Error and Stability Estimates of a Least-Squares Variational Kernel-Based Method for Second Order Elliptic PDEs

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Abstract

We consider a least-squares variational kernel-based method for numerical solution of second order elliptic partial differential equations on a multi-dimensional domain. In this setting it is not assumed that the differential operator is self-adjoint or positive definite as it should be in the Rayleigh-Ritz setting. However, the new scheme leads to a symmetric and positive definite algebraic system of equations. Moreover, the resulting method does not rely on certain subspaces satisfying the boundary conditions. The trial space for discretization is provided via standard kernels that reproduce the Sobolev spaces as their native spaces. The error analysis of the method is given, but it is partly subjected to an inverse inequality on the boundary which is still an open problem. The condition number of the final linear system is approximated in terms of the smoothness of the kernel and the discretization quality. Finally, the results of some computational experiments support the theoretical error bounds.

Keywords: Meshfree methods, Least-squares principles, Radial basis functions, Inverse inequalities, Error estimates.

AMS subject classifications: 65N12, 65N15, 65D15, 65N99.

1 Introduction

It is of interest to extend the theory of least-squares methods for numerical treatment of elliptic systems. Some advantageous features are obtained via Least-Squares Principles (LSP) because of using the artificial energy functional to provide a Rayleigh-Ritz-like setting; see [3, 7]. One of the most attractive features of the least-squares methods is that the choice of

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approximating spaces is not subject to the Ladyzhenskaya-Babuska-Brezzi (LBB) condition [15]. Indeed, the computation of stationary points, that is the paradigm of mixed-Galerkin methods, demands strict compatibility LBB condition for continuous and discrete spaces, if stable and accurate approximations are desired. Furthermore, standard and mixed-Galerkin methods usually produce nonsymmetric systems of algebraic equations which must then be solved by direct or non-robust iterative methods, while least-squares methods involve only symmetric and positive definite systems. The other motivation to extend the least-squares methods to PDE problems of general boundary conditions, including nonhomogeneous ones, is a greatly facilitated treatment with boundary conditions because their residuals can be incorporated into the least-squares functional [7].

The theory of least-squares methods in numerical solution of elliptic boundary value problems was considered in Bramble and Schatz [13, 14] and Bramble and Nitsche [12]. An extension to an elliptic equation of order $2m$ was given in [14], and an important simplifications in the analysis was presented in [4]. Also, a least-squares theory was developed for an elliptic system of Petrovsky type in [53] and for elliptic systems of Agmon-Douglis-Nirenberg (ADN) type in [3]. We refer the reader to the survey articles [5, 6] and books [7, 28] for more details.

In this paper we develop a least-squares method for numerical solution of second order elliptic boundary value problems via reproducing kernels of Sobolev spaces $H^\tau(\Omega)$, $\Omega \subset \mathbb{R}^d$, for some $\tau > d/2$, where Ω is a domain on which the PDE is posed. In particular we use radial basis function (RBF) approximations. Although, we focus on the second order elliptic boundary value problems, a generalization to higher order equations can be done in an obvious way. The method involves the minimization of a least-squares functional that consists of a weighted sum of the residuals occurring in the differential equation and the boundary conditions. One of the additional advantages is that the method provides a more accurate solution than one might be expected from the approximating space. The method requires an approximating space consisting of functions which are smooth enough to lie in the domain of the elliptic operator. Thus, for the success of the least-squares method it is crucial to choose proper function spaces in which the boundary value problem is well-posed.

In a natural way, straightforward least-squares methods for second or higher order differential equations require finite dimensional subspaces of $H^k(\Omega)$, $k \geq 2$. It is well known in the theory of finite element methods that the construction of such subspaces is much more difficult than those of $H^1(\Omega)$. Because the latter only need to be at most continuous whereas, in practice, the former have to consist of k times differentiable functions. The $C^k(\Omega)$ regularity requirement complicates finite element spaces in several ways. First, it cannot be satisfied unless the reference polynomial space is of a sufficiently high degree. Second, unisolvency sets of $C^k(\Omega)$ elements include both values of a function and its derivatives. This fact greatly complicates the construction of bases and the assembly of the matrix problem. Finally, $C^k(\Omega)$ elements are not necessarily affine equivalent because affine mappings do not necessarily preserve the normal direction. To overcome this problem in finite element methods, the given problem is converted into a first order system. Since in kernel-based method one can simply construct arbitrary smooth approximation spaces, converting the problem into a first order system of equations may not be actually required. Thus, we directly apply RBF-based least-squares methods on the original equation. However, it is a good idea for a future study to apply the method on the corresponding first order system of equations because smoothness of the basis functions affects stability of the final system. On the other hand, the construction of

the finite dimensional subspaces using RBFs is independent of the problem dimension and an extension to high dimensional problems is straightforward.

RBFs are powerful tools in multi-variable approximation and there exist substantial interest and effort for developing these basis functions. This approximation is based on unrelated centers for the discretization process while most other methods are relied on underlying meshes. In the present paper, we restrict ourselves to RBFs that reproduce Sobolev spaces as their native spaces. We collect some few necessary results on RBFs while the whole theory is extensively presented in [52].

Both kernel-based collocation and Galerkin methods were investigated for solving PDEs. The unsymmetric collocation method was introduced by Kansa [29], in 1990. The linear system arising from this method may not be solvable in general; see Hon and Schaback [26]. By changing the setting, a convergence analysis was given by Schaback [45]. The symmetric collocation method was initially investigated by Wu [56] and Narcowich and Ward [37]. The analysis of this method was investigated in [21, 22, 52], and recently in [18]. In [49] an analysis for a meshless Galerkin method for a second order elliptic problem with natural boundary conditions was given and a finite element like convergent estimate was obtained. A Petrov-Galerkin kernel-based method was given and analyzed in [32]. Also, a meshless method for numerical solution of PDEs by using the Hermite-Birkhoff interpolation with radial basis is presented in see [58].

Despite of numerous theoretical and computational advantages of LSPs, there has not been a substantial effort devoted to investigating the least-squares variational kernel-based approaches for solving PDEs; the subject that will be considered in this paper.

The paper is organized as follows. In Section 2, some few notations and some auxiliary results are introduced. A short summary of the theory of RBFs approximation with a focus on those basis functions which generate Sobolev spaces is given. The rest of this section is devoted to introduce the necessary technical framework of continuous and discrete least-squares principles for numerical solution of second order differential equations. Section 3 has three parts. In the first part, the approximate solution is defined to be the minimizer of a mesh-dependent least-squares functional that is a weighted sum of the least-squares residuals of differential equation and boundary conditions. In the second part, the error analysis of the method is given. The analysis is partly based on an unproven inverse inequality on the boundary that demands a new research study. In the last part, the condition number of the final matrix is estimated. Finally, in Section 4 some numerical results are reported to verify the theoretical bounds of the preceding section.

2 Notations and Auxiliary Results

In this paper, Ω will denote a simply connected bounded region in $\mathbb{R}^d$ with a sufficiently smooth boundary $\partial\Omega$, and C will be considered a generic positive constant whose meaning and value changes with context. For $s \geq 0$, we use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ with corresponding inner products denoted by $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\partial\Omega}$ and norms by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$, respectively; see, e.g., [1], for details. For $s < 0$, the spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ are identified with the duals of $H^{-s}(\Omega)$ and $H^{-s}(\partial\Omega)$, respectively. A

norm for $f \in H^s(\Omega)$, with $s < 0$, is defined by

$$\|f\|_{s,\Omega} = \sup_{u \in H^{-s}(\Omega)} \frac{(f, u)_{0,\Omega}}{\|u\|_{-s,\Omega}}.$$

For $s < 0$, the norm on $H^s(\partial\Omega)$ can be defined similarly; see [23] for more details.

A function f defined on Ω is said to be *Lipschitz continuous* if for some constant C, there holds the inequality

$$|f(x) - f(y)| \le C \|x - y\|, \qquad \forall x, y \in \Omega.$$

In this formula, $\|x - y\|$ denotes the standard Euclidean distance between x and y. More generally, a function f is said to be *Hölder continuous* with exponent $\beta \in (0, 1]$ if for some constant C,

$$|f(x) - f(y)| \le C\|x - y\|^{\beta}, \qquad \forall x, y \in \Omega.$$

The Hölder space $C^{0,\beta}(\overline{\Omega})$ is defined to be the subspace of $C(\overline{\Omega})$ functions that are Hölder continuous with the exponent β. For $l \in \mathbb{Z}_+$ and $\beta \in (0, 1]$, we similarly define the Hölder space

$$C^{l,\beta}(\overline{\Omega}) = \left\{ f \in C^l(\overline{\Omega}) \,\middle|\, D^\alpha f \in C^{0,\beta}(\overline{\Omega}), \ \ |\alpha| = l \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The partial derivative operator D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial(x^1)^{\alpha_1} \cdots \partial(x^d)^{\alpha_d}}.$$

where $(x^1, \ldots, x^d)^T \in \mathbb{R}^d$.

2.1   Approximation by RBFs

For a given function space $H^\tau(\Omega)$ on bounded domain $\Omega \subset \mathbb{R}^d$, we define the finite dimensional kernel-based meshless trial spaces $U_{\Phi,X} \subset H^\tau(\Omega)$ by

$$U_{\Phi,X} := \text{span} \left\{ \Phi\left(\cdot - x_j \right) \ : \ x_j \in X \right\},$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a radial basis function and

$$X = \{x_1, \ldots, x_N\},$$

will always be a finite subset of Ω, with the points all assumed to be distinct. There are two useful quantities associated with X. The first is the mesh norm for X related to Ω, called *fill distance*, given by

$$h_{X,\Omega} := \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|,$$

where norm $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. In other words, the largest ball in Ω that does not contain a data site has radius at most $h_{X,\Omega}$. The second is the *separation radius*,

$$q_X := \frac{1}{2} \min_{x_j \neq x_k} \|x_j - x_k\|.$$

It is easy to see that if Ω is connected, we have $h_{X,\Omega} \geq q_X$. A sequence of set points $\{X_k\}$ is called *quasi-uniform* if there exists a uniform constant $\delta > 0$ such that $q_{X_k} \geq \delta\, h_{X_k,\Omega}$ for all k. In particular, the quantity $\rho_X := h_{X,\Omega}/q_X$ is commonly referred as the *mesh ratio* of X.

Definition 2.1. A continuous and even function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be positive definite if for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{x_1, \ldots, x_N\}$ in $\mathbb{R}^d$, and all $\alpha \in \mathbb{R}^N \backslash \{0\}$ the quadratic form $\sum_{j,k=1}^N \alpha_j \alpha_k \Phi(x_j - x_k)$ is strictly positive.

The RBF interpolant of a continuous functions u on a set X is denoted by $I_X u$ and is given by

$$I_X u := \sum_{j=1}^N b_j \Phi(\cdot - x_j),$$

where the coefficient vector b is determined by enforcing the interpolation conditions $I_X u(x_k) = u(x_k)$ for $k = 1, \ldots, N$. If Φ is a positive definite kernel then the interpolation matrix $B = (\Phi(x_k - x_j))$ is positive definite and the problem is uniquely solvable.

It is known that (see for example [52]) a function $\Phi \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is positive definite if and only if it is bounded and its Fourier transform is nonnegative and nonvanishing. Our convention for the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is

$$\widehat{f}(\omega) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\omega^T x} dx, \quad \omega \in \mathbb{R}^d.$$

In this paper we will further assume that Φ has an algebraically decaying Fourier transform. To be more precise, we assume that

$$C_1(1 + \|\omega\|^2)^{-\tau} \leq \widehat{\Phi}(\omega) \leq C_2(1 + \|\omega\|^2)^{-\tau}, \quad \omega \in \mathbb{R}^d, \tag{1}$$

where C_1 and C_2 are constants and $\tau > d/2$. By this assumption the *native space*

$$\mathcal{N}_\Phi(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \cap C(\mathbb{R}^d) \ : \ \widehat{f}/\sqrt{\widehat{\Phi}} \in L^2(\mathbb{R}^d) \right\},$$

with the inner product

$$(f, g)_{\mathcal{N}_\Phi(\mathbb{R}^d)} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)} d\omega,$$

is identical with the Sobolev space $H^\tau(\mathbb{R}^d)$ and their norms are equivalent [52]. Note that the inner product in $H^\tau(\mathbb{R}^d)$ is defined by

$$(f, g)_{\tau, \mathbb{R}^d} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}(1 + \|\omega\|^2)^\tau d\omega, \quad f, g \in H^\tau(\mathbb{R}^d).$$

If we assume that Ω has a Lipschitz boundary to ensure the existence of a continuous extension operator $E_\Omega : H^\tau(\Omega) \to H^\tau(\mathbb{R}^d)$ then the native space $\mathcal{N}_\Phi(\Omega)$ is norm-equivalent to $H^\tau(\Omega)$ [52].

It is well known that RBF interpolants are also the best approximants in the following sense

$$\min_{v \in U_{\Phi,X}} \|u - v\|_{\mathcal{N}_\Phi(\Omega)} = \|u - I_X u\|_{\mathcal{N}_\Phi(\Omega)}.$$

Hence, if the native space coincides with an appropriate Sobolev space, the norm of $u - I_X u$ can be bounded by the norm of the target function u in Sobolev spaces. Since the smoothness of u is unknown in general, we have to look for convergence results where Φ can be chosen independent of the smoothness of u; i.e., the error estimates include situations in which u does not belong to the native space of the RBF. In [33, 34, 35] the Sobolev type error estimates for positive real τ, for functions inside or outside the native space were derived.

Theorem 2.2. *Suppose a positive definite kernel Φ satisfying (1), with $\tau \geq k > d/2$, and let a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be given. Furthermore, let $X \subset \Omega$ has mesh norm $h_{X,\Omega}$. Then there exists a function $v^h \in U_{\Phi,X}$, a constant C independent of u and $h_{X,\Omega}$ such that*

$$\|u - v^h\|_{r,\Omega} \leq C\, h_{X,\Omega}^{k-r} \|u\|_{k,\Omega}, \quad 0 \leq r \leq k,$$

and

$$\|u - I_X u\|_{r,\Omega} \leq C\, h_{X,\Omega}^{k-r} \|u\|_{k,\Omega}, \quad 0 \leq r \leq k,$$

for all $u \in H^k(\Omega)$.

2.2 CLSP for second order PDEs

For a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, we consider the following second order elliptic operator

$$Lu(x) = -\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j}(x) + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x^i}(x) + c(x)u(x) = f(x), \quad x \in \Omega, \qquad (2)$$

$$Bu(x) = u(x) = g(x), \quad x \in \partial\Omega, \qquad (3)$$

where $u \in H^2(\Omega)$, $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$.

If we assume that U, V and W are Hilbert spaces and problem (2)-(3) is well-posed so that it has a unique solution for all smooth data f and g and there exist positive constants C_1 and C_2 such that

$$C_1\|u\|_U \leq \|Lu\|_V + \|Bu\|_W \leq C_2\|u\|_U. \qquad (4)$$

This relation is called *energy balance* which is fundamental to least-squares methods because it defines a proper norm-equivalence between solution space U and data Space $V \times W$.

In order to achieve high order convergence, the regularity of u needs to be higher than what is strictly required by the problem itself. Here we assume for a real $k \geq 2$,

$$U := H^k(\Omega), \quad V := H^{k-2}(\Omega), \quad W := H^{k-1/2}(\partial\Omega).$$

By using an Agmon-Douglas-Nirenberg (ADN) setting [2], the left inequality in (4) can be proved for the above L and B operators. See [38, 39] for details and proofs. Problem (2)-(3) is well-posed if and only if the boundary operator B complements L in a proper way. As specified in [2], this is equivalent to an algebraic condition, called the *complementing condition*, on the principal parts of L and B. However, we shall not state these conditions here as they are somewhat complicated and are not needed in the continuation. But in what follows we assume L is uniformly elliptic in the sense of ADN in $\overline{\Omega}$ and B satisfies the complementing condition.

Lemma 2.3. *Let $k \geq 2$ be real and assume Ω is a bounded domain such that $\partial\Omega \in C^k$. Furthermore, assume that the a_{ij}, b_i, and c are in $C^{k-2}\left(\overline{\Omega}\right)$. If $f \in H^{k-2}\left(\Omega\right)$ and $g \in H^{k-1/2}\left(\partial\Omega\right)$ then every solution $u \in H^2\left(\Omega\right)$ is indeed in $H^k\left(\Omega\right)$. Also, there exists a constant $C > 0$ independent of u, f and g such that for every solution $u \in H^k\left(\Omega\right)$ we have*

$$\|u\|_{k,\Omega} \leq C\left(\|f\|_{k-2,\Omega} + \|g\|_{k-1/2,\partial\Omega}\right). \tag{5}$$

Moreover, the a priori bound (5) can be also extended to all real values $k < 2$.

Throughout the paper and in what follows, whenever we assume $u \in H^k(\Omega)$, for some real $k \geq 2$, is the unique solution of (2)-(3) (perhaps in a weak sense), then Ω is assumed to be a bounded domain in $\mathbb{R}^d$ such that $\partial\Omega \in C^k$. Furthermore, the functions a_{ij}, b_i and c in (2) are assumed to be of class $C^{k-2}\left(\overline{\Omega}\right)$.

Lemma 2.3 yields the inverse of mapping $T : H^k\left(\Omega\right) \to H^{k-2}\left(\Omega\right) \times H^{k-1/2}\left(\partial\Omega\right)$ defined by $Tu = (Lu, u)$ which is continuous for all real k. To extend this a priori estimate to the energy balance, we need the trace theorem that relates the Sobolev norms of functions on the interior of Ω with the Sobolev norms of their restrictions to the boundary $\partial\Omega$ [1].

Theorem 2.4. *(Trace Theorem) Assume that $\partial\Omega \in C^{\ell,1}$ for some $\ell \geq 0$ and $1/2 < k \leq \ell+1$. Then, the trace operator $\pi : H^k(\Omega) \to H^{k-1/2}(\partial\Omega)$, where $\pi u := u\,|_{\partial\Omega}$, is bounded. This means there exists a positive constant C such that for all $u \in H^k\left(\Omega\right)$*

$$\|\pi u\|_{k-1/2,\partial\Omega} \leq C\|u\|_{k,\Omega}.$$

By applying Lemma 2.3 and Theorem 2.4 we simply have the following theorem.

Theorem 2.5. *For real $q \geq 0$, let Ω be a bounded domain such that $\partial\Omega \in C^{q+2}$. Furthermore, assume that the coefficients of L are of class $C^q\left(\overline{\Omega}\right)$. Then the mapping $T : H^{q+2}\left(\Omega\right) \to H^q\left(\Omega\right) \times H^{q+3/2}\left(\partial\Omega\right)$ defined by $Tu = (Lu, u)$ is a homeomorphism, and the norms $\|\cdot\|_{q+2,\Omega}$ and $\|L\cdot\|_{q,\Omega} + \|\cdot\|_{q+3/2,\partial\Omega}$ are equivalent; i.e., there exists a constant $C > 0$ independent of u such that*

$$C^{-1}\|u\|_{q+2,\Omega} \leq \|Lu\|_{q,\Omega} + \|u\|_{q+3/2,\partial\Omega} \leq C\|u\|_{q+2,\Omega}. \tag{6}$$

This relation defines a proper norm-equivalence between solution space $H^{q+2}\left(\Omega\right)$ and data Space $H^q\left(\Omega\right) \times H^{q+3/2}\left(\partial\Omega\right)$. Moreover, the energy balance (6) allows to define a well-posed *continuous least-squares principle* (CLSP) for (2)-(3) by energy functional

$$J_q\left(u; f, g\right) := \frac{1}{2}\left(\|Lu - f\|^2_{q,\Omega} + \|u - g\|^2_{q+3/2,\partial\Omega}\right). \tag{7}$$

The corresponding CLSP is given by the pair $\left\{H^{q+2}\left(\Omega\right), J_q\right\}$, which corresponds to an unconstrained minimization problems

$$\min_{u \in H^{q+2}(\Omega)} J_q\left(u; f, g\right). \tag{8}$$

From Theorem 2.5 the least-squares functional $J_q(\cdot; 0, 0)$ defines a norm-equivalent property for $\|\cdot\|_{q+2,\Omega}$, i.e. , obviously one sees that the functional $J_q\left(\cdot; 0, 0\right)$ is equivalent to $\|\cdot\|_{q+2,\Omega}$ in the sense that

$$\frac{1}{4}C_1^2\|u\|^2_{q+2,\Omega} \leq J_q\left(u; 0, 0\right) \leq \frac{1}{2}C_2^2\|u\|^2_{q+2,\Omega}. \tag{9}$$

Therefore, according to [5, Theorem 2.5], for all real values $q \geq 0$, problem (8) has a unique minimizer $u \in H^{q+2}(\Omega)$ that depends continuously on the data $(f,g) \in H^q(\Omega) \times H^{q+3/2}(\partial\Omega)$. Moreover, it is not difficult to see that a minimizer of (2)-(3) solves (8) and conversely; i.e., the problems (2)-(3) and (8) are equivalent in the sense that $u \in H^{q+2}(\Omega)$ is a solution of (8) if and only if it is also a solution, perhaps in a generalized sense, of (2)-(3).

The Euler-Lagrange equation for (8) is then given by the variational problem:

$$\text{seek } u \in H^{q+2}(\Omega) \text{ such that } Q_q(u,v) = F_q(v), \quad \forall v \in H^{q+2}(\Omega),$$

where

$$Q_q(u,v) = (Lu, Lv)_{q,\Omega} + (u,v)_{q+3/2,\partial\Omega}, \quad \text{and} \quad F_q(v) = (Lv, f)_{q,\Omega} + (v, g)_{q+3/2,\partial\Omega}.$$

We notice that the *energy inner product* associated with $\{H^{q+2}(\Omega), J_q\}$ is given by $((\cdot, \cdot))_q : H^{q+2}(\Omega) \times H^{q+2}(\Omega) \to \mathbb{R}$, where

$$((u,v))_q := Q_q(u,v),$$

and *energy norm* is defined by $|||u|||_q := ((u,u))_q^{1/2} = [2J_q(u)]^{1/2}$. The norm-equivalence property

$$C_1 \|u\|_{q+2,\Omega} \leq |||u|||_q \leq C_2 \|u\|_{q+2,\Omega}, \tag{10}$$

holds by (6). The special case $q = 0$ in (7) gives rise to the CLSP

$$J_0(u; f, g) = \frac{1}{2} \left(\|Lu - f\|_{0,\Omega}^2 + \|u - g\|_{3/2,\partial\Omega}^2 \right),$$

where its associated energy balance for all $u \in H^2(\Omega)$ is

$$C_1 \|u\|_{2,\Omega} \leq \|Lu\|_{0,\Omega} + \|u\|_{3/2,\partial\Omega} \leq C\|u\|_{2,\Omega}. \tag{11}$$

In what follows we may write $J_q(u)$ instead of $J_q(u; 0, 0)$, J_q instead of $J_q(\cdot; 0, 0)$ and Q_q instead of $Q_q(\cdot, \cdot)$.

A least-squares discretization can be defined by choosing a family of finite subspaces $U^h \subset H^{q+2}(\Omega)$ parameterized by h tending to zero and then restricting the unconstrained minimization problem (8) to the subspaces. Thus, the approximation $u^h \in U^h$ to the solution $u \in H^{q+2}(\Omega)$ of (2)-(3) is the solution of the following problem

$$\text{seek } u^h \in U^h \text{ such that } J_q(u^h; f, g) \leq J_q(v^h; f, g), \quad \forall u^h \in U^h. \tag{12}$$

This process leads to a discrete variational form given by

$$\text{seek } u^h \in U^h \text{ such that } Q_q(u^h, v^h) = F_q(v^h), \quad \forall v^h \in U^h. \tag{13}$$

If we choose a basis $\{\phi_j\}_{j=1}^N$ and assume $u^h = \sum_{j=1}^N c_j \phi_j$ for some constants $\{c_j\}_{j=1}^N$, then the discretized problem (13) is equivalent to the linear system

$$Ac = b \tag{14}$$

where A is a symmetric matrix with entries $a_{ij} = Q_q(\phi_i, \phi_j)$ and b is a N vector with $b_i = F_q(\phi_i)$ for all $i, j = 1, \dots, N$.

Note that, in the above setting, one does not assumed that L is positive definite and self-adjoint, while in Rayleigh-Ritz setting does. However, not only LSP preserves all attractive features of a Rayleigh-Ritz setting but also it does not have some Rayleigh-Ritz restrictions. More precisely, the CLSP $\{H^{q+2}(\Omega), J_q\}$ defines an external Rayleigh-Ritz principle for (2)-(3).

The pair $\{U^h, J_q\}$ is called *discrete least-squares principle* (DLSP) where $U^h \subset H^{q+2}(\Omega)$ and J_q is given by (7). Although the CLSP describes a mathematically well-posed variational setting, its associated DLSP $\{U^h, J_q\}$ may describe an algorithmically infeasible setting. For instance, the least-squares functional may contain inner products in fractional-order Sobolev spaces on the boundary that are inconvenient for actual implementations. Practical issues may force us to abandon the DLSP setting described above and consider instead another pair for DLSP, denoted by $\{U^h, J^h\}$, where

$$J^h(u) = \frac{1}{2}\left(h^{-t}\|Lu\|_{\tilde{V}}^2 + h^{-s}\|Bu\|_{\tilde{W}}^2\right)$$

with proper (computationally feasible) Sobolev spaces $\tilde{V}$ and $\tilde{W}$ and nonnegative powers t and s [3, 6, 7, 8]. The pair $\{U^h, J^h\}$ is called the *data-weighted DLSP*. Also, a weighted least-squares strong-form based on RBF collocation is given in [18].

3 RBFs discretization and error estimation

Up to here we are given a least-squares functional which is equivalent to a combination of Sobolev norms, but these norms might be inconvenient from the computational point of view. To circumvent this flaw, when this functional is restricted to a finite subspace, we can use the fact that all norms on a finite-dimensional space are equivalent. Thus, essentially all norms can be replaced by L^2-norms weighted by some respective equivalence constants. In this section we try to introduce a mesh-dependent least-squares functional by using RBFs where the residual of each equation is measured in the L^2-norm multiplied by a weight determined by the equation index and the mesh parameter h. As some earlier work for weighted least-squares methods we refer the reader to [3, 8].

3.1 Weighted discretization of CLSP by RBFs

Recall the kernel Φ satisfying (1) for $\tau \geq q + 2 > d/2$ to form the data dependent trial space $U_{\Phi, X}$ for a quasi-uniform set X. Throughout the paper, τ (the smoothness index of Φ) satisfies $\tau > d/2$ and it is fixed. Assuming $h = h_{X, \Omega}$, we define the convex data-weighted functional

$$J^h(u; f, g) := \frac{1}{2}\left(\|Lu - f\|_{0,\Omega}^2 + h^{-3}\|u - g\|_{0,\partial\Omega}^2\right), \quad u \in H^{q+2}(\Omega), \tag{15}$$

for $q \geq 0$. The corresponding data-weighted DLSP $\{U_{\Phi, X}, J^h\}$ then leads to the unconstrained minimization problem

$$\text{seek } u^h \in U_{\Phi, X} \text{ such that } J^h(u^h; f, g) \leq J^h(v^h; f, g) \quad \forall v^h \in U_{\Phi, X}. \tag{16}$$

The Euler-Lagrange equation for (16) is given by the variational problem

$$\text{seek } u^h \in U_{\Phi,X} \text{ such that } Q^h(u^h, v^h) = F^h(v^h) \quad \forall v^h \in U_{\Phi,X}, \tag{17}$$

where

$$Q^h(u, v) = (Lu, Lv)_{0,\Omega} + h^{-3}(u, v)_{0,\partial\Omega},$$

and

$$F^h(v) = (Lv, f)_{0,\Omega} + h^{-3}(v, g)_{0,\partial\Omega}.$$

The bilinear form $Q^h(\cdot, \cdot)$ defines an inner product $((\cdot, \cdot)) : U_{\Phi,X} \times U_{\Phi,X} \to \mathbb{R}$ by

$$((u, v)) := Q^h(u, v)$$

which is called the *data-weighted discrete energy inner product*. The *data-weighted discrete energy norm* is then defined via $|||u||| := \sqrt{((u, u))} = \sqrt{2J^h(u)}$. Moreover, the discrete energy inner product and norm can be extended to all functions in $H^{q+2}(\Omega)$.

If we define $\phi_j = \Phi(\cdot - x_j)$ and $A_{kj} = Q^h(\phi_k, \phi_j)$ and $b_k = F^h(\phi_k)$ then the final linear system $Ac = b$ gives the solution vector c for $u^h = \sum_{j=1}^N c_j \Phi(\cdot - x_j)$.

3.2 Error Estimates

First of all, we are interested in finding elements from $U_{\Phi,X} \subset H^{q+2}(\Omega)$ which are closest to $u \in H^{q+2}(\Omega)$. More precisely, we are interested in the minimization problem

$$\inf_{v \in U_{\Phi,X}} |||u - v|||.$$

Since $U_{\Phi,X}$ is convex and J^h is a strictly convex functional then from [57, Theorem 38.C], J^h has at most one minimum on $U_{\Phi,X}$. On the other hand, (15) guarantees that the discrete energy norm can be extended to all smooth functions $u \in H^{q+2}(\Omega)$ for $q \geq 0$, and since $U_{\Phi,X} \subset H^{q+2}(\Omega)$, we have

$$((u, v^h)) = F^h(v^h), \quad \forall v^h \in U_{\Phi,X}.$$

From (17) we also have $((u^h, v^h)) = F^h(v^h)$ for all $v^h \in U_{\Phi,X}$. Subtraction gives

$$((u - u^h, v^h)) = 0, \quad \forall v^h \in U_{\Phi,X}. \tag{18}$$

Therefore, for all $v^h \in U_{\Phi,X}$ we can write

$$\begin{aligned}
|||u - u^h|||^2 &= ((u - u^h, u - u^h)) \\
&= ((u - u^h, u - v^h + v^h - u^h)) \\
&= ((u - u^h, u - v^h)) \\
&\leq (|||u - u^h|||)(|||u - v^h|||).
\end{aligned}$$

This leads that the minimizer of the data-dependent DLSP $\{U_{\Phi,X}, J^h(\cdot)\}$ is the best approximation of the minimizer of CLSP $\{H^{q+2}(\Omega), J_q(\cdot)\}$ out of subspace $U_{\Phi,X}$ in the discrete energy norm.

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Theorem 3.1. *Let $q \geq 0$ be given and $u \in H^{q+2}(\Omega)$ be the unique solution of (2)-(3). Then, $\{U_{\Phi,X}, J^h(\cdot)\}$ has at most one minimizer $u^h \in U_{\Phi,X}$. Also the minimizer $u^h \in U_{\Phi,X}$ is the orthogonal projection of u with respect to the discrete energy norm; i.e.,*

$$\inf_{v^h \in U_{\Phi,X}} |||u - v^h||| = |||u - u^h|||. \tag{19}$$

Note that the orthogonality (18) yields the Pythagorean law $|||u - u^h|||^2 + |||u^h|||^2 = |||u|||^2$ giving immediately the stability bounds $|||u - u^h||| \leq |||u|||$ and $|||u^h||| \leq |||u|||$ in the discrete energy norm.

We give the error analysis of the method in two parts. In the first part some segment of the error bound can be obtained from the analysis of least squares methods given in [3]. The approximation space $U_{\Phi,X}$ should possess an optimality property with respect to some pairs (r, k). This property is addressed in the following lemma which is a direct consequence of Theorem 2.2 and a result on approximation in scales of Banach spaces [9].

Lemma 3.2. *Under the assumptions on X, Ω, Φ, and $h < 1$ made in Theorem 2.2, with $\tau \geq k > d/2$, for all $u \in H^k(\Omega)$ there exist a function $v^h \in U_{\Phi,X}$ and a constant $C > 0$ independent of h and u such that for all $0 \leq r \leq k$*

$$\inf_{v^h \in U_{\Phi,X}} \sum_{i=0}^{r} h^i \|u - v^h\|_{i,\Omega} \leq C h^k \|u\|_{k,\Omega}. \tag{20}$$

Lemma 3.3. *Assume $u \in H^k(\Omega)$ is given as the unique minimizer of $\{H^{q+2}(\Omega), J_q(\cdot)\}$ and $u^h \in U_{\Phi,X}$ indicates the unique minimizer of $\{U_{\Phi,X}, J^h(\cdot)\}$ for some $q \geq 0$. Then, there exists a constant $C > 0$ such that for all s with $\tau \geq k \geq s \geq q + 2 > d/2$ we have*

$$|||u - u^h||| \leq C h^{s-2} \|u\|_{s,\Omega}.$$

Proof. Let $v^h \in U_{\Phi,X}$. Using the definition of $||| \cdot |||$ we have

$$|||u - v^h||| = \left(\|L(u - v^h)\|_{0,\Omega}^2 + h^{-3} \|u - v^h\|_{0,\partial\Omega}^2 \right)^{1/2}$$

$$\leq C \left(\|u - v^h\|_{2,\Omega} + h^{-3/2} \|u - v^h\|_{0,\partial\Omega} \right)$$

$$\leq C \left(\|u - v^h\|_{2,\Omega} + h^{-1} \|u - v^h\|_{1,\Omega} + h^{-2} \|u - v^h\|_{0,\Omega} \right),$$

where in the last inequality , for $0 < h \leq 1$ the inequality (see [10, 25])

$$\|v\|_{0,\partial\Omega} \leqslant C(\eta^{-1} \|v\|_{0,\Omega} + \eta \|v\|_{1,\Omega}) \tag{21}$$

is used for $\|u - v^h\|_{0,\partial\Omega}$ with $\eta = h^{1/2}$. Therefore,

$$\inf_{v^h \in U_{\Phi,X}} |||u - v^h||| \leq C h^{-2} \inf_{v^h \in U_{\Phi,X}} \sum_{i=0}^{2} h^i \|u - v^h\|_{i,\Omega}.$$

Using Lemma 3.2 and (19), the desired bound is obtained. □

According to Lemma 3.2, $U_{\Phi,X}$ approximates *optimally* with respect to (r,k) for $u \in H^k(\Omega)$ and $0 \leqslant r \leqslant k$ in the sense of [9, 3]. Thus, we may do some modifications to the statement and the proof of [3, Theorem 4.1] to tune the following result for the kernel-based least-squares method.

Theorem 3.4. *Assume $u \in H^k(\Omega)$ is given as the unique minimizer of $\{H^{q+2}(\Omega), J_q(\,\cdot\,)\}$ and $u^h \in U_{\Phi,X}$ indicates the unique minimizer of $\{U_{\Phi,X}, J^h(\,\cdot\,)\}$ for some $q \geq 0$. There exists a constant $C > 0$ such that for all s with $\tau \geq k \geq s \geq q + 2 > d/2$ we have*

$$
\begin{aligned}
\|L(u - u^h)\|_{t,\Omega} &\leq C\,h^{s-t-2}\|u\|_{s,\Omega}, & 2 - k \leq t \leq 0, \\
\|u - u^h\|_{t,\partial\Omega} &\leq C\,h^{s-t-1/2}\,\|u\|_{s,\Omega}, & \tfrac{1}{2} - k \leq t \leq 0.
\end{aligned}
$$

Proof. Let $f_1 \in H^{p-2}(\Omega)$ and $g_1 \in H^{p-1/2}(\partial\Omega)$ be given for a fixed p such that $\tau \geq k \geq p > d/2$ and $p \geq q + 2$. From Theorem 2.5, there exists a function $\varphi \in H^p(\Omega)$ that satisfies

$$
\begin{cases}
L\varphi = f_1, & \text{in } \Omega, \\
\varphi = g_1, & \text{on } \partial\Omega.
\end{cases}
\tag{22}
$$

For $v^h \in U_{\Phi,X}$, from (18) and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
((u - u^h, \varphi)) &= ((u - u^h, \varphi - v^h)) \\
&\leq (|||u - u^h|||)(|||\varphi - v^h|||).
\end{aligned}
$$

Lemma 3.3 then yields

$$
\begin{aligned}
((u - u^h, \varphi)) &\leq (|||u - u^h|||)(\inf_{v^h \in U_{\Phi,X}} |||\varphi - v^h|||) \\
&\leq C(h^{s-2}\|u\|_{s,\Omega})(h^{p-2}\|\varphi\|_{p,\Omega}) \\
&= Ch^{s+p-4}\|u\|_{s,\Omega}\|\varphi\|_{p,\Omega},
\end{aligned}
$$

for all s with $\tau \geq k \geq s > d/2$. Now, we can apply Lemma 2.3 for problem (22) to obtain

$$
((u - u^h, \varphi)) \leq Ch^{s+p-4}\|u\|_{s,\Omega}\left\{\|f_1\|_{p-2,\Omega} + \|g_1\|_{p-1/2,\partial\Omega}\right\}.
$$

From the definition of the discrete energy inner product we have

$$
(L(u - u^h), f_1)_{0,\Omega} + h^{-3}(u - u^h, g_1)_{0,\partial\Omega} \leq Ch^{s+p-4}\|u\|_{s,\Omega}\left\{\|f_1\|_{p-2,\Omega} + \|g_1\|_{p-1/2,\partial\Omega}\right\}.
\tag{23}
$$

In particular, let $g_1 = 0$ in (23) to get

$$
(L(u - u^h), f_1)_{0,\Omega} \leq Ch^{s+p-4}\|u\|_{s,\Omega}\|f_1\|_{p-2,\Omega}.
$$

Consequently,

$$
\|L(u - u^h)\|_{-(p-2),\Omega} = \sup_{f_1 \in H^{p-2}(\Omega)} \frac{(L(u - u^h), f_1)_{0,\Omega}}{\|f_1\|_{p-2,\Omega}} \leq Ch^{s+p-4}\|u\|_{s,\Omega}.
\tag{24}
$$

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In particular, for $p = k$ in (24) we obtain

$$\|L(u - u^h)\|_{-(k-2),\Omega} \le C\, h^{s+k-4}\, \|u\|_{s,\Omega}. \tag{25}$$

Also, the definition of the discrete energy norm implies

$$\|L(u - u^h)\|_{0,\Omega} \le \||u - u^h\||,$$

leading to

$$\|L(u - u^h)\|_{0,\Omega} \le C\, h^{s-2}\, \|u\|_{s,\Omega}, \tag{26}$$

by applying Lemma 3.3. Bounds (25) and (26) give the estimates for $t = 0$ and $t = -(k-2)$, respectively. We can use the interpolation theorem on Sobolev spaces (see [1, Chapter 4] or [30, Proposition 2.3]) to get

$$\|L(u - u^h)\|_{t,\Omega} \le C\Big(\|L(u - u^h)\|_{-(k-2),\Omega}\Big)^{\theta}\Big(\|L(u - u^h)\|_{0,\Omega}\Big)^{1-\theta}, \tag{27}$$

where $t = -(k-2)\theta$, for $0 \le \theta \le 1$. Inserting estimates (25) and (26) into (27) yields

$$\|L(u - u^h)\|_{t,\Omega} \le C\, h^{s-t-2}\|u\|_{s,\Omega}, \quad 2-k \le t \le 0,$$

for $2 - k \le t \le 0$.

Now, it is possible to choose $f_1 = 0$ in (23) to get

$$(u - u^h, g_1)_{0,\partial\Omega} \le C\, h^{s+p-1}\|u\|_{s,\Omega}\|g_1\|_{p-1/2,\partial\Omega}.$$

Hence,

$$\|u - u^h\|_{-(p-1/2),\partial\Omega} \le C\, h^{s+p-1}\, \|u\|_{s,\Omega}. \tag{28}$$

Now, let $p = k$ in (28) to obtain

$$\|u - u^h\|_{-(k-1/2),\partial\Omega} \le C\, h^{s+p-1}\, \|u\|_{s,\Omega}. \tag{29}$$

Furthermore, from the definition of the discrete energy norm we have

$$\|u - u^h\|_{0,\partial\Omega} \le h^{3/2}\||u - u^h\||.$$

leading to

$$\|u - u^h\|_{0,\partial\Omega} \le C\, h^{s-1/2}\, \|u\|_{s,\Omega} \tag{30}$$

after applying Lemma 3.3. Now, using the interpolation theorem on trace Sobolev spaces [1, 30] we have

$$\|u - u^h\|_{t+3/2,\partial\Omega} \le C(\|u - u^h\|_{-(k-1/2),\partial\Omega})^{\theta}(\|u - u^h\|_{0,\partial\Omega})^{1-\theta}, \quad 0 \le \theta \le 1, \tag{31}$$

where $t + 3/2 = -(k-1/2)\theta$. Inserting (29) and (30) to (31) gives

$$\|u - u^h\|_{t+3/2,\partial\Omega} \le C\, h^{s-t-2}\, \|u\|_{s,\Omega}, \quad -k-1 \le t \le -\tfrac{3}{2}.$$

$\square$

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Theorem 3.5. *Assume that $u \in H^k(\Omega)$ is the unique solution of (2)-(3) in the CLSP $\left\{H^{q+2}(\Omega), J_q(\cdot)\right\}$ such that $\tau \geq k > d/2$ and $k \geq \max\{q+2, 4\}$ for some real $q \geq 0$. Also, assume that $u^h \in U_{\Phi, X}$ is the unique minimizer of $\{U_{\Phi, X}, J^h(\cdot)\}$. Then, there exists a constant $C > 0$ such that for all s with $k \geq s \geq q + 2 > d/2$ we have*

$$\|u - u^h\|_{t,\Omega} \leq C\, h^{s-t}\, \|u\|_{s,\Omega}, \qquad 4 - k \leq t \leq \tfrac{1}{2}.$$

Proof. Using Lemma 2.3 we have

$$\|u - u^h\|_{t+2,\Omega} \leq C \left(\|L(u - u^h)\|_{t,\Omega} + \|u - u^h\|_{t+3/2,\partial\Omega}\right),$$

for all real $t \leq k - 2$. Hence, Theorem 3.4 yields

$$\|u - u^h\|_{t+2,\Omega} \leq C\, h^{s-t-2}\, \|u\|_{s,\Omega}, \quad 2 - k \leq t \leq -\tfrac{3}{2},$$

or equivalently

$$\|u - u^h\|_{t,\Omega} \leq C\, h^{s-t}\, \|u\|_{s,\Omega}, \quad 4 - k \leq t \leq \tfrac{1}{2}.$$

$\square$

The last bound of Theorem 3.5 measures the error in Sobolev norms $\|\cdot\|_{t,\Omega}$ for $4 - k \leq t \leq 1/2$ that also includes the error estimation in the L_2 norm because $k \geq 4$. The reminder parts of this section are devoted to prove a norm-equivalent property and to extend the above error analysis to higher order Sobolev norms on the left hand side. However, our results are hanged on an *open problem* that will be stated after some auxiliary lemmas form the kernel approximation theory.

The proof of the following *inverse inequality* of Bernstein type can be found in [18].

Lemma 3.6. *Assume a kernel Φ satisfying (1) with $\tau > d/2$ is given. Suppose the domain Ω is a bounded Lipschitz region satisfying an interior cone condition. Then for all $u^h \in U_{\Phi, X}$ and all finite sets $X = \{x_1, \ldots, x_N\} \subset \Omega$ with separation distance q_X, there is a constant C depending only on Φ, Ω and μ such that for all $d/2 < \mu \leq \tau$ we have*

$$\|u^h\|_{\tau,\Omega} \leq C\, q_X^{-\tau+\mu}\|u^h\|_{\mu,\Omega}. \tag{32}$$

By applying Theorem 2.5, Lemma 3.6 with $\mu = 2$ and for quasi-uniform sets X (i.e. $q_X \approx h_{X,\Omega}$), and by invoking an interpolation argument we can prove that for all $u^h \in U_{\Phi, X}$ there exists a constant $C > 0$, independent of u^h, such that for all q with $0 \leq q \leq \tau - 2$,

$$\|Lu^h\|_{q,\Omega} \leq Ch_{X,\Omega}^{-q} \left(\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega}\right) \tag{33}$$

where $\Omega \subset \mathbb{R}^d$ with $d \leq 3$.

We also need a *sampling inequality* or *zeros lemma* to support our argument. A variation of zeros lemma that holds for fractional Sobolev norms on both sides of inequality has been proved in [51]. Also, see [34] for older versions.

Lemma 3.7. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Let $r, k \in \mathbb{R}$ satisfy $k > d/2$ and $0 \leq r \leq k$. If $u \in H^k(\Omega)$ satisfies $u|_X = 0$, then for any discrete sets $X \subset \Omega$ with sufficiently small mesh norm $h_{X,\Omega}$, there exists a constant C that depends only on Ω and k such that*

$$\|u\|_r \leq Ch_{X,\Omega}^{k-r}\, \|u\|_k, \quad 0 \leq r \leq k.$$

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The weighted DLSP setting uses the L^2-norm for the boundary part while CLSP involves the boundary norm $\|\cdot\|_{q+3/2,\partial\Omega}$. Thus, we need an inverse inequality that relates $\|\cdot\|_{3/2,\partial\Omega}$ and $\|\cdot\|_{0,\partial\Omega}$ for approximating function $u^h \in U_{\Phi,X}$. The proof of such boundary inverse inequality seems to need a counterpart inverse inequality in Ω that is still an open problem. Thus we conjecture

> **Conjecture A.** For all finite quasi-uniform set $X \subset \Omega \subset \mathbb{R}^d$, with sufficiently small fill distance h, there exists a constant C, depending only on Ω, $\partial\Omega$, and Φ such that for all $u \in U_{\Phi,X}$,
> $$\|u\|_{3/2,\partial\Omega} \leq C\, h^{-3/2} \|u\|_{0,\partial\Omega}. \tag{34}$$

Apart form Theorem 3.5, Conjecture A will be used to prove the error bound for the least-squares method in $\|\cdot\|_{t,\Omega}$ for $0 < t \leq k$ when $u \in H^k(\Omega)$; see Theorem 3.12 below. Also, it supports our estimation for the lower bound of the smallest eigenvalue of the final matrix in section 3.3.

Using Conjecture A, in the following we show that the weighted least-squares functional satisfies a data-dependent energy balance.

Lemma 3.8. *For all real q with $0 \leq q \leq \tau - 2$, there exists a positive constant C independent of u^h such that for all $u^h \in U_{\Phi,X}$ with $0 < h \leq 1$ and $d \leq 3$ the inequality*

$$C^{-1} h^{2q} \|u^h\|_{q+2,\Omega}^2 \leq Q^h(u^h, u^h) \tag{35}$$

holds.

Proof. Let $q = \tau - 2$. Using Lemma 2.3 and bound (33) we have

$$
\begin{aligned}
\|u^h\|_{\tau,\Omega}^2 &\leq C \left(\|Lu^h\|_{\tau-2,\Omega} + \|u^h\|_{\tau-1/2,\partial\Omega} \right)^2 \\
&\leq C \left(\|Lu^h\|_{\tau-2,\Omega} + \|u^h\|_{\tau,\Omega} \right)^2 && \text{(Theorem 2.4)} \\
&\leq C\, h^{-2(\tau-2)} \left(\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega} + \|u^h\|_{2,\Omega} \right)^2 && \text{(using (33) and Lemma 3.6)} \\
&\leq C\, h^{-2(\tau-2)} \left(\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega} \right)^2 && \text{(using (11))} \\
&\leq C\, h^{-2(\tau-2)} \left(\|Lu^h\|_{0,\Omega}^2 + \|u^h\|_{3/2,\partial\Omega}^2 \right) \\
&= C\, h^{-2(\tau-2)} \left(\|Lu^h\|_{0,\Omega}^2 + h^{-3} \|u^h\|_{0,\partial\Omega}^2 \right) && \text{(using (34))} \\
&= C\, h^{-2(\tau-2)} Q^h(u^h, u^h).
\end{aligned}
$$

Now, by using the interpolation theorem for $q = 0$ and $q = \tau - 2$ we get

$$
\begin{aligned}
\|u^h\|_{q+2,\Omega} &\leq C \left(\|u^h\|_{\tau,\Omega} \right)^{1-\theta} \left(\|u^h\|_{2,\Omega} \right)^{\theta} \\
&\leq C \left(h^{2-\tau} \left[\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega} \right] \right)^{1-\frac{2-\tau+q}{2-\tau}} \left(\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega} \right)^{\frac{2-\tau+q}{2-\tau}} \\
&\leq C h^{-q} \left(\|Lu^h\|_{0,\Omega} + \|u^h\|_{3/2,\partial\Omega} \right),
\end{aligned}
$$

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where $q + 2 = (1 - \theta)\tau + 2\theta$, for $0 \le \theta \le 1$. Hence, for all $0 \le q \le \tau - 2$,

$$
\begin{aligned}
\|u^h\|_{q+2,\Omega}^2 &\le C h^{-2q}(\|Lu^h\|_{0,\Omega}^2 + \|u^h\|_{3/2,\partial\Omega}^2) \\
&\le C h^{-2q}(\|Lu^h\|_{0,\Omega}^2 + h^{-3}\|u^h\|_{0,\partial\Omega}^2) \quad \text{(using (34))} \\
&= C h^{-2q} Q^h(u^h, u^h)
\end{aligned}
$$

$\square$

Lemma 3.9. *There exist a positive constant C, independent of u^h, such that for all $u^h \in U_{\Phi,X}$ with $0 < h \le 1$ we have*

$$
Q^h(u^h, u^h) \le C h^{-4} \|u^h\|_{2,\Omega}^2.
$$

Proof. From (11) for some constant $C > 0$ we get

$$
\begin{aligned}
Q^h(u^h, u^h) &= \|Lu^h\|_{0,\Omega}^2 + h^{-3}\|u^h\|_{0,\partial\Omega}^2 \\
&\le \left(\|Lu^h\|_{0,\Omega} + h^{-3/2}\|u^h\|_{0,\partial\Omega} \right)^2 \\
&\le C\left(\|u^h\|_{2,\Omega} + h^{-3/2}\|u^h\|_{0,\partial\Omega} \right)^2.
\end{aligned}
$$

Moreover, for $0 < h \le 1$ by using the inequality (21), we obtain for $\eta = h^{1/2}$

$$
\begin{aligned}
Q^h(u^h, u^h) &\le C\left\{ \|u^h\|_{2,\Omega} + h^{-3/2}\left(h^{-1/2}\|u^h\|_{0,\Omega} + h^{1/2}\|u^h\|_{1,\Omega} \right) \right\}^2 \\
&= C\left(\|u^h\|_{2,\Omega} + h^{-1}\|u^h\|_{1,\Omega} + h^{-2}\|u^h\|_{0,\Omega} \right)^2 \\
&\le C h^{-4} \|u^h\|_{2,\Omega}^2.
\end{aligned}
$$

$\square$

Lemmas 3.8 and 3.9 show that the discrete energy inner product $((\,\cdot\,,\,\cdot\,)) : U_{\Phi,X} \times U_{\Phi,X} \to \mathbb{R}$ and the discrete energy norm $||| \cdot |||$ are related to the Sobolev norms of the solution space.

Corollary 3.10. *Assume $0 \le q \le \tau - 2$ is given and $0 < h \le 1$ is sufficiently small. Then, for all $u^h \in U_{\Phi,X}$ there exists a constant $C > 0$, independent of u^h, such that*

$$
C^{-1} h^q |||u^h|||_q \le |||u^h||| \le C h^{-2} |||u^h|||_0 \le C h^{-2} |||u^h|||_q.
$$

In the following, we aim to extend our error analysis in Theorem 3.5 to other (positive) Sobolev norms.

Lemma 3.11. *Suppose that Φ satisfies (1) with $\tau \ge k > d/2$, and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Furthermore, let $X \subset \Omega$ be a discrete set of centers with sufficiently small mesh norm $h = h_{X,\Omega}$. If $u \in H^k(\Omega)$ and the RBF interpolant of u on X is given by $I_X u$ then there exists a constant $C > 0$ such that for all s with $\tau \ge k \ge s \ge q + 2 > d/2$ we have*

$$
|||u - I_X u||| \le C h^{s-2} \|u - I_X u\|_s.
$$

Proof. Using the definition of $||| \cdot |||$ we can write

$$\begin{aligned}
|||u - I_X u||| &= \left(\|L(u - I_X u)\|_{0,\Omega}^2 + h^{-3} \|u - I_X u\|_{0,\partial\Omega}^2 \right)^{1/2} \\
&\leq C \left(\|u - I_X u\|_{2,\Omega} + h^{-3/2} \|u - I_X u\|_{0,\partial\Omega} \right) \\
&\leq C \left(\|u - I_X u\|_{2,\Omega} + h^{-1} \|u - I_X u\|_{1,\Omega} + h^{-2} \|u - I_X u\|_{0,\Omega} \right) \\
&\leq C(h^{s-2} \|u - I_X u\|_s + h^{-1} h^{s-1} \|u - I_X u\|_s + h^{-2} h^s \|u - I_X u\|_s) \\
&= C h^{s-2} \|u - I_X u\|_s,
\end{aligned}$$

which completes the proof. In the third line above the bound (21) is applied with $\eta = h^{1/2}$, and in the fourth line Lemma 3.7 is used. $\qquad\square$

If $u \in H^k(\Omega)$ is the unique minimizer of $\{H^{q+2}(\Omega), J_q\}$ for $q \geq 0$ and $u^h \in U_{\Phi,X}$ is the unique minimizer of $\{U_{\Phi,X}, J^h\}$ then by using the fact that u^h is the best approximation in the discrete energy norm, and under the assumptions and notations of Lemma 3.11 we have

$$|||u - u^h||| \leq |||u - I_X u||| \leq C h^{s-2} \|u - I_X u\|_{s,\Omega},$$

which turns the error of PDE solution into the error of pure interpolation problem. By applying the bound $\|u - I_X u\|_{s,\Omega} \leq \|u\|_{s,\Omega}$ from Theorem 2.2 we obtain

$$|||u - u^h||| \leq C h^{s-2} \|u\|_{s,\Omega}, \quad d/2 < s \leq k \leq \tau, \tag{36}$$

for true solution $u \in H^k(\Omega)$. Altogether, we have the following theorem.

Theorem 3.12. *Assume that Φ satisfies (1) for $\tau > d/2$ and $u \in H^k(\Omega)$ is the unique solution of (2)-(3), where k is a real number such that $\tau \geq k > d/2$ for $d \leq 3$, and $k \geq \max\{q+2, 4\}$ for some real $q \geq 0$. Moreover, assume that $u^h \in U_{\Phi,X}$ is the unique minimizer of $\{U_{\Phi,X}, J^h\}$. Then*

$$\|u - u^h\|_{t,\Omega} \leq C h^{k-t} \|u\|_{k,\Omega}, \quad 0 \leq t \leq k.$$

Proof. Let us first assume $t = k$. Then the inequality

$$\|u - u^h\|_{k,\Omega} \leq \|u - I_X u\|_{k,\Omega} + \|u^h - I_X u\|_{k,\Omega},$$

suggests that we can focus on the difference $u^h - I_X u \in U_{\Phi,X}$ and on the difference $u - I_X u \in H^k(\Omega)$ where $I_X u$ denotes the unique interpolant of the exact solution u from the trial space $U_{\Phi,X} \subset H^\tau(\Omega)$. For the first norm on the right hand side, from Theorem 2.2 we have

$$\|u - I_X u\|_{k,\Omega} \leq C \|u\|_{k,\Omega}.$$

Since the result in Theorem 3.8 only applies to functions in the trial space, for $d \leq 3$, we obtain

$$\|u^h - I_X u\|_{k,\Omega} \leq C h^{2-k} |||u^h - I_X u||| \leq C h^{2-k} (|||u - I_X u||| + |||u - u^h|||).$$

Using the discussions right before the theorem for $s = k$, the right hand side can be bounded by $C \|u\|_{k,\Omega}$ to get

$$\|u^h - I_X u\|_{k,\Omega} \leq C \|u\|_{k,\Omega}.$$

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Combining the recent bounds, we obtain the following stability bound

$$\|u - u^h\|_{k,\Omega} \leq C\|u\|_{k,\Omega}.$$

On the other hand, putting $s = k$ and $t = 0$ in Theorem 3.5 yields

$$\|u - u^h\|_{0,\Omega} \leq C\,h^k\|u\|_{k,\Omega}.$$

Henceforth, for $0 \leq t \leq k$, using the interpolation theorem we have

$$\|u - u^h\|_{t,\Omega} \leq C\left(\|u - u^h\|_{0,\Omega}\right)^{1-\theta}\left(\|u - u^h\|_{k,\Omega}\right)^{\theta}$$
$$\leq C\left(h^k\|u\|_{k,\Omega}\right)^{1-t/k}\left(\|u\|_{k,\Omega}\right)^{t/k}$$
$$= C\,h^{k-t}\|u\|_{k,\Omega},$$

where $t = \theta k$ and $0 \leq \theta \leq 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3   Condition Numbers

In this subsection we estimate the condition number of the presented least-squares method. Under reasonable assumptions, the condition number of the discrete least-squares matrix is controlled by the mesh size h and regularity parameter τ. Since the final matrix A is positive definite, its condition number can be defined as

$$\mathrm{cond}_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and smallest eigenvalue of A, respectively. An appropriate way to bound $\lambda_{\min}$ is the use of an inverse inequality in the trial space to turn the conditioning of the PDE matrix back to one of the approximation theory. To this end, we review a lemma from [52] that computes a lower bound for the smallest eigenvalue of interpolation matrix by kernel Φ.

Lemma 3.13. *Assume $B_{\Phi,X} = (\Phi(x_j - x_k))_{j,k=1}^N$ is the usual interpolation matrix. Then the minimum eigenvalue of $B_{\Phi,X}$ can be bounded by*

$$\lambda_{\min}(B_{\Phi,X}) \geqslant Cq_X^{2\tau-d}. \tag{37}$$

Lemma 3.14. *Suppose that Φ satisfies (1) for $\tau > d/2$, $X \subset \Omega \subset \mathbb{R}^d$ is quasi-uniform and $h = h_{X,\Omega}$ is sufficiently small. The minimum eigenvalue of A, for $d \leq 3$, can be bounded by*

$$\lambda_{\min}(A) \geqslant Ch^{4\tau-d-4}.$$

Proof. An appropriate formula for $\lambda_{\min}(A)$ is

$$\lambda_{\min}(A) = \min_{0 \neq \xi \in \mathbb{R}^N} \frac{\xi^T A\xi}{\|\xi\|^2}.$$

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For a given $\xi \in \mathbb{R}^N$ assume that $u^h = \sum_{j=1}^{N} \xi_j \Phi(\cdot - x_j)$. Thus, by using (35) and inverse inequality (32) we deduce

$$
\begin{aligned}
\xi^T A \xi = ((u^h, u^h)) &= |||u^h|||^2 \\
&\geqslant Ch^{2\tau-4} \|u^h\|_{\tau,\Omega}^2 \qquad (\text{ Theorem 3.8}) \\
&\geqslant Ch^{2\tau-4} \|u^h\|_{\Phi}^2 \\
&= Ch^{2\tau-4} \xi^T B_{\Phi,X} \xi \\
&\geqslant Ch^{2\tau-4} h^{2\tau-d} \|\xi\|^2, \qquad (\text{using bound (37)})
\end{aligned}
$$

which shows that $\lambda_{\min}(A) \geqslant Ch^{4\tau-d-4}$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

To bound $\lambda_{\max}(A)$, we need some results about derivatives of positive definite functions. In [16, 17, 31] it is proved that certain derivatives of positive definite functions are also positive (or negative) definite. Authors show that some simple conditions on even order derivatives of positive definite functions at the origin strongly determine their global properties. In particular, they show that the derivatives of a smooth positive definite function can be estimated in terms of the even order derivatives at the origin. Proposition 3.2 of [31] prove that if Φ is a positive definite function of class C^{2n} in some neighborhood of the origin, for some positive integer n, then for each $|\alpha| \leq n$ the function $(-1)^{|\alpha|} D^{2\alpha}\Phi$ is positive definite of class $C^{2(n-|\alpha|)}(\mathbb{R}^d)$. Also, the following inequality holds for $|\alpha|, |\beta| \leq n$,

$$
|D^{\alpha+\beta}\Phi(x-\cdot)|^2 \leq (-1)^{|\alpha+\beta|} D^{2\alpha}\Phi(0) D^{2\beta}\Phi(0), \quad x \in \mathbb{R}^d.
$$

Therefore, if $\Phi \in C^{2n}(\Omega)$ for $n \geq 1$, then

$$
|D^{\alpha}\Phi(x-\cdot)|^2 \leq -\Phi(0) D^{2\alpha}\Phi(0) \text{ for } |\alpha| = 1,
$$

and for $|\gamma| = 2$

$$
|D^{\gamma}\Phi(x-\cdot)|^2 = |D^{\alpha+\beta}\Phi(x-\cdot)|^2 \leq D^{2\alpha}\Phi(0) D^{2\beta}\Phi(0) \text{ for } |\alpha| = |\beta| = 1.
$$

Henceforth,

$$
\sum_{|\alpha|\leq 2} \|D^{\alpha}\Phi(x-\cdot)\|_{\infty,\Omega}^2 \leq \sum_{|\alpha|=|\beta|=1} \max\{\Phi(0), -\Phi(0)D^{2\alpha}\Phi(0), D^{2\alpha}\Phi(0)D^{2\beta}\Phi(0)\}. \tag{38}
$$

Lemma 3.15. *Suppose that $X \subset \Omega \subset \mathbb{R}^d$ is quasi-uniform and $h = h_{X,\Omega}$ is sufficiently small. Also, assume that $\Phi \in C^{2n}(\Omega)$ for some $n \geq 1$. Then the maximum eigenvalue of A can be bounded by*

$$
\lambda_{\max}(A) \leqslant Ch^{-d-4}.
$$

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Proof. We can employ (38) and the inequality $|\sum_{j=1}^{N} \xi_j|^2 \leq N \sum_{j=1}^{N} |\xi_j|^2$, to deduce that

$$
\begin{aligned}
\|u^h\|_{2,\Omega}^2 &= \sum_{|\alpha| \leq 2} \int_\Omega [D^\alpha u^h]^2 dx \\
&= \sum_{|\alpha| \leq 2} \int_\Omega \Big[\sum_j \xi_j D^\alpha \Phi(x - x_j) \Big]^2 dx \\
&\leq \sum_{|\alpha| \leq 2} \|D^\alpha \Phi(x - x_j)\|_{\infty,\Omega}^2 \times \int_\Omega \Big| \sum_j \xi_j \Big|^2 dx \\
&\leq \sum_{|\alpha| \leq 2} \|D^\alpha \Phi(x - x_j)\|_{\infty,\Omega}^2 \times \Big| \sum_j \xi_j \Big|^2 \times \mathrm{vol}(\Omega) \\
&\leq \mathrm{vol}(\Omega) \times \Big| \sum_j \xi_j \Big|^2 \times \sum_{|\alpha|=|\beta|=1} \max\{\Phi(0), (-1)\Phi(0)D^{2\alpha}\Phi(0), D^{2\alpha}\Phi(0)D^{2\beta}\Phi(0)\} \\
&\leq C \Big| \sum_j \xi_j \Big|^2 \leq CN \sum_j |\xi_j|^2 \leq Ch^{-d}\|\xi\|^2. \qquad (\text{Since } N = \mathcal{O}(h^{-d}))
\end{aligned}
$$

Thus we can conclude

$$
\xi^T A \xi = |||u^h|||^2 \leq Ch^{-4}\|u^h\|_{2,\Omega}^2 \leq Ch^{-4}h^{-d}\|\xi\|^2
$$

which gives the desired bound. $\qquad \square$

Corollary 3.16. *For $d \leq 3$, the condition number of the final linear system of the least-squares kernel-based method is bounded by*

$$
\mathrm{cond}_2(A) \leqslant Ch^{-4\tau}.
$$

4 Numerical Examples

In this section, results of some numerical experiments are reported to verify the theoretical bounds of the preceding sections. The convergence of the numerical solution u^h toward the true solution u are investigated and the rates of convergence are estimated numerically. Besides, the condition number of the final system is estimated. The computational rate of convergence p is approximated in two successive levels h_1 and h_2 via

$$
p = \log \frac{\|u - u^{h_1}\|_{t,\Omega}}{\|u - u^{h_2}\|_{t,\Omega}} \Big/ \log \frac{h_1}{h_2},
$$

for $t = 0, 2$. Moreover, to estimate H^2 error and convergence rates we use

$$
\|u - u^h\|_{2,\Omega} \approx |||u - u^h|||.
$$

From Theorems 3.5 and 3.12, one finds that the theoretical rates are $k - t$ for $u \in H^k(\Omega)$.

The following test problem in $\mathbb{R}^2$ is considered

$$Lu = -\frac{\partial^2 u}{\partial(x^1)^2} - \frac{\partial^2 u}{\partial(x^2)^2} + \frac{\partial u}{\partial x^1} + \frac{\partial u}{\partial x^2} + u = f, \quad \text{in } \Omega.$$

Assume that the exact solution is given by $u^*(x) := \|x\|_2^\kappa$, over the computational domain $\Omega = \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$. To form the trial space $U_{\Phi,X}$, we employ the Whittle-Matérn-Sobolev kernel

$$\Phi(x) = (\varepsilon\|x\|)^{\tau-d/2} K_{\tau-d/2}(\varepsilon\|x\|) \quad \text{for all } x \in \mathbb{R}^d, \quad \tau > d/2,$$

where K_β is the modified Bessel function of the second kind of order β, and $\varepsilon > 0$ is a shape parameter. This kernel has the Fourier transform $\widehat{\Phi}(\omega) = C(1 + \|\varepsilon\omega\|^2)^{-\tau}$ and its native space is identical with $H^\tau(\mathbb{R}^d)$.

Discretization is done by using a series set of trial points $X \subset \Omega \cup \partial\Omega$ with different fill-distance h. However, in this scheme it is not mandatory to locate some trial points on $\partial\Omega$. We note that, to enforce the boundary conditions in some collocation methods such as the symmetric kernel-based method of [21, 22, 56], some trial points have to be qualitatively located on the boundary. All reported errors in $L^2(\Omega)$ norm are RMS errors approximated by using the fixed set of 7668 equidistant points in Ω. Also, the error is analyzed in $\|\cdot\|_{0,\partial\Omega}$ with 1000 equidistant points on $\partial\Omega$. All integrals in DLSP variational form are computed via the Gauss-Legendre quadrature rule with sufficient number of integration points in angular and radial directions. We do not employ any special technique to deal with the problem of ill-conditioning.

For a nonnegative integer k, it is well-known that $u^* \in H^k(\Omega)$ if $\kappa > k - d/2$. In our numerical example, we set $\kappa = 4$ to have $u^* \in H^k(\Omega)$ for $k < 5$. We set $q = 0$ in the least-squares approach, $\varepsilon = 10$ for shape parameter, and $\tau = 3, 4, 5, 6$ for kernel function Φ. From Theorems 3.5 and 3.12 the parameters τ and k should satisfy $\tau \geq k \geq 4$. Thus the cases $\tau = 3, 4$ exclude the requirements of the theory, but yet perform the L^2 convergence; see Table 1. For brevity, the notation $e^h = u^* - u^h$ is used. In cases $\tau = 5, 6$, the table contains also the theoretical orders of Theorems 3.5 and 3.12 and the bound (36). In all columns, except that of $\|\|e^h\|\|$, the numerical orders at finer levels are better than the expected theoretical orders. Note that the orders do not improve when going from $\tau = 5$ to $\tau = 6$ because of the limitation caused by the smoothness of true solution u^*.

According to the results of subsection 3.3, there is a direct relation between the smoothness of trial kernel and the conditioning of final system, where a higher smoothness leads to a larger condition number. Since in this paper we do not focus on preconditioning techniques, for small values of h the results suffer from a severe ill-conditioning, specially for higher values $\tau = 5, 6$. This is the reason why fewer rows are reported in this cases. Table 2 shows the condition numbers of the final linear systems together with the numerical orders. In all cases, as $h \to 0$, the approximate rate of conditioning of the final matrix is of $\mathcal{O}(h^{-4\tau})$ as proven in Corollary 3.16. Unsatisfactory results for small values of h and higher values of τ are obtained.

Table 1: Approximate errors and orders for DLSP using Whittle-Matérn-Sobolev kernel with $\varepsilon = 10$.

h	$\|e^h\|_{0,\Omega}$	order	$\|e^h\|_{0,\partial\Omega}$	order	$\|Le^h\|_{0,\Omega}$	order	$\|\|e^h\|\|$	order
				$\tau = 3$				
0.25	4.0771e-01	-	9.4412e-01	-	9.3784e+00	-	6.9802e+01	-
$h/2$	1.3016e+00	-1.6747	2.5902e-01	1.8659	1.7612e+01	-0.9092	1.5023e+02	-1.1058
$h/4$	1.4516e+00	0.1574	3.3942e-02	2.9319	1.6565e+01	0.0884	1.5559e+02	-0.0506
$h/6$	1.1913e+00	0.4874	1.3580e-02	2.2093	1.4866e+01	0.2670	2.0259e+02	-0.6510
$h/8$	7.9384e-01	1.4111	7.8229e-03	1.9172	1.1875e+01	0.7806	2.6821e+02	-0.9753
$h/10$	4.9996e-01	2.0721	4.7968e-03	2.1920	9.1629e+00	1.1621	3.1616e+02	-0.7369
$h/12$	4.3891e-01	0.7142	3.0758e-03	2.4374	8.6532e+00	0.3139	3.4881e+02	-0.5391
$h/14$	2.8627e-01	2.7725	2.0804e-03	2.5366	6.8777e+00	1.4898	3.7222e+02	-0.4216
Theory		-		-		-		-
				$\tau = 4$				
0.25	3.5863e-01	-	8.4776e-01	-	1.0102e+01	-	6.3975e+01	-
$h/2$	1.2905e+00	-1.8473	1.4867e-01	2.5013	1.5338e+01	-0.6024	9.1456e+01	-0.5156
$h/4$	7.7511e-01	0.7354	2.5646e-02	2.5353	1.1615e+01	0.4011	1.1666e+02	-0.3512
$h/6$	3.6300e-01	1.8709	8.9105e-03	2.6073	7.8091e+00	0.9791	1.3099e+02	-0.2857
$h/8$	1.1878e-01	3.8834	3.4407e-03	3.3077	4.4627e+00	1.9528	1.1720e+02	0.3867
$h/10$	4.3416e-02	4.5102	1.3764e-03	4.1058	2.6917e+00	2.2558	9.0872e+01	1.1445
$h/12$	2.5797e-02	2.8553	7.4864e-04	3.3402	2.0585e+00	1.4708	8.4852e+01	0.3705
$h/14$	1.1324e-02	5.3411	4.0271e-04	4.0222	1.3795e+00	2.5968	7.2102e+01	1.0562
Theory		-		-		-		-
				$\tau = 5$				
0.25	4.5055e-01	-	6.9867e-01	-	1.0848e+01	-	5.5563e+01	-
$h/2$	1.0304e+00	-1.1935	1.0777e-01	2.6967	1.3647e+01	-0.3312	6.8825e+01	-0.3088
$h/4$	2.9679e-01	1.7957	1.7418e-02	2.6293	7.1083e+00	0.9410	7.8402e+01	-0.1889
$h/6$	6.0793e-02	3.9104	3.8090e-03	3.7492	3.2229e+00	1.9508	5.5880e+01	0.8368
$h/8$	1.0943e-02	5.9607	9.6698e-04	4.7656	1.4055e+00	2.8846	3.3091e+01	1.8212
$h/10$	2.6877e-03	6.2919	2.8314e-04	5.5042	4.0552e-01	3.0887	1.8827e+01	2.5275
Theory		5		4.5		3		3
				$\tau = 6$				
0.25	5.4390e-01	-	5.9587e-01	-	1.1171e+01	-	4.9306e+01	-
$h/2$	7.7146e-01	-0.5043	8.1924e-02	2.8626	1.1857e+01	-0.0860	5.3803e+01	-0.1259
$h/4$	8.6708e-02	3.1534	9.9357e-03	3.0436	3.9212e+00	1.5964	4.4618e+01	0.2701
$h/6$	8.2390e-03	5.8049	1.3425e-03	4.9366	1.2420e+00	2.8355	1.9801e+01	2.0037
$h/8$	9.3344e-04	7.5700	2.3671e-04	6.0326	4.3282e-01	3.6644	8.1892e+00	3.0690
Theory		5		4.5		3		3

Table 2: Condition numbers and their orders of DLSP matrix for various values of τ

h	$\tau = 3$	order	$\tau = 4$	order	$\tau = 5$	order	$\tau = 6$	order
0.25	2.1442e+00	-	2.6245e+00	-	2.7575e+00	-	2.2632e+00	-
$h/2$	6.1697e+00	-1.5248	5.0880e+01	-4.2770	5.7017e+02	-7.6919	5.9607e+03	-11.3629
$h/4$	2.3461e+03	-8.5708	3.4348e+05	-12.7208	4.9499e+07	-16.4056	6.8306e+09	-20.1281
$h/6$	1.5702e+05	-10.3674	1.1296e+08	-14.2939	8.1722e+10	-18.2732	5.8350e+10	-22.3261
$h/8$	3.6020e+06	-10.8901	8.0450e+09	-14.8280	1.8394e+13	-18.8227	4.2756e+16	-22.9309
$h/10$	4.1449e+07	-10.9480	2.2430e+11	-14.9138	1.2496e+15	-18.9050	-	-
$h/12$	4.0986e+08	-12.5677	3.3015e+12	-14.7496	-	-	-	-
$h/14$	2.1131e+09	-10.6396	3.3761e+13	-15.0822	-	-	-	-

5   Conclusion

A least-squares variational kernel-based method for solving the general second order elliptic problem with nonhomogenous Dirichlet boundary conditions is given in this paper. One of the attractive features of the method is that the approximating space is not subject to the LBB condition. Besides, the discretization yields a positive definite system while the original PDE may not be symmetric at all. The approximation space is formed via kernels that reproduce Sobolev spaces as their native spaces. We show that the DLSP formulations using sufficiently smooth kernels, which reproduce $H^\tau(\Omega)$, can converge at the optimal rate in $H^t(\Omega)$-norm, with $4 - k \leq t \leq k$, where $\tau \geq k \geq 4$. The condition number of the final least-squares system is also estimated in terms of the smoothness of the basis function and the discretization parameter. Some parts of our analysis are subjected to a *conjecture* that demands an independent study in the theory of kernel approximations. See (34). Finally, we have reported some numerical results to confirm the theoretical bounds. As a downside, the condition numbers grow at hight algebraic rates for smooth trial kernels. This paper does not concern special approaches or any preconditioning technique to overcome this problem. However, we suggest some possible approaches here. The stability estimates may be greatly improved if a similar theory could be derived for *polyharmonic kernels* by scaling the points with mesh norm h and carrying the computation in the blown-up situation. More details can be found in [27, 11]. As another possibility, one can use the *localized bases for kernel space* [24] instead of the global basis $U_{\Phi,X}$ to improve the condition numbers. The use of "greedy algorithms" in trial space will be another possible approach [50, 46]. The compactly supported kernels in a multiscale setting can also be used to improve the conditioning at the price of a more computational cost [20]. Finally, the application of the method on the corresponding first order system of equations needs less smooth basis functions leading to a great improvement in the numerical conditioning. Since all the proposals above are rather involved and contain their own technical details, we do not peruse them further and leave them for future studies.

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