (3.4)$$

$$+ \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; v_\tau^h - u_{n\tau}^{hk} \right) d\Gamma \geq (f_n, v^h - u_n^{hk})_V \quad \forall v^h \in U^h.$$

For the existence and uniqueness result of the fully-discrete scheme P_V^{hk} , we need to prove that, with $\{u_j^{hk}\}_{j \leq n-1}$ known, u_n^{hk} is uniquely determined by (3.2)–(3.4). In fact, we only need to consider a history-dependent variational-hemivariational inequality: Find $u_n^{hk} \in U^h$ such that for all $v^h \in U^h$, there holds

$$\begin{aligned} & (\varepsilon \varepsilon(u_n^{hk}), \varepsilon(v^h - u_n^{hk}))_Q + (p(u_{nv}^{hk}), v_v^h - u_{nv}^{hk})_{L^2(\Gamma_3)} + \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; v_\tau^h - u_{n\tau}^{hk} \right) d\Gamma \\ & \geq (f_n, v^h - u_n^{hk})_V - \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_v^{h,+} - u_{nv}^{hk,+} \right)_{L^2(\Gamma_3)} \\ & - \left(\sum_{j=1}^n k_j G(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}), \varepsilon(v^h - u_n^{hk}) \right)_Q - (\sigma_0^{hk} - \varepsilon \varepsilon(u_0^{hk}), \varepsilon(v^h - u_n^{hk}))_Q. \end{aligned}$$

By a discrete analogue of the argument in the proof of Theorem 1, we conclude that this inequality has a unique solution $u_n^{hk} \in U^h$, which implies the unique solvability of our Problem P_V^{hk} .

In order to derive the error estimate, we first prove the following theorem.

Theorem 3. Let $\{u_n^{hk}\}_{n=0}^N$, $\{\sigma_n^{hk}\}_{n=0}^N$ and $\{\kappa_n^{hk}\}_{n=0}^N$ be the unique solution of Problem P_V^{hk} , then there exists a constant $c > 0$ such that

$$\|\sigma_n^{hk}\|_Q + \|u_n^{hk}\|_V + \|\kappa_n^{hk}\|_Y \leq c, \quad 0 \leq n \leq N.$$

Proof. From $H(\mathcal{G})(i)$:

$$\|\mathcal{G}(\sigma, \varepsilon, \kappa)\|_Q \leq \|\mathcal{G}(0, 0, 0)\|_Q + \sqrt{3}L_G(\|\sigma\|_Q + \|\varepsilon\|_Q + \|\kappa\|_Y).$$

From $H(G)(i)$:

$$\|G(\sigma, \varepsilon, \kappa)\|_Y \leq \|G(0, 0, 0)\|_Y + \sqrt{3}L_G(\|\sigma\|_Q + \|\varepsilon\|_Q + \|\kappa\|_Y).$$

From $H(j_\tau)(iv)$:

$$\int_{\Gamma_3} j_\tau^0(r, u_{n\tau}^{hk}; -u_{n\tau}^{hk})d\Gamma + \int_{\Gamma_3} j_\tau^0(r, 0; u_{n\tau}^{hk})d\Gamma \leq m_j \|u_{n\tau}^{hk}\|_{L^2(\Gamma_3; \mathbb{R}^d)}^2 \leq m_j \|\gamma\|^2 \|u_n^{hk}\|_V^2.$$

From $H(j_\tau)(iii)$ and property (2.1):

$$\begin{aligned} & \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1, \tau}^{hk}\|_{\mathbb{R}^d}, 0; u_{n\tau}^{hk} \right) d\Gamma \\ & \geq - \int_{\Gamma_3} (\tilde{c}_0 + \tilde{c}_{1r} \sum_{j=1}^n k_j \|u_{j-1, \tau}^{hk}\|_{\mathbb{R}^d}) \|u_{n\tau}^{hk}\|_{\mathbb{R}^d} d\Gamma \\ & \geq -\tilde{c}_0 \int_{\Gamma_3} \|u_n^{hk}\|_{\mathbb{R}^d} d\Gamma - \tilde{c}_{1r} \int_{\Gamma_3} \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_{\mathbb{R}^d} \right) \|u_n^{hk}\|_{\mathbb{R}^d} d\Gamma \\ & \geq -\tilde{c}_0 \sqrt{\text{meas}(\Gamma_3)} \|\gamma\| \|u_n^{hk}\|_V - \tilde{c}_{1r} \|\gamma\|^2 \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_V \right) \|u_n^{hk}\|_V. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1, \tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; -u_{n\tau}^{hk} \right) d\Gamma \\ & \leq m_j \|\gamma\|^2 \|u_n^{hk}\|_V^2 - \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=0}^n k_j \|u_{j-1, \tau}^{hk}\|_{\mathbb{R}^d}, 0; u_{n\tau}^{hk} \right) d\Gamma \\ & \leq m_j \|\gamma\|^2 \|u_n^{hk}\|_V^2 + \tilde{c}_0 \sqrt{\text{meas}(\Gamma_3)} \|\gamma\| \|u_n^{hk}\|_V + \tilde{c}_{1r} \|\gamma\|^2 \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_V \right) \|u_n^{hk}\|_V. \end{aligned}$$

From (3.2), combined with $H(\mathcal{E})(i)$ and the definition of the norm in space V , we have

$$\begin{aligned} \|\sigma_n^{hk}\|_Q & \leq \sum_{j=1}^n k_j \|\mathcal{G}(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk})\|_Q + \|\sigma_0^{hk}\|_Q + \|\mathcal{E}\varepsilon(u_0^{hk})\|_Q + \|\mathcal{E}\varepsilon(u_n^{hk})\|_Q \\ & \leq c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y) + c \|\mathcal{G}(0, 0, 0)\|_Q + \|\sigma_0^{hk}\|_Q + c \|u_0^{hk}\|_V + c \|u_n^{hk}\|_V \\ & \leq c \|u_n^{hk}\|_V + c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y) + c. \end{aligned}$$

From (3.3), similar arguments lead to

$$\|\kappa_n^{hk}\|_Y \leq c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y) + c.$$

Take $v^h = 0 \in U^h$ and use (3.2) in (3.4):

$$\begin{aligned} & (\mathcal{E}\varepsilon(u_n^{hk}), \varepsilon(u_n^{hk}))_Q + (p(u_{nv}^{hk}), u_{nv}^{hk})_{L^2(\Gamma_3)} \leq -(\sigma_0^{hk} - \mathcal{E}\varepsilon(u_0^{hk}), \varepsilon(u_n^{hk}))_Q \\ & - \left(\sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}), \varepsilon(u_n^{hk}) \right)_Q - \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1, v}^{hk, +}, u_{nv}^{hk, +} \right)_{L^2(\Gamma_3)} \end{aligned}$$

$$+ \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}, u_{n\tau}^{hk}; -u_{n\tau}^{hk} \right) d\Gamma + (f_n, u_n^{hk})_V.$$

From $H(\mathcal{E})(i)(ii)$, $H(p)(ii)(iv)$ and $H(b)$, we have:

$$\begin{aligned} m_\varepsilon \|u_n^{hk}\|_V^2 &\leq \left(\|\sigma_0^{hk}\|_Q + c \|u_0^{hk}\|_V + c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y) \right. \\ &\quad \left. + c \|\mathcal{G}(0, 0, 0)\|_Q \right) \|u_n^{hk}\|_V + \max_{t \in [0, T]} \|b(t)\|_{L^\infty(\Gamma_3)} \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_V \right) \|\gamma\|^2 \|u_n^{hk}\|_V \\ &\quad + m_j \|\gamma\|^2 \|u_n^{hk}\|_V^2 + \tilde{c}_0 \sqrt{\text{meas}(\Gamma_3)} \|\gamma\| \|u_n^{hk}\|_V \\ &\quad + \tilde{c}_{1,r} \|\gamma\|^2 \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_V \right) \|u_n^{hk}\|_V + \|f_n\|_V \|u_n^{hk}\|_V. \end{aligned}$$

From condition (2.6), we can deduce

$$\|u_n^{hk}\|_V \leq c + c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y).$$

As a result,

$$\|u_n^{hk}\|_V + \|\sigma_n^{hk}\|_Q + \|\kappa_n^{hk}\|_Y \leq c + c \sum_{j=1}^n k_j (\|\sigma_{j-1}^{hk}\|_Q + \|u_{j-1}^{hk}\|_V + \|\kappa_{j-1}^{hk}\|_Y).$$

Using Lemma 2, we finish the proof. \square

Now we proceed to derive error estimate. Take the unique solution of Problem P_V at time $t = t_n$, i.e. set $t = t_n$ in (2.3), (2.4) and (2.5):

$$\sigma_n = \int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds + \sigma_0 - \varepsilon \varepsilon(u_0) + \varepsilon \varepsilon(u_n), \quad (3.5)$$

$$\kappa_n = \int_0^{t_n} G(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds + \kappa_0, \quad (3.6)$$

$$\begin{aligned} (\sigma_n, \varepsilon(v) - \varepsilon(u_n))_Q + \left(\int_0^{t_n} b(t_n - s) u_v^+(s) ds, v_v^+ - u_{nv}^+ \right)_{L^2(\Gamma_3)} + (p(u_{nv}), v_v - u_{nv})_{L^2(\Gamma_3)} \\ + \int_{\Gamma_3} j_\tau^0 \left(\int_0^{t_n} \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_{n\tau}; v_\tau - u_{n\tau} \right) d\Gamma \geq (f_n, v - u_n)_V \quad \forall v \in U. \end{aligned} \quad (3.7)$$

Subtracting Eq. (3.2) from Eq. (3.5), we obtain:

$$\begin{aligned} \sigma_n - \sigma_n^{hk} &= \sigma_n - \mathcal{P}_{Q^h} \sigma_n + \mathcal{P}_{Q^h} \sigma_n - \sigma_n^{hk} \\ &= (I_Q - \mathcal{P}_{Q^h})(\sigma_n - \sigma_0) + \mathcal{P}_{Q^h} \varepsilon \varepsilon(u_n - u_n^{hk}) + [\sigma_0 - \sigma_0^{hk} - \mathcal{P}_{Q^h} \varepsilon \varepsilon(u_0 - u_0^{hk})] \\ &\quad + \mathcal{P}_{Q^h} \left[\int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) \right] \\ &\quad + \mathcal{P}_{Q^h} \sum_{j=1}^n k_j [\mathcal{G}(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) - \mathcal{G}(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk})], \end{aligned}$$

where I_Q is the identity operator defined on Q . Denote

$$I_n = \left\| \int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) \right\|_Q, \quad n = 1, \dots, N.$$

Use assumptions $H(\mathcal{E})(i)$, $H(\mathcal{G})(i)$ and the property of projection, we can derive the following inequality:

$$\begin{aligned} \|\sigma_n - \sigma_n^{hk}\|_Q &\leq \|(I_Q - \mathcal{P}_{Q^h})(\sigma_n - \sigma_0)\|_Q + c \|u_n - u_n^{hk}\|_V + \|\sigma_0 - \sigma_0^{hk}\|_Q \\ &\quad + c \|u_0 - u_0^{hk}\|_V + I_n + c \sum_{j=1}^n k_j (\|\sigma_{j-1} - \sigma_{j-1}^{hk}\|_Q + \|u_{j-1} - u_{j-1}^{hk}\|_V + \|\kappa_{j-1} - \kappa_{j-1}^{hk}\|_Y). \end{aligned} \quad (3.8)$$

Similarly, subtracting Eq. (3.3) from Eq. (3.6) and denoting

$$L_n = \left\| \int_0^{t_n} G(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j G(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) \right\|_Y, \quad n = 1, \dots, N,$$

we can derive the following inequality:

$$\begin{aligned} & \|\kappa_n - \kappa_n^{hk}\|_Y \\ &= \|\kappa_n - \mathcal{P}_{Y^h} \kappa_n + \mathcal{P}_{Y^h} \kappa_n - \kappa_n^{hk}\|_Y \\ &\leq \|(I_Y - \mathcal{P}_{Y^h})(\kappa_n - \kappa_0)\|_Y + \|\kappa_0 - \kappa_0^{hk}\|_Y + L_n \\ &\quad + c \sum_{j=1}^n k_j (\|\sigma_{j-1} - \sigma_{j-1}^{hk}\|_Q + \|u_{j-1} - u_{j-1}^{hk}\|_V + \|\kappa_{j-1} - \kappa_{j-1}^{hk}\|_Y), \end{aligned} \quad (3.9)$$

where I_Y is the identity operator defined on Y .

Combining (3.5) and (3.7) with $v = u_n^{hk} \in U^h \subset U$, we have

$$\begin{aligned} & (\varepsilon \varepsilon(u_n), \varepsilon(u_n - u_n^{hk}))_Q \leq \left(\int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds, \varepsilon(u_n^{hk} - u_n) \right)_Q \\ & + \left(\sigma_0 - \varepsilon \varepsilon(u_0), \varepsilon(u_n^{hk} - u_n) \right)_Q + \left(\int_0^{t_n} b(t_n - s) u_v^+(s) ds, u_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} \\ & + (p(u_{nv}), u_{nv}^{hk} - u_{nv})_{L^2(\Gamma_3)} + \int_{\Gamma_3} j_\tau^0 \left(\int_0^{t_n} \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_{n\tau}; u_{n\tau}^{hk} - u_{n\tau} \right) d\Gamma - (f_n, u_n^{hk} - u_n)_V. \end{aligned} \quad (3.10)$$

Combining (3.2) and (3.4) with $v^h = v_n^h \in U^h$, we have

$$\begin{aligned} & -(\varepsilon \varepsilon(u_n^{hk}), \varepsilon(u_n - u_n^{hk}))_Q \\ &= -(\varepsilon \varepsilon(u_n^{hk}), \varepsilon(u_n - v_n^h))_Q - (\varepsilon \varepsilon(u_n^{hk}), \varepsilon(v_n^h - u_n^{hk}))_Q \\ &\leq -(\varepsilon \varepsilon(u_n^{hk}), \varepsilon(u_n - v_n^h))_Q + \left(\sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}), \varepsilon(v_n^h - u_n^{hk}) \right)_Q \\ &+ \left(\sigma_0^{hk} - \varepsilon \varepsilon(u_0^{hk}), \varepsilon(v_n^h - u_n^{hk}) \right)_Q + (p(u_{nv}^{hk}), v_{nv}^h - u_{nv}^{hk})_{L^2(\Gamma_3)} \\ &+ \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_{nv}^{h,+} - u_{nv}^{hk,+} \right)_{L^2(\Gamma_3)} \\ &+ \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; v_{n\tau}^h - u_{n\tau}^{hk} \right) d\Gamma - (f_n, v_n^h - u_n^{hk})_V. \end{aligned} \quad (3.11)$$

Combining (3.10), (3.11), (3.5) and using property (2.2), we obtain the following inequality:

$$\begin{aligned} & m_\varepsilon \|u_n - u_n^{hk}\|_V^2 \\ &\leq (\varepsilon \varepsilon(u_n - u_n^{hk}), \varepsilon(u_n - u_n^{hk}))_Q \\ &\leq (\varepsilon \varepsilon(u_n - u_n^{hk}), \varepsilon(u_n - v_n^h))_Q + \int_{\Gamma_3} j_\tau^0 \left(\int_0^{t_n} \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_{n\tau}; u_{n\tau}^{hk} - u_{n\tau} \right) d\Gamma \\ &+ \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; u_{n\tau} - u_{n\tau}^{hk} \right) d\Gamma + \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; v_{n\tau}^h - u_{n\tau} \right) d\Gamma \\ &+ (\sigma_0 - \sigma_0^{hk} - \varepsilon \varepsilon(u_0 - u_0^{hk}), \varepsilon(u_n^{hk} - v_n^h))_Q + (p(u_{nv}), u_{nv}^{hk} - u_{nv})_{L^2(\Gamma_3)} \\ &+ (p(u_{nv}^{hk}), v_{nv}^h - u_{nv}^{hk})_{L^2(\Gamma_3)} + \left(\int_0^{t_n} b(t_n - s) u_v^+(s) ds, u_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} \\ &+ \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_{nv}^{h,+} - u_{nv}^{hk,+} \right)_{L^2(\Gamma_3)} \\ &+ \left(\int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}^{hk}, \varepsilon(u_{j-1}^{hk}), \kappa_{j-1}^{hk}), \varepsilon(u_n^{hk} - v_n^h) \right)_Q \end{aligned} \quad (3.12)$$

$$+ (f_n, u_n - v_n^h)_V - (\sigma_n, \varepsilon(u_n - v_n^h))_Q.$$

From $H(j_\tau)$, (2.1) and Theorem 3, we get the following two inequalities:

$$\begin{aligned} & \int_{\Gamma_3} j_\tau^0 \left(\int_0^{t_n} \|u_\tau(s)\|_{\mathbb{R}^d} ds, u_{n\tau}; u_{n\tau}^{hk} - u_{n\tau} \right) d\Gamma + \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; u_{n\tau} - u_{n\tau}^{hk} \right) d\Gamma \\ & \leq \tilde{m}_r \int_{\Gamma_3} \left| \int_0^{t_n} \|u_\tau(s)\|_{\mathbb{R}^d} ds - \sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d} \right| \cdot \|u_{n\tau} - u_{n\tau}^{hk}\|_{\mathbb{R}^d} d\Gamma + m_j \int_{\Gamma_3} \|u_{n\tau} - u_{n\tau}^{hk}\|_{\mathbb{R}^d}^2 d\Gamma \\ & \leq c \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u(s) - u_{j-1}\|_V ds \cdot \|u_n - u_n^{hk}\|_V \\ & \quad + c \left(\sum_{j=1}^n k_j \|u_{j-1} - u_{j-1}^{hk}\|_V \right) \|u_n - u_n^{hk}\|_V + m_j \|\gamma\|^2 \|u_n - u_n^{hk}\|_V^2; \\ & \int_{\Gamma_3} j_\tau^0 \left(\sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d}, u_{n\tau}^{hk}; v_{n\tau}^h - u_{n\tau} \right) d\Gamma \\ & \leq \int_{\Gamma_3} [\tilde{c}_0 + \tilde{c}_{1r} \sum_{j=1}^n k_j \|u_{j-1,\tau}^{hk}\|_{\mathbb{R}^d} + \tilde{c}_1 \|u_{n\tau}^{hk}\|_{\mathbb{R}^d}] \cdot \|v_{n\tau}^h - u_{n\tau}\|_{\mathbb{R}^d} d\Gamma \\ & \leq \tilde{c}_0 \sqrt{\text{meas}(\Gamma_3)} \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)} + \tilde{c}_{1r} \|\gamma\| \left(\sum_{j=1}^n k_j \|u_{j-1}^{hk}\|_V \right) \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \\ & \quad + \tilde{c}_1 \|\gamma\| \cdot \|u_n^{hk}\|_V \cdot \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \\ & \leq c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)}. \end{aligned}$$

From $H(p)$ and Theorem 3, we have:

$$\begin{aligned} & (p(u_{nv}), u_{nv}^{hk} - u_{nv})_{L^2(\Gamma_3)} + (p(u_{nv}^{hk}), v_{nv}^h - u_{nv}^{hk})_{L^2(\Gamma_3)} \\ & = -(p(u_{nv}) - p(u_{nv}^{hk}), u_{nv} - u_{nv}^{hk})_{L^2(\Gamma_3)} + (p(u_{nv}^{hk}), v_{nv}^h - u_{nv})_{L^2(\Gamma_3)} \\ & \leq L_p \|\gamma\| \|u_n^{hk}\|_V \cdot \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \\ & \leq c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)}. \end{aligned}$$

From $H(b)$ and Theorem 3, we have:

$$\begin{aligned} & \left(\int_0^{t_n} b(t_n - s) u_v^+(s) ds, u_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} + \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_{nv}^{h,+} - u_{nv}^{hk,+} \right)_{L^2(\Gamma_3)} \\ & \leq \left| \left(\int_0^{t_n} b(t_n - s) u_v^+(s) ds - \sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^+, u_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} \right| \\ & \quad + \left| \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) (u_{j-1,v}^+ - u_{j-1,v}^{hk,+}), u_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} \right| \\ & \quad + \left| \left(\sum_{j=1}^n k_j b(t_n - t_{j-1}) u_{j-1,v}^{hk,+}, v_{nv}^{hk,+} - u_{nv}^+ \right)_{L^2(\Gamma_3)} \right| \\ & \leq c \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n - s) u(s) - b(t_n - t_{j-1}) u_{j-1} ds \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} \cdot \|u_n^{hk} - u_n\|_V \\ & \quad + c \left(\sum_{j=1}^n k_j \|u_{j-1} - u_{j-1}^{hk}\|_V \right) \|u_n^{hk} - u_n\|_V + c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)}. \end{aligned}$$

Denote $R_n(u_n, v_n^h) = (f_n, u_n - v_n^h)_V - (\sigma_n, \varepsilon(u_n - v_n^h))_Q$. Then (3.12) can be written as:

$$\begin{aligned} & (m_{\mathcal{E}} - m_j \|\gamma\|^2) \|u_n - u_n^{hk}\|_V^2 \\ & \leq c \|u_n - v_n^h\|_V \|u_n - u_n^{hk}\|_V + c \left(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u(s) - u_{j-1}\|_V ds \right) \cdot \|u_n - u_n^{hk}\|_V \\ & + c \left(\sum_{j=1}^n k_j \|u_{j-1} - u_{j-1}^{hk}\|_V \right) \|u_n - u_n^{hk}\|_V \\ & + c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)} + c (\|\sigma_0 - \sigma_0^{hk}\|_Q + \|u_0 - u_0^{hk}\|_V) \|u_n^{hk} - v_n^h\|_V \\ & + c \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n - s)u(s) - b(t_n - t_{j-1})u_{j-1} ds \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} \cdot \|u_n^{hk} - u_n\|_V \\ & + c \left[I_n + \sum_{j=1}^n k_j (\|\sigma_{j-1} - \sigma_{j-1}^{hk}\|_Q + \|u_{j-1} - u_{j-1}^{hk}\|_V + \|\kappa_{j-1} - \kappa_{j-1}^{hk}\|_Y) \right] \cdot \|u_n^{hk} - v_n^h\|_V + |R_n(u_n, v_n^h)|. \end{aligned}$$

After some manipulations, we have the following inequality.

$$\begin{aligned} & \|u_n - u_n^{hk}\|_V \tag{3.13} \\ & \leq c \|u_n - v_n^h\|_V + c \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u(s) - u_{j-1}\|_V ds + c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)}^{1/2} \\ & + c (\|\sigma_0 - \sigma_0^{hk}\|_V + \|u_0 - u_0^{hk}\|_V) \\ & + c \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n - s)u(s) - b(t_n - t_{j-1})u_{j-1} ds \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} + c I_n \\ & + c \sum_{j=1}^n k_j (\|\sigma_{j-1} - \sigma_{j-1}^{hk}\|_Q + \|u_{j-1} - u_{j-1}^{hk}\|_V + \|\kappa_{j-1} - \kappa_{j-1}^{hk}\|_Y) + c |R_n(u_n, v_n^h)|^{1/2}. \end{aligned}$$

Combine (3.8), (3.9) and (3.13), we obtain:

$$\begin{aligned} & \|\sigma_n - \sigma_n^{hk}\|_Q + \|u_n - u_n^{hk}\|_V + \|\kappa_n - \kappa_n^{hk}\|_Y \tag{3.14} \\ & \leq \|(I_Q - \mathcal{P}_Q^h)(\sigma_n - \sigma_0)\|_Q + \|(I_Y - \mathcal{P}_Y^h)(\kappa_n - \kappa_0)\|_Y \\ & + c (\|\sigma_0 - \sigma_0^{hk}\|_Q + \|u_0 - u_0^{hk}\|_V + \|\kappa_0 - \kappa_0^{hk}\|_Y) + c I_n + c L_n \\ & + c \sum_{j=1}^n k_j (\|\sigma_{j-1} - \sigma_{j-1}^{hk}\|_Q + \|u_{j-1} - u_{j-1}^{hk}\|_V + \|\kappa_{j-1} - \kappa_{j-1}^{hk}\|_Y) \\ & + c \|u_n - v_n^h\|_V + c \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u(s) - u_{j-1}\|_V ds + c \|v_n^h - u_n\|_{L^2(\Gamma_3; \mathbb{R}^d)}^{1/2} \\ & + c \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n - s)u(s) - b(t_n - t_{j-1})u_{j-1} ds \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} + c |R_n(u_n, v_n^h)|^{1/2}. \end{aligned}$$

We further assume that

$$b \in W^{1,1}(0, T; L^2(\Gamma_3)), u \in W^{1,1}(0, T; V), \sigma \in W^{1,1}(0, T; Q), \kappa \in W^{1,1}(0, T; Y). \tag{3.15}$$

Then it is easy to see that:

$$\begin{aligned} & \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u(s) - u_{j-1}\|_V ds = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\| \int_{t_{j-1}}^s \frac{d}{d\tau}(u(\tau)) d\tau \right\|_V ds \\ & \leq k \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|u'(\tau)\|_V d\tau = k \int_0^{t_n} \|u'(\tau)\|_V d\tau \leq k \|\dot{u}\|_{L^1(0, T; V)} \leq ck ; \\ & \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n - s)u(s) - b(t_n - t_{j-1})u_{j-1} ds \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^s \left\| \frac{d}{d\tau} (b(t_n - s)u(\tau)) \right\|_{L^2(\Gamma_3; \mathbb{R}^d)} d\tau ds \\
&\leq k \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (\|\dot{b}(t_n - \tau)\|_{L^2(\Gamma_3)} \|\gamma\| \|u(\tau)\|_V + \|b(t_n - \tau)\|_{L^2(\Gamma_3)} \|\gamma\| \|\dot{u}(\tau)\|_V) d\tau \\
&\leq ck (\|\dot{b}\|_{L^1(0,T;L^2(\Gamma_3))} \|u\|_{L^1(0,T;V)} + \|b\|_{L^1(0,T;L^2(\Gamma_3))} \|\dot{u}\|_{L^1(0,T;V)}) \leq ck; \\
I_n &= \left\| \int_0^{t_n} \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j \mathcal{G}(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) \right\|_Q \\
&\leq c \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (\|\sigma(s) - \sigma_{j-1}\|_Q + \|u(s) - u_{j-1}\|_V + \|\kappa(s) - \kappa_{j-1}\|_Y) ds \\
&\leq ck (\|\dot{\sigma}\|_{L^1(0,T;Q)} + \|\dot{u}\|_{L^1(0,T;V)} + \|\dot{\kappa}\|_{L^1(0,T;Y)}) \leq ck; \\
L_n &= \left\| \int_0^{t_n} G(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds - \sum_{j=1}^n k_j G(\sigma_{j-1}, \varepsilon(u_{j-1}), \kappa_{j-1}) \right\|_Y \\
&\leq ck (\|\dot{\sigma}\|_{L^1(0,T;Q)} + \|\dot{u}\|_{L^1(0,T;V)} + \|\dot{\kappa}\|_{L^1(0,T;Y)}) \leq ck.
\end{aligned}$$

Noting Lemma 2, we can conclude the following important Theorem.

Theorem 4. Let (u, σ, κ) be the solution of Problem P_V and $\{u_n^{hk}, \sigma_n^{hk}, \kappa_n^{hk}\}_{n=1}^N$ be the solution of Problem P_V^{hk} . Then under the assumptions (3.15), we have the error estimate

$$\begin{aligned}
&\max_{1 \leq n \leq N} (\|\sigma_n - \sigma_n^{hk}\|_Q + \|u_n - u_n^{hk}\|_V + \|\kappa_n - \kappa_n^{hk}\|_Y) \\
&\leq ck + c (\|\sigma_0 - \sigma_0^{hk}\|_Q + \|u_0 - u_0^{hk}\|_V + \|\kappa_0 - \kappa_0^{hk}\|_Y) \\
&\quad + c \max_{1 \leq n \leq N} \left(\|(I_Q - \mathcal{P}_{Q^h})(\sigma_n - \sigma_0)\|_Q + \|(I_Y - \mathcal{P}_{Y^h})(\kappa_n - \kappa_0)\|_Y \right. \\
&\quad \left. + \inf_{v_n^h \in U^h} \{ \|u_n - v_n^h\|_V + \|u_n - v_n^h\|_{L^2(\Gamma_3; \mathbb{R}^d)}^{1/2} + |R_n(u_n, v_n^h)|^{1/2} \} \right).
\end{aligned} \tag{3.16}$$

Theorem 4 is the basis of error analysis for the fully discrete scheme. For further study, we assume that Ω is a polygonal or polyhedral domain. Then we express the boundary as unions of closed flat components with disjoint interiors:

$$\bar{\Gamma}_k = \bigcup_{i=1}^{I_k} \Gamma_{k,i}, \quad 1 \leq k \leq 3.$$

Let \mathcal{T}^h be a regular finite element partition on Ω in such a way that if a side of an element lies on the boundary Γ_k , $1 \leq k \leq 3$, then the side is entirely in one of the subsets $\bar{\Gamma}_{k,i}$, $1 \leq i \leq I_k$. We now specify the finite element spaces V^h , Q^h and Y^h . According to assumption (3.1), we naturally use continuous linear elements for the finite element space V^h and piecewise constants for Q^h and Y^h .

Theorem 5. Assume g is concave on each line segment of Γ_3 . Let (u, σ, κ) be the solution of Problem P_V and $\{u_n^{hk}, \sigma_n^{hk}, \kappa_n^{hk}\}_{n=1}^N$ be the solution of Problem P_V^{hk} . Assume (3.15) and

$$\begin{aligned}
&u \in C(0, T; H^2(\Omega; \mathbb{R}^d)), \quad \sigma \in C(0, T; H^1(\Omega; \mathbb{S}^d)), \quad \kappa \in C(0, T; H^1(\Omega; \mathbb{R}^m)), \\
&u|_{\Gamma_{3,i}} \in C(0, T; H^2(\Gamma_{3,i}; \mathbb{R}^d)), \quad 1 \leq i \leq I.
\end{aligned}$$

The initial values $u_0^h \in U^h$, $\sigma_0^h \in Q^h$, $\kappa_0^h \in Y^h$ are chosen in such a way that

$$\|\sigma_0 - \sigma_0^h\|_Q \leq ch, \quad \|u_0 - u_0^h\|_V \leq ch, \quad \|\kappa_0 - \kappa_0^h\|_Y \leq ch.$$

Then we have the optimal order error estimate

$$\max_{1 \leq n \leq N} (\|\sigma_n - \sigma_n^{hk}\|_Q + \|u_n - u_n^{hk}\|_V + \|\kappa_n - \kappa_n^{hk}\|_Y) \leq c(k + h). \tag{3.17}$$

Proof. For $t \in [0, T]$, let $\Pi^h u(t) \in V^h$ be the piecewise linear interpolant of $u(t)$. Since g is concave on each line segment of Γ_3 , we have $\Pi^h u(t) \in U$. Then we have the error estimates [21]:

$$\|u(t) - \Pi^h u(t)\|_V \leq ch \|u(t)\|_{H^2(\Omega; \mathbb{R}^d)},$$

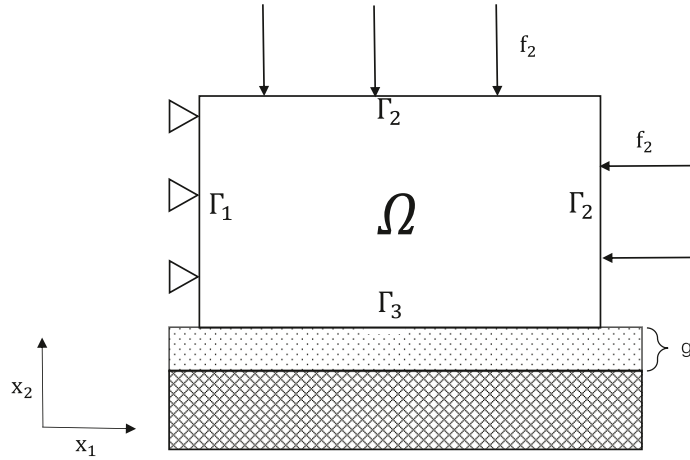


Fig. 1. Initial configuration of the contact.

$$\begin{aligned} \|u(t) - \Pi^h u(t)\|_{L^2(\Gamma_3; \mathbb{R}^d)} &\leq ch^2 \sum_{i=1}^I \|u(t)\|_{H^2(\Gamma_{3,i}; \mathbb{R}^d)}, \\ \|(I_Q - \mathcal{P}_{Q^h})\sigma(t)\|_Q &\leq ch \|\sigma(t)\|_{H^1(\Omega; \mathbb{S}^d)}, \\ \|(I_Y - \mathcal{P}_{Y^h})\kappa(t)\|_Y &\leq ch \|\kappa(t)\|_{H^1(\Omega; \mathbb{R}^m)}. \end{aligned}$$

Thus

$$\begin{aligned} \|(I_Q - \mathcal{P}_{Q^h})(\sigma_n - \sigma_0)\|_Q &\leq ch \|\sigma_n - \sigma_0\|_{H^1(\Omega; \mathbb{S}^d)}, \\ \|(I_Y - \mathcal{P}_{Y^h})(\kappa_n - \kappa_0)\|_Y &\leq ch \|\kappa_n - \kappa_0\|_{H^1(\Omega; \mathbb{R}^m)}. \end{aligned}$$

Noting that $\sigma(t) \in H^1(\Omega; \mathbb{S}^d)$ implies $\sigma(t)v \in L^2(\Gamma; \mathbb{R}^d)$ [11]. Using integration by parts, we have

$$\begin{aligned} &|R(t; u(t), \Pi^h u(t))| \\ &= |(f(t), u(t) - \Pi^h u(t))_V - (\sigma(t), \varepsilon(u(t) - \Pi^h u(t)))_Q| \\ &= \left| \int_{\Gamma_3} (\sigma(t)v) \cdot (u(t) - \Pi^h u(t)) d\Gamma \right| \\ &\leq c \|u(t) - \Pi^h u(t)\|_{L^2(\Gamma_3; \mathbb{R}^d)}. \end{aligned}$$

Combining above inequalities, we obtain the error estimate (3.17). \square

4. Numerical results

In this section, some numerical results are presented to show the solution of Problem P_V^{hk} . We use iterative scheme which is based on the primal–dual active set approach to solve the discrete problem. More details can be found in [22].

Note that in Problem P_V , three history-dependent terms are included in the variational–hemivariational inequality, the $\int_0^t \mathcal{G}(\sigma(s), \varepsilon(u(s)), \kappa(s)) ds$ in rate-type viscoplastic constitutive law (2.3), the memory term ξ and the accumulated slip term in frictional function j_τ . In the following numerical example, we only take into account the first history-dependent term in order to simplify the computation. Then the variational–hemivariational inequality (2.5) is simplified to

$$\begin{aligned} &(\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q + (p(u_v(t)), v_v - u_v(t))_{L^2(\Gamma_3)} + \int_{\Gamma_3} j_\tau^0(u_\tau(t); v_\tau - u_\tau(t)) d\Gamma \\ &\geq (f_0(t), v - u(t))_{L^2(\Omega; \mathbb{R}^d)} + (f_2(t), v - u(t))_{L^2(\Gamma_2; \mathbb{R}^d)} \quad \forall v \in U. \end{aligned}$$

The physical setting of the contact is depicted in Fig. 1. Let $\Omega = (0, L_1) \times (0, L_2)$ be the rectangle with a boundary Γ which is divided into three parts

$$\Gamma_1 = \{0\} \times [0, L_1], \Gamma_2 = (\{L_1\} \times [0, L_2]) \cup ((0, L_1) \times \{L_2\}), \Gamma_3 = (0, L_1) \times \{0\}.$$

The domain Ω represents the cross-section of a three-dimensional viscoplastic body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed.

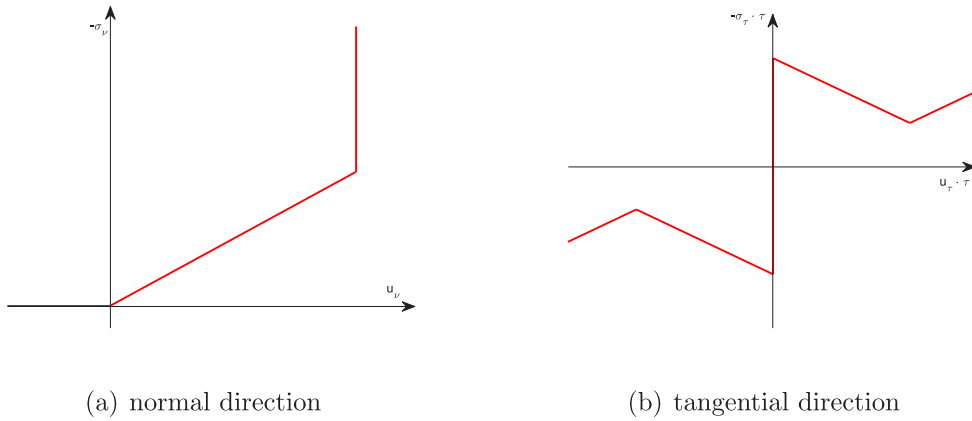


Fig. 2. Graph of the multivalued boundary condition.

Contact boundary conditions on Γ_3 are characterized as follows.

$$-\sigma_v = \begin{cases} 0 & \text{if } u_v \leq 0, \\ c_v u_v & \text{if } u_v \in (0, g), \\ [c_v g, +\infty] & \text{if } u_v = g, \end{cases} \quad (4.1)$$

$$-\sigma_\tau \cdot \tau = \begin{cases} [-b, b]\tau & \text{if } \|u_\tau\| = 0, \\ (b - c_\tau^1 \|u_\tau\|) \frac{u_\tau}{\|u_\tau\|} & \text{if } 0 < \|u_\tau\| < u_\tau^1, \\ (c_\tau^2 (\|u_\tau\| - u_\tau^1) + (b - c_\tau^1 u_\tau^1)) \frac{u_\tau}{\|u_\tau\|} & \text{if } \|u_\tau\| \geq u_\tau^1. \end{cases} \quad (4.2)$$

To better appreciate the character, we show the dependence of $-\sigma_v$ as a function of the normal displacement u_v related to (4.1), and the dependence of $-\sigma_\tau \cdot \tau$ as a function of the tangential displacement $u_\tau \cdot \tau$ related to (4.2), see Fig. 2. So we have the multivalued term in both tangential direction and normal direction. Moreover in the tangential direction, the boundary relation is non-monotone.

Remark. Note that $-\sigma_v$ can take the value from $[c_v g, +\infty]$ when $u_v = g$. In order to better observe the monotone behavior in the normal direction, we made some modification when plotting the contact interface forces $-\sigma_v$. If u_v reaches g , no matter what the value of the contact interface force $-\sigma_v$ is, we plot it as $c_v g$.

The elasticity tensor \mathcal{E} is in the form of

$$(\mathcal{E}\tau)_{ij} = \frac{E\varsigma}{1-\varsigma^2}(\tau_{11} + \tau_{22})\delta_{ij} + \frac{E}{1+\varsigma}\tau_{ij}, \quad 1 \leq i, j \leq 2.$$

The notation E denotes the Young modulus, ς denotes the Poisson ratio of the material and δ_{ij} denotes the Kronecker symbol. The viscoplastic constitutive function \mathcal{G} was assumed to be of the Perzyna type [6]:

$$\mathcal{G}(\sigma, \varepsilon(u), \kappa) = -\frac{1}{2\lambda_1}\mathcal{E}(\sigma - \mathcal{P}_{K(\kappa)}\sigma),$$

where $\lambda_1 > 0$ is the viscosity coefficient. $\mathcal{P}_{K(\kappa)}$ is the orthogonal projection operator (with respect to the norm $\|\tau\| = (\mathcal{E}\tau, \tau)^{1/2}$) over the convex subset $K \subset \mathbb{S}^2$ defined by

$$K(\kappa) = \{\tau \in \mathbb{S}^2 : \|\tau\|_{VM} \leq \kappa\},$$

where $\|\cdot\|_{VM}$ represents the von Mises stress norm

$$\|\tau\|_{VM}^2 = \tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2.$$

Moreover, the function G in Eq. (1.2) is given by

$$G(\sigma, \varepsilon(u), \kappa) = \frac{2}{3\lambda_2} \|\sigma - \mathcal{P}_{K(\kappa)}\sigma\|_{VM}.$$

We take the following data in our numerical experiments:

$$L_1 = 1 \text{ m}, \quad L_2 = 1 \text{ m}, \quad T = 1 \text{ s},$$

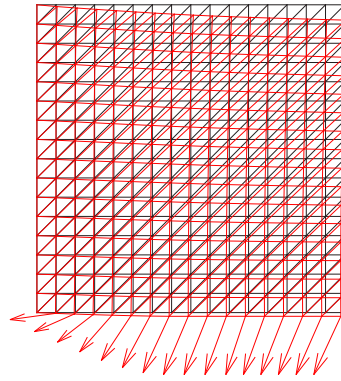
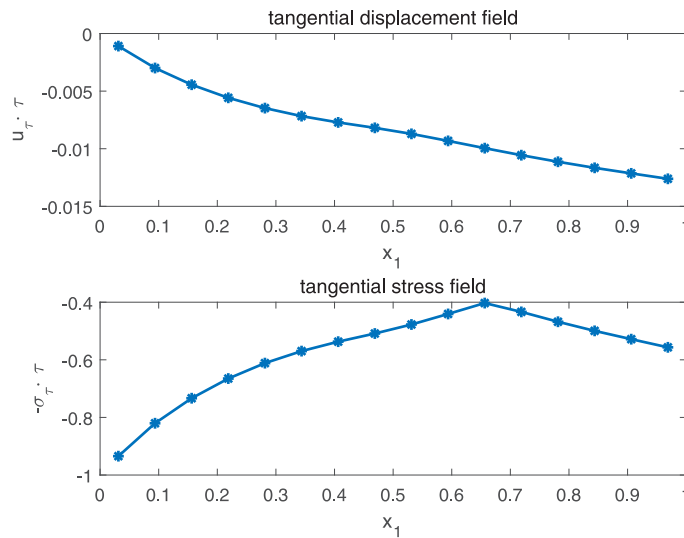


Fig. 3. Deformed meshes and contact interface forces.

Fig. 4. Tangential displacement and stress on contact boundary Γ_3 .

$$\begin{aligned}
 E &= 1000 \text{ N/m}^2, & \varsigma &= 0.4, & \lambda_1 &= 100 \text{ N s/m}^2, & \lambda_2 &= 100 \text{ s}, \\
 g &= 0.012 \text{ m}, & c_v &= 100 \text{ N/m}^2, \\
 b &= 1 \text{ N/m}, & c_\tau^1 &= c_\tau^2 = 60 \text{ N/m}^2, & u_\tau^1 &= 0.01 \text{ m},
 \end{aligned}$$

$$f_0(t) = (0, 0) \text{ N/m}^2, \quad f_2(t) = \begin{cases} (0, -8) \text{ N/m} & \text{on } (0, L_1) \times \{L_2\}, \\ (-3.2, 0) \text{ N/m} & \text{on } \{L_1\} \times [0, L_2], \end{cases}$$

$$\sigma_0 = 0 \text{ N/m}^2, \quad u_0 = 0 \text{ m}, \quad \kappa_0 = 0 \text{ N/m}^2.$$

We use a uniform partition of the time interval $[0, T]$ with time step being $k = \frac{1}{16}$. The spatial step is $h = \frac{1}{16}$.

Our results are presented in Figs. 3–5 and Table 1. They are described in the following.

First, in Fig. 3, the deformed configuration as well as the contact interface forces is plotted at $T = 1$ s. In the normal direction on Γ_3 , part of the nodes are in the status of normal compliance and part of them reach the bound g . When the nodes do not reach the bound, we can observe that the interface forces increase with respect to the penetration. This agrees with the theory since in normal direction we are in the monotone case.

In the tangential direction on Γ_3 , we have $0 > u_\tau \cdot \tau > -u_\tau^1$ for part of the nodes and $u_\tau \cdot \tau \leq -u_\tau^1$ for the others, see Fig. 4. It is shown that $|\sigma_\tau \cdot \tau|$ is firstly decreasing and then increasing with respect to $|u_\tau \cdot \tau|$. This also agrees with the theory since in tangential direction we are in the non-monotone case.

We also show the VM norm of stress σ and the value of internal state κ in Fig. 5.

Use the numerical solution with $h = \frac{1}{128}$, $k = \frac{1}{128}$ as the ‘exact’ solution. The numerical error is defined by

$$\text{Error} := \max_{0 \leq n \leq N} \{ \|\sigma_n - \sigma_n^{hk}\|_Q + \|u_n - u_n^{hk}\|_V + \|\kappa_n - \kappa_n^{hk}\|_Y \}.$$

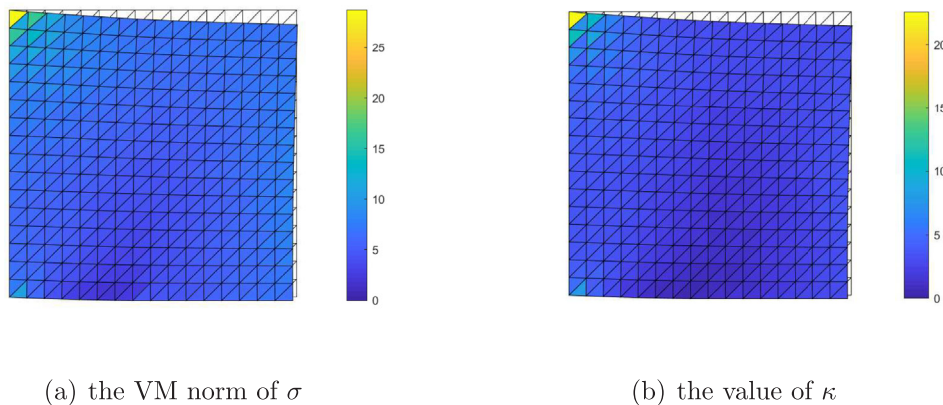


Fig. 5. Graph of the $\|\sigma\|_{VM}$ and κ .

Table 1

Convergence rate with respect to spatial step h and time step τ .

h	1/8	1/16	1/32	1/64
τ	1/8	1/16	1/32	1/64
Error	23.1363	19.6575	13.2976	6.1820
Convergence rate	–	0.2351	0.5639	1.1050

The numerical results are presented in Table 1, where the dependence of the Error with respect to h and k are showed. Asymptotic first-order convergence for spatial and time discretization is observed, which illustrates the previous theoretical results.

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