# A GENERALIZED WEAK GALERKIN METHOD FOR OSEEN EQUATION 


#### Abstract

In this work, the authors introduce a generalized weak Galerkin (gWG) finite element method for the time-dependent Oseen equation. The generalized weak Galerkin method is based on a new framework for approximating the gradient operator. Both a semi-discrete and a fully-discrete numerical scheme are developed and analyzed for their convergence, stability, and error estimates. A generalized inf-sup condition is developed to assist the convergence analysis. The backward Euler discretization is employed in the design of the fully-discrete scheme. Error estimates of optimal order are established mathematically, and they are validated numerically with some benchmark examples.


Key words. Generalized weak gradient, Oseen equation, backward Euler, error estimates.

AMS subject classifications. Primary 65N30; Secondary 65N50

1. Introduction. Navier-Stokes equation has been used to describe incompressible viscous flow and forms the backbone of fluid mechanics [25]. One of the difficulties in the equation is the nonlinear convective acceleration term which involves the product of the velocity with its gradient. One approximation to the Navier-Stokes equation is the Stokes equation where this convective term is omitted. Oseen equation, in comparison to Stokes flow, includes a partial inclusion of the convective acceleration and a reactive term in the momentum equations [11, 22]. Oseen equation is also considered as an auxiliary problem to linearize Navier-Stokes equation [23, 15, 31, 26,

The research on the development, analysis and application of the finite element method for the Oseen equation continues to receive attention in the computational sciences community. In [16], the framework of finite element method for Oseen equation was established and some stabilized schemes were analyzed. A general a posteriori error estimator was designed for problems with incompressibility constraint and shown to be valid for the Oseen equation in [2]. The local discontinuous Galerkin method using mixed setting has been introduced for the steady Oseen equation in [12] and the optimal convergence was derived. A stabilized finite element method was derived for Oseen equation in [5, 6] through the introduction of a stabilization parameter and mesh dependent term. In [8, with the estimates independent of local Reynolds number, the interior penalty finite element method was discussed for steady-state Oseen equation. The space-time discontinuous Galerkin discretization was introduced and analyzed for the evolutionary Oseen equation with time-dependent flow domain in [27]. By using subgrid scale methods, a stabilized finite element formulation was presented for Oseen equation in [13, which also proved the stability and optimal convergence. For optimal control of the Oseen equation, SUPG/PSPG were analyzed in [7], and a priori analysis was provided. In view of stabilization tensor, the hybridizable

[^0]discontinuous Galerkin was provided for steady Oseen equation in [10. A pressure stabilized finite element scheme was estimated for the evolutionary Oseen equation in [21. Least-squares finite element method was presented in [9] where the convergence in various norms was proved. Based on a residual type a posteriori error estimator and pressure projection method, a stabilized finite volume method was analyzed in [20]. In [17, a divergence-conforming discontinuous method for steady Oseen equation was established and a fully robust posteriori error estimator was derived. Based on the skew-symmetry scheme and two grid discretizations, local and parallel finite element algorithms were established for the evolutionary Oseen equation in [14], and error estimates for semi-discrete and fully-discrete were derived. A multiscale hybrid-mixed method was presented for Oseen equation in [3], and a posteriori error estimator was analyzed for the adaptive method.

Weak Galerkin method was first introduced for elliptic problem in [28, 29, and weak function was presented for the definition of weak gradient operator. Weak Galerkin method has been used for Stokes equation in [30] with the use of weak gradient and weak divergence. A weak Galerkin method with skew-symmetry scheme was introduced in [18] for steady-state Oseen equation by using a weak convective operator. By introducing weak trilinear term, a skew-symmetry weak Galerkin method was analyzed for Navier-Stokes equation in [19], and linearized Oseen approximation was employed in numerical experiments. For extending the finite element piecewise polynomial spaces to arbitrary finite dimensional spaces, the generalized weak Galerkin method (gWG) has been introduced with new generalized weak gradient definition. In particular, gWG has been established for steady Stokes equation in [24] with the finite element spaces of arbitrary combination of polynomials. Based on the same spaces as [24], we consider the steady and evolutionary Oseen equation by using gWG scheme in this work. The main contribution of this work includes the development of a scheme that is proposed from the equation directly without the skew-symmetry terms, and the coercivity of bilinear form which is derived for the uniqueness and existence of solutions.

This work is organized as follows. In Section 2 generalized weak gradient and the related weak divergence are presented. The semi-discrete and fully-discrete gWG schemes for the evolutionary Oseen equation are established with backward Euler for time discretization. In Section 3 the gWG scheme is introduced for steady-state Oseen equation firstly, and optimal convergence orders are obtained. Then, for evolutionary case, the error estimate is presented by introducing an elliptic projection operator in Section 4 Finally, numerical examples for benchmark problems are shown to verify the theoretical results in Section 5 .

In this article, we adopt the notations in [1] for Sobolev spaces and denote generic nonnegative constants by $C$ which is independent of the mesh size and time step.
2. gWG for Evolutionary Oseen Equation. We consider the evolutionary Oseen equation that seeks the unknown fluid velocity $\mathbf{u}(t):=\mathbf{u}(\cdot, t)$ and the pressure $p(t):=p(\cdot, t)$ such that

$$
\begin{align*}
\rho \mathbf{u}_{t}-\mu \Delta \mathbf{u}+\rho(\beta \cdot \nabla) \mathbf{u}+\nabla p & =\mathbf{f}, & & \text { in } \Omega \times(0, \bar{T}], \\
\nabla \cdot \mathbf{u} & =0, & & \text { in } \Omega \times(0, \bar{T}],  \tag{2.1}\\
\mathbf{u} & =\mathbf{g}, & & \text { on } \partial \Omega \times(0, \bar{T}], \\
\mathbf{u}(0) & =\mathbf{g}_{2}, & & \text { in } \Omega,
\end{align*}
$$

where $\Omega$ is an open bounded polygonal or polyhedral domain in $\mathbb{R}^{d}, d=2,3$. The external force field $\mathbf{f}$, the admitted flux $\mathbf{g}$ across $\partial \Omega$ and the initial velocity $\mathbf{g}_{2}$ are given functions. $\beta$ is a given convection field, $\mu>0$ is the constant kinematic viscosity and $\rho \geq 0$ is a given scalar function. When $\rho=0$, (2.1) is steady-state Stokes equation which has been analyzed with gWG in [24, and we consider $\rho>0$ in this paper.

The weak form of problem (2.1) reads as follows: find $\mathbf{u}(t) \in\left[H^{1}(\Omega)\right]^{d}$ and $p(t) \in L_{0}^{2}(\Omega)$ satisfying $\mathbf{u}(t)=\mathbf{g}$ on $\partial \Omega$ and $\mathbf{u}(0)=\mathbf{g}_{2}$ in $\Omega$ such that

$$
\begin{align*}
\rho\left(\mathbf{u}_{t}, \mathbf{v}\right)+\mu(\nabla \mathbf{u}, \nabla \mathbf{v})+\rho((\beta \cdot \nabla) \mathbf{u}, \mathbf{v})-(p, \nabla \cdot \mathbf{v}) & =(\mathbf{f}, \mathbf{v}), & & \forall \mathbf{v} \in\left[H_{0}^{1}(\Omega)\right]^{d},  \tag{2.2}\\
(\nabla \cdot \mathbf{u}, q) & =0, & & \forall q \in L_{0}^{2}(\Omega) .
\end{align*}
$$

Here, we denote the spaces

$$
\begin{aligned}
{\left[H_{0}^{1}(\Omega)\right]^{d} } & =\left\{\mathbf{v} \in\left[H^{1}(\Omega)\right]^{d}, \mathbf{v}=\mathbf{0} \text { on } \partial \Omega\right\} \\
L_{0}^{2}(\Omega) & =\left\{q \in L^{2}(\Omega), \int_{\Omega} q d x=0\right\}
\end{aligned}
$$

Denote the partition using polygon or polyhedra of domain $\Omega$ by $\mathcal{T}_{h}$ which satisfy the shape regular conditions in [29]. For each $T \in \mathcal{T}_{h}, h_{T}$ is its diameter and $h=$ $\max _{T \in \mathcal{T}_{h}} h_{T}$ is mesh size of the partition $\mathcal{T}_{h}$. Let $\partial T$ be the edges of element $T$ and $\mathcal{E}_{I}$ the set of interior edges. Denote by $P_{k}(T)$ the polynomials space of degree less than or equal to $k$ in all variables on element $T$. We consider the finite element spaces

$$
\begin{aligned}
\mathbf{V}_{h} & :=\left\{\left(\mathbf{v}_{0}, \mathbf{v}_{b}\right):\left.\mathbf{v}_{0}\right|_{T} \in\left[P_{k}(T)\right]^{d},\left.\mathbf{v}_{b}\right|_{\partial T} \in\left[P_{j}(\partial T)\right]^{d}, \forall T \in \mathcal{T}_{h}\right\} \\
W_{h} & :=\left\{q \in L_{0}^{2}(\Omega):\left.q\right|_{T} \in P_{n}(T), \forall T \in \mathcal{T}_{h}\right\}
\end{aligned}
$$

where $k, j, n \geq 0$ are arbitrary non-negative integers. Denote the subspace of $\mathbf{V}_{h}$ with vanishing boundary condition by $\mathbf{V}_{h}^{0}$, i.e. $\mathbf{V}_{h}^{0}=\left\{\mathbf{v} \in \mathbf{V}_{h}, \mathbf{v}_{b}=\mathbf{0}\right.$ on $\left.\partial \Omega\right\}$.

We define the local $L^{2}$ projection operators $Q_{0}:\left[L^{2}(T)\right]^{d} \rightarrow\left[P_{k}(T)\right]^{d}$ and $Q_{b}$ : $\left[L^{2}(\partial T)\right]^{d} \rightarrow\left[P_{j}(\partial T)\right]^{d}$. Denote by $Q_{h}=\left\{Q_{0}, Q_{b}\right\}$ the projection operator to the space of weak functions and by $Q_{h}^{p}$ the $L^{2}$ projection operator onto $W_{h}$.

Next, based on the weak function and projection operators, we define a weak gradient.

Definition 2.1. For each $\mathbf{v} \in \mathbf{V}_{h}$, a generalized discrete weak gradient operator $\nabla_{w} \mathbf{v}$ is defined as follows

$$
\nabla_{w} \mathbf{v}=\nabla \mathbf{v}_{0}+\delta_{w} \mathbf{v}
$$

where $\left.\delta_{w} \mathbf{v}\right|_{T} \in\left[P_{l}(T)\right]^{d \times d}$ is given by

$$
\left(\delta_{w} \mathbf{v}, \phi\right)_{T}=\left\langle\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}, \phi \cdot \mathbf{n}\right\rangle_{\partial T}, \forall \phi \in\left[P_{l}(T)\right]^{d \times d}
$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial T$.
In order to present the numerical scheme for Oseen equation, we also need to define a weak divergence operator.

Definition 2.2. For each $\mathbf{v} \in \mathbf{V}_{h}$, we define the weak divergence $\nabla_{w} \cdot \mathbf{v} \in P_{m}(T)$ such that

$$
\left(\nabla_{w} \cdot \mathbf{v}, \psi\right)_{T}=-\left(\mathbf{v}_{0}, \nabla \psi\right)_{T}+\left\langle\mathbf{v}_{b}, \psi \cdot \mathbf{n}\right\rangle_{\partial T}, \forall \psi \in P_{m}(T)
$$

Based on the weak formulation (2.2) and the definition of weak differential operators, the semi-discrete gWG scheme for the evolutionary Oseen equation (2.1) reads as follows: For $t \in(0, \bar{T}]$, find $\mathbf{u}_{h}(t) \in \mathbf{V}_{h}$ and $p_{h}(t) \in W_{h}$ satisfying $\mathbf{u}_{b}(t)=Q_{b} \mathbf{g}$ on $\partial \Omega$ and $\mathbf{u}_{h}(0)=Q_{h} \mathbf{g}_{2}$ in $\Omega$ such that

$$
\begin{array}{ll}
\rho\left(\mathbf{u}_{h t}, \mathbf{v}_{h}\right)+\mu\left(\nabla_{w} \mathbf{u}_{h}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{u}_{h}, \mathbf{v}_{0}\right) & \\
-\left(p_{h}, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)=\left(\mathbf{f}, \mathbf{v}_{0}\right), & \forall \mathbf{v}_{h} \in \mathbf{V}_{h}^{0},  \tag{2.3}\\
\left(\nabla_{w} \cdot \mathbf{u}_{h}, q_{h}\right)+s_{2}\left(p_{h}, q_{h}\right)=0, & \forall q_{h} \in W_{h} .
\end{array}
$$

Here, the stabilization terms are defined as follows

$$
\begin{aligned}
s_{1}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right) & =\zeta \sum_{T \in \mathcal{T}_{h}} h_{T}^{\gamma}\left\langle Q_{b}\left(\mathbf{u}_{b}-\mathbf{u}_{0}\right), Q_{b}\left(\mathbf{v}_{b}-\mathbf{v}_{0}\right)\right\rangle_{\partial T}, \\
s_{2}\left(p_{h}, q_{h}\right) & =\sigma \sum_{e \in \mathcal{E}_{I}} h_{e}^{\alpha}\left\langle\llbracket p_{h} \rrbracket, \llbracket q_{h} \rrbracket\right\rangle_{e},
\end{aligned}
$$

where $\llbracket q_{h} \rrbracket=\left.q_{h}\right|_{\partial T_{1} \cap e}-\left.q_{h}\right|_{\partial T_{2} \cap e}$ denotes the jump on interior edge $e$ shared by the elements $T_{1}$ and $T_{2}$. Assume that $|\gamma|<\infty,|\alpha|<\infty$. Here $\zeta$ is a positive parameter, and $\sigma$ is zero or one.

For simplicity, we consider uniform time stepping $\tau=\bar{T} / N$ with $N$ being the total time steps. Denote by $t^{\bar{n}}=\bar{n} * \tau$ the $\bar{n}$-th time level.

By replacing the time derivative by backward difference quotient which is backward Euler discretization, the fully-discrete gWG scheme seeks $\mathbf{u}^{\bar{n}+1} \in \mathbf{V}_{h}$ and $p^{\bar{n}+1} \in W_{h}$ satisfying the boundary condition $\mathbf{u}_{b}^{\bar{n}+1}=Q_{b} \mathbf{g}$ on $\partial \Omega$ and initial condition $\mathbf{u}^{0}=Q_{h} \mathbf{g}_{2}$ in $\Omega$ such that

$$
\begin{array}{ll}
\rho\left(\bar{\partial}_{t} \mathbf{u}^{\bar{n}+1}, \mathbf{v}_{h}\right)+\mu\left(\nabla_{w} \mathbf{u}^{\bar{n}+1}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{u}^{\bar{n}+1}, \mathbf{v}_{0}\right) & \\
-\left(p^{\bar{n}+1}, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(\mathbf{u}^{\bar{n}+1}, \mathbf{v}_{h}\right)=\left(\mathbf{f}^{\bar{n}+1}, \mathbf{v}_{0}\right), & \forall \mathbf{v}_{h} \in \mathbf{V}_{h}^{0},  \tag{2.4}\\
\left(\nabla_{w} \cdot \mathbf{u}^{\bar{n}+1}, q_{h}\right)+s_{2}\left(p^{\bar{n}+1}, q_{h}\right)=0, & \forall q_{h} \in W_{h} .
\end{array}
$$

Here, the backward difference quotient is denoted by $\bar{\partial}_{t} \mathbf{u}^{\bar{n}+1}=\frac{\mathbf{u}^{\bar{n}+1}-\mathbf{u}^{\bar{n}}}{\tau}$.
Before coming to the estimates of gWG schemes (2.4) and (2.3) for evolutionary case, let us analyze the gWG for steady-state Oseen equation.
3. gWG for Steady-State Oseen Equation. The steady-state Oseen equation seeks an unknown velocity $\mathbf{u}$ and a pressure $p$ such that

$$
\begin{align*}
-\mu \Delta \mathbf{u}+\rho(\beta \cdot \nabla) \mathbf{u}+\nabla p & =\mathbf{f}, & & \text { in } \Omega, \\
\nabla \cdot \mathbf{u} & =0, & & \text { in } \Omega,  \tag{3.1}\\
\mathbf{u} & =\mathbf{g}, & & \text { on } \partial \Omega .
\end{align*}
$$

By a weak solution we mean a pair of $\mathbf{u} \in\left[H^{1}(\Omega)\right]^{d}$ and $p \in L_{0}^{2}(\Omega)$ satisfying $\mathbf{u}=\mathbf{g}$ on $\partial \Omega$ and the following equations

$$
\begin{align*}
\mu(\nabla \mathbf{u}, \nabla \mathbf{v})+\rho((\beta \cdot \nabla) \mathbf{u}, \mathbf{v})-(p, \nabla \cdot \mathbf{v}) & =(\mathbf{f}, \mathbf{v}), & & \forall \mathbf{v} \in\left[H_{0}^{1}(\Omega)\right]^{d},  \tag{3.2}\\
(\nabla \cdot \mathbf{u}, q) & =0, & & \forall q \in L_{0}^{2}(\Omega) .
\end{align*}
$$

The gWG scheme for the steady-state Oseen equation (3.1) seeks $\mathbf{u}_{h} \in \mathbf{V}_{h}$ and $p_{h} \in$ $W_{h}$ satisfying $\mathbf{u}_{b}=Q_{b} \mathbf{g}$ on $\partial \Omega$ and the following equations

$$
\begin{align*}
& \mu\left(\nabla_{w} \mathbf{u}_{h}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{u}_{h}, \mathbf{v}_{0}\right)-\left(p_{h}, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)=\left(\mathbf{f}, \mathbf{v}_{0}\right), \\
& \left(\nabla_{w} \cdot \mathbf{u}_{h}, q_{h}\right)+s_{2}\left(p_{h}, q_{h}\right)=0, \tag{3.3}
\end{align*}
$$

for all $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ and $q_{h} \in W_{h}$.
For convenience, we introduce the following energy semi-norm in $\mathbf{V}_{h}$ :

$$
\left\|\mathbf{v}_{h}\right\|^{2}:=\left(\nabla_{w} \mathbf{v}_{h}, \nabla_{w} \mathbf{v}_{h}\right)+s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right), \quad v_{h} \in \mathbf{V}_{h}
$$

Lemma 3.1. $\left\|\|\cdot\|\right.$ defines a norm in the subspace $\mathbf{V}_{h}^{0}$ consisting of functions with vanishing boundary value.

Proof. For $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ with $\left\|\mathbf{v}_{h}\right\|=0$, we see that $\nabla_{w} \mathbf{v}_{h}=\nabla \mathbf{v}_{0}+\delta_{w} \mathbf{v}_{h}=0$ on each element $T$ and $Q_{b} \mathbf{v}_{0}=\mathbf{v}_{b}$ on each $\partial T$. By using the definition of weak gradient, we have

$$
\left(\delta_{w} \mathbf{v}_{h}, \delta_{w} \mathbf{v}_{h}\right)_{T}=\left\langle\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}, \delta_{w} \mathbf{v}_{h} \cdot \mathbf{n}\right\rangle_{\partial T}=0
$$

It follows that $\delta_{w} \mathbf{v}_{h}=0$ and hence, $\nabla \mathbf{v}_{0}=0$, which implies that $\mathbf{v}_{0}$ is a constant on each element. Thus, we have $Q_{b} \mathbf{v}_{0}=\mathbf{v}_{b}$ on each $\partial T$. Together with $\mathbf{v}_{b}=0$ on $\partial \Omega$, we obtain $\mathbf{v}_{h} \equiv 0$.

In addition, for simplicity, we denote $b\left(\mathbf{v}_{h}, q_{h}\right):=\left(\nabla_{w} \cdot \mathbf{v}_{h}, q_{h}\right)$.
Lemma 3.2. 24 Assume that $n \leq \min \{m, k+1\}$. For each $q_{h} \in W_{h}$, there exists $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ such that

$$
\begin{align*}
b\left(\mathbf{v}_{h}, q_{h}\right) & \geq \frac{1}{2}\left\|q_{h}\right\|^{2}-C h^{1-\alpha} \sum_{e \in \mathcal{E}_{I}} h_{e}^{\alpha}\left\|\llbracket q_{h} \rrbracket\right\|_{e}^{2}  \tag{3.4}\\
\left\|\mathbf{v}_{h}\right\| & \leq C\left(1+h^{\frac{1+\gamma}{2}}\right)\left\|q_{h}\right\|
\end{align*}
$$

Next, we set $s=\min \{j, l\}$ and introduce a local $L^{2}$ projection operator $\boldsymbol{Q}_{s}$ : $\left[L^{2}(T)\right]^{d \times d} \rightarrow\left[P_{S}(T)\right]^{d \times d}$. From the construction of the generalized weak gradient operator, we have the following results.

Lemma 3.3. 24 For any $\mathbf{v} \in \mathbf{V}_{h}$ and $\mathbf{w} \in\left[H^{1}(T)\right]^{d}$ on the element $T \in \mathcal{T}_{h}$, one has for each $\phi \in\left[\overline{P_{s}(T)}\right]^{d \times d}$

$$
\begin{aligned}
\left(\nabla_{w} \mathbf{v}, \phi\right)_{T} & =-\left(\mathbf{v}_{0}, \nabla \cdot \phi\right)_{T}+\left\langle\mathbf{v}_{b}, \phi \cdot \mathbf{n}\right\rangle_{\partial T} \\
\left(\nabla_{w} Q_{h} \mathbf{w}, \phi\right)_{T} & =(\nabla \mathbf{w}, \phi)_{T}+\left(\left(I-Q_{0}\right) \mathbf{w}, \nabla \cdot \phi\right)_{T}
\end{aligned}
$$

The following is a useful result for the analysis on convergence and solution existence of the gWG scheme.

Lemma 3.4. Assume $\gamma \leq 0$ and $-\frac{\nabla \cdot \beta}{2} \geq C_{0} \geq 0$. There exists a constant $\zeta=\frac{C\|\beta\|_{\infty}+\|\beta\|_{\infty}^{2}}{2 \rho}$ such that

$$
\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right) \geq C_{0}\left\|\mathbf{v}_{0}\right\|^{2}-\frac{1}{2 \rho} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)-\epsilon\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}
$$

for all $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$.
Proof. From the definition of the weak gradient, we have for each $\mathbf{w}_{h} \in V_{h}^{0}$

$$
\begin{aligned}
& \left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{w}_{0}\right)_{T}=\left(\nabla \mathbf{v}_{0}+\delta_{w} \mathbf{v}_{h}, \beta \cdot \mathbf{w}_{0}^{t}\right)_{T} \\
= & -\left(\mathbf{v}_{0}, \nabla \cdot\left(\beta \cdot \mathbf{w}_{0}^{t}\right)_{T}+\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{w}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}+\left(\delta_{w} \mathbf{v}_{h}, \beta \cdot \mathbf{w}_{0}^{t}\right)_{T}\right. \\
= & -\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{w}_{0}\right)_{T}-\left(\beta \cdot \mathbf{v}_{0}^{t}, \nabla \mathbf{w}_{0}\right)_{T}+\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{w}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}+\left(\delta_{w} \mathbf{v}_{h}, \beta \cdot \mathbf{w}_{0}^{t}\right)_{T} \\
= & -\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{w}_{0}\right)_{T}-\left(\beta \cdot \mathbf{v}_{0}^{t}, \nabla_{w} \mathbf{w}_{h}\right)_{T}+\left(\beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{w}_{h}\right)_{T} \\
& +\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{w}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}+\left(\delta_{w} \mathbf{v}_{h}, \beta \cdot \mathbf{w}_{0}^{t}\right)_{T} .
\end{aligned}
$$

From Definition 2.1 and $\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{v}_{b}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}=\left\langle\mathbf{v}_{b}, \beta \cdot \mathbf{v}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}$, we have by taking $\mathbf{w}_{h}=\mathbf{v}_{h}$ in the above identity,

$$
\begin{aligned}
\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right)_{T}= & -\frac{1}{2}\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{v}_{0}\right)_{T}+\left\langle\mathbf{Q}_{s}\left(\beta \cdot \mathbf{v}_{0}^{t}\right) \cdot \mathbf{n}, \mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\rangle_{\partial \mathcal{T}} \\
& +\frac{1}{2}\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{v}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}+\left(\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{v}_{h}\right)_{T} \\
= & -\frac{1}{2}\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{v}_{0}\right)_{T}-\left\langle\mathbf{Q}_{s}\left(\beta \cdot \mathbf{v}_{0}^{t}\right) \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle_{\partial \mathcal{T}} \\
& +\frac{1}{2}\left\langle\mathbf{v}_{0}, \beta \cdot \mathbf{v}_{0}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}+\left(\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{v}_{h}\right)_{T} \\
= & -\frac{1}{2}\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{v}_{0}\right)_{T}+\left\langle\left(I-\mathbf{Q}_{s}\right)\left(\beta \cdot \mathbf{v}_{0}^{t}\right) \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle_{\partial \mathcal{T}} \\
& +\left(\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{v}_{h}\right)_{T}-\frac{1}{2}\left\langle\mathbf{v}_{0}-\mathbf{v}_{b}, \beta \cdot\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right)^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}} \\
& +\frac{1}{2}\left\langle\mathbf{v}_{b}, \beta \cdot \mathbf{v}_{b}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}} .
\end{aligned}
$$

By summing over all elements $T \in \mathcal{T}_{h}$ and noticing $\sum_{T \in \mathcal{T}_{h}}\left\langle\mathbf{v}_{b}, \beta \cdot \mathbf{v}_{b}^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}}=0$ we obtain

$$
\begin{aligned}
\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right)_{\mathcal{T}_{h}}= & -\frac{1}{2}\left(\mathbf{v}_{0}, \nabla \cdot \beta \mathbf{v}_{0}\right)_{\mathcal{T}_{h}}+\left\langle\left(I-\mathbf{Q}_{s}\right)\left(\beta \cdot \mathbf{v}_{0}^{t}\right) \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle_{\partial \mathcal{T}_{h}} \\
& +\left(\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{v}_{h}\right)_{\mathcal{T}_{h}}-\frac{1}{2}\left\langle\mathbf{v}_{0}-\mathbf{v}_{b}, \beta \cdot\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right)^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}_{h}}
\end{aligned}
$$

Observe the following estimate from the property of $L^{2}$ projections:

$$
\begin{equation*}
\left\|\nabla \mathbf{v}_{0}\right\|_{T} \leq\left\|\nabla_{w} \mathbf{v}_{h}\right\|_{T}+\left\|\delta_{w} \mathbf{v}_{h}\right\|_{T} \leq\left\|\nabla_{w} \mathbf{v}_{h}\right\|_{T}+C h^{-\frac{1}{2}}\left\|\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\|_{\partial T} \tag{3.5}
\end{equation*}
$$

and

$$
\begin{align*}
\left\|\mathbf{v}_{b}-\mathbf{v}_{0}\right\|_{\partial T} & \leq\left\|\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\|_{\partial T}+\left\|Q_{b} \mathbf{v}_{0}-\mathbf{v}_{0}\right\|_{\partial T} \leq\left\|\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\|_{\partial T}+C h^{\frac{1}{2}}\left\|\nabla \mathbf{v}_{0}\right\|_{T}  \tag{3.6}\\
& \leq C\left\|\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\|_{\partial T}+C h^{\frac{1}{2}}\left\|\nabla_{w} \mathbf{v}_{h}\right\|_{T}
\end{align*}
$$

From the Cauchy-Schwarz inequality, (3.5), (3.6), the construction of $\delta_{w} \mathbf{v}_{h}$, and the trace inequality, we obtain

$$
\begin{aligned}
& \left|\left\langle\left(I-\mathbf{Q}_{s}\right)\left(\beta \cdot \mathbf{v}_{0}^{t}\right) \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle_{\partial \mathcal{T}_{h}}+\left(\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}, \delta_{w} \mathbf{v}_{h}\right)_{\mathcal{T}_{h}}\right| \\
\leq & C\|\beta\|_{\infty} \sum_{T \in \mathcal{T}_{h}} h^{\frac{1}{2}}\left\|\nabla \mathbf{v}_{0}\right\|_{T}\left(\left\|\mathbf{v}_{0}-\mathbf{v}_{b}\right\|_{\partial T}+\left\|Q_{b} \mathbf{v}_{0}-\mathbf{v}_{b}\right\|_{\partial T}\right) \\
\leq & C h\left\|\nabla \mathbf{v}_{0}\right\|^{2}+C h\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}+\|\beta\|_{\infty}^{2}\left\|Q_{b} \mathbf{v}_{0}-\mathbf{v}_{b}\right\|_{\partial T}^{2} \\
\leq & C h\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}+\frac{\|\beta\|_{\infty}^{2}}{\zeta} h^{-\gamma} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right) .
\end{aligned}
$$

Next, using the Cauchy-Schwarz inequality and (3.6) again leads to

$$
\begin{aligned}
\left|\left\langle\mathbf{v}_{0}-\mathbf{v}_{b}, \beta \cdot\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right)^{t} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{T}_{h}}\right| & \leq \sum_{T \in \mathcal{T}_{h}}\|\beta\|_{\infty}\left\|\mathbf{v}_{b}-\mathbf{v}_{0}\right\|_{\partial T}^{2} \\
& \leq C h\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}+\frac{C\|\beta\|_{\infty}}{\zeta} h^{-\gamma} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)
\end{aligned}
$$

By combining the above inequality with $\gamma \leq 0$, we get

$$
\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right) \geq C_{0}\left\|\mathbf{v}_{0}\right\|^{2}-C h\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}-\left(C\|\beta\|_{\infty}+\|\beta\|_{\infty}^{2}\right) \frac{1}{\zeta} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)
$$

By choosing $\left(C\|\beta\|_{\infty}+\|\beta\|_{\infty}^{2}\right) \frac{1}{\zeta}=\frac{1}{2 \rho}$, i.e $\zeta=\frac{C\|\beta\|_{\infty}+\|\beta\|_{\infty}^{2}}{2 \rho}$, we arrive at

$$
\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right) \geq C_{0}\left\|\mathbf{v}_{0}\right\|^{2}-C h\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}-\frac{1}{2 \rho} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)
$$

When $h \rightarrow 0$, we have $\epsilon=C h \rightarrow 0$. This completes the proof of the lemma.
Using Lemmas 3.1, 3.2 and 3.4, we obtain the following theorem.
Theorem 3.5. Assume $\nabla \cdot \beta \leq 0$. The generlized weak Galerkin method (3.3) has one and only one solution for sufficiently small meshsize $h$.

From the assumption, there exists $C_{0}$ such that $-\frac{\nabla \cdot \beta}{2} \geq C_{0} \geq 0$. It follows that

$$
\begin{align*}
& \mu\left(\nabla_{w} \mathbf{v}_{h}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \mathbf{v}_{0}\right)+s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right) \\
\geq & (\mu-\epsilon)\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}+C_{0} \rho\left\|\mathbf{v}_{0}\right\|^{2}+\frac{1}{2} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)  \tag{3.7}\\
\geq & C_{1}\left\|\nabla_{w} \mathbf{v}_{h}\right\|^{2}+C_{0} \rho\left\|\mathbf{v}_{0}\right\|^{2}+\frac{1}{2} s_{1}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)
\end{align*}
$$

where $C_{1}>0$ for sufficiently small meshsize $h$. The coercivity estimate (3.7) is a key component behind the proof of Theorem 3.5. The proof of Theorem 3.5 is routine and details are left to interested readers.
3.1. Error Equations of gWG for Steady-State Oseen Equation. In this subsection, we shall derive the error equations for the velocity and pressure approximations. Denote the velocity error by $e_{h}^{u}=Q_{h} \mathbf{u}-\mathbf{u}_{h}$ and the pressure error by $e_{h}^{p}=Q_{h}^{p} p-p_{h}$.

From the properties of projection and Lemma3.3, we obtain the following identity [24,

$$
\begin{align*}
\left(\nabla_{w} Q_{h} \mathbf{u}, \nabla_{w} \mathbf{v}_{h}\right)= & -\left(\Delta \mathbf{u}, \mathbf{v}_{0}\right)-\left\langle\left(I-\boldsymbol{Q}_{s}\right) \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{b}-\mathbf{v}_{0}\right\rangle \\
& +\left(\left(\boldsymbol{Q}_{s}-I\right) \nabla \mathbf{u}, \nabla \mathbf{v}_{0}\right)+\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \boldsymbol{Q}_{s} \nabla_{w} \mathbf{v}_{h}\right)  \tag{3.8}\\
& +\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\boldsymbol{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)
\end{align*}
$$

For each $\mathbf{v}_{h} \in \mathbf{V}_{h}$, it follows from the definition of the generalized weak gradient and Lemma 3.3 that

$$
\begin{align*}
& \left(\left(\beta \cdot \nabla_{w}\right) Q_{h} \mathbf{u}, \mathbf{v}_{0}\right)=\left(\nabla_{w} Q_{h} \mathbf{u}, \beta \mathbf{v}_{0}^{t}\right) \\
= & \left(\nabla_{w} Q_{h} \mathbf{u}, \mathbf{Q}_{s}\left(\beta \mathbf{v}_{0}^{t}\right)\right)+\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right) \\
= & \left(\nabla \mathbf{u}, \mathbf{Q}_{s}\left(\beta \mathbf{v}_{0}^{t}\right)\right)+\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s}\left(\beta \mathbf{v}_{0}^{t}\right)\right)+\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right)  \tag{3.9}\\
= & \left(\nabla \mathbf{u}, \beta \mathbf{v}_{0}^{t}\right)-\left(\nabla \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right)+\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s}\left(\beta \mathbf{v}_{0}^{t}\right)\right) \\
& +\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right) .
\end{align*}
$$

With $k-1 \leq n \leq m$, we note that [24]

$$
\begin{equation*}
\left(Q_{h}^{p} p, \nabla_{w} \cdot \mathbf{v}_{h}\right)=-\left(\nabla p, \mathbf{v}_{0}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle\left(I-Q_{h}^{p}\right) p,\left(\mathbf{v}_{b}-\mathbf{v}_{0}\right) \cdot \mathbf{n}\right\rangle_{\partial T} \tag{3.10}
\end{equation*}
$$

Combining (3.8) -(3.10) leads to

$$
\begin{align*}
& \mu\left(\nabla_{w} Q_{h} \mathbf{u}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) Q_{h} \mathbf{u}, \mathbf{v}_{0}\right)-\left(Q_{h}^{p} p, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(Q_{h} \mathbf{u}, \mathbf{v}_{h}\right) \\
= & \left(\mathbf{f}, \mathbf{v}_{0}\right)+L_{u, p}^{1}\left(\mathbf{v}_{h}\right), \tag{3.11}
\end{align*}
$$

where

$$
\begin{aligned}
L_{u, p}^{1}\left(\mathbf{v}_{h}\right):= & \mu\left(\left\langle\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle-\left(\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u}, \nabla \mathbf{v}_{0}\right)\right. \\
& \left.+\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s} \nabla_{w} \mathbf{v}_{h}\right)+\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)\right) \\
& +\rho\left(-\left(\nabla \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right)+\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s}\left(\beta \mathbf{v}_{0}^{t}\right)\right)\right. \\
& \left.+\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right)\right)-\left\langle\left(I-Q_{h}^{p}\right) p,\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right) \cdot \mathbf{n}\right\rangle+s_{1}\left(Q_{h} \mathbf{u}, \mathbf{v}_{h}\right) .
\end{aligned}
$$

From (3.3) and (3.11), we arrive at the following error equations

$$
\begin{align*}
& \mu\left(\nabla_{w} e_{h}^{u}, \nabla_{w} \mathbf{v}_{h}\right)+\left(\left(\beta \cdot \nabla_{w}\right) e_{h}^{u}, \mathbf{v}_{0}\right)-\left(e_{h}^{p}, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(e_{h}^{u}, \mathbf{v}_{h}\right)=L_{u, p}^{1}\left(\mathbf{v}_{h}\right),  \tag{3.12}\\
& \left(\nabla_{w} \cdot e_{h}^{u}, q_{h}\right)+s_{2}\left(e_{h}^{p}, q_{h}\right)=L_{u, p}^{2}\left(q_{h}\right)
\end{align*}
$$

for all $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ and $q_{h} \in W_{h}$, where

$$
L_{u, p}^{2}\left(q_{h}\right)=-\left\langle\left(I-Q_{b}\right) \mathbf{u}, q_{h} \cdot \mathbf{n}\right\rangle+s_{2}\left(Q_{h}^{p} p, q_{h}\right)
$$

3.2. Convergence of gWG for Steady-State Oseen Equation. In this subsection, our goal is to establish the error estimates in $L^{2}$ norm and energy norm for the gWG approximations for steady-state Oseen equation.

Lemma 3.6. Assume $\mathbf{u} \in\left[H^{i}(\Omega)\right]^{d}$ where $i=\max \{s+2, k+1, j+1\}$ and $p \in H^{n+1}(\Omega)$. There exists a constant $C$ such that

$$
\begin{align*}
\left|L_{u, p}^{1}\left(\mathbf{v}_{h}\right)\right| \leq & C\left(\left(1+h^{\frac{-\gamma-1}{2}}\right)\left(h^{s+1}\|\mathbf{u}\|_{s+2}+h^{k}\|\mathbf{u}\|_{k+1}+h^{n+1}\|p\|_{n+1}\right)\right. \\
& \left.+h^{\frac{\gamma+1}{2}} h^{k}\|\mathbf{u}\|_{k+1}\right)\left\|\mathbf{v}_{h}\right\|  \tag{3.13}\\
\left|L_{u, p}^{2}\left(q_{h}\right)\right| \leq & C\left(h^{j} h^{\frac{1-\alpha}{2}}\|\mathbf{u}\|_{j+1}+h^{n+1} h^{\frac{\alpha-1}{2}}\|p\|_{n+1}\right) s_{2}^{\frac{1}{2}}\left(q_{h}, q_{h}\right)
\end{align*}
$$

for all $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ and $q_{h} \in W_{h}$,
Proof. From the Cauchy-Schwarz inequality, trace inequality, and approximation properties of $L^{2}$ projections, we have

$$
\begin{aligned}
\left|\left\langle\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle\right| & \leq C \sum_{T \in \mathcal{T}_{h}}\left\|\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u}\right\|_{\partial T}\left\|\mathbf{v}_{0}-\mathbf{v}_{b}\right\|_{\partial T} \\
& \leq C h^{s+1}\left(1+h^{\frac{-\gamma-1}{2}}\right)\|\mathbf{u}\|_{s+2}\left\|\mathbf{v}_{h}\right\|
\end{aligned}
$$

where we have used $\left\|\mathbf{v}_{0}-\mathbf{v}_{b}\right\|_{\partial T} \leq C h^{\frac{1}{2}}\left(1+h^{\frac{-\gamma-1}{2}}\right)\left\|\mathbf{v}_{h}\right\|$. Analogously, we have

$$
\left|\left(\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u}, \nabla \mathbf{v}_{0}\right)\right| \leq C\left\|\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u}\right\|\left\|\nabla \mathbf{v}_{0}\right\| \leq C h^{s+1}\left(1+h^{\frac{-\gamma-1}{2}}\right)\|\mathbf{u}\|_{s+2}\left\|\mathbf{v}_{h}\right\|,
$$

where we have used the following inequalities

$$
\left\|\delta_{w} \mathbf{v}_{h}\right\|^{2}=\left\langle\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}, \delta_{w} \mathbf{v}_{h}\right\rangle \leq C \sum_{T \in \mathcal{T}_{h}} h^{\frac{\gamma}{2}}\left\|\mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\|_{\partial T} h^{\frac{-\gamma-1}{2}}\left\|\delta_{w} \mathbf{v}_{h}\right\|_{T}
$$

and $\left\|\nabla \mathbf{v}_{0}\right\| \leq\left\|\nabla_{w} \mathbf{v}_{h}\right\|+\left\|\delta_{w} \mathbf{v}_{h}\right\| \leq C\left(1+\frac{-\gamma-1}{2}\right)\left\|\mathbf{v}_{h}\right\|$.
Next, from the inverse inequality we have

$$
\left|\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s} \nabla_{w} \mathbf{v}_{h}\right)\right| \leq C h^{k}\|\mathbf{u}\|_{k+1}\left\|\mathbf{v}_{h}\right\|
$$

By using the definition of generalized weak gradient, we obtain

$$
\begin{align*}
& \left|\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)\right|=\left|\left(\nabla Q_{0} \mathbf{u}+\delta_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)\right| \\
\leq & C\left(\left\|\left(I-\mathbf{Q}_{s}\right) \nabla Q_{0} \mathbf{u}\right\|+\left\|\delta_{w} Q_{h} \mathbf{u}\right\|\right)\left\|\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right\| \\
\leq & C\left(h^{s+1}\|\mathbf{u}\|_{s+2}+h^{-1}\left\|\left(I-Q_{0}\right) \mathbf{u}\right\|\right)\left\|\mathbf{v}_{h}\right\|  \tag{3.14}\\
\leq & C\left(h^{s+1}\|\mathbf{u}\|_{s+2}+h^{k}\|\mathbf{u}\|_{k+1}\right)\left\|\mathbf{v}_{h}\right\|,
\end{align*}
$$

where we have used the fact that $\left\|\delta_{w} Q_{h} \mathbf{u}\right\| \leq C h^{-1}\left\|\left(I-Q_{0}\right) \mathbf{u}\right\|$.
Next, the following estimates hold true:

$$
\begin{aligned}
\left|\left(\nabla \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}\right)\right| & =\left|\left(\left(I-\mathbf{Q}_{s}\right) \nabla \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \cdot \mathbf{v}_{0}^{t}\right)\right| \\
& \leq C h^{s+1}\|\mathbf{u}\|_{s+2} h\left\|\nabla \mathbf{v}_{0}\right\| \\
& \leq C h^{s+1}\|\mathbf{u}\|_{s+2} h\left(1+h^{\frac{-\gamma-1}{2}}\right)\left\|\mathbf{v}_{h}\right\| .
\end{aligned}
$$

Similar to the above inequality and (3.14), we have

$$
\begin{aligned}
& \left|\left(\left(I-Q_{0}\right) \mathbf{u}, \nabla \cdot \mathbf{Q}_{s} \beta \mathbf{v}_{0}^{t}\right)\right| \leq C h^{k+1}\|\mathbf{u}\|_{k+1}\left(1+h^{\frac{-\gamma-1}{2}}\right)\left\|\mathbf{v}_{h}\right\| \\
& \left|\left(\nabla_{w} Q_{h} \mathbf{u},\left(I-\mathbf{Q}_{s}\right) \beta \mathbf{v}_{0}^{t}\right)\right| \leq C\left(h^{s+1}\|\mathbf{u}\|_{s+2}+h^{k}\|\mathbf{u}\|_{k+1}\right) h\left(1+h^{\frac{-\gamma-1}{2}}\right)\left\|\mathbf{v}_{h}\right\| .
\end{aligned}
$$

From the construction of the stabilizer, we get

$$
\begin{aligned}
& \left|\left\langle\left(I-Q_{h}^{p}\right) p,\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right) \cdot \mathbf{n}\right\rangle\right| \leq C\left(1+h^{\frac{-\gamma-1}{2}}\right) h^{n+1}\|p\|_{n+1}\left\|\mathbf{v}_{h}\right\| \\
& \left|s_{1}\left(Q_{h} \mathbf{u}, \mathbf{v}_{h}\right)\right|=\left|\sum_{T \in \mathcal{T}_{h}} \zeta h^{\gamma}\left\langle Q_{b}\left(I-Q_{0}\right) \mathbf{u}, \mathbf{v}_{b}-Q_{b} \mathbf{v}_{0}\right\rangle_{\partial T}\right| \leq C h^{\frac{\gamma+1}{2}} h^{k}\|\mathbf{u}\|_{k+1}\left\|\mathbf{v}_{h}\right\| .
\end{aligned}
$$

Based on the above inequalities, we obtain the desired estimate for $L_{u, p}^{1}\left(\mathbf{v}_{h}\right)$.
Finally, from the following estimates

$$
\begin{aligned}
\left|\left\langle\left(I-Q_{b}\right) \mathbf{u}, q_{h} \cdot \mathbf{n}\right\rangle\right| & \leq C h^{j}\|\mathbf{u}\|_{j+1} h^{\frac{-\alpha+1}{2}} s_{2}\left(q_{h}, q_{h}\right)^{\frac{1}{2}} \\
\left|s_{2}\left(Q_{h}^{p} p, q_{h}\right)\right| & \leq C h^{n+1}\|p\|_{n+1} h^{\frac{\alpha-1}{2}} s_{2}\left(q_{h}, q_{h}\right)^{\frac{1}{2}}
\end{aligned}
$$

we may deduce the desired estimate for $L_{u, p}^{2}\left(q_{h}\right)$ as well. This completes the proof of the lemma.

Theorem 3.7. Let $(\mathbf{u}, p)$ be the exact solution of (3.2) and $\left(\mathbf{u}_{h}, p_{h}\right)$ be its numerical approximation arising from the $\mathrm{g} W G$ scheme (3.3). Assume that the velocity $\mathbf{u} \in\left[H^{i}(\Omega)\right]^{d}, i=\max \{s+2, k+1, j+1\}$ and the pressure $p \in H^{n+1}(\Omega)$. Let $k-1 \leq n \leq \max \{m, k+1\}$ be satisfied. Then we have

$$
\begin{align*}
& \left\|e_{h}^{u}\right\|^{2} \leq C R_{1} \\
& \left\|e_{h}^{p}\right\|^{2} \leq C\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) R_{1} \tag{3.15}
\end{align*}
$$

where

$$
\begin{aligned}
R_{1}= & h^{2(s+1)}\left(1+h^{-\gamma-1}\right)\|\mathbf{u}\|_{s+2}^{2}+h^{2 k}\left(1+h^{-\gamma-1}+h^{\gamma+1}\right)\|\mathbf{u}\|_{k+1}^{2} \\
& +h^{2 j} h^{1-\alpha}\|\mathbf{u}\|_{j+1}^{2}+h^{2(n+1)}\left(1+h^{-\gamma-1}+h^{\alpha-1}\right)\|p\|_{n+1}^{2}
\end{aligned}
$$

Moreover, when $\gamma=-1$ and $\alpha=1$, optimal order of convergence is achieved.
Proof. By choosing $\mathbf{v}_{h}=e_{h}^{u}$ and $q_{h}=e_{h}^{p}$ in (3.12), we have

$$
\mu\left\|\nabla_{w} e_{h}^{u}\right\|^{2}+\rho\left(\left(\beta \cdot \nabla_{w}\right) e_{h}^{u}, e_{h}^{u}\right)+s_{1}\left(e_{h}^{u}, e_{h}^{u}\right)+s_{2}\left(e_{h}^{p}, e_{h}^{p}\right)=L_{u, p}^{1}\left(e_{h}^{u}\right)+L_{u, p}^{2}\left(e_{h}^{p}\right)
$$

From Lemma 3.4 or the coercivity (3.7) and the estimate in (3.13), we have

$$
C_{1}\left\|\nabla_{w} e_{h}^{u}\right\|^{2}+C_{0} \rho\left\|e_{h 0}^{u}\right\|^{2}+\frac{1}{2} s_{1}\left(e_{h}^{u}, e_{h}^{u}\right)+s_{2}\left(e_{h}^{p}, e_{h}^{p}\right) \leq C R_{1},
$$

which gives rise to the error estimate for the velocity approximation in the energy norm.

Next, by using Lemma 3.2 and (3.12), for $e_{h}^{p}$, there exists a $\mathbf{v}_{h} \in \mathbf{V}_{h}^{0}$ such that

$$
\begin{aligned}
& \left\|e_{h}^{p}\right\|^{2} \leq b\left(\mathbf{v}_{h}, e_{h}^{p}\right)+C h^{1-\alpha} s_{2}\left(e_{h}^{p}, e_{h}^{p}\right) \\
\leq & \mu\left(\nabla_{w} e_{h}^{u}, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) e_{h}^{u}, \mathbf{v}_{0}\right)+s_{1}\left(e_{h}^{u}, \mathbf{v}_{h}\right)-L_{u, p}^{1}\left(\mathbf{v}_{h}\right)+C h^{1-\alpha} s_{2}\left(e_{h}^{p}, e_{h}^{p}\right)
\end{aligned}
$$

Thus, from the estimate (3.13), we obtain the desired estimate for $\left\|e_{h}^{p}\right\|$ in the $L^{2}$ norm. $\square$

In order to estimate the velocity in $L^{2}$ norm, we consider the dual problem for the steady-state Oseen equation (3.1): Seek $(\phi, \xi)$ such that

$$
\begin{align*}
-\mu \Delta \phi-\rho(\beta \cdot \nabla) \phi-\rho \nabla \cdot \beta \phi+\nabla \xi & =e_{h 0}^{u}, & & \text { in } \Omega, \\
\nabla \cdot \phi & =0, & & \text { in } \Omega,  \tag{3.16}\\
\phi & =0, & & \text { on } \partial \Omega .
\end{align*}
$$

Assume that the dual problem satisfies the following $H^{2} / H^{1}$ regularity assumption:

$$
\|\phi\|_{2}+\|\xi\|_{1} \leq C\left\|e_{h 0}^{u}\right\|
$$

THEOREM 3.8. Let $(\mathbf{u}, p)$ be the exact solution of (3.2) and $\left(\mathbf{u}_{h}, p_{h}\right)$ be its numerical approximation arising from the $\mathrm{g} W G$ scheme (3.3). Assume that the velocity $\mathbf{u} \in\left[H^{i}(\Omega)\right]^{d}, i=\max \{s+2, k+1, j+1\}$ and the pressure $p \in H^{n+1}(\Omega)$. Let $k-1 \leq n \leq \max \{m, k+1\}$ and $j \geq 1$ be satisfied. Then there exists a constant such that

$$
\begin{equation*}
\left\|e_{h 0}^{u}\right\|^{2} \leq C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) R_{1} \tag{3.17}
\end{equation*}
$$

Moreover, when $\gamma=-1$ and $\alpha=1$, the above error estimate yields the optimal order of convergence in $L^{2}$.

Proof. First, notice that

$$
\left(Q_{0} \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)=\left(\phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)-\left(\left(I-Q_{0}\right) \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)
$$

Next, from the definition of the generalized weak gradient we have

$$
\begin{aligned}
& \left(\phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)=\left(\beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right) \\
= & \left(\mathbf{Q}_{s} \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right)+\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right) \\
= & \left(\mathbf{Q}_{s} \beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left\langle\mathbf{Q}_{s} \beta \phi^{t} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle+\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right) \\
= & \left(\beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left\langle\mathbf{Q}_{s} \beta \phi^{t} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle \\
& +\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right) .
\end{aligned}
$$

It follows that
(3.18)

$$
\begin{aligned}
& \left(Q_{0} \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)=\left(\beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left\langle\mathbf{Q}_{s} \beta \phi^{t} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle \\
& +\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right)-\left(\left(I-Q_{0}\right) \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right) \\
= & -\left(\nabla \cdot\left(\beta \phi^{t}\right), \mathbf{v}_{0}\right)+\left\langle\beta \phi^{t}, \mathbf{v}_{0} \cdot \mathbf{n}\right\rangle-\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla \mathbf{v}_{0}\right)-\left\langle\mathbf{Q}_{s} \beta \phi^{t} \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle \\
& +\left(\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}, \nabla_{w} \mathbf{v}_{h}\right)-\left(\left(I-Q_{0}\right) \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)
\end{aligned}
$$

From the property of $L^{2}$ projections, (3.8), (3.10), and (3.18), we have (3.19)

$$
\begin{aligned}
& \mu\left(\nabla_{w} Q_{h} \phi, \nabla_{w} \mathbf{v}_{h}\right)+\rho\left(Q_{0} \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)-\left(Q_{h}^{p} \xi, \nabla_{w} \cdot \mathbf{v}_{h}\right)+s_{1}\left(Q_{h} \phi, \mathbf{v}_{h}\right) \\
= & \left(e_{h 0}^{u}, \mathbf{v}_{0}\right)+\rho\left\langle\beta \phi^{t}, \mathbf{v}_{0} \cdot \mathbf{n}\right\rangle+\mu\left(\left\langle\left(I-\mathbf{Q}_{s}\right) \nabla \phi \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle-\left(\left(I-\mathbf{Q}_{s}\right) \nabla \phi, \nabla \mathbf{v}_{0}\right)\right. \\
& \left.+\left(\left(I-Q_{0}\right) \phi, \nabla \cdot \mathbf{Q}_{s} \nabla_{w} \mathbf{v}_{h}\right)+\left(\nabla_{w} Q_{h} \phi,\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)\right)-\rho\left(\left(\nabla \mathbf{v}_{0},\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}\right)\right. \\
& \left.-\left(\nabla_{w} \mathbf{v}_{h},\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}\right)+\left\langle\mathbf{v}_{0}-\mathbf{v}_{b}, \mathbf{Q}_{s}\left(\beta \phi^{t}\right) \cdot \mathbf{n}\right\rangle+\left(\left(I-Q_{0}\right) \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)\right) \\
& -\left\langle\left(I-Q_{h}^{p}\right) \xi,\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right) \cdot \mathbf{n}\right\rangle+s_{1}\left(Q_{h} \phi, \mathbf{v}_{h}\right) \\
= & \left(e_{h 0}^{u}, \mathbf{v}_{0}\right)+L_{\phi, \xi}^{3}\left(\mathbf{v}_{h}\right)
\end{aligned}
$$

where we have used $\left\langle\beta \phi^{t}, \mathbf{v}_{b} \cdot \mathbf{n}\right\rangle=0$ and $\left\langle\beta \phi^{t}, \mathbf{v}_{0} \cdot \mathbf{n}\right\rangle=\left\langle\mathbf{v}_{0}, \beta \phi^{t} \cdot \mathbf{n}\right\rangle$. Here $L_{\phi, \xi}^{3}\left(\mathbf{v}_{h}\right)$ is defined as follows:

$$
\begin{aligned}
& L_{\phi, \xi}^{3}\left(\mathbf{v}_{h}\right)=\mu\left(\left\langle\left(I-\mathbf{Q}_{s}\right) \nabla \phi \cdot \mathbf{n}, \mathbf{v}_{0}-\mathbf{v}_{b}\right\rangle-\left(\left(I-\mathbf{Q}_{s}\right) \nabla \phi, \nabla \mathbf{v}_{0}\right)\right. \\
& \left.+\left(\left(I-Q_{0}\right) \phi, \nabla \cdot \mathbf{Q}_{s} \nabla_{w} \mathbf{v}_{h}\right)+\left(\nabla_{w} Q_{h} \phi,\left(I-\mathbf{Q}_{s}\right) \nabla_{w} \mathbf{v}_{h}\right)\right)-\rho\left(\left(\nabla \mathbf{v}_{0},\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}\right)\right. \\
& \left.-\left(\nabla_{w} \mathbf{v}_{h},\left(I-\mathbf{Q}_{s}\right) \beta \phi^{t}\right)+\left\langle\mathbf{v}_{0}-\mathbf{v}_{b},\left(\mathbf{Q}_{s}-I\right)\left(\beta \phi^{t}\right) \cdot \mathbf{n}\right\rangle+\left(\left(I-Q_{0}\right) \phi,\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}\right)\right) \\
& -\left\langle\left(I-Q_{h}^{p}\right) \xi,\left(\mathbf{v}_{0}-\mathbf{v}_{b}\right) \cdot \mathbf{n}\right\rangle+s_{1}\left(Q_{h} \phi, \mathbf{v}_{h}\right) .
\end{aligned}
$$

Taking $\mathbf{v}_{h}=e_{h}^{u}$ in (3.19) and using the error equation (3.12) lead to

$$
\begin{align*}
& \left\|e_{h 0}^{u}\right\|^{2}=\mu\left(\nabla_{w} Q_{h} \phi, \nabla_{w} e_{h}^{u}\right)+\rho\left(Q_{0} \phi,\left(\beta \cdot \nabla_{w}\right) e_{h}^{u}\right)-\left(Q_{h}^{p} \xi, \nabla_{w} \cdot e_{h}^{u}\right) \\
& +s_{1}\left(Q_{h} \phi, e_{h}^{u}\right)-L_{\phi, \xi}^{3}\left(e_{h}^{u}\right) \\
= & L_{u, p}^{1}\left(Q_{h} \phi\right)-L_{u, p}^{2}\left(Q_{h} \xi\right)+s_{2}\left(e_{h}^{p}, Q_{h} \xi\right)+\left(e_{h}^{p}, \nabla_{w} \cdot Q_{h} \phi\right)-L_{\phi, \xi}^{3}\left(e_{h}^{u}\right)  \tag{3.20}\\
= & L_{u, p}^{1}\left(Q_{h} \phi\right)-L_{u, p}^{2}\left(Q_{h} \xi\right)+s_{2}\left(e_{h}^{p}, Q_{h} \xi\right)+\left\langle\left(I-Q_{b}\right) \phi, e_{h}^{p} \cdot \mathbf{n}\right\rangle-L_{\phi, \xi}^{3}\left(e_{h}^{u}\right),
\end{align*}
$$

where we also used the assumption of $n \leq k+1$. By employing $\left\|\nabla_{w} Q_{h} \phi\right\|_{1} \leq C\|\phi\|_{2}$, we deduce

$$
\begin{aligned}
\left|L_{u, p}^{1}\left(Q_{h} \phi\right)\right| & \leq C\left(h^{s+1}\|\mathbf{u}\|_{s+2}+h^{k}\|\mathbf{u}\|_{k+1}+h^{n+1}\|p\|_{n+1}\right) h\|\phi\|_{2}, \\
\left|L_{u, p}^{2}\left(Q_{h} \xi\right)\right| & \leq C\left(h^{j+1}\|\mathbf{u}\|_{j+1}+h^{n+1} h^{\alpha}\|p\|_{n+1}\right)\|\xi\|_{1} \\
\left|\left\langle\left(I-Q_{b}\right) \phi, e_{h}^{p} \cdot \mathbf{n}\right\rangle\right| & \leq C h\|\phi\|_{2}\left\|e_{h}^{p}\right\| \\
\left|s_{2}\left(e_{h}^{p}, Q_{h} \xi\right)\right| & \leq C h\left\|e_{h}^{p}\right\|\|\xi\|_{1}
\end{aligned}
$$

Similar to the estimate in (3.13), we have

$$
\left|L_{\phi, \xi}^{3}\left(e_{h}^{u}\right)\right| \leq C h\left(\|\phi\|_{2}+\|\xi\|_{1}\right)\left\|e_{h}^{u}\right\|
$$

Thus, from the regularity of dual problem, we get

$$
\begin{aligned}
\left\|e_{h 0}^{u}\right\| \leq & C\left(h^{s+2}\|\mathbf{u}\|_{s+2}+h^{k+1}\|\mathbf{u}\|_{k+1}+h^{j+1}\|\mathbf{u}\|_{j+1}+h^{n+1}\left(h+h^{\alpha}\right)\|p\|_{n+1}\right. \\
& \left.+\left(h+h^{\alpha}\right)\left\|e_{h}^{p}\right\|+h\left\|e_{h}^{u}\right\|\right)
\end{aligned}
$$

By combining the above inequality with Theorem 3.7 we obtain the desired error estimate for the velocity approximation in $L^{2}$ norm. $\square$

Remark 1. When $n \leq j$, the parameter $\mu$ can be chosen as $\mu=0$; i.e. without including the stability term $s_{2}$ in the $\mathrm{g} W G$ scheme (3.3). It should be pointed out that both the inf-sup condition and the convergence are established by assuming $k-1 \leq$ $n \leq \max \{m, k+1\}$. In particular, the velocity estimate in $L^{2}$ norm holds true with $j \geq 0$. A proof for this last claim can be given by following the above process; interested readers are referred to [24] for details.
4. gWG for Evolutionary Oseen Equation. Now, we return to consider the evolutionary case. For simplicity, we introduce the following bilinear form

$$
a\left(\mathbf{v}_{h}, \chi_{h}\right)=\mu\left(\nabla_{w} \mathbf{v}_{h}, \nabla_{w} \chi_{h}\right)+\rho\left(\left(\beta \cdot \nabla_{w}\right) \mathbf{v}_{h}, \chi_{0}\right)+s_{1}\left(\mathbf{v}_{h}, \chi_{h}\right)
$$

We further introduce the projection $\left(E_{h} \mathbf{v}, E_{h}^{p} q\right)$ defined by the following equations:

$$
\begin{align*}
a\left(E_{h} \mathbf{v}, \chi_{h}\right)-\left(E_{h}^{p} q, \nabla_{w} \cdot \chi_{h}\right) & =\left(-\mu \Delta \mathbf{v}+\rho(\beta \cdot \nabla) \mathbf{v}+\nabla q, \chi_{0}\right), & & \forall \chi_{h} \in \mathbf{V}_{h}^{0}  \tag{4.1}\\
\left(\nabla_{w} \cdot E_{h} \mathbf{v}, \psi\right)+s_{2}\left(E_{h}^{p} q, \psi\right) & =(\nabla \cdot \mathbf{v}, \psi), & & \forall \psi \in W_{h}
\end{align*}
$$

It is easy to see that (4.1) is actually the generalized weak Galerkin scheme for the following problem: Find (v,q) such that

$$
\begin{align*}
-\mu \nabla \cdot(\nabla \mathbf{v})+\rho(\beta \cdot \nabla) \mathbf{v}+\nabla q & =\mathbf{f}^{*}, & & \text { in } \Omega \\
\nabla \cdot \mathbf{v} & =0, & & \text { in } \Omega  \tag{4.2}\\
\mathbf{v} & =\mathbf{g}, & & \text { on } \partial \Omega .
\end{align*}
$$

From Theorems 3.15 and 3.17 we have the following results.
Lemma 4.1. Let $\left(E_{h} \mathbf{v}, E_{h}^{p} q\right)$ be defined by (4.1). Then there exists a constant such that

$$
\begin{align*}
\left\|Q_{h} \mathbf{v}-E_{h} \mathbf{v}\right\|^{2} & \leq C R_{1} \\
\left\|Q_{h}^{p} q-E_{h}^{p} q\right\|^{2} & \leq C\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) R_{1}  \tag{4.3}\\
\left\|Q_{h} \mathbf{v}-E_{h} \mathbf{v}\right\|^{2} & \leq C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) R_{1}
\end{align*}
$$

4.1. Convergence of Semi-discrete Scheme. To estimate the convergence of the evolutionary case (2.3), we denote the error terms of semi-discrete scheme by $e_{h}^{u}=Q_{h} \mathbf{u}-\mathbf{u}_{h}=\eta_{h}^{u}+\theta_{h}^{u}$ and $e_{h}^{p}=Q_{h}^{p} p-p_{h}=\eta_{h}^{p}+\theta_{h}^{p}$, where $\eta_{h}^{u}=Q_{h} \mathbf{u}-E_{h} \mathbf{u}$, $\theta_{h}^{u}=E_{h} \mathbf{u}-\mathbf{u}_{h}$ and $\eta_{h}^{p}=Q_{h}^{p} p-E_{h}^{p} p, \theta_{h}^{p}=E_{h}^{p} p-p_{h}$.

From the initial condition in gWG (2.3), we deduce $e_{h}^{u}(0)=0$.
ThEOREM 4.2. Let $\left(\mathbf{u}_{h}, p_{h}\right)$ and $(\mathbf{u}, p)$ be the numerical solution of $g W G$ (2.3) and exact solution of (2.2), respectively. Assume that the velocity $\mathbf{u}, \mathbf{u}_{t}, \mathbf{u}_{t t} \in\left[H^{i}(\Omega)\right]^{d}$, $i=\max \{s+2, k+1, j+1\}$ and the pressure $p, p_{t}, p_{t t} \in H^{n+1}(\Omega)$. Let $k-1 \leq n \leq$
$\max \{m, k+1\}$ and $j \geq 1$. Then there exists a constant such that

$$
\begin{align*}
\left\|e_{h}^{u}(t)\right\|^{2} \leq & \left\|\theta_{h}^{u}(0)\right\|^{2}+C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right)\left(R_{1}(t)+\int_{0}^{t} R_{2}(\bar{s}) d \bar{s}\right) \\
\left\|e_{h}^{u}(t)\right\|^{2} \leq & \left\|\theta_{h}^{u}(0)\right\|^{2}+C R_{1}(t) \\
& +C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) \int_{0}^{t}\left(R_{2}(\bar{s})+R_{3}(\bar{s})\right) d \bar{s}  \tag{4.4}\\
\left\|e_{h}^{p}(t)\right\|^{2} \leq & C\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right)\left(R_{1}(t)+\left(1+h^{1+\gamma}+h^{1-\alpha}\right)\left(h^{2}+h^{2 \alpha}\right)\left(R_{2}(t)\right.\right. \\
& \left.+\int_{0}^{t}\left(R_{2}(\bar{s})+R_{3}(\bar{s})\right) d \bar{s}+\mu\left\|\theta_{h}^{u}(0)\right\|^{2}+\frac{\rho}{2}\left\|\theta_{h t}^{u}(0)\right\|^{2}\right) .
\end{align*}
$$

Here,

$$
\begin{aligned}
R_{1}(t)= & h^{2(s+1)}\left(1+h^{-\gamma-1}\right)\|\mathbf{u}(t)\|_{s+2}^{2}+h^{2 k}\left(1+h^{-\gamma-1}+h^{\gamma+1}\right)\|\mathbf{u}(t)\|_{k+1}^{2} \\
& +h^{2(n+1)}\left(1+h^{-\gamma-1}+h^{\alpha-1}\right)\|p(t)\|_{n+1}^{2}+h^{2 j} h^{1-\alpha}\|\mathbf{u}(t)\|_{j+1}^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
R_{2}(\bar{s})= & h^{2(s+1)}\left(1+h^{-\gamma-1}\right)\left\|\mathbf{u}_{t}(\bar{s})\right\|_{s+2}^{2}+h^{2 k}\left(1+h^{-\gamma-1}+h^{\gamma+1}\right)\left\|\mathbf{u}_{t}(\bar{s})\right\|_{k+1}^{2} \\
& +h^{2(n+1)}\left(1+h^{-\gamma-1}+h^{\alpha-1}\right)\left\|p_{t}(\bar{s})\right\|_{n+1}^{2}+h^{2 j} h^{1-\alpha}\left\|\mathbf{u}_{t}(\bar{s})\right\|_{j+1}^{2} \\
R_{3}(\bar{s})= & h^{2(s+1)}\left(1+h^{-\gamma-1}\right)\left\|\mathbf{u}_{t t}(\bar{s})\right\|_{s+2}^{2}+h^{2 k}\left(1+h^{-\gamma-1}+h^{\gamma+1}\right)\left\|\mathbf{u}_{t t}(\bar{s})\right\|_{k+1}^{2} \\
& +h^{2(n+1)}\left(1+h^{-\gamma-1}+h^{\alpha-1}\right)\left\|p_{t t}(\bar{s})\right\|_{n+1}^{2}+h^{2 j} h^{1-\alpha}\left\|\mathbf{u}_{t t}(\bar{s})\right\|_{j+1}^{2}
\end{aligned}
$$

Moreover, when $\gamma=-1$ and $\alpha=1$, optimal order of convergence is obtained.
Proof. Recall that we have obtained the estimate for $\eta_{h}^{u}$ and $\eta_{h}^{p}$ in Lemma 4.1, It remains to bound $\theta_{h}^{u}$ and $\theta_{h}^{p}$. To this end, note that for $\chi_{h} \in \mathbf{V}_{h}^{0}$ and $\psi \in W_{h}$, we have

$$
\begin{align*}
\rho\left(\theta_{h t}^{u}, \chi_{0}\right)+a\left(\theta_{h}^{u}, \chi_{h}\right)-\left(\theta_{h}^{p}, \nabla_{w} \cdot \chi_{h}\right) & =-\rho\left(\eta_{h t}^{u}, \chi_{0}\right) \\
\left(\nabla_{w} \cdot \theta_{h}^{u}, \psi\right)+s_{2}\left(\theta_{h}^{p}, \psi\right) & =0 \tag{4.5}
\end{align*}
$$

By choosing $\chi_{h}=\theta_{h}^{u}$ and $\psi=\theta_{h}^{p}$ in (4.5), we have

$$
\begin{aligned}
& \frac{\rho}{2} \frac{d}{d t}\left\|\theta_{h}^{u}\right\|^{2}+\mu\left\|\nabla_{w} \theta_{h}^{u}\right\|^{2}+\rho\left(\left(\beta \cdot \nabla_{w}\right) \theta_{h}^{u}, \theta_{h}^{u}\right)+s_{1}\left(\theta_{h}^{u}, \theta_{h}^{u}\right)+s_{2}\left(\theta_{h}^{p}, \theta_{h}^{p}\right) \\
= & -\rho\left(\eta_{h t}^{u}, \theta_{h}^{u}\right)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, Lemma 3.4, and integrating over ( $0, t$ ), we arrive at

$$
\begin{align*}
& \frac{\rho}{2}\left\|\theta_{h}^{u}(t)\right\|^{2}+\int_{0}^{t}\left(C_{1}\left\|\nabla_{w} \theta_{h}^{u}\right\|^{2}+C_{0} \rho\left\|\theta_{h}^{u}\right\|^{2}+\frac{1}{2} s_{1}\left(\theta_{h}^{u}, \theta_{h}^{u}\right)+s_{2}\left(\theta_{h}^{p}, \theta_{h}^{p}\right)\right) d \bar{s}  \tag{4.6}\\
\leq & \frac{\rho}{2}\left\|\theta_{h}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}
\end{align*}
$$

The error estimate for velocity in $L^{2}$ norm is obtained at once from (4.6) and Theorem 4.2 .

$$
\begin{aligned}
\left\|e_{h}^{u}(t)\right\|^{2} & \leq\left\|\eta_{h}^{u}(t)\right\|^{2}+\left\|\theta_{h}^{u}(t)\right\|^{2} \\
& \leq\left\|\theta_{h}^{u}(0)\right\|^{2}+\left\|\eta_{h}^{u}(t)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}
\end{aligned}
$$

To estimate the velocity in energy norm, we have the following error equation by differentiating the equation (4.5)

$$
\begin{aligned}
\rho\left(\theta_{h t t}^{u}, \chi_{0}\right)+a\left(\theta_{h t}^{u}, \chi_{h}\right)-\left(\theta_{h t}^{p}, \nabla_{w} \cdot \chi_{h}\right) & =-\rho\left(\eta_{h t t}^{u}, \chi_{0}\right), \\
\left(\nabla_{w} \cdot \theta_{h t}^{u}, \psi\right)+s_{2}\left(\theta_{h t}^{p}, \psi\right) & =0 .
\end{aligned}
$$

Taking $\chi_{h}=\theta_{h t}^{u}$ and $\psi=\theta_{h t}^{p}$ in the above equation yields

$$
\begin{aligned}
& \frac{\rho}{2} \frac{d}{d t}\left\|\theta_{h t}^{u}\right\|^{2}+\mu\left\|\nabla_{w} \theta_{h t}^{u}\right\|^{2}+\rho\left(\left(\beta \cdot \nabla_{w}\right) \theta_{h t}^{u}, \theta_{h t}^{u}\right)+s_{1}\left(\theta_{h t}^{u}, \theta_{h t}^{u}\right)+s_{2}\left(\theta_{h t}^{p}, \theta_{h t}^{p}\right) \\
= & -\rho\left(\eta_{h t t}^{u}, \theta_{h t}^{u}\right) .
\end{aligned}
$$

By integrating over $(0, t)$, we have

$$
\begin{align*}
& \frac{\rho}{2}\left\|\theta_{h t}^{u}(t)\right\|^{2}+\int_{0}^{t}\left(C_{1}\left\|\nabla_{w} \theta_{h t}^{u}\right\|^{2}+C_{0} \rho\left\|\theta_{h t}^{u}\right\|^{2}+\frac{1}{2} s_{1}\left(\theta_{h t}^{u}, \theta_{h t}^{u}\right)+s_{2}\left(\theta_{h t}^{p}, \theta_{h t}^{p}\right)\right) d \bar{s} \\
\leq & \frac{\rho}{2}\left\|\theta_{h t}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s} . \tag{4.7}
\end{align*}
$$

Note the following equation

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left(\mu\left\|\nabla_{w} \theta_{h}^{u}\right\|^{2}+s_{1}\left(\theta_{h}^{u}, \theta_{h}^{u}\right)+s_{2}\left(\theta_{h}^{p}, \theta_{h}^{p}\right)\right) \\
= & \left(\mu \nabla_{w} \theta_{h t}^{u}, \nabla_{w} \theta_{h}^{u}\right)+s_{1}\left(\theta_{h t}^{u}, \theta_{h}^{u}\right)+s_{2}\left(\theta_{h t}^{p}, \theta_{h}^{p}\right) .
\end{aligned}
$$

Then, it follows from (4.6) and (4.7) that

$$
\begin{align*}
& \quad \mu\left\|\nabla_{w} \theta_{h}^{u}(t)\right\|^{2}+s_{1}\left(\theta_{h}^{u}(t), \theta_{h}^{u}(t)\right)+s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right)  \tag{4.8}\\
& \leq \mu\left\|\nabla_{w} \theta_{h}^{u}(0)\right\|^{2}+s_{1}\left(\theta_{h}^{u}(0), \theta_{h}^{u}(0)\right)+s_{2}\left(\theta_{h}^{p}(0), \theta_{h}^{p}(0)\right)+\int_{0}^{t}\left(\mu\left\|\nabla_{w} \theta_{h t}^{u}\right\|^{2}\right. \\
& \left.\quad+s_{1}\left(\theta_{h t}^{u}, \theta_{h t}^{u}\right)+s_{2}\left(\theta_{h t}^{p}, \theta_{h t}^{p}\right)\right) d \bar{s}+\int_{0}^{t}\left(\mu\left\|\nabla_{w} \theta_{h}^{u}\right\|^{2}+s_{1}\left(\theta_{h}^{u}, \theta_{h}^{u}\right)+s_{2}\left(\theta_{h}^{p}, \theta_{h}^{p}\right)\right) d \bar{s} \\
& \leq \mu\left\|\nabla_{w} \theta_{h}^{u}(0)\right\|^{2}+s_{1}\left(\theta_{h}^{u}(0), \theta_{h}^{u}(0)\right)+s_{2}\left(\theta_{h}^{p}(0), \theta_{h}^{p}(0)\right) \\
& \quad+\rho\left\|\theta_{h}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+\rho\left\|\theta_{h t}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}
\end{align*}
$$

which, together with (4.8), leads to the error estimate for the velocity in energy norm:

$$
\begin{aligned}
\left\|e_{h}^{u}(t)\right\|^{2} \leq & \leq\left\|\eta_{h}^{u}(t)\right\|^{2}+\left\|\theta_{h}^{u}(t)\right\|^{2} \\
& \leq\left\|\eta_{h}^{u}(t)\right\|^{2}+\left\|\theta_{h}^{u}(0)\right\|^{2} \\
& +\rho\left\|\theta_{h}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+\rho\left\|\theta_{h t}^{u}(0)\right\|^{2}+C \int_{0}^{t}\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}
\end{aligned}
$$

In the following, we shall estimate the pressure in $L^{2}$ by using the inf-sup condition in Lemma 3.2 and the equations (4.5):

$$
\begin{aligned}
& \left\|\theta_{h}^{p}(t)\right\|^{2} \leq 2 b\left(\mathbf{v}_{h}, \theta_{h}^{p}(t)\right)+C h^{1-\alpha} s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right) \\
= & 2 \rho\left(\theta_{h t}^{u}(t), \mathbf{v}_{0}\right)+2 a\left(\theta_{h}^{u}(t), \mathbf{v}_{h}\right)-2 \rho\left(\eta_{h t}^{u}(t), \mathbf{v}_{0}\right)+C h^{1-\alpha} s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right) \\
\leq & C\left(\left\|\theta_{h t}^{u}(t)\right\|+\left\|\eta_{h t}^{u}(t)\right\|+\left\|\theta_{h}^{u}(t)\right\|\right)\left\|\mathbf{v}_{h}\right\|+C h^{1-\alpha} s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right) \\
\leq & C\left(\left\|\theta_{h t}^{u}(t)\right\|+\left\|\eta_{h t}^{u}(t)\right\|+\left\|\theta_{h}^{u}(t)\right\|\right)\left(1+h^{\frac{1-\alpha}{2}}\right)\left\|\theta_{h}^{p}(t)\right\|+C h^{1-\alpha} s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right) .
\end{aligned}
$$

Here we have used the fact $\left\|\mathbf{v}_{0}\right\| \leq C\left\|\mathbf{v}_{h}\right\|$. Hence, from the Cauchy-Schwarz inequality and (4.7), (4.6), (4.8) we have

$$
\begin{align*}
& \left\|\theta_{h}^{p}(t)\right\|^{2} \leq C\left(1+h^{1+\gamma}\right)\left(\left\|\theta_{h t}^{u}(t)\right\|^{2}+\left\|\eta_{h t}^{u}(t)\right\|^{2}+\left\|\theta_{h}^{u}(t)\right\|^{2}\right)+C h^{1-\alpha} s_{2}\left(\theta_{h}^{p}(t), \theta_{h}^{p}(t)\right)  \tag{4.9}\\
& \leq C\left(1+h^{1+\gamma}+h^{1-\alpha}\right)\left(\int_{0}^{t}\left(\left\|\eta_{h t}^{u}\right\|^{2}+\left\|\eta_{h t}^{u}\right\|^{2}\right) d \bar{s}+\left\|\eta_{h t}^{u}(t)\right\|^{2}\right. \\
& \left.\quad+\mu\left\|\theta_{h}^{u}(0)\right\|^{2}+\frac{\rho}{2}\left\|\theta_{h t}^{u}(0)\right\|^{2}\right)
\end{align*}
$$

Similar to the estimate for the velocity, we have

$$
\left\|e_{h}^{p}(t)\right\|^{2} \leq\left\|\eta_{h}^{p}(t)\right\|^{2}+\left\|\theta_{h}^{p}(t)\right\|^{2},
$$

which, together with the estimates in Theorem 4.2, completes the convergence proof for the semi-discrete scheme.
4.2. Convergence of Fully-discrete Scheme. Now, we turn our attention to the fully-discrete scheme (2.4). The error term is denoted by

$$
e_{u}^{\bar{n}}=Q_{h} \mathbf{u}\left(t^{\bar{n}}\right)-\mathbf{u}^{\bar{n}}=\eta_{u}^{\bar{n}}+\theta_{u}^{\bar{n}},
$$

where $\eta_{u}^{\bar{n}}=Q_{h} \mathbf{u}\left(t^{\bar{n}}\right)-E_{h} \mathbf{u}\left(t^{\bar{n}}\right)$ and $\theta_{u}^{\bar{n}}=E_{h} \mathbf{u}\left(t^{\bar{n}}\right)-\mathbf{u}^{\bar{n}}$. Similarly, we write $e_{p}^{\bar{n}}=$ $Q_{h}^{p} p\left(t^{\bar{n}}\right)-p^{\bar{n}}=\eta_{p}^{\bar{n}}+\theta_{p}^{\bar{n}}$, and $\eta_{p}^{\bar{n}}=Q_{h}^{p} p\left(t^{\bar{n}}\right)-E_{h}^{p} p\left(t^{\bar{n}}\right), \theta_{p}^{\bar{n}}=E_{h}^{p} p\left(t^{\bar{n}}\right)-p^{\bar{n}}$.

Theorem 4.3. Let $\left(\mathbf{u}^{\bar{n}+1}, p^{\bar{n}+1}\right)$ and $(\mathbf{u}, p)$ be the numerical solution of $g W G$ (2.4) and exact solution of (2.2), respectively. Assume that the velocity $\mathbf{u}, \mathbf{u}_{t}, \mathbf{u}_{t t} \in$ $\left[H^{i}(\Omega)\right]^{d}, i=\max \{s+2, k+1, j+1\}$ and the pressure $p, p_{t}, p_{t t} \in H^{n+1}(\Omega)$. Let $k-1 \leq n \leq \max \{m, k+1\}$ and $j \geq 1$. Then there exists a constant such that

$$
\begin{align*}
\left\|e_{u}^{N+1}\right\|^{2} & \leq\left\|\theta_{u}^{0}\right\|^{2}+C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s}+C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right)  \tag{4.10}\\
& \left(R_{1}\left(t^{N+1}\right)+\int_{t^{0}}^{t^{N+1}} R_{2}(s) d \bar{s}\right), \\
\left\|e_{u}^{N+1}\right\|^{2} & \leq \mu\left\|\nabla_{w} \theta_{u}^{0}\right\|^{2}+s_{1}\left(\theta_{u}^{0}, \theta_{u}^{0}\right)+s_{2}\left(\theta_{p}^{0}, \theta_{p}^{0}\right)+\rho\left\|\bar{\partial}_{t} \theta_{u}^{1}\right\|^{2}+\rho\left\|\theta_{u}^{0}\right\|^{2}+C R_{1}\left(t^{N+1}\right) \\
& +C\left(h^{2}+h^{2 \alpha}\right)\left(1+h^{\gamma+1}+h^{2(1-\alpha)}\right) \int_{t^{0}}^{t^{N+1}} R_{2}(\bar{s})+R_{3}(\bar{s}) d \bar{s} \\
& +C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2}+\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s},
\end{align*}
$$

and

$$
\begin{align*}
& \left\|e_{p}^{N+1}\right\|^{2} \leq C\left(1+h^{1+\gamma}+h^{1-\alpha}+h^{2(1-\alpha)}\right)\left(\left\|\theta_{u}^{0}\right\|^{2}+s_{2}\left(\theta_{p}^{0}, \theta_{p}^{0}\right)+\rho\left\|\bar{\partial}_{t} \theta_{u}^{1}\right\|^{2}+\rho\left\|\theta_{u}^{0}\right\|^{2}\right.  \tag{4.11}\\
& +R_{1}\left(t^{N+1}\right)+\int_{t^{0}}^{t^{N+1}} R_{2}(\bar{s})+R_{3}(\bar{s}) d \bar{s}+\tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2}+\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s} \\
& \left.+\frac{1}{\tau^{\prime}} \int_{t^{N}}^{t^{N+1}} R_{2}(\bar{s}) d \bar{s}+\tau^{\prime} \int_{t^{N}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s}\right),
\end{align*}
$$

where $\tau^{\prime}=t^{N+1}-t^{N}$.

Moreover, when $\gamma=-1$ and $\alpha=1$, optimal order of convergence is obtained.
Proof. Observe that the estimate for $\eta_{u}^{\bar{n}+1}$ and $\eta_{p}^{\bar{n}+1}$ has been established in Lemma 4.1. Thus, it suffices to handle $\theta_{u}^{\bar{n}+1}$ and $\theta_{p}^{\bar{n}+1}$. Note the following equations for all $\chi_{h} \in \mathbf{V}_{h}^{0}$ and $\psi \in W_{h}$

$$
\begin{align*}
& \rho\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \chi_{0}\right)+a\left(\theta_{u}^{\bar{n}+1}, \chi_{h}\right)-\left(\theta_{p}^{\bar{n}+1}, \nabla_{w} \cdot \chi_{h}\right) \\
= & \rho\left(\bar{\partial}_{t} \eta_{u}^{\bar{n}+1}, \chi_{0}\right)+\rho\left(\bar{\partial}_{t} \mathbf{u}\left(t^{\bar{n}+1}\right)-\mathbf{u}_{t}\left(t^{\bar{n}+1}\right), \chi_{0}\right)  \tag{4.12}\\
& \left(\nabla_{w} \cdot \theta_{u}^{\bar{n}+1}, \psi\right)+s_{2}\left(\theta_{p}^{\bar{n}+1}, \psi\right)=0 .
\end{align*}
$$

By choosing $\chi_{h}=\theta_{u}^{\bar{n}+1}$ and $\psi=\theta_{p}^{\bar{n}+1}$ in (4.12) and using Lemma 3.4, we obtain

$$
\begin{aligned}
& \frac{\rho}{2 \tau}\left(\left\|\theta_{u}^{\bar{n}+1}\right\|^{2}-\left\|\theta_{u}^{\bar{n}}\right\|^{2}\right)+C_{1}\left\|\nabla_{w} \theta_{u}^{\bar{n}+1}\right\|^{2}+C_{0} \rho\left\|\theta_{u}^{\bar{n}+1}\right\|^{2} \\
& +\frac{1}{2} s_{1}\left(\theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right) \\
\leq & C\left\|\bar{\partial}_{t} \eta_{u}^{\bar{n}+1}\right\|^{2}+C\left\|\bar{\partial}_{t} \mathbf{u}\left(t^{\bar{n}+1}\right)-\mathbf{u}_{t}\left(t^{\bar{n}+1}\right)\right\|^{2} \\
\leq & C\left\|\frac{1}{\tau} \int_{t^{\bar{n}}}^{t^{\bar{n}+1}} \eta_{h t}^{u} d \bar{s}\right\|^{2}+C\left\|\frac{1}{\tau} \int_{t^{\bar{n}}}^{t^{\bar{n}+1}}\left(t^{\bar{n}}-\bar{s}\right) \mathbf{u}_{t t} d \bar{s}\right\|^{2} \\
\leq & C \frac{1}{\tau} \int_{t^{\bar{n}}}^{t^{\bar{n}+1}}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+C \tau \int_{t^{\bar{n}}}^{t^{\bar{n}+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s} .
\end{aligned}
$$

Thus, summing the above inequality from 0 to $N$ yields

$$
\begin{align*}
& \rho\left\|\theta_{u}^{N+1}\right\|^{2}+\tau \sum_{\bar{n}=0}^{N}\left(2 C_{1}\left\|\nabla_{w} \theta_{u}^{\bar{n}+1}\right\|^{2}+2 C_{0} \rho\left\|\theta_{u}^{\bar{n}+1}\right\|^{2}+s_{1}\left(\theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)\right. \\
& \left.+2 s_{2}\left(\theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right)\right)  \tag{4.13}\\
& \leq \rho\left\|\theta_{u}^{0}\right\|^{2}+C \int_{t^{0}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s} .
\end{align*}
$$

Thus,

$$
\begin{aligned}
\left\|e_{u}^{N+1}\right\|^{2} & \leq\left\|\eta_{u}^{N+1}\right\|^{2}+\left\|\theta_{u}^{N+1}\right\|^{2} \\
& \leq\left\|\eta_{u}^{N+1}\right\|^{2}+\left\|\theta_{u}^{0}\right\|^{2}+C \int_{t^{0}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s}
\end{aligned}
$$

Combining (4.13) with Lemma 4.1 yields the estimate for the velocity in $L^{2}$ norm.
In order to estimate the velocity in energy norm, we rewrite (4.12) as follows

$$
\begin{aligned}
& \rho\left(\bar{\partial}_{t t} \theta_{u}^{\bar{n}+1}, \chi_{0}\right)+a\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \chi_{h}\right)-\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \nabla_{w} \cdot \chi_{h}\right) \\
= & \rho\left(\bar{\partial}_{t t} \eta_{u}^{\bar{n}+1}, \chi_{0}\right)+\rho\left(\bar{\partial}_{t t} \mathbf{u}\left(t^{\bar{n}+1}\right)-\bar{\partial}_{t} \mathbf{u}_{t}\left(t^{\bar{n}+1}\right), \chi_{0}\right) \\
& \left(\nabla_{w} \cdot \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \psi\right)+s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \psi\right)=0
\end{aligned}
$$

where $\bar{\partial}_{t t} \theta_{u}^{\bar{n}+1}=\frac{\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}-\bar{\partial}_{t} \theta_{u}^{\bar{n}}}{\tau}$. Taking $\chi_{h}=\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}$ and $\psi=\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}$ in the above
identity leads to

$$
\begin{aligned}
& \frac{\rho}{2 \tau}\left(\left\|\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}-\left\|\bar{\partial}_{t} \theta_{u}^{\bar{n}}\right\|^{2}\right)+C_{1}\left\|\nabla_{w} \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}+C_{0} \rho\left\|\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2} \\
& +\frac{1}{2} s_{1}\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{p}^{\bar{n}+1}\right) \\
\leq & C\left\|\bar{\partial}_{t t} \eta_{u}^{\bar{n}+1}\right\|^{2}+C\left\|\bar{\partial}_{t t} \mathbf{u}\left(t^{\bar{n}+1}\right)-\bar{\partial}_{t} \mathbf{u}_{t}\left(t^{\bar{n}+1}\right)\right\|^{2} \\
\leq & C \frac{1}{\tau} \int_{t^{\bar{n}-1}}^{t^{\bar{n}+1}}\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}+C \tau \int_{t^{\bar{n}-1}}^{t^{\bar{n}+1}}\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s} .
\end{aligned}
$$

Here, we have used

$$
\left\|\bar{\partial}_{t t} \eta_{u}^{\bar{n}+1}\right\|^{2} \leq \frac{1}{\tau} \int_{t^{\bar{n}-1}}^{t^{\bar{n}+1}}\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}
$$

and

$$
\left\|\bar{\partial}_{t t} \mathbf{u}\left(t^{\bar{n}+1}\right)-\bar{\partial}_{t} \mathbf{u}_{t}\left(t^{\bar{n}+1}\right)\right\|^{2} \leq \tau \int_{t^{\bar{n}-1}}^{t^{\bar{n}+1}}\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s}
$$

Summing $\bar{n}$ from 0 to $N$ gives rise to

$$
\begin{align*}
& \quad \rho\left(\left\|\bar{\partial}_{t} \theta_{u}^{N+1}\right\|^{2}-\left\|\bar{\partial}_{t} \theta_{u}^{1}\right\|^{2}\right)+\tau \sum_{\bar{n}=0}^{N}\left(2 C_{1}\left\|\nabla_{w} \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}+2 C_{0} \rho\left\|\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}\right. \\
& \left.\quad+s_{1}\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right)+2 s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{p}^{\bar{n}+1}\right)\right)  \tag{4.14}\\
& \leq C \int_{t^{0}}^{t^{N+1}}\left\|\eta_{h t t}^{u}\right\|^{2} d s+C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t t}\right\|^{2} d s
\end{align*}
$$

On the other side, we have by using the Cauchy-Schwarz inequality

$$
\begin{aligned}
& \frac{1}{2 \tau}\left(\mu\left\|\nabla_{w} \theta_{u}^{\bar{n}+1}\right\|^{2}-\mu\left\|\nabla_{w} \theta_{u}^{\bar{n}}\right\|^{2}+s_{1}\left(\theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)-s_{1}\left(\theta_{u}^{\bar{n}}, \theta_{u}^{\bar{n}}\right)\right. \\
& \left.+s_{2}\left(\theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right)-s_{2}\left(\theta_{p}^{\bar{n}}, \theta_{p}^{\bar{n}}\right)\right) \\
\leq & \mu\left(\nabla_{w} \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \nabla_{w} \theta_{u}^{\bar{n}+1}\right)+s_{1}\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right) \\
\leq & \frac{1}{2}\left(\mu\left\|\nabla_{w} \theta_{u}^{\bar{n}+1}\right\|^{2}+s_{1}\left(\theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right)\right) \\
& +\frac{1}{2}\left(\mu\left\|\nabla_{w} \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}+s_{1}\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{p}^{\bar{n}+1}\right)\right) .
\end{aligned}
$$

Summing the above inequality from 0 to $N$ leads to

$$
\begin{aligned}
& \mu\left\|\nabla_{w} \theta_{u}^{N+1}\right\|^{2}-\mu\left\|\nabla_{w} \theta_{u}^{0}\right\|^{2}+s_{1}\left(\theta_{u}^{N+1}, \theta_{u}^{N+1}\right)-s_{1}\left(\theta_{u}^{0}, \theta_{u}^{0}\right) \\
& +s_{2}\left(\theta_{p}^{N+1}, \theta_{p}^{N+1}\right)-s_{2}\left(\theta_{p}^{0}, \theta_{p}^{0}\right) \\
\leq & \tau \sum_{\bar{n}=0}^{N}\left(\mu\left\|\nabla_{w} \theta_{u}^{\bar{n}+1}\right\|^{2}+s_{1}\left(\theta_{u}^{\bar{n}+1}, \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\theta_{p}^{\bar{n}+1}, \theta_{p}^{\bar{n}+1}\right)\right) \\
& +\tau \sum_{\bar{n}=0}^{N}\left(\mu\left\|\nabla_{w} \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right\|^{2}+s_{1}\left(\bar{\partial}_{t} \theta_{u}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{u}^{\bar{n}+1}\right)+s_{2}\left(\bar{\partial}_{t} \theta_{p}^{\bar{n}+1}, \bar{\partial}_{t} \theta_{p}^{\bar{n}+1}\right)\right) .
\end{aligned}
$$

Together with (4.13) and (4.14), we obtain

$$
\begin{align*}
& \quad \mu\left\|\nabla_{w} \theta_{u}^{N+1}\right\|^{2}+s_{1}\left(\theta_{u}^{N+1}, \theta_{u}^{N+1}\right)+s_{2}\left(\theta_{p}^{N+1}, \theta_{p}^{N+1}\right) \\
& \leq \mu\left\|\nabla_{w} \theta_{u}^{0}\right\|^{2}+s_{1}\left(\theta_{u}^{0}, \theta_{u}^{0}\right)+s_{2}\left(\theta_{p}^{0}, \theta_{p}^{0}\right)+\rho\left\|\bar{\partial}_{t} \theta_{u}^{1}\right\|^{2}+\rho\left\|\theta_{u}^{0}\right\|^{2}  \tag{4.15}\\
& \quad+C \int_{t^{0}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2}+\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}+C \tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2}+\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s}
\end{align*}
$$

Note that $\left\|e_{u}^{N+1}\right\|^{2} \leq\left\|\eta_{u}^{N+1}\right\|^{2}+\left\|\theta_{u}^{N+1}\right\|^{2}$. Thus, by combining the above inequality and Lemma 4.1 we obtain the desired estimate for the velocity in energy norm.

To estimate the pressure in $L^{2}$ norm, we first use Lemma 3.2 and (4.12) to obtain

$$
\begin{aligned}
& \quad\left\|\theta_{p}^{N+1}\right\|^{2} \leq 2 b\left(\mathbf{v}_{h}, \theta_{p}^{N+1}\right)+C h^{1-\alpha} s_{2}\left(\theta_{p}^{N+1}, \theta_{p}^{N+1}\right) \\
& = \\
& \quad \rho\left(\bar{\partial}_{t} \theta_{u}^{N+1}, \mathbf{v}_{0}\right)+a\left(\theta_{u}^{N+1}, \mathbf{v}_{h}\right)-\rho\left(\bar{\partial}_{t} \eta_{u}^{N+1}, \mathbf{v}_{0}\right)-\rho\left(\bar{\partial}_{t} \mathbf{u}\left(t^{N+1}\right)-\mathbf{u}_{t}\left(t^{N+1}\right), \mathbf{v}_{0}\right) \\
& \quad+C h^{1-\alpha} s_{2}\left(\theta_{p}^{N+1}, \theta_{p}^{N+1}\right)
\end{aligned}
$$

Next from the Cauchy-Schwarz inequality and (4.13), (4.14), (4.15), we have

$$
\begin{aligned}
\left\|\theta_{p}^{N+1}\right\|^{2} \leq & C\left(1+h^{1+\gamma}\right)\left(\left\|\bar{\partial}_{t} \theta_{u}^{N+1}\right\|^{2}+\left\|\theta_{u}^{N+1}\right\|^{2}+\left\|\theta_{u}^{N+1}\right\|^{2}\right. \\
& \left.+\frac{1}{\tau^{\prime}} \int_{t^{N}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+\tau^{\prime} \int_{t^{N}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d \bar{s}\right)+C h^{1-\alpha} s_{2}\left(\theta_{p}^{N+1}, \theta_{p}^{N+1}\right) \\
\leq & C\left(1+h^{1+\gamma}+h^{1-\alpha}\right)\left(\mu\left\|\nabla_{w} \theta_{u}^{0}\right\|^{2}+s_{1}\left(\theta_{u}^{0}, \theta_{u}^{0}\right)+s_{2}\left(\theta_{p}^{0}, \theta_{p}^{0}\right)+\rho\left\|\bar{\partial}_{t} \theta_{u}^{1}\right\|^{2}\right. \\
& +\rho\left\|\theta_{u}^{0}\right\|^{2}+\int_{t^{0}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2}+\left\|\eta_{h t t}^{u}\right\|^{2} d \bar{s}+\tau^{2} \int_{t^{0}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2}+\left\|\mathbf{u}_{t t t}\right\|^{2} d \bar{s} \\
& \left.+\frac{1}{\tau^{\prime}} \int_{t^{N}}^{t^{N+1}}\left\|\eta_{h t}^{u}\right\|^{2} d \bar{s}+\tau^{\prime} \int_{t^{N}}^{t^{N+1}}\left\|\mathbf{u}_{t t}\right\|^{2} d s\right)
\end{aligned}
$$

where $\tau^{\prime}=t^{N+1}-t^{N}$. This completes the desired estimate for the pressure approximation by using Lemma 4.1 and the inequality $\left\|e_{p}^{N+1}\right\|^{2} \leq\left\|\eta_{p}^{N+1}\right\|^{2}+\left\|\theta_{p}^{N+1}\right\|^{2}$.

Remark 2. For the case of $n \leq j$, the gWG scheme (2.4) for the evolutionary equation can be formulated without including the stabilizer $s_{2}$; i.e. $\sigma=0$. The corresponding error estimate can be obtained without any difficulty by following the above procedure.
5. Numerical Experiments. In this section, we will present two computational examples to verify the convergence theory established in previous sections. One of the two examples is a steady-state Oseen equation and the other is an evolutionary one. We employ the elements $\left(\left[P_{1}\right]^{2},\left[P_{0}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{0}, P_{0}\right)$ and $\left(\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right)$ in the numerical implementation. The parameters are set as follows: $\mu=1, \rho=1$ and $\zeta=1$. The numerical experiments are based on a regular family $\mathcal{T}_{h}$ of triangulations for the domain.

Example 1. Consider the steady-state Oseen equation in domain $\Omega=(0,1)^{2}$. The exact solution is given by $\mathbf{u}=\left[x^{2} y ;-x y^{2}\right]$ and $p=(2 x-1)(2 y-1)$. The convective vector is set as $\beta=[-x+\sin x \sin y ; \cos x \cos y]$ so that $\nabla \cdot \beta=-1$.

The results are shown in Tables 5.1 and 5.2. For the $\left(\left[P_{1}\right]^{2},\left[P_{0}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{0}, P_{0}\right)$ element, we observe a convergence of order $O(h)$ for the velocity in energy norm and
the pressure in $L^{2}$ norm. The convergence is shown to be of order $O\left(h^{2}\right)$ for the velocity in $L^{2}$ norm in Table 5.1 For the $\left(\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right)$ element, we obtained the expected order of convergence for both the velocity and the pressure in various norms, as demonstrated in Table 5.2. The numerical results are in great consistency with the convergence theory developed in previous sections.

| h | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{p}\right\\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | order | error | order | error | order |
| $1 / 8$ | $9.5305 \mathrm{e}-02$ |  | $4.6205 \mathrm{e}-03$ |  | $1.3175 \mathrm{e}-01$ |  |
| $1 / 16$ | $4.9781 \mathrm{e}-02$ | 0.93 | $1.3090 \mathrm{e}-03$ | 1.81 | $6.4353 \mathrm{e}-02$ | 1.03 |
| $1 / 32$ | $2.5309 \mathrm{e}-02$ | 0.97 | $3.4292 \mathrm{e}-04$ | 1.93 | $3.0501 \mathrm{e}-02$ | 1.07 |
| $1 / 64$ | $1.2727 \mathrm{e}-02$ | 0.99 | $8.7119 \mathrm{e}-05$ | 1.97 | $1.4586 \mathrm{e}-02$ | 1.06 |

Table 5.1: Example 1 Convergence with elements $\left(\left[P_{1}\right]^{2},\left[P_{0}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{0}, P_{0}\right): \gamma=1$ and $\sigma=0$, i.e. without the stabilizer $s_{2}$

| h | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{p}\right\\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | order | error | order | error | order |
| $1 / 8$ | $8.4499 \mathrm{e}-03$ |  | $3.3478 \mathrm{e}-04$ |  | $1.1235 \mathrm{e}-02$ |  |
| $1 / 16$ | $2.1246 \mathrm{e}-03$ | 1.99 | $4.1883 \mathrm{e}-05$ | 2.99 | $2.8109 \mathrm{e}-03$ | 1.99 |
| $1 / 32$ | $5.3256 \mathrm{e}-04$ | 1.99 | $5.2382 \mathrm{e}-06$ | 2.99 | $7.0305 \mathrm{e}-04$ | 1.99 |
| $1 / 64$ | $1.3331 \mathrm{e}-04$ | 1.99 | $6.5497 \mathrm{e}-07$ | 2.99 | $1.7581 \mathrm{e}-04$ | 1.99 |

Table 5.2: Example 1 Convergence with elements $\left(\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right): \gamma=1$ and $\sigma=0$, i.e. without the stabilizer $s_{2}$

Example 2. Consider the evolutionary Oseen equation with domain $(0,1)^{2} \times$ $(0, \bar{T}]$. The exact solution is chosen as $\mathbf{u}=e^{-t}\left[x^{2} y ;-x y^{2}\right]$ and $p=\sin t(2 x-1)(2 y-1)$. The convective vector is set as $\beta=[-x+\sin x \sin y ; \cos x \cos y]$ and $\bar{T}=1$.

For the evolutionary case, the convergence depends on the mesh size $h$ and the time step $\tau$. Tables 5.3 and 5.4 illustrate the order of convergence in terms of the mesh size $h$. It can be seen that the expected convergence order of $O(h)$ is illustrated for the velocity in energy norm and the pressure in $L^{2}$ norm. For the velocity in $L^{2}$ norm, Table 5.3 shows a clear convergence of $O\left(h^{2}\right)$. For the $\left(\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right)$ element, Table 5.4 shows a convergence order of $O\left(h^{2}\right)$ for the velocity in energy norm and the pressure in $L^{2}$ norm. The optimal order of convergence $O\left(h^{3}\right)$ was clearly seen in Table 5.4 for the velocity in $L^{2}$ norm.

Table 5.5 demonstrates the convergence of the numerical scheme in time discretization. As the backward Euler scheme was implemented in the fully-discrete scheme, the table shows a convergence of order $O(\tau)$ in time with expectation. All the results are consistent with the theory developed in this paper.
6. Conclusions. In this paper, a gWG method was introduced and analyzed for the evolutionary Oseen equation by using generalized weak gradient for velocity. This new numerical scheme was designed without using the usual skew-symmetric formulation for the convective term. Error estimates of optimal order were established for the new scheme with arbitrary combination of polynomials. The backward Euler discretization was employed in the fully discrete scheme. Numerical experiments were conducted to validate the theory developed in this paper.

| h | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{p}\right\\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | order | error | order | error | order |
| $1 / 8$ | $5.6155 \mathrm{e}-02$ |  | $2.9496 \mathrm{e}-03$ |  | $1.0792 \mathrm{e}-01$ |  |
| $1 / 16$ | $3.0217 \mathrm{e}-02$ | 0.89 | $8.7776 \mathrm{e}-04$ | 1.74 | $5.3055 \mathrm{e}-02$ | 1.02 |
| $1 / 32$ | $1.5530 \mathrm{e}-02$ | 0.96 | $2.3377 \mathrm{e}-04$ | 1.90 | $2.5267 \mathrm{e}-02$ | 1.07 |
| $1 / 64$ | $7.8384 \mathrm{e}-03$ | 0.98 | $5.9719 \mathrm{e}-05$ | 1.96 | $1.2118 \mathrm{e}-02$ | 1.06 |

Table 5.3: Example 2 Convergence with elements $\left(\left[P_{1}\right]^{2},\left[P_{0}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{0}, P_{0}\right): \tau=h^{2}, \gamma=1$ and $\sigma=0$, i.e. without the stabilizer $s_{2}$

| h | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{p}\right\\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | order | error | order | error | order |
| $1 / 4$ | $2.6240 \mathrm{e}-02$ |  | $2.0985 \mathrm{e}-03$ |  | $3.7737 \mathrm{e}-02$ |  |
| $1 / 8$ | $6.6697 \mathrm{e}-03$ | 1.97 | $2.6389 \mathrm{e}-04$ | 2.99 | $9.4291 \mathrm{e}-03$ | 2.00 |
| $1 / 16$ | $1.6786 \mathrm{e}-03$ | 1.99 | $3.3072 \mathrm{e}-05$ | 2.99 | $2.3582 \mathrm{e}-03$ | 1.99 |
| $1 / 32$ | $4.2090 \mathrm{e}-04$ | 1.99 | $4.1448 \mathrm{e}-06$ | 2.99 | $5.8976 \mathrm{e}-04$ | 1.99 |

Table 5.4: Example2 Convergence with elements $\left(\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right): \tau=h^{2}, \gamma=1$ and $\sigma=0$, i.e. without the stabilizer $s_{2}$

| $\tau$ | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{\mathbf{u}}\right\\|$ |  | $\left\\|e_{h}^{p}\right\\|$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | order | error | order | error | order |
| $1 / 4$ | $5.9875 \mathrm{e}-04$ |  | $7.6289 \mathrm{e}-05$ |  | $3.2232 \mathrm{e}-03$ |  |
| $1 / 8$ | $2.8739 \mathrm{e}-04$ | 1.05 | $3.6495 \mathrm{e}-05$ | 1.06 | $1.5441 \mathrm{e}-03$ | 1.06 |
| $1 / 16$ | $1.4248 \mathrm{e}-04$ | 1.01 | $1.7854 \mathrm{e}-05$ | 1.03 | $7.5657 \mathrm{e}-04$ | 1.02 |
| $1 / 32$ | $7.4106 \mathrm{e}-05$ | 0.94 | $8.8315 \mathrm{e}-06$ | 1.01 | $3.7568 \mathrm{e}-04$ | 1.00 |

Table 5.5: Example 2 Convergence with elements ( $\left.\left[P_{2}\right]^{2},\left[P_{1}\right]^{2},\left[P_{1}\right]^{2 \times 2}, P_{1}, P_{1}\right): h=1 / 128, \gamma=1$ and $\sigma=0$, i.e. without the stabilizer $s_{2}$

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