# HIGH ORDER MORLEY ELEMENTS FOR BIHARMONIC EQUATIONS ON POLYTOPAL PARTITIONS 

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#### Abstract

This paper introduces an extension of the Morley element for approximating solutions to biharmonic equations. Traditionally limited to piecewise quadratic polynomials on triangular elements, the extension leverages weak Galerkin finite element methods to accommodate higher degrees of polynomials and the flexibility of general polytopal elements. By utilizing the Schur complement of the weak Galerkin method, the extension allows for fewest local degrees of freedom while maintaining sufficient accuracy and stability for the numerical solutions. The numerical scheme incorporates locally constructed weak tangential derivatives and weak second order partial derivatives, resulting in an accurate approximation of the biharmonic equation. Optimal order error estimates in both a discrete $H^{2}$ norm and the usual $L^{2}$ norm are established to assess the accuracy of the numerical approximation. Additionally, numerical results are presented to validate the developed theory and demonstrate the effectiveness of the proposed extension.


Key words. weak Galerkin, finite element method, Morley element, biharmonic equation, weak tangential derivative, weak second order partial derivative, polytopal partitions.

AMS subject classifications. Primary 65N30, 65N12, 65N15; Secondary 35B45, 35J50.

1. Introduction. This paper is concerned with the new development of high order Morley elements for the biharmonic equation by using the weak Galerkin (WG) method. For simplicity, we consider the biharmonic equation that seeks an unknown function $u$ satisfying

$$
\begin{align*}
\Delta^{2} u & =g, & & \text { in } \Omega \\
u & =\zeta, & & \text { on } \partial \Omega  \tag{1.1}\\
\frac{\partial u}{\partial \mathbf{n}} & =\xi, & & \text { on } \partial \Omega
\end{align*}
$$

where $\Omega \subset \mathbb{R}^{d}(d=2,3)$ is a bounded polytopal domain with Lipschitz continuous boundary $\partial \Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial \Omega$.

A weak formulation of (1.1) seeks $u \in H^{2}(\Omega)$ satisfying $\left.u\right|_{\partial \Omega}=\zeta$ and $\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial \Omega}=\xi$

[^0]such that
\[

$$
\begin{equation*}
\sum_{i, j=1}^{d}\left(\partial_{i j}^{2} u, \partial_{i j}^{2} v\right)=(g, v), \quad \forall v \in H_{0}^{2}(\Omega) \tag{1.2}
\end{equation*}
$$

\]

where $H_{0}^{2}(\Omega)=\left\{v \in H^{2}(\Omega):\left.v\right|_{\partial \Omega}=0,\left.\nabla v\right|_{\partial \Omega}=\mathbf{0}\right\}$.
The $H^{2}$-conforming finite element method for the biharmonic equation is wellknown but requires a $C^{1}$-continuity of piecewise polynomials on simplicial elements, which poses practical difficulties. To address this issue, various nonconforming finite element methods were introduced. Among these methods, the Morley element has the fewest degrees of freedom on each triangular element, making it not only a popular research topic but also a practically useful method. Previous works such as [17, 26, 27] extended the Morley element to higher dimensions. Other works, including [16, 28, 29, 8, [15], proposed generalizations of the Morley element for different types of meshes. Parallel algorithms and multigrid methods for the Morley element were developed in [6, 7, 18, 19. Since then, rapid progress has been made in various numerical methods for the biharmonic equation on polytopal meshes, such as discontinuous Galerkin finite element methods [3, 14, 30, virtual element methods [1, 4], and weak Galerkin methods [31, 21, 22, 13, 2, 5, 9, 10, 32. The WG finite element method was first proposed for second-order elliptic problems in [20]. The WG method is a natural generalization of classical finite element methods as it relaxes the continuity requirement for the approximating functions. This weak continuity of the numerical approximation allows for high flexibility in constructing weak finite elements with any desired order of convergence. To the best of our knowledge, no high-order extension has been developed that combines the advantages of the Morley element, including its minimal degrees of freedom, with the ability to handle general polytopal partitions.

The objective of this paper is to present a high-order generalization of the Morley element using the weak Galerkin method. Inspired by the de Rham complexes for weak Galerkin spaces [24, we propose innovations to the original weak finite element procedures. These innovations involve the introduction of additional approximating functions defined on the ( $d-2$ )-dimensional sub-polytopes and ( $d-1$ )-dimensional subpolytopes of $d$-dimensional polytopal elements, resulting in a reduction of the degrees of freedom. To enhance the numerical scheme, we incorporate a locally designed weak tangential derivative operator and a weak second-order partial derivative operator. Furthermore, we establish optimal order error estimates for the resulting numerical approximations in both the energy norm and the $L^{2}$ norm.

The main contributions of this paper can be summarized as follows. Firstly, unlike the original Morley element, the proposed WG extension allows for higher-order polynomial approximation with the local minimum number of degrees of freedom, while also being applicable to general polytopal elements. This extension broadens the scope of problems that can be effectively addressed. Secondly, in comparison to existing results on WG methods, we introduce a novel technique within the WG framework that significantly reduces the number of unknowns. This advancement enhances the efficiency and computational feasibility of the method. Finally, the versatility of the new WG method enables its application to various modeling problems, including those that involve the Hessian operator in their weak formulation.

The paper is structured as follows. In Section 2 we provide a review of the definitions of the discrete weak tangential derivative and the discrete weak second-
order partial derivatives. Section 3 presents the weak Galerkin scheme and introduces its Schur complement. Section 4 establishes the solution existence and uniqueness for this new scheme. Section 5 is devoted to the derivation of an error equation for the weak Galerkin scheme, providing insights into the accuracy of the method. In Section [6] we present some technical results that are utilized in the subsequent section. Section 7 is dedicated to establishing error estimates for the numerical approximation, considering both the energy norm and the $L^{2}$ norm. Finally, in Section 8, we present numerical results that demonstrate the effectiveness of the developed theory.

This paper will follow the standard notations for the Sobolev space. For an open bounded domain $D \subset \mathbb{R}^{d}$ with Lipschitz continuous boundary $\partial D$, we denote by $\|\cdot\|_{s, D}$ and $|\cdot|_{s, D}$ the norm and semi-norm in the Sobolev space $H^{s}(D)$ for any $s \geq 0$. When $s=0$, we use $(\cdot, \cdot)$ and $|\cdot|_{D}$ to denote the usual integral inner product and semi-norm, respectively. The subscript will be omitted when $D=\Omega$. Moreover, we use " $A \lesssim B$ " to denote the inequality " $A \leq C B$ " where $C$ stands for a generic constant independent of the meshsize or the functions appearing in the inequality.
2. Discrete weak derivatives. Let $\mathcal{T}_{h}$ be a polytopal partition of $\Omega$ satisfying the shape regular assumptions described in [23. For $T \in \mathcal{T}_{h}$, denote by $\partial T$ the boundary of $T$ consisting of $(d-1)$-dimensional polytopal elements (called "face" for simplicity). For each face $\mathcal{F} \subset \partial T$, denote by $\partial \mathcal{F}$ the boundary of $\mathcal{F}$ consisting of $(d-2)$-dimensional polytopal elements (called "edge" for simplicity). Denote by $\mathcal{F}_{h}$ the set of all faces for all elements in $\mathcal{T}_{h}$ and $\mathcal{F}_{h}^{0}=\mathcal{F}_{h} \backslash \partial \Omega$ the set of all interior faces. Analogously, denote by $\mathcal{E}_{h}$ the set of all edges for all elements in $\mathcal{T}_{h}$ and $\mathcal{E}_{h}^{0}=\mathcal{E}_{h} \backslash \partial \Omega$ the set of all interior edges. Moreover, denote by $h_{T}$ the meshsize of $T$ and $h=\max _{T \in \mathcal{T}_{h}} h_{T}$ the meshsize of $\mathcal{T}_{h}$. For any given integer $r \geq 0$, denote by $P_{r}(T)$ and $P_{r}(\partial T)$ the space of polynomials on $T$ and $\partial T$ with degrees no more than $r$, respectively.

For each element $T \in \mathcal{T}_{h}$, we introduce a weak function $v=\left\{v_{0}, v_{b, e}, v_{b, f}, v_{n} \mathbf{n}_{f}\right\}$, where $v_{0}$ represents the value of $v$ in the interior of $T, v_{b, e}$ and $v_{b, f}$ represent the values of $v$ on the edge $e$ and face $\mathcal{F}$ respectively, $\mathbf{n}_{f}$ is the unit outward normal vector to $\mathcal{F}$, and $v_{n}$ represents the normal derivative of $v$ on $\partial T$ along the direction $\mathbf{n}_{f}$.

For any given integer $k \geq 3$, denote by $V_{k}(T)$ the discrete space of local weak functions given by

$$
\begin{gathered}
V_{k}(T)=\left\{\left\{v_{0}, v_{b, e}, v_{b, f}, v_{n} \mathbf{n}_{f}\right\}: v_{0} \in P_{k}(T), v_{b, e} \in P_{k-2}(e), v_{b, f} \in P_{k-3}(\mathcal{F}),\right. \\
\\
\left.v_{n} \in P_{k-2}(\mathcal{F}), \mathcal{F} \subset \partial T, e \subset \partial \mathcal{F}\right\} .
\end{gathered}
$$

It should be pointed out that $v_{b, e}=$ const from problems in 2D.
On each face $\mathcal{F}$, we introduce a finite element space $\mathcal{W}_{k-2}(\mathcal{F})$ as polynomial vectors of degree $k-2$ tangential to $\mathcal{F}$ :

$$
\mathcal{W}_{k-2}(\mathcal{F})=\left\{\boldsymbol{\psi}: \boldsymbol{\psi} \in\left[P_{k-2}(\mathcal{F})\right]^{d}, \boldsymbol{\psi} \cdot \mathbf{n}_{f}=0\right\}
$$

Definition 2.1. 24](Discrete weak tangential derivative) The discrete weak tangential derivative for any weak function $v \in V_{k}(T)$, denoted by $\nabla_{w, \boldsymbol{\tau}, k-2, T} v$, is defined as the unique polynomial in $\mathcal{W}_{k-2}(\mathcal{F})$ satisfying

$$
\begin{equation*}
\left\langle\nabla_{w, \boldsymbol{\tau}, k-2, T} v, \boldsymbol{\psi} \times \boldsymbol{n}_{f}\right\rangle_{\mathcal{F}}=-\left\langle v_{b, f},(\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{n}_{f}\right\rangle_{\mathcal{F}}+\left\langle v_{b, e}, \boldsymbol{\psi} \cdot \boldsymbol{\tau}\right\rangle_{\partial \mathcal{F}} \tag{2.1}
\end{equation*}
$$

for all $\boldsymbol{\psi} \in \mathcal{W}_{k-2}(\mathcal{F})$. Here, $\boldsymbol{\tau}$ represents the tangential unit vector on $\partial \mathcal{F}$ that is set such that $\boldsymbol{\tau}$ and $\boldsymbol{n}_{f}$ obey the right hand rule.

With the normal derivative $v_{n}$ and the discrete weak tangential derivative $\nabla_{w, \boldsymbol{\tau}, k-2, T} v$, we can define the weak gradient of $v$ on the face $\mathcal{F}$ as follows:

$$
\begin{equation*}
\boldsymbol{v}_{\boldsymbol{g}}=v_{n} \mathbf{n}_{f}+\nabla_{w, \boldsymbol{\tau}, k-2, T} v \tag{2.2}
\end{equation*}
$$

Definition 2.2. [21] (Discrete weak second order partial derivative) For any $v \in V_{k}(T)$, the discrete weak second order partial derivative, denoted by $\partial_{i j, w, k-2, T}^{2} v$, is defined as a unique polynomial in $P_{k-2}(T)$ satisfying

$$
\begin{equation*}
\left(\partial_{i j, w, k-2, T}^{2} v, \varphi\right)_{T}=\left(v_{0}, \partial_{j i}^{2} \varphi\right)_{T}-\left\langle v_{b, f} n_{i}, \partial_{j} \varphi\right\rangle_{\partial T}+\left\langle v_{g i}, \varphi n_{j}\right\rangle_{\partial T} \tag{2.3}
\end{equation*}
$$

for any $\varphi \in P_{k-2}(T)$. Here, $\mathbf{n}_{f}=\left(n_{1}, \ldots, n_{d}\right)$ represents the unit outward normal vector to $\partial T$, and $v_{g i}$ is the $i$-th component of the vector $\boldsymbol{v}_{\boldsymbol{g}}$ given in (2.2).

By utilizing the integration by parts to the first term on the right-hand side of (2.3) we obtain

$$
\begin{equation*}
\left(\partial_{i j, w, k-2, T}^{2} v, \varphi\right)_{T}=\left(\partial_{i j}^{2} v_{0}, \varphi\right)_{T}+\left\langle\left(v_{0}-v_{b, f}\right) n_{i}, \partial_{j} \varphi\right\rangle_{\partial T}-\left\langle\partial_{i} v_{0}-v_{g i}, \varphi n_{j}\right\rangle_{\partial T} \tag{2.4}
\end{equation*}
$$

for any $\varphi \in P_{k-2}(T)$.
3. Weak Galerkin schemes. We construct a global finite element space $V_{h}$ by patching $V_{k}(T)$ over all the elements $T \in \mathcal{T}_{h}$ through common values $v_{b, e}$ on $\mathcal{E}_{h}^{0}, v_{b, f}$ and $v_{n} \mathbf{n}_{f}$ on $\mathcal{F}_{h}^{0}$; i.e.,

$$
V_{h}=\left\{v=\left\{v_{0}, v_{b, e}, v_{b, f}, v_{n} \mathbf{n}_{f}\right\}:\left.v\right|_{T} \in V_{k}(T), T \in \mathcal{T}_{h}\right\}
$$

Denote by $V_{h}^{0}$ the subspace of $V_{h}$ given by

$$
V_{h}^{0}=\left\{v: v \in V_{h},\left.v_{b, e}\right|_{e}=0,\left.v_{b, f}\right|_{\mathcal{F}}=0,\left.v_{n}\right|_{\mathcal{F}}=0, e \subset \partial \Omega, \mathcal{F} \subset \partial \Omega\right\}
$$

For convenience, denote by $\nabla_{w, \boldsymbol{\tau}} v$ the discrete weak tangential derivative $\nabla_{w, \boldsymbol{\tau}, k-2, T} v$ and $\partial_{i j, w}^{2} v$ the discrete weak second order partial derivative $\partial_{i j, w, k-2, T}^{2} v$; i.e.,

$$
\left.\left(\nabla_{w, \boldsymbol{\tau}} v\right)\right|_{T}=\nabla_{w, \boldsymbol{\tau}, k-2, T}\left(\left.v\right|_{T}\right),\left.\quad\left(\partial_{i j, w}^{2} v\right)\right|_{T}=\partial_{i j, w, k-2, T}^{2}\left(\left.v\right|_{T}\right), \quad v \in V_{h}
$$

Denote by $Q_{b}, Q_{f}$ and $Q_{n}$ the usual $L^{2}$ projection operators onto $P_{k-2}(e)$, $P_{k-3}(\mathcal{F})$ and $P_{k-2}(\mathcal{F})$, respectively. In $V_{h} \times V_{h}$, we introduce the following bilinear forms:

$$
\begin{aligned}
\left(\partial_{w}^{2} w, \partial_{w}^{2} v\right) & =\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j, w}^{2} w, \partial_{i j, w}^{2} v\right)_{T} \\
s(w, v)= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\langle Q_{b} w_{0}-w_{b, e}, Q_{b} v_{0}-v_{b, e}\right\rangle_{\partial \mathcal{F}} \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\langle Q_{f} w_{0}-w_{b, f}, Q_{f} v_{0}-v_{b, f}\right\rangle_{\partial T} \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\langle Q_{n}\left(\nabla w_{0}\right) \cdot \boldsymbol{n}_{f}-w_{n}, Q_{n}\left(\nabla v_{0}\right) \cdot \boldsymbol{n}_{f}-v_{n}\right\rangle_{\partial T} \\
& +\delta_{k, 3} \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\langle Q_{n} D_{\boldsymbol{\tau}} w_{0}-\nabla_{w, \boldsymbol{\tau}} w, Q_{n} D_{\boldsymbol{\tau}} v_{0}-\nabla_{w, \boldsymbol{\tau}} v\right\rangle_{\partial T} \\
a_{s}(w, v) & =\left(\partial_{w}^{2} w, \partial_{w}^{2} v\right)+s(w, v)
\end{aligned}
$$

where $Q_{n} D_{\boldsymbol{\tau}} w_{0}=Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla w_{0} \times \mathbf{n}_{f}\right)\right)$ and $\delta_{k, 3}$ is the usual Kronecker's delta with value 1 when $k=3$ and value 0 otherwise.

Weak Galerkin Algorithm 1. A numerical approximation for the model equation (1.1) based on the weak formulation (1.2) can be obtained by seeking $u_{h}=$ $\left\{u_{0}, u_{b, e}, u_{b, f}, u_{n} \mathbf{n}_{f}\right\} \in V_{h}$ satisfying $u_{b, e}=Q_{b} \zeta$ on $e \subset \partial \Omega, u_{b, f}=Q_{f} \zeta$ and $u_{n}=Q_{n} \xi$ on $\mathcal{F} \subset \partial \Omega$ and the following equation

$$
\begin{equation*}
a_{s}\left(u_{h}, v\right)=\left(g, v_{0}\right), \quad \forall v \in V_{h}^{0} \tag{3.1}
\end{equation*}
$$

One may apply the Schur complement approach to the weak Galerkin scheme (3.1), yielding an equivalent formulation with reduced number of unknowns in the resulting linear system. More specifically, the Schur complement for (1.1) seeks $u_{h}=$ $\left\{D\left(u_{b, e}, u_{b, f}, u_{n}, g\right), u_{b, e}, u_{b, f}, u_{n} \mathbf{n}_{f}\right\} \in V_{h}$ such that $u_{b, e}=Q_{b} \zeta$ on $e \subset \partial \Omega, u_{b, f}=$ $Q_{f} \zeta$ and $u_{n}=Q_{n} \xi$ on $\mathcal{F} \subset \partial \Omega$ satisfying

$$
\begin{equation*}
a_{s}\left(\left\{D\left(u_{b, e}, u_{b, f}, u_{n}, g\right), u_{b, e}, u_{b, f}, u_{n} \mathbf{n}_{f}\right\}, v\right)=0 \tag{3.2}
\end{equation*}
$$

for all $v=\left\{0, v_{b, e}, v_{b, f}, v_{n} \mathbf{n}_{f}\right\} \in V_{h}^{0}$, where $u_{0}=D\left(u_{b, e}, u_{b, f}, u_{n}, g\right)$ is obtained by solving the following equation

$$
\begin{equation*}
a_{s}\left(\left\{u_{0}, u_{b, e}, u_{b, f}, u_{n} \mathbf{n}_{f}\right\}, v\right)=\left(g, v_{0}\right) \tag{3.3}
\end{equation*}
$$

for all $v=\left\{v_{0}, 0,0, \mathbf{0}\right\} \in V_{h}^{0}$.
Remark 3.1. The weak Galerkin scheme (3.1) is equivalent to its Schur complement (3.2)-(3.3). The proof is similar to that in [11]. As an illustration, when $k=3$, the degrees of freedom on a pentagonal element and a hexahedral element are shown in Figure 3.1.


Fig. 3.1. Local degrees of freedom for the finite element space $V_{3}(T)$ on a pentagonal element (left) and a hexahedral element (right).
4. Solution existence and uniqueness. On each element $T \in \mathcal{T}_{h}$, denote by $Q_{0}$ the usual $L^{2}$ projection operator onto $P_{k}(T)$. For any $\phi \in H^{2}(\Omega)$, let

$$
Q_{h} \phi=\left\{Q_{0} \phi, Q_{b} \phi, Q_{f} \phi, Q_{n}\left(\nabla \phi \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}\right\}
$$

Similarly, denote by $\mathbb{Q}_{h}$ the $L^{2}$ projection operator onto $P_{k-2}(T)$.

Lemma 4.1. For any $\phi \in H^{2}(T)$, the following commutative property holds true

$$
\begin{align*}
\nabla_{w, \boldsymbol{\tau}} Q_{h} \phi & =Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla \phi \times \mathbf{n}_{f}\right)\right)  \tag{4.1}\\
\partial_{i j, w}^{2}\left(Q_{h} \phi\right) & =\mathbb{Q}_{h}\left(\partial_{i j}^{2} \phi\right), \quad i, j=1, \ldots, d \tag{4.2}
\end{align*}
$$

Proof. First of all, the identity (4.1) has been established in [24. Hence, the gradient representation (2.2) for $\left(Q_{h} \phi\right)_{g}$ has the following form

$$
\begin{gather*}
\left(Q_{h} \phi\right)_{g}=Q_{n}\left(\nabla \phi \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}+Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla \phi \times \mathbf{n}_{f}\right)\right)  \tag{4.3}\\
=Q_{n}(\nabla \phi) \tag{4.4}
\end{gather*}
$$

In other words, the weak gradient of $Q_{h} \phi$ is the $L^{2}$ projection of the classical gradient of $\phi$ on each face $\mathcal{F} \subset \partial T$. Thus, from (2.3) and the usual integration by parts we obtain

$$
\begin{aligned}
& \left(\partial_{i j, w}^{2}\left(Q_{h} \phi\right), \varphi\right)_{T} \\
= & \left(Q_{0} \phi, \partial_{j i}^{2} \varphi\right)_{T}-\left\langle Q_{f} \phi n_{i}, \partial_{j} \varphi\right\rangle_{\partial T}+\left\langle Q_{n}(\nabla \phi)_{i}, \varphi n_{j}\right\rangle_{\partial T} \\
= & \left(\phi, \partial_{j i}^{2} \varphi\right)_{T}-\left\langle\phi n_{i}, \partial_{j} \varphi\right\rangle_{\partial T}+\left\langle(\nabla \phi)_{i}, \varphi n_{j}\right\rangle_{\partial T} \\
= & \left(\partial_{i j}^{2} \phi, \varphi\right)_{T} \\
= & \left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} \phi\right)^{2}, \varphi\right)_{T}
\end{aligned}
$$

for all $\varphi \in P_{k-2}(T)$. This verifies the identity (4.2).
Observe that the bilinear form $a_{s}(v, v)$ induces a semi-norm in the finite element space $V_{h}$ given by

$$
\begin{equation*}
\|v\|=\left(a_{s}(v, v)\right)^{1 / 2} \tag{4.5}
\end{equation*}
$$

Lemma 4.2. The semi-norm $\|v\| \|$ defined by (4.5) is a norm in the subspace $V_{h}^{0}$.
Proof. It suffices to show that $\|v\| \|=0$ implies $v=0$. To this end, assume $\|v\|=0$ for some $v \in V_{h}^{0}$. From (4.5) we have $\partial_{w}^{2} v=0$ and $s(v, v)=0$, which implies $\partial_{i j, w}^{2} v=0$ for $i, j=1, \ldots, d$ on each $T, Q_{b} v_{0}=v_{b, e}$ on each $\partial \mathcal{F}, Q_{f} v_{0}=v_{b, f}$ and $Q_{n}\left(\nabla v_{0}\right) \cdot \mathbf{n}_{f}=v_{n}$ on each $\partial T$. Thus, on each element $T \in \mathcal{T}_{h}$ we have $Q_{h} v_{0}=v$ so that by using (4.2)

$$
\partial_{i j}^{2} v_{0}=\mathbb{Q}_{h} \partial_{i j}^{2} v_{0}=\partial_{i j, w}^{2}\left(Q_{h} v_{0}\right)=\partial_{i j, w}^{2} v=0, \quad i, j=1, \ldots, d .
$$

Hence, $\nabla v_{0}=$ const on each $T \in \mathcal{T}_{h}$. Note that on each face $\mathcal{F} \in \partial T$ we have

$$
\nabla v_{0}=\left(\nabla v_{0} \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}+\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)
$$

which, together with $Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)=\nabla_{w, \boldsymbol{\tau}} v$ and $Q_{n}\left(\nabla v_{0}\right) \cdot \mathbf{n}_{f}=v_{n}$, gives rise to $\nabla v_{0}=v_{n} \mathbf{n}_{f}+\nabla_{w, \boldsymbol{\tau}} v$ on each face $\mathcal{F} \in \mathcal{F}_{h}$ and hence $\nabla v_{0} \in C^{0}(\Omega)$. Next, with $v_{b, f}=0$ on each $\mathcal{F} \subset \partial \Omega$ and $v_{b, e}=0$ on each $e \subset \partial \Omega$ we have from (2.1) that $\nabla_{w, \boldsymbol{\tau}} v=0$ on each $\mathcal{F} \subset \partial \Omega$. This, together with $v_{n}=0$ on each $\mathcal{F} \subset \partial \Omega$, gives $\nabla v_{0}=0$ on $\mathcal{F} \subset \partial \Omega$ and further $\nabla v_{0}=0$ in the domain $\Omega$ since $\nabla v_{0}=$ const on each $T$ and $\nabla v_{0} \in C^{0}(\Omega)$. Hence, $v_{n}=0$ on each $\mathcal{F}$ and $v_{0}=$ const on each $T$. This further leads to $v_{0}=Q_{b} v_{0}=v_{b, e}$ on each $\partial \mathcal{F}$ and $v_{0}=Q_{f} v_{0}=v_{b, f}$ on each $\partial T$, and hence $v_{0} \in C^{0}(\Omega)$. From $v_{b, e}=0$ on $e \subset \partial \Omega$ and $v_{b, f}=0$ on each $\mathcal{F} \subset \partial \Omega$ we have
$v_{0}=0$ in $\Omega$. Finally, from $v_{b, e}=Q_{b} v_{0}$ on each $\partial \mathcal{F}$ and $v_{b, f}=Q_{f} v_{0}$ on each $\partial T$ we have $v_{b, e}=0$ on each $\partial \mathcal{F}$ and $v_{b, f}=0$ on each $\partial T$. This completes the proof of the lemma.

LEmma 4.3. The weak Galerkin scheme (3.1) has one and only one numerical approximation.

Proof. It suffices to verify the uniqueness of the numerical approximation. To this end, assume that $u_{h}^{(1)}$ and $u_{h}^{(2)}$ are two solutions of (3.1). It is clear that

$$
\begin{equation*}
a_{s}\left(u_{h}^{(1)}-u_{h}^{(2)}, v\right)=0, \quad \forall v \in V_{h}^{0} \tag{4.6}
\end{equation*}
$$

By letting $v=u_{h}^{(1)}-u_{h}^{(2)} \in V_{h}^{0}$ in (4.6) we obtain

$$
\left\|u_{h}^{(1)}-u_{h}^{(2)}\right\|=0
$$

which implies $u_{h}^{(1)}=u_{h}^{(2)}$ from Lemma 4.2. This completes the proof of the lemma.
5. Error equations. Let $u$ be the exact solution of the model equation (1.1) and $u_{h} \in V_{h}$ be the numerical solution of the WG scheme (3.1), respectively. Denote by

$$
\begin{equation*}
e_{h}=Q_{h} u-u_{h} \tag{5.1}
\end{equation*}
$$

the error function between the $L^{2}$ projection of the exact solution and its WG finite element approximation $u_{h}$.

Lemma 5.1. The error function $e_{h}$ defined in (5.1) satisfies the following error equation

$$
\begin{equation*}
a_{s}\left(e_{h}, v\right)=\zeta_{u}(v), \quad \forall v \in V_{h}^{0} \tag{5.2}
\end{equation*}
$$

where $\zeta_{u}(v)$ is given by

$$
\begin{align*}
\zeta_{u}(v)= & s\left(Q_{h} u, v\right)+\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle v_{0}-v_{b, f}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T}  \tag{5.3}\\
& +\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{i} v_{0}-v_{g i},\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right) n_{j}\right\rangle_{\partial T}
\end{align*}
$$

Proof. Let $v \in V_{h}^{0}$. On any face $\mathcal{F} \subset \partial \Omega$, we have $v_{b, f}=0$ and $v_{b, e}=0$ on $e \subset \partial \mathcal{F}$. Thus, from (2.1) we have

$$
\left\langle\nabla_{w, \boldsymbol{\tau}} v, \boldsymbol{\psi} \times \boldsymbol{n}_{f}\right\rangle_{\mathcal{F}}=-\left\langle v_{b, f},(\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{n}_{f}\right\rangle_{\mathcal{F}}+\left\langle v_{b, e}, \boldsymbol{\psi} \cdot \boldsymbol{\tau}\right\rangle_{\partial \mathcal{F}}=0
$$

for any $\boldsymbol{\psi} \in \mathcal{W}_{k-2}(\mathcal{F})$. Hence, $\nabla_{w, \boldsymbol{\tau}} v=0$ on $\partial \Omega$. This, together with (2.2) and $v_{n}=0$ on $\partial \Omega$, gives rise to $\boldsymbol{v}_{g}=0$ on $\partial \Omega$.

By testing the model equation (1.1) against $v_{0}$ and then using the usual integration
by parts, we have

$$
\begin{align*}
\left(g, v_{0}\right)= & \sum_{T \in \mathcal{T}_{h}}\left(\Delta^{2} u, v_{0}\right)_{T} \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j}^{2} u, \partial_{i j}^{2} v_{0}\right)_{T}-\left\langle\partial_{i j}^{2} u, \partial_{i} v_{0} n_{j}\right\rangle_{\partial T}+\left\langle\partial_{j}\left(\partial_{i j}^{2} u\right) n_{i}, v_{0}\right\rangle_{\partial T}  \tag{5.4}\\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j}^{2} u, \partial_{i j}^{2} v_{0}\right)_{T}-\left\langle\partial_{i j}^{2} u,\left(\partial_{i} v_{0}-v_{g i}\right) n_{j}\right\rangle_{\partial T} \\
& +\left\langle\partial_{j}\left(\partial_{i j}^{2} u\right) n_{i}, v_{0}-v_{b, f}\right\rangle_{\partial T}
\end{align*}
$$

where we used the fact that

$$
\begin{aligned}
& \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{i j}^{2} u, v_{g i} n_{j}\right\rangle_{\partial T}=0 \\
& \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{j}\left(\partial_{i j}^{2} u\right) n_{i}, v_{b, f}\right\rangle_{\partial T}=0
\end{aligned}
$$

and $v_{b, f}=0, \mathbf{v}_{g}=0$ on $\mathcal{F} \subset \partial \Omega$.
To handle the first term on last line in (5.4), we choose $\varphi=\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right) \in P_{k-2}(T)$ in (2.4) and then use Lemma 4.1 to obtain

$$
\begin{align*}
\left(\partial_{i j}^{2} v_{0}, \partial_{i j}^{2} u\right)_{T}= & \left(\partial_{i j}^{2} v_{0}, \mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)_{T} \\
= & \left(\partial_{i j, w}^{2} v, \mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)_{T}-\left\langle\left(v_{0}-v_{b, f}\right) n_{i}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)\right\rangle_{\partial T} \\
& +\left\langle\partial_{i} v_{0}-v_{g i}, \mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right) n_{j}\right\rangle_{\partial T}  \tag{5.5}\\
= & \left(\partial_{i j, w}^{2} v, \partial_{i j, w}^{2} Q_{h} u\right)_{T}-\left\langle\left(v_{0}-v_{b, f}\right) n_{i}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)\right\rangle_{\partial T} \\
& +\left\langle\partial_{i} v_{0}-v_{g i}, \mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right) n_{j}\right\rangle_{\partial T} .
\end{align*}
$$

Substituting (5.5) into (5.4) gives

$$
\begin{align*}
\left(g, v_{0}\right)= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j, w}^{2} v, \partial_{i j, w}^{2} Q_{h} u\right)_{T}+\left\langle v_{0}-v_{b, f}, \partial_{j}\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right) n_{i}\right\rangle_{\partial T}  \tag{5.6}\\
& +\left\langle\partial_{i} v_{0}-v_{g i},\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{j}\right\rangle_{\partial T}
\end{align*}
$$

Subtracting (3.1) from (5.6) gives rise to Lemma 5.1.
6. Technical results. Note that for any $T \in \mathcal{T}_{h}$ and $\phi \in H^{1}(T)$, the following trace inequality [23] holds true:

$$
\begin{equation*}
\|\phi\|_{\partial T}^{2} \lesssim h_{T}^{-1}\|\phi\|_{T}^{2}+h_{T}\|\nabla \phi\|_{T}^{2} \tag{6.1}
\end{equation*}
$$

If $\phi$ is a polynomial on the element $T \in \mathcal{T}_{h}$, we have from the inverse inequality that

$$
\begin{equation*}
\|\phi\|_{\partial T}^{2} \lesssim h_{T}^{-1}\|\phi\|_{T}^{2} \tag{6.2}
\end{equation*}
$$

Lemma 6.1. Assume that $\mathcal{T}_{h}$ is a finite element partition satisfying the regular assumptions described in [23]. Then, for any $0 \leq s \leq 2$, the following error estimates
[23, 12] hold true:

$$
\begin{gather*}
\sum_{T \in \mathcal{T}_{h}} h_{T}^{2 s}\left\|\phi-Q_{0} \phi\right\|_{s, T}^{2} \lesssim h^{2(k+1)}\|\phi\|_{k+1}^{2}  \tag{6.3}\\
\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}^{2 s}\left\|\partial_{i j}^{2} \phi-\mathbb{Q}_{h}\left(\partial_{i j}^{2} \phi\right)\right\|_{s, T}^{2} \lesssim h^{2(k-1)}\|\phi\|_{k+1}^{2}
\end{gather*}
$$

Lemma 6.2. For any $v \in V_{h}$, there holds

$$
\begin{equation*}
\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i=1}^{d} h_{T}^{-1}\left\|Q_{n}\left(\partial_{i} v_{0}\right)-v_{g i}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \lesssim\|v\| . \tag{6.5}
\end{equation*}
$$

Proof. From $\nabla v_{0}=\left(\nabla v_{0} \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}+\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)$ and (2.2), we have

$$
\begin{align*}
& \sum_{T \in \mathcal{T}_{h}} \sum_{i=1}^{d} h_{T}^{-1}\left\|Q_{n}\left(\partial_{i} v_{0}\right)-v_{g i}\right\|_{\partial T}^{2} \\
= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\nabla v_{0}\right)-\boldsymbol{v}_{\boldsymbol{g}}\right\|_{\partial T}^{2} \\
= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\nabla v_{0} \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}+Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)-\left(v_{n} \mathbf{n}_{f}+\nabla_{w, \boldsymbol{\tau}} v\right)\right\|_{\partial T}^{2}  \tag{6.6}\\
\lesssim & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\nabla v_{0} \cdot \mathbf{n}_{f}\right)-v_{n}\right\|_{\partial T}^{2}+h_{T}^{-1}\left\|Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)-\nabla_{w, \boldsymbol{\tau}} v\right\|_{\partial T}^{2} \\
\lesssim & \|v\|^{2}+\sum_{\mathcal{F} \in \mathcal{F}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)-\nabla_{w, \boldsymbol{\tau}} v\right\|_{\mathcal{F}}^{2} .
\end{align*}
$$

Next, from (2.1) and the Stokes Theorem we have

$$
\begin{aligned}
& \left|\left\langle Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)-\nabla_{w, \boldsymbol{\tau}} v, \boldsymbol{\psi} \times \mathbf{n}_{f}\right\rangle_{\mathcal{F}}\right| \\
= & \left|\left\langle Q_{f} v_{0}-v_{b, f},(\nabla \times \boldsymbol{\psi}) \cdot \mathbf{n}_{f}\right\rangle_{\mathcal{F}}+\left\langle v_{b, e}-Q_{b} v_{0}, \boldsymbol{\psi} \cdot \boldsymbol{\tau}\right\rangle_{\partial \mathcal{F}}\right| \\
\leq & \left\|Q_{f} v_{0}-v_{b, f}\right\|_{\mathcal{F}}\|\nabla \times \boldsymbol{\psi}\|_{\mathcal{F}}+\left\|v_{b, e}-Q_{b} v_{0}\right\|_{\partial \mathcal{F}}\|\boldsymbol{\psi}\|_{\partial \mathcal{F}} \\
\lesssim & \left\|Q_{f} v_{0}-v_{b, f}\right\|_{\mathcal{F}} h_{T}^{-1}\|\boldsymbol{\psi}\|_{\mathcal{F}}+\left\|v_{b, e}-Q_{b} v_{0}\right\|_{\partial \mathcal{F}} h_{T}^{-\frac{1}{2}}\|\boldsymbol{\psi}\|_{\mathcal{F}}
\end{aligned}
$$

for all $\psi \in \mathcal{W}_{k-2}(\mathcal{F})$. Hence,

$$
\left\|Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla v_{0} \times \mathbf{n}_{f}\right)\right)-\nabla_{w, \boldsymbol{\tau}} v\right\|_{\mathcal{F}} \lesssim h_{T}^{-1}\left\|Q_{f} v_{0}-v_{b, f}\right\|_{\mathcal{F}}+h_{T}^{-\frac{1}{2}}\left\|v_{b, e}-Q_{b} v_{0}\right\|_{\partial \mathcal{F}}
$$

Substituting the above estimate into (6.6) gives rise to the desired inequality (6.5).
Lemma 6.3. For any $v \in V_{h}$, there yields

$$
\begin{equation*}
\sum_{T \in \mathcal{T}_{h}}\left|v_{0}\right|_{2, T}^{2} \lesssim\|v\|^{2} \tag{6.7}
\end{equation*}
$$

Proof. By taking $\varphi=\partial_{i j}^{2} v_{0} \in P_{k-2}(T)$ in (2.4) we have

$$
\begin{aligned}
& \left(\partial_{i j, w}^{2} v, \partial_{i j}^{2} v_{0}\right)_{T} \\
= & \left(\partial_{i j}^{2} v_{0}, \partial_{i j}^{2} v_{0}\right)_{T}+\left\langle\left(v_{0}-v_{b, f}\right) n_{i}, \partial_{j}\left(\partial_{i j}^{2} v_{0}\right)\right\rangle_{\partial T}-\left\langle\partial_{i} v_{0}-v_{g i}, \partial_{i j}^{2} v_{0} n_{j}\right\rangle_{\partial T} \\
= & \left(\partial_{i j}^{2} v_{0}, \partial_{i j}^{2} v_{0}\right)_{T}+\left\langle\left(Q_{f} v_{0}-v_{b, f}\right) n_{i}, \partial_{j}\left(\partial_{i j}^{2} v_{0}\right)\right\rangle_{\partial T}-\left\langle Q_{n}\left(\partial_{i} v_{0}\right)-v_{g i}, \partial_{i j}^{2} v_{0} n_{j}\right\rangle_{\partial T}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{T \in \mathcal{T}_{h}}\left|v_{0}\right|_{2, T}^{2} \lesssim & \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\|\partial_{i j, w}^{2} v\right\|_{T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\|\partial_{i j}^{2} v_{0}\right\|_{T}^{2}\right)^{\frac{1}{2}} \\
& +\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{f} v_{0}-v_{b, f}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}^{3}\left\|\partial_{j}\left(\partial_{i j}^{2} v_{0}\right)\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
& +\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i=1}^{d} h_{T}^{-1}\left\|Q_{n}\left(\partial_{i} v_{0}\right)-v_{g i}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}\left\|\partial_{i j}^{2} v_{0}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
\lesssim & \|v\|\left(\sum_{T \in \mathcal{T}_{h}}\left|v_{0}\right|_{2, T}^{2}\right)^{\frac{1}{2}} .
\end{aligned}
$$

This completes the proof of the lemma.
Lemma 6.4. Let $k \geq 3$. For any $v \in V_{h}$ and $\varphi \in H^{k+1}(\Omega)$, there holds

$$
\begin{align*}
& \mid \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle v_{0}-Q_{f} v_{0}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} \varphi\right)-\partial_{i j}^{2} \varphi\right) n_{i}\right\rangle_{\partial T} \mid  \tag{6.8}\\
& \lesssim h^{k-1}\|\varphi\|_{k+1}\|v\|  \tag{6.9}\\
&\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(D_{\tau} Q_{0} \varphi\right)-\nabla_{w, \boldsymbol{\tau}} Q_{h} \varphi\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \lesssim h^{k-1}\|\varphi\|_{k+1}
\end{align*}
$$

Proof. We first note the following identity

$$
\begin{aligned}
J & :=\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle v_{0}-Q_{f} v_{0}, \partial_{j}\left(\partial_{i j}^{2} \varphi-\mathbb{Q}_{h}\left(\partial_{i j}^{2} \varphi\right)\right) n_{i}\right\rangle_{\partial T} \\
& =\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle v_{0}-Q_{f} v_{0}, \partial_{j} \partial_{i j}^{2} \varphi n_{i}\right\rangle_{\partial T} \\
& =\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle v_{0}-Q_{f} v_{0},\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi n_{i}\right\rangle_{\partial T} \\
& =\sum_{\mathcal{F} \in \mathcal{F}_{h}} \sum_{i, j=1}^{d}\left\langle\left[v_{0}\right]-Q_{f}\left[v_{0}\right],\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi n_{i}\right\rangle_{\mathcal{F}}
\end{aligned}
$$

For $k>3$, the finite element space on face $\mathcal{F}$ consists of linear functions so that

$$
\left|\left\langle\left[v_{0}\right]-Q_{f}\left[v_{0}\right],\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi n_{i}\right\rangle_{\mathcal{F}}\right| \leq C h^{2}\left\|\left[v_{0}\right]\right\|_{2, \mathcal{F}}\left\|\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi\right\|_{0, \mathcal{F}}
$$

which can be used to derive the desired inequality (6.8) without any difficulty. Here $\left[v_{0}\right]=\left.v_{0}\right|_{T_{L} \cap \mathcal{F}}-\left.v_{0}\right|_{T_{R} \cap \mathcal{F}}$ is the jump of $v_{0}$ on the face $\mathcal{F}$ shared by two adjacent elements $T_{L}$ and $T_{R}$.

For the case of $k=3$, the finite element space on face $\mathcal{F}$ consists of constants only so that

$$
\left|\left\langle\left[v_{0}\right]-Q_{f}\left[v_{0}\right],\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi n_{i}\right\rangle_{\mathcal{F}}\right| \leq C h\left\|\left[D_{\boldsymbol{\tau}} v_{0}\right]\right\|_{0, \mathcal{F}}\left\|\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi\right\|_{0, \mathcal{F}},
$$

where $D_{\boldsymbol{\tau}} v_{0}$ stands for the tangential derivative on $\mathcal{F}$. It follows from the trace inequalities (6.1)-(6.2) and the inverse inequality that

$$
\begin{aligned}
|J| \lesssim & \left(\sum_{\mathcal{F} \in \mathcal{F}_{h}} h_{T}^{2}\left\|\left[D_{\tau} v_{0}\right]\right\|_{\mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}^{-1}\left\|\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi\right\|_{T}^{2}+h_{T}\left|\left(I-Q_{f}\right) \partial_{j} \partial_{i j}^{2} \varphi\right|_{1, T}^{2}\right)^{\frac{1}{2}} \\
\lesssim & \left(\sum_{\mathcal{F} \in \mathcal{F}_{h}} h_{T}^{2}\left\|\left[D_{\tau} v_{0}\right]-Q_{n}\left(\left[D_{\tau} v_{0}\right]\right)\right\|_{\mathcal{F}}^{2}+h_{T}^{2}\left\|Q_{n}\left(\left[D_{\boldsymbol{\tau}} v_{0}\right]\right)\right\|_{\mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{2 k-5}\|\varphi\|_{k+1, T}^{2}\right)^{\frac{1}{2}} \\
& \lesssim\left(\sum_{\mathcal{F} \in \mathcal{F}_{h}} h_{T}^{4}\left\|\left[D_{\tau \tau} v_{0}\right]\right\|_{\mathcal{F}}^{2}+h_{T}^{2}\left\|Q_{n}\left(\left[D_{\boldsymbol{\tau}} v_{0}\right]\right)-\left[\nabla_{w, \boldsymbol{\tau}} v\right]\right\|_{\mathcal{F}}^{2}\right)^{\frac{1}{2}} h^{k-\frac{5}{2}}\|\varphi\|_{k+1} \\
\lesssim & \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{3}\left|v_{0}\right|_{2, T}^{2}+\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|Q_{n} D_{\tau} v_{0}-\nabla_{w, \tau} v\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} h^{k-\frac{5}{2}}\|\varphi\|_{k+1} \\
\lesssim & \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{3}\left|v_{0}\right|_{2, T}^{2}+h^{3}\|v\|^{2}\right)^{\frac{1}{2}} h^{k-\frac{5}{2}}\|\varphi\|_{k+1} \\
\lesssim & h^{k-1}\|\varphi\|_{k+1}\|v\|,
\end{aligned}
$$

which completes the proof of (6.8).
To verify (6.9), we recall that $Q_{n} D_{\tau} w_{0}=Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla w_{0} \times \mathbf{n}_{f}\right)\right)$. Hence, from (4.1), the trace inequality (6.1), and (6.3) we arrive at

$$
\begin{aligned}
& \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(D_{\tau} Q_{0} \varphi\right)-\nabla_{w, \tau} Q_{h} \varphi\right\|_{\partial T}^{2} \\
= & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla Q_{0} \varphi \times \mathbf{n}_{f}\right)\right)-Q_{n}\left(\mathbf{n}_{f} \times\left(\nabla \varphi \times \mathbf{n}_{f}\right)\right)\right\|_{\partial T}^{2} \\
\lesssim & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\mathbf{n}_{f} \times\left(\nabla Q_{0} \varphi \times \mathbf{n}_{\mathcal{F}}\right)-\mathbf{n}_{f} \times\left(\nabla \varphi \times \mathbf{n}_{f}\right)\right\|_{\partial T}^{2} \\
\lesssim & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\nabla Q_{0} \varphi-\nabla \varphi\right\|_{\partial T}^{2} \\
\lesssim & h^{k-1}\|\varphi\|_{k+1} .
\end{aligned}
$$

This completes the proof of the lemma.
7. Error estimates. The following is an error estimate for the numerical scheme (3.1) with respect to the natural "energy" norm.

Theorem 7.1. Let $u$ be the exact solution of the equation (1.1) and $u_{h} \in V_{h}$ be its numerical approximation arising from the WG scheme (3.1). Under the assumtion of $u \in H^{k+1}(\Omega)$, the following error estimate holds true:

$$
\begin{equation*}
\left\|e_{h}\right\| \lesssim h^{k-1}\|u\|_{k+1} . \tag{7.1}
\end{equation*}
$$

Proof. By taking $v=e_{h} \in V_{h}^{0}$ in (5.2) we have

$$
\begin{align*}
\left\|e_{h}\right\|^{2}= & \zeta_{u}\left(e_{h}\right) \\
= & s\left(Q_{h} u, e_{h}\right)+\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle e_{0}-e_{b, f}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T}  \tag{7.2}\\
& +\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{i} e_{0}-e_{g i},\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right) n_{j}\right\rangle_{\partial T} \\
= & I_{1}+I_{2}+I_{3}
\end{align*}
$$

For $I_{1}$, we have from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\left|I_{1}\right|= & \left|s\left(Q_{h} u, e_{h}\right)\right| \\
\leq & \sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left|\left\langle Q_{b}\left(Q_{0} u\right)-Q_{b} u, Q_{b} e_{0}-e_{b, e}\right\rangle_{\partial \mathcal{F}}\right| \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left|\left\langle Q_{f}\left(Q_{0} u\right)-Q_{f} u, Q_{f} e_{0}-e_{b, f}\right\rangle_{\partial T}\right| \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left|\left\langle Q_{n}\left(\nabla Q_{0} u\right) \cdot \mathbf{n}_{f}-Q_{n}\left(\nabla u \cdot \mathbf{n}_{f}\right), Q_{n}\left(\nabla e_{0}\right) \cdot \mathbf{n}_{f}-e_{n}\right\rangle_{\partial T}\right| \\
& +\delta_{k, 3} \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left|\left\langle Q_{n} D_{\boldsymbol{\tau}} Q_{0} u-\nabla_{w, \boldsymbol{\tau}} Q_{h} u, Q_{n} D_{\boldsymbol{\tau}} e_{0}-\nabla_{w, \boldsymbol{\tau}} e_{h}\right\rangle_{\partial T}\right| \\
\lesssim & \left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|Q_{0} u-u\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|Q_{b} e_{0}-e_{b, e}\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
& +\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{0} u-u\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{f} e_{0}-e_{b, f}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
& +\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\nabla Q_{0} u-\nabla u\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n}\left(\nabla e_{0}\right) \cdot \mathbf{n}_{f}-e_{n}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
& +\delta_{k, 3}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|Q_{n} D_{\boldsymbol{\tau}} Q_{0} u-\nabla_{w, \boldsymbol{\tau}} Q_{h} u\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| .
\end{aligned}
$$

Next, using the trace inequality (6.1) and the estimates (6.3) and (6.9), we arrive at

$$
\begin{aligned}
&\left|I_{1}\right| \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{0} u-u\right\|_{\mathcal{F}}^{2}+h_{T}^{-1}\left\|\nabla Q_{0} u-\nabla u\right\|_{\mathcal{F}}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| \\
&+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-4}\left\|Q_{0} u-u\right\|_{T}^{2}+h_{T}^{-2}\left\|\nabla Q_{0} u-\nabla u\right\|_{T}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| \\
&+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|\nabla Q_{0} u-\nabla u\right\|_{T}^{2}+\left|Q_{0} u-u\right|_{2, T}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| \\
&+\delta_{k, 3} h^{2}\|u\|_{4}\left\|e_{h}\right\| \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-4}\left\|Q_{0} u-u\right\|_{T}^{2}+h_{T}^{-2}\left\|\nabla Q_{0} u-\nabla u\right\|_{T}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| \\
&+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|\nabla Q_{0} u-\nabla u\right\|_{T}^{2}+\left|Q_{0} u-u\right|_{2, T}^{2}\right)^{\frac{1}{2}}\left\|e_{h}\right\| \\
&+\delta_{k, 3} h^{2}\|u\|_{4}\left\|e_{h}\right\| \\
& \lesssim h^{k-1}\|u\|_{k+1}\left\|e_{h}\right\|+\delta_{k, 3} h^{2}\|u\|_{4}\left\|e_{h}\right\| \\
& \lesssim h^{k-1}\|u\|_{k+1}\left\|e_{h}\right\| .
\end{aligned}
$$

For the term $I_{2}$, we have from (6.8), the Cauchy-Schwarz inequality, and the trace inequality (6.1) that

$$
\begin{align*}
\left|I_{2}\right|= & \left|\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle e_{0}-e_{b, f}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T}\right| \\
= & \mid \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle e_{0}-Q_{f} e_{0}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \\
& +\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle Q_{f} e_{0}-e_{b, f}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \mid  \tag{7.4}\\
\lesssim & h^{k-1}\|u\|_{k+1}\left\|e_{h}\right\|+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{f} e_{0}-e_{b, f}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}^{3}\left\|\partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right)\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
\lesssim & h^{k-1}\|u\|_{k+1}\left\|e_{h}\right\| .
\end{align*}
$$

As to $I_{3}$, we have from the Cauchy-Schwarz inequality, Lemmas $6.2 \sqrt{6.3}$, the trace
inequality (6.1), and (6.4) that

$$
\begin{align*}
\left|I_{3}\right|= & \left|\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{i} e_{0}-e_{g i},\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right) n_{j}\right\rangle_{\partial T}\right| \\
\lesssim & \left(\sum_{T \in \mathcal{T}_{h}} \sum_{i=1}^{d} h_{T}^{-1}\left\|Q_{n}\left(\partial_{i} e_{0}\right)-e_{g i}\right\|_{\partial T}^{2}+h_{T}^{-1}\left\|\partial_{i} e_{0}-Q_{n}\left(\partial_{i} e_{0}\right)\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}\left\|\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
\lesssim & \left(\left\|e_{h}\right\|^{2}+\sum_{T \in \mathcal{T}_{h}}\left|\partial_{i} e_{0}\right|_{1, T}^{2}\right)^{\frac{1}{2}}  \tag{7.5}\\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\|\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right\|_{T}^{2}+h_{T}^{2}\left\|\nabla\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)\right\|_{T}^{2}\right)^{\frac{1}{2}} \\
\lesssim & \left(\left\|e_{h}\right\|^{2}+\sum_{T \in \mathcal{T}_{h}}\left|e_{0}\right|_{2, T}^{2}\right)^{\frac{1}{2}} h^{k-1}\|u\|_{k+1} \\
\lesssim & h^{k-1}\|u\|_{k+1}\left\|e_{h}\right\| .
\end{align*}
$$

Substituting (7.3)-(7.5) into (7.2) gives rise to (7.1). This completes the proof of the theorem.

To establish an optimal order error estimate for the numerical solution in the $L^{2}$ norm, we consider the dual problem that seeks $\Phi$ satisfying

$$
\begin{align*}
\Delta^{2} \Phi & =e_{0}, & & \text { in } \Omega \\
\Phi & =0, & & \text { on } \partial \Omega  \tag{7.6}\\
\frac{\partial \Phi}{\partial \mathbf{n}} & =0, & & \text { on } \partial \Omega
\end{align*}
$$

Assume that the problem (7.6) has the $H^{4}$-regularity in the sense that there exists a constant $C$ such that

$$
\begin{equation*}
\|\Phi\|_{4} \leq C\left\|e_{0}\right\| \tag{7.7}
\end{equation*}
$$

Theorem 7.2. Let $u \in H^{k+1}(\Omega)$ be the exact solution of the problem (1.1) and $u_{h} \in V_{h}$ be its numerical solution arising from the $W G$ scheme (3.1). Under the $H^{4}$-regularity assumption (7.7), we have the following error estimate

$$
\left\|e_{0}\right\| \lesssim h^{k+1}\|u\|_{k+1} .
$$

Proof. First, using (2.1) with $e_{b, f}=0$ on each $\mathcal{F} \subset \partial \Omega$ and $e_{b, e}=0$ on each $e \subset \partial \Omega$ gives $\nabla_{w, \tau} e_{h}=0$ on each $\mathcal{F} \subset \partial \Omega$. This, together with $e_{n}=0$ on $\partial \Omega$ and (2.2), gives $\mathbf{e}_{g}=0$ on $\partial \Omega$. Next, we test the dual problem (7.6) against $e_{0}$ and use
the integration by parts to obtain

$$
\begin{align*}
\left\|e_{0}\right\|^{2}= & \left(\Delta^{2} \Phi, e_{0}\right) \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j}^{2} \Phi, \partial_{i j}^{2} e_{0}\right)_{T}-\left\langle\partial_{i j}^{2} \Phi, \partial_{i} e_{0} n_{j}\right\rangle_{\partial T}+\left\langle\partial_{j}\left(\partial_{i j}^{2} \Phi\right) n_{i}, e_{0}\right\rangle_{\partial T} \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j}^{2} \Phi, \partial_{i j}^{2} e_{0}\right)_{T}-\left\langle\partial_{i j}^{2} \Phi,\left(\partial_{i} e_{0}-e_{g i}\right) n_{j}\right\rangle_{\partial T}  \tag{7.8}\\
& +\left\langle\partial_{j}\left(\partial_{i j}^{2} \Phi\right) n_{i}, e_{0}-e_{b, f}\right\rangle_{\partial T}
\end{align*}
$$

where we have used $\sum_{T \in \mathcal{T}_{h}}\left\langle\partial_{i j}^{2} \Phi, e_{g i} n_{j}\right\rangle_{\partial T}=0$ and $\sum_{T \in \mathcal{T}_{h}}\left\langle\partial_{j}\left(\partial_{i j}^{2} \Phi\right) n_{i}, e_{b, f}\right\rangle_{\partial T}=0$ since $e_{b, f}=0$ and $\mathbf{e}_{g}=0$ on $\partial \Omega$.

Analogues to (5.5), we have

$$
\begin{aligned}
\left(\partial_{i j}^{2} \Phi, \partial_{i j}^{2} e_{0}\right)_{T}= & \left(\partial_{i j, w}^{2} e_{h}, \partial_{i j, w}^{2} Q_{h} \Phi\right)_{T}+\left\langle\left(e_{b, f}-e_{0}\right) n_{i}, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} \Phi\right)\right)\right\rangle_{\partial T} \\
& +\left\langle\partial_{i} e_{0}-e_{g i}, \mathbb{Q}_{h}\left(\partial_{i j}^{2} \Phi\right) n_{j}\right\rangle_{\partial T}
\end{aligned}
$$

which, together with (7.8) and (5.2)-(5.3), leads to

$$
\begin{align*}
\left\|e_{0}\right\|^{2}= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left(\partial_{i j, w}^{2} e_{h}, \partial_{i j, w}^{2} Q_{h} \Phi\right)_{T}+\left\langle\left(\partial_{i} e_{0}-e_{g i}\right) n_{j}, \mathbb{Q}_{h}\left(\partial_{i j}^{2} \Phi\right)-\partial_{i j}^{2} \Phi\right\rangle_{\partial T} \\
& +\left\langle\left(e_{0}-e_{b, f}\right) n_{i}, \partial_{j}\left(\partial_{i j}^{2} \Phi-\mathbb{Q}_{h}\left(\partial_{i j}^{2} \Phi\right)\right)\right\rangle_{\partial T}  \tag{7.9}\\
= & \zeta_{u}\left(Q_{h} \Phi\right)-\zeta_{\Phi}\left(e_{h}\right) \\
= & \sum_{i=1}^{3} J_{i}-\zeta_{\Phi}\left(e_{h}\right)
\end{align*}
$$

where $J_{i}$ are given as in (5.3) with $v=Q_{h} \Phi$.
The rest of the proof amounts to the estimate for each of the four terms on the last line in (7.9).

For $J_{1}$, we have from Cauchy-Schwarz inequality, (6.9), the trace inequality (6.1),
(6.3), and the $H^{4}$ regularity assumption (7.7) that

$$
\begin{align*}
&\left|J_{1}\right| \\
&= \sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\langle Q_{b}\left(Q_{0} u\right)-Q_{b} u, Q_{b}\left(Q_{0} \Phi\right)-Q_{b} \Phi\right\rangle_{\partial \mathcal{F}} \\
&+h_{T}^{-3}\left\langle Q_{f}\left(Q_{0} u\right)-Q_{f} u, Q_{f}\left(Q_{0} \Phi\right)-Q_{f} \Phi\right\rangle_{\partial T} \\
&+h_{T}^{-1}\left\langle Q_{n}\left(\nabla Q_{0} u\right) \cdot \mathbf{n}_{f}-Q_{n}\left(\nabla u \cdot \mathbf{n}_{f}\right), Q_{n}\left(\nabla Q_{0} \Phi\right) \cdot \mathbf{n}_{f}-Q_{n}\left(\nabla \Phi \cdot \mathbf{n}_{f}\right)\right\rangle_{\partial T} \\
&+\delta_{k, 3} \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\langle Q_{n} D_{\tau} Q_{0} u-\nabla_{w, \tau} Q_{h} u, Q_{n} D_{\tau} Q_{0} \Phi-\nabla_{w, \tau} Q_{h} \Phi\right\rangle_{\partial T} \mid \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|Q_{0} u-u\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-2}\left\|Q_{0} \Phi-\Phi\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}}  \tag{7.10}\\
&+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{0} u-u\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-3}\left\|Q_{0} \Phi-\Phi\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
&+\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\nabla Q_{0} u-\nabla u\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\nabla Q_{0} \Phi-\nabla \Phi\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
&+\delta_{k, 3} h^{4}\|u\|_{4}\|\Phi\|_{4} \\
& \lesssim h^{k+1}\|u\|_{k+1}\|\Phi\|_{4} \\
& \lesssim h^{k+1}\|u\|_{k+1}\left\|e_{0}\right\| .
\end{align*}
$$

For the term $J_{2}$, we have

$$
\begin{aligned}
J_{2}= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle Q_{0} \Phi-Q_{f} \Phi, \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\left(Q_{0} \Phi-\Phi\right)+\left(\Phi-Q_{f} \Phi\right), \partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle Q_{0} \Phi-\Phi, \partial_{j}\left(\left(\mathbb{Q}_{h}-I\right) \partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \\
& +\left\langle\Phi-Q_{f} \Phi, \partial_{j}\left(\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T} \\
= & \sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle Q_{0} \Phi-\Phi, \partial_{j}\left(\mathbb{Q}_{h} \partial_{i j}^{2} u-\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T},
\end{aligned}
$$

where we used the fact that $\sum_{T \in \mathcal{T}_{h}}\left\langle\Phi-Q_{f} \Phi, \partial_{j}\left(\partial_{i j}^{2} u\right) n_{i}\right\rangle_{\partial T}=0$. It follows that

$$
\begin{align*}
\left|J_{2}\right| & \lesssim\left(\sum_{T \in \mathcal{T}_{h}}\left\|Q_{0} \Phi-\Phi\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\|\partial_{j}\left(\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)-\partial_{i j}^{2} u\right)\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}  \tag{7.11}\\
& \lesssim h^{k+1}\|\Phi\|_{4}\|u\|_{k+1} \\
& \lesssim h^{k+1}\|u\|_{k+1}\left\|e_{0}\right\| .
\end{align*}
$$

For the term $J_{3}$, we note that the weak gradient of the $L^{2}$ projection of a smooth function is the same as the $L^{2}$ projection of its classical gradient on the boundary of
each element, see (4.3). Hence,

$$
J_{3}=\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d}\left\langle\partial_{i} Q_{0} \Phi-Q_{n}\left(\partial_{i} \Phi\right),\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right) n_{j}\right\rangle_{\partial T}
$$

It follows from the Cauchy-Schwarz inequality, the trace inequality (6.1), Lemma 6.1, and the regularity assumption (7.7) that

$$
\begin{align*}
\left|J_{3}\right| & \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\nabla Q_{0} \Phi-\nabla \Phi\right\|_{T}^{2}+h_{T}\left|\nabla Q_{0} \Phi-\nabla \Phi\right|_{1, T}^{2}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{T \in \mathcal{T}_{h}} \sum_{i, j=1}^{d} h_{T}^{-1}\left\|\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right\|_{T}^{2}+h_{T}\left\|\nabla\left(\partial_{i j}^{2} u-\mathbb{Q}_{h}\left(\partial_{i j}^{2} u\right)\right)\right\|_{T}^{2}\right)^{\frac{1}{2}}  \tag{7.12}\\
& \lesssim h^{k+1}\|\Phi\|_{4}\|u\|_{k+1} \\
& \lesssim h^{k+1}\|u\|_{k+1}\left\|e_{0}\right\| .
\end{align*}
$$

To deal with the last term, using the same arguments as in (7.3)-(7.5) with $u=\Phi$ and then combining (7.1) with (7.7), there yields

$$
\begin{align*}
\left|\zeta_{\Phi}\left(e_{h}\right)\right| & \lesssim h^{2}\|\Phi\|_{4}\left\|e_{h}\right\| \\
& \lesssim h^{k+1}\|u\|_{k+1}\|\Phi\|_{4}  \tag{7.13}\\
& \lesssim h^{k+1}\|u\|_{k+1}\left\|e_{0}\right\| .
\end{align*}
$$

Finally, substituting (7.10)-(7.13) into (7.9) completes the proof of the theorem. $\square$

We further introduce the following measure for the numerical solutions on element boundaries:

$$
\begin{aligned}
\left\|e_{b, e}\right\|_{\mathcal{E}_{h}} & =\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|e_{b, e}\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
\left\|e_{b, f}\right\|_{\mathcal{F}_{h}} & =\left(\sum_{T \in \mathcal{T}_{h}} h_{T}\left\|e_{b, f}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}} \\
\left\|e_{n}\right\|_{\mathcal{F}_{h}} & =\left(\sum_{T \in \mathcal{T}_{h}} h_{T}\left\|e_{n}\right\|_{\partial T}^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

Theorem 7.3. Under the assumptions of Theorem 7.2, there holds

$$
\begin{align*}
\left\|e_{b, e}\right\|_{\mathcal{E}_{h}} & \lesssim h^{k+1}\|u\|_{k+1}  \tag{7.14}\\
\left\|e_{b, f}\right\|_{\mathcal{F}_{h}} & \lesssim h^{k+1}\|u\|_{k+1}  \tag{7.15}\\
\left\|e_{n}\right\|_{\mathcal{F}_{h}} & \lesssim h^{k}\|u\|_{k+1} \tag{7.16}
\end{align*}
$$

Proof. From the triangular inequality, the trace inequality (6.2), (4.5), Theorems
7.1 and 7.2, there holds

$$
\begin{aligned}
\left\|e_{b, e}\right\|_{\mathcal{E}_{h}} & =\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|e_{b, e}\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{2}\left\|Q_{b} e_{0}\right\|_{\partial \mathcal{F}}^{2}+h_{T}^{2}\left\|e_{b, e}-Q_{b} e_{0}\right\|_{\partial \mathcal{F}}^{2}\right)^{\frac{1}{2}} \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{2} h_{T}^{-1}\left\|e_{0}\right\|_{\partial T}^{2}+h_{T}^{2} h_{T}^{2}\left\|e_{h}\right\|^{2}\right)^{\frac{1}{2}} \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T} h_{T}^{-1}\left\|e_{0}\right\|_{T}^{2}+h_{T}^{4} h_{T}^{2(k-1)}\|u\|_{k+1}^{2}\right)^{\frac{1}{2}} \\
& \lesssim h^{k+1}\|u\|_{k+1},
\end{aligned}
$$

which completes the proof for (7.14).
The proof for (7.15) and (7.16) can be obtained by using a similar argument. $\square$
8. Numerical experiments. In this section, the numerical scheme (3.1) will be implemented to verify the convergence theory established in the previous sections. To this end, we first solve the biharmonic equation (1.1) on the unit square $\Omega=(0,1)^{2}$, where $g$ and the boundary conditions are chosen so that the exact solution is

$$
\begin{equation*}
u(x, y)=2^{8}\left(x-x^{2}\right)^{2}\left(y-y^{2}\right)^{2} \tag{8.1}
\end{equation*}
$$

Test Example 1. We take the square as the initial mesh, and subdivide each square into four to get subsequent meshes, as shown in Figure 8.1. One can see from Table 8.1 that the optimal rates of convergence are obtained in the usual $L^{2}$ and $H^{2}$-like triple-bar norm for $P_{3}, P_{4}$ and $P_{5}$ WG methods.


Fig. 8.1. The first three levels of square grids used in Table 8.1 computation.
Test Example 2. We take the uniform triangular meshes, as shown in Figure 8.2. One can see from Table 8.2 that optimal rates of convergence are demonstrated in the usual $L^{2}$ and $H^{2}$-like triple-bar norm for $P_{3}, P_{4}$ and $P_{5}$ WG methods.

Test Example 3. We take polygonal meshes shown as in Figure 8.3. Table 8.3 illustrates the corresponding numerical results which clearly demonstrate optimal rates of convergence in the usual $L^{2}$ and $H^{2}$-like triple-bar norms for $P_{3}, P_{4}$ and $P_{5}$ WG methods.

Table 8.1
The error profile for solving (8.1) on square grids shown in Figure 8.1

| Grid | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate | $\left\\|\left\\|Q_{h} u-u_{h}\right\\|\right.$ | Rate |
| :---: | :---: | :---: | :---: | :---: |
|  | The $P_{3}$ WG finite element |  |  |  |
| 5 | $0.1486 \mathrm{E}-02$ | 3.90 | $0.9339 \mathrm{E}+00$ | 1.95 |
| 6 | $0.9595 \mathrm{E}-04$ | 3.95 | $0.2373 \mathrm{E}+00$ | 1.98 |
| 7 | $0.6092 \mathrm{E}-05$ | 3.98 | $0.5981 \mathrm{E}-01$ | 1.99 |
| The $P_{4}$ WG finite element |  |  |  |  |
| 3 | $0.3791 \mathrm{E}-01$ | 3.85 | $0.3692 \mathrm{E}+01$ | 2.86 |
| 4 | $0.1330 \mathrm{E}-02$ | 4.83 | $0.4803 \mathrm{E}+00$ | 2.94 |
| 5 | $0.4232 \mathrm{E}-04$ | 4.97 | $0.6068 \mathrm{E}-01$ | 2.98 |
| The $P_{5}$ WG finite element |  |  |  |  |
| 2 | $0.2460 \mathrm{E}+00$ | 5.05 | $0.1823 \mathrm{E}+02$ | 5.21 |
| 3 | $0.5110 \mathrm{E}-02$ | 5.59 | $0.9983 \mathrm{E}+00$ | 4.19 |
| 4 | $0.8558 \mathrm{E}-04$ | 5.90 | $0.5589 \mathrm{E}-01$ | 4.16 |



FIG. 8.2. The first three levels of triangular grids used in Table 8.2 computation.
Table 8.2
The error profile for solving 8.1) on triangular grids shown in Figure 8.2.

| Grid | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate |
| :---: | :---: | :---: | :---: | :---: |
| The $P_{3}$ WG finite element |  |  |  |  |
| 5 | $0.8263 \mathrm{E}-03$ | 3.97 | $0.7030 \mathrm{E}+00$ | 1.98 |
| 6 | $0.5190 \mathrm{E}-04$ | 3.99 | $0.1764 \mathrm{E}+00$ | 2.00 |
| 7 | $0.3252 \mathrm{E}-05$ | 4.00 | $0.4414 \mathrm{E}-01$ | 2.00 |
| The $P_{4}$ WG finite element |  |  |  |  |
| 4 | $0.6526 \mathrm{E}-03$ | 4.86 | $0.2874 \mathrm{E}+00$ | 2.88 |
| 5 | $0.2088 \mathrm{E}-04$ | 4.97 | $0.3666 \mathrm{E}-01$ | 2.97 |
| 6 | $0.6563 \mathrm{E}-06$ | 4.99 | $0.4606 \mathrm{E}-02$ | 2.99 |
| The $P_{5} \mathrm{WG}$ finite element |  |  |  |  |
| 3 | $0.2622 \mathrm{E}-02$ | 5.59 | $0.3941 \mathrm{E}+00$ | 3.65 |
| 4 | $0.4362 \mathrm{E}-04$ | 5.91 | $0.2601 \mathrm{E}-01$ | 3.92 |
| 5 | $0.6929 \mathrm{E}-06$ | 5.98 | $0.1649 \mathrm{E}-02$ | 3.98 |

Test Example 4. We solve the biharmonic equation (1.1) on the unit cubic domain $\Omega=(0,1)^{3}$, where $g$ and the boundary conditions are chosen so that the exact solution is given by

$$
\begin{equation*}
u(x, y, z)=2^{12}\left(x-x^{2}\right)^{2}\left(y-y^{2}\right)^{2}\left(z-z^{2}\right)^{2} . \tag{8.2}
\end{equation*}
$$



Fig. 8.3. The first two levels of quadrilateral-pentagon-hexagon grids used in Table 8.3 computation.

TABLE 8.3
The error profile for solving (8.1) on polygonal grids shown in Figure 8.3 ,

| Grid | $\left\\|Q_{h} u-u_{h}\right\\| \quad$ Rate | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate |  |
| :---: | :---: | :---: | :---: | :---: |
| The $P_{3}$ WG finite element |  |  |  |  |
| 4 | $0.5052 \mathrm{E}-02$ | 3.92 | $0.1724 \mathrm{E}+01$ | 1.94 |
| 5 | $0.3207 \mathrm{E}-03$ | 3.98 | $0.4355 \mathrm{E}+00$ | 1.98 |
| 6 | $0.1999 \mathrm{E}-04$ | 4.00 | $0.1092 \mathrm{E}+00$ | 2.00 |
| The $P_{4}$ WG finite element |  |  |  |  |
| 3 | $0.4732 \mathrm{E}-02$ | 4.96 | $0.9861 \mathrm{E}+00$ | 3.04 |
| 4 | $0.1473 \mathrm{E}-03$ | 5.01 | $0.1213 \mathrm{E}+00$ | 3.02 |
| 5 | $0.4974 \mathrm{E}-05$ | 4.89 | $0.1510 \mathrm{E}-01$ | 3.01 |
| The $P_{5}$ WG finite element |  |  |  |  |
| 1 | $0.1201 \mathrm{E}+01$ | 0.00 | $0.3857 \mathrm{E}+02$ | 0.00 |
| 2 | $0.1468 \mathrm{E}-01$ | 6.36 | $0.1683 \mathrm{E}+01$ | 4.52 |
| 3 | $0.2705 \mathrm{E}-03$ | 5.76 | $0.9122 \mathrm{E}-01$ | 4.21 |

In this test, we use the uniform cube meshes shown as in Figure 8.4. The results from the $P_{3}$ and $P_{4}$ WG methods are shown in Table8.4. The optimal order of convergence is achieved in all cases.


FIG. 8.4. The first three levels of cube grids used in the computation of Table 8.4.

Table 8.4
The error profile for solving (8.2) on cube grids shown in Figure 8.4 .

| Grid | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate | $\left\\|Q_{h} u-u_{h}\right\\|$ | Rate |
| :---: | :---: | :---: | :---: | :---: |
|  | The $P_{3}$ WG finite element |  |  |  |
| 2 | $0.8474 \mathrm{E}-01$ | 6.2 | $0.1633 \mathrm{E}+01$ | 4.5 |
| 3 | $0.2583 \mathrm{E}-02$ | 5.0 | $0.1846 \mathrm{E}+00$ | 3.1 |
| 4 | $0.2063 \mathrm{E}-03$ | 3.6 | $0.4861 \mathrm{E}-01$ | 1.9 |
| The $P_{4}$ WG finite element |  |  |  |  |
| 2 | $0.2247 \mathrm{E}-01$ | 9.8 | $0.1373 \mathrm{E}+01$ | 6.9 |
| 3 | $0.4988 \mathrm{E}-03$ | 5.5 | $0.1079 \mathrm{E}+00$ | 3.7 |
| 4 | $0.1705 \mathrm{E}-04$ | 4.9 | $0.1009 \mathrm{E}-01$ | 3.4 |

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