(1)

HYBRID MIXED DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR INCOMPRESSIBLE WORMHOLE PROPAGATION PROBLEM

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ABSTRACT. Wormhole propagation plays a very important role in the product enhancement of oil and gas reservoir. A new combined hybrid mixed finite element method is proposed to solve incompressible wormhole propagation problem with discontinuous Galerkin finite element procedure, in which, the new hybrid mixed finite element algorithm is established for pressure equation, while the discontinuous Galerkin finite element method is considered for concentration equation, and then the porosity function is computed straightly by the approximate value of the concentration. This new combined method can keep local mass balance, meantime it also keeps the boundedness of the porosity. The convergence of the proposed method is analyzed and the optimal error estimate is derived. Finally, numerical examples are presented to verify the validity of the algorithm and the correctness of the theoretical results.

1. INTRODUCTION

The acid treatment of carbonate reservoirs is a widely practiced oil and gas well stimulation technique. In fact, when acids are injected into oil production wells, chemical reactions cause the dissolution of the material near the wellbore to result into flow channels. Such flow channels look like worm holes that they are usually called wormholes. Because of its important role in the product enhancement of oil and gas reservoir, the wormhole propagations have been a topic of key interest for research during recent decades. The theoretical researches on numerical methods for these problems have extensive practicability and important significance.

Here, we will construct a new combined numerical procedure to solve the incompressible wormhole propagation problem which is usually described by the following nonlinear partial differential equations (see [1, 2, 3]):

$$\begin{cases} \frac{\partial \phi}{\partial t} = \frac{\alpha k_c a_v (c_f - c_s)}{\rho_s}, & x \in \Omega, \quad 0 \le t \le T, \\ \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = f, & x \in \Omega, \quad 0 \le t \le T, \\ \mathbf{u} = \frac{-k(\phi)}{\mu} \nabla p, & x \in \Omega, \quad 0 \le t \le T, \\ \frac{\partial (\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u} c_f) = \nabla \cdot (\phi \mathbf{D}(\mathbf{u}) \nabla c_f) + k_c a_v (c_s - c_f) + f_I c_I - f_P c_f, \end{cases}$$

and the corresponding initial-boundary conditions are considered as follows:

(2)
$$\begin{cases} \phi(x,0) = \phi_0(x), \quad c_f(x,0) = c_f^0(x), \quad x \in \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - c_f \mathbf{u}) \cdot \mathbf{n} = 0, \quad x \in \partial\Omega, \quad 0 \le t \le T, \end{cases}$$

where $\Omega \subset R^d (d = 2, 3)$ denotes a bounded polygonal/polyhedral domain; α is the dissolving constant of the acid; ρ_s is the density of the rock; **n** is the unit outward normal vector to $\partial\Omega$; the functions p and **u** denote the pressure and Darcy velocity; ϕ and k are the porosity and permeability of rocks, and μ is the viscosity of fluid; a_v is the interfacial area available for reaction; f, f_I and f_P are the external volumetric flow rate, the injection flow rate and the production flow rate, respectively; c_f , c_s and c_I are the concentrations of acid in the fluid phase, the fluid-solid interface and the injected flow, respectively; Diffusion coefficient $\mathbf{D}(\mathbf{u}) = \phi[d_m I + |\mathbf{u}|(d_I E(\mathbf{u}) + d_t E^{\perp}(\mathbf{u}))]$ comes from two aspects: small molecule diffusion of oil field scale problem, and speed-related diffusion in petroleum engineering, where the matrix $E(\mathbf{u}) = (u_i u_j/|\mathbf{u}|^2)_{d\times d}$ and

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 $E^{\perp}(\mathbf{u}) = I - E(\mathbf{u}); k_c$ is the local mass-transfer coefficient. In the case of first order kinetic reaction, the concentrations c_s and c_f satisfy the relationship:

$$c_s = \frac{c_f}{1 + k_s/k_c}$$

where k_s is the kinetic constant for reaction. The quantitative relationship between the coefficients k, v and ϕ is as follows:

(4)
$$\frac{k}{k_0} = \frac{\phi}{\phi_0} \left(\frac{\phi(1-\phi_0)}{\phi_0(1-\phi)}\right)^2, \quad \frac{a_v}{a_0} = \frac{1-\phi}{1-\phi_0}$$

where k_0 , a_0 , and ϕ_0 are the initial values for k, a_v , ϕ .

Many numerical methods have been constructed for simulating the wormhole propagation. In [3], Kou etc. used a classical mixed finite element procedure to establish a fully conservative method for incompressible wormhole problem; And then, they considered a parallel algorithm for wormhole problem in [4] under the Darcy-Brinkman-Forchheimer framework. In [2, 5], Rui and Li studied the block-centered finite difference methods with or without the method of characteristics for wormhole propagation. Zhang etc. established a combined splitting mixed finite element method for compressible wormhole propagation with the method of the characteristics in [6]. And Guo etc. studied the local discontinuous Galerkin finite element method for incompressible problem in [7].

Generally speaking, the mixed finite element methods can be used to obtain more accurate approximation of the velocity function. However, the classical mixed element methods usually result into some saddle point problems, in which the coefficient matrix of the mixed system loses the symmetric positive definite property and the finite element spaces require the LBB condition. By introducing the Lagrange multiplier, a symmetric and positive definite system is obtained by hybrid mixed element method ([8, 9, 10, 11, 12, 13]). Therefore, the hybrid technique will be considered for the pressure and velocity equations. In the procedure, the velocity and pressure are eliminated by use of the Lagrange multiplier, so that they can be solved element by element. In addition, the resulted global mixed system only involves the degrees of freedom with the Lagrange multiplier, so this technique can significantly improve the computational efficiency.

As we know, the concentration equation is usually characterized as convection-dominant, the traditional Galerkin finite element method is not well applied, in particular for the discontinuous problems. The discontinuous Galerkin (DG) methods in [14, 15, 16] were introduced. They have several advantages over other types of finite element methods. For example, test functions across the finite element interfaces do not explicitly impose continuity constraints. As a result, the finite element spaces allow highly nonuniform and unstructured meshes. These methods have been more and more widely used, such as [17, 18]. Zhu and his coauthors studied the discontinuous Galerkin finite element methods for nonlinear reaction diffusion equations in [19, 20], and then extended these techniques to the dipolar Bose-Einstein condensation and Lagrangian compressible Euler equations in [21, 22]. Based on the traditional mixed finite element methods and DGFE methods, the combined mixed DG methods were proposed to solve the compressible and incompressible miscible displacement problems in [23, 24, 25, 26]. However, there is little research on the discontinuous Galerkin methods for wormhole propagation.

The focus of this article is to combine the discontinuous Galerkin finite element method with the hybrid mixed technique to simulate the incompressible wormhole propagation. In the combined method, a new hybrid mixed finite element (HMFE) procedure is constructed to solve pressure equation, and the symmetric interior penalty discontinuous Galerkin (SIPDG) procedure is proposed to solve the concentration equation, then the porosity is computed straightly by the approximated value of the concentration. Compared with other existing combined methods, the proposed method not only keeps mass balance locally, but also keeps the boundedness of the porosity, especially it can deal well with the discontinuous case. The consistency and stability of the proposed method are analyzed, and then the corresponding error estimate is given under the case that the diffusion coefficient includes the molecular diffusion and dispersion, unlike the ones in [2, 5, 7] where only molecular diffusion was considered. Finally, numerical examples are presented to verify the validity of the algorithm and the correctness of the theoretical results.

For the convenience of analysis, we make the following assumptions:

Assumption 1.1. Assume that the parameters μ , k_c , k_s , α , ρ_s are positive constants, and that ϕ_0 , $\frac{k(\phi)}{\mu}$ and $f(\cdot,t)$ are bounded as follows:

(5)
$$0 < a_* \le \frac{k(\phi)}{\mu} \le a^*, \quad 0 < \phi_0 < 1, \quad |f(\cdot, t)| \le C,$$

where a_* , a^* and C are some positive constants. And we also assume that the diffusion coefficient $\mathbf{D}(\mathbf{u})$ satisfies the uniformly positive definiteness and Lipschitz continuousness

$$\mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c \ge D_* |\nabla c|^2$$

and

(6)
$$\|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})\|_{[L^2]^d} \le D^* \|\mathbf{u} - \mathbf{v}\|_{[L^2]^d},$$

where D* and D_* are two positive constants independent of \mathbf{u} and \mathbf{v} and c.

Assumption 1.2. Assume that the solution (p, \mathbf{u}, c_f) of the system (1) has the regularities as follows:

(a)
$$p \in L^2(0,T; H^{k+2}(\Omega)), \quad \phi_0 \le \phi \le C_1 < 1,$$

(b)
$$\mathbf{u} \in L^{\infty}(0,T; H^{k+1}(\Omega)) \cap L^{\infty}(0,T; L^{\infty}(\Omega)),$$

(7)

(c)
$$c_f \in H^1(0,T; H^{k+1}(\Omega)) \cap L^{\infty}(0,T; W^{1,\infty}(\Omega)),$$

(d) $\frac{\partial c_f}{\partial t} \in L^2(0,T; H^{k+1}(\Omega)) \cap L^{\infty}(0,T; L^{\infty}(\Omega)).$

Moreover, we only consider the homogeneous boundary condition case in this article for simplification. For nonhomogeneous boundary value problem, we can use some simple technique to transform it into homogeneous boundary value problem, so our method proposed later is still valid.

2. The formulation of HMDG method

In order to illustrate our method, we first give a uniform regular partition of Ω , denoted by $\mathcal{T}_h = \{K_1, K_2, ..., K_n\}$. We denote $\mathcal{E}_h = \bigcup_{K \in \mathcal{T}_h} \{e | e \in \partial K\}$ to be the set of all cell edges and $\mathcal{E}_h^i = \mathcal{E}_h \setminus \partial \Omega$ as all the interior ones. Furthermore, let $h_e = diam(e)$ for all $e \in \mathcal{E}_h$. Introduce the following piecewise Sobolev spaces associated with \mathcal{T}_h

$$H^{s}(\mathcal{T}_{h}) = \{ v \in L^{2}(\Omega) : v |_{K} \in H^{s}(K), K \in \mathcal{T}_{h} \}, \quad s \ge 0.$$

We also define the following spaces on \mathcal{E}_h :

$$L^{2}(\mathcal{E}_{h}) = \{ v \in L^{2}(e), \forall e \in \mathcal{E}_{h} \}, \quad L^{2}(\mathcal{E}_{h}^{i}) = \{ v \in L^{2}(e), \forall e \in \mathcal{E}_{h}^{i} \}.$$

For $e \in \mathcal{E}_h^i$, denote by n_e a fixed unit normal direction. For $e \in \partial \Omega$, $n_e = \mathbf{n}$. We define averages $\{\cdot\}$ and jumps $[\cdot]$:

$$\{v\} = \frac{1}{2}[(v|_{K^i})|_e + (v|_{K^j})|_e], \quad [v] = (v|_{K^i})|_e - (v|_{K^j})|_e, \quad \text{on} \quad e \in \mathcal{E}_h^i.$$

In particular, if $e \in \partial \Omega$, $\{v\} = v|_e$, $[v] = v|_e$. Meantime, we define inner products as follows:

$$(\cdot, \cdot)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (\cdot, \cdot)_K, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}_h^i} = \sum_{e \in \mathcal{E}_h^i} \langle \cdot, \cdot \rangle_e,$$

and the norms $\|\cdot\|_{\mathcal{T}_h} = \sqrt{(\cdot, \cdot)_{\mathcal{T}_h}}$ and $|\cdot|_{\partial \mathcal{T}_h} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{E}_h}}$.

Introduce the discrete approximate spaces denoted by Ψ_h, Λ_h, Π_h and Σ_h as follows:

$$\Psi_{h} = \{ v \in H^{k}(\mathcal{T}_{h}) : v|_{K} \in P_{k}(K), K \in \mathcal{T}_{h} \},$$

$$\Lambda_{h} = \{ v \in L^{2}(\mathcal{T}_{h}) : v|_{K} \in P_{k}(K), K \in \mathcal{T}_{h} \},$$

$$\Pi_{h} = \{ v \in [H^{k}(\mathcal{T}_{h})]^{d} : v|_{K} \in RT_{k}(K), K \in \mathcal{T}_{h} \},$$

$$\Sigma_{h} = \{ v \in L^{2}(\mathcal{E}_{h}) : v|_{e} \in P_{k}(e), e \in \mathcal{E}_{h} \},$$

where $P_k(K), P_k(e)$ are the spaces of polynomial functions of degree at most k for each $K \in \mathcal{T}_h$ and each $e \in \mathcal{E}_h$, respectively, $RT_k(K) = [P_k(K)]^d \oplus xP_k(K)$ denotes the Raviart-Thomas mixed finite element space as in [9, 27, 28].

Set

$$\kappa = \frac{k_c k_s a_0}{(k_c + k_s)(1 - \phi_0)}.$$

Using (3) and (4), we can rewrite (1) in the following equivalent form:

 $\partial \phi$

(8)

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{\alpha \kappa}{\rho_s} (1 - \phi) c_f, \\ \frac{\alpha \kappa}{\rho_s} (1 - \phi) c_f + \nabla \cdot \mathbf{u} &= f, \quad \mathbf{u} = \frac{-k(\phi)}{\mu} \nabla p, \\ \frac{\partial (\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u} c_f) &= \nabla \cdot (\phi \mathbf{D}(\mathbf{u}) \nabla c_f) - \kappa (1 - \phi) c_f + f_I c_I - f_P c_f,. \end{aligned}$$

Next, we will formulate our method for wormhole propagation.

For the discretization of the porosity, we consider the similar technique as in [3]. The discrete porosity is point-wise defined and can be stated as follows:

(9)
$$\frac{\partial \phi_h}{\partial t} = \frac{\alpha \kappa}{\rho_s} (1 - \phi_h) \bar{c}_h,$$

where $\bar{c}_h = \max(0, \min(c_h, 1)), c_h$ is a given approximation of the concentration c_f .

2.1. Hybrid mixed finite element scheme for pressure and velocity. In this subsection, we give the hybrid mixed finite element (HMFE) method for pressure and velocity. which can be written as below:

Algorithm 1. For given approximate values of ϕ_h and c_h , seek $(p_h, \mathbf{u}_h, \lambda_h) \in \Lambda_h \times \Pi_h \times \Sigma_h$ such that

(10)
$$\sum_{K\in\mathcal{T}_{h}} (\frac{\alpha\kappa}{\rho_{s}}(1-\phi_{h})\bar{c}_{h}, v_{h})_{K} + \sum_{K\in\mathcal{T}_{h}} (\nabla\cdot\mathbf{u}_{h}, v_{h})_{K} = \sum_{K\in\mathcal{T}_{h}} (f, v_{h})_{K}, \quad \forall v_{h} \in \Lambda_{h},$$
$$\sum_{K\in\mathcal{T}_{h}} (a(\phi_{h})\mathbf{u}_{h}, \omega_{h})_{K} - \sum_{K\in\mathcal{T}_{h}} (p_{h}, \nabla\cdot\omega_{h})_{K} + \sum_{e\in\mathcal{E}_{h}} \langle\lambda_{h}, [\omega_{h}] \cdot n_{e}\rangle_{e} = 0, \quad \forall \omega_{h} \in \Pi_{h},$$
$$\sum_{e\in\mathcal{E}_{h}^{i}} \langle\mu_{h}, [\mathbf{u}_{h}] \cdot n_{e}\rangle_{e} = 0, \quad \forall \mu_{h} \in \Sigma_{h},$$

where $a(\phi_h) = \frac{\mu}{k(\phi_h)}$ and $\bar{c}_h = \max(0, \min(c_h, 1))$.

Define the bilinear form:

$$B_{\mathbf{u}}((\mathbf{u}_{h}, p_{h}, \lambda_{h}), (\omega_{h}, v_{h}, \mu_{h})) =: (\mathbf{u}_{h}, \nabla v_{h})_{\mathcal{T}_{h}} + (\nabla p_{h}, \omega_{h})_{\mathcal{T}_{h}} + (a(\phi_{h})\mathbf{u}_{h}, \omega_{h})_{\mathcal{T}_{h}} + \langle \lambda_{h} - p_{h}, [\omega_{h}] \cdot n_{e} \rangle_{\mathcal{E}_{h}} + \langle [\mathbf{u}_{h}] \cdot n_{e}, \mu_{h} - v_{h} \rangle_{\mathcal{E}_{h}^{i}}, \\B_{\phi}(c_{h}; \phi_{h}, v_{h}) := -(\frac{\alpha\kappa}{\rho_{s}}(1 - \phi_{h})\bar{c}_{h}, v_{h})_{\mathcal{T}_{h}}.$$

We can rewrite (10) into the following equivalent form:

Algorithm 2 (HMFE Algorithm). For given ϕ_h and c_h , find $(\mathbf{u}_h, p_h, \lambda_h) \in \Pi_h \times \Lambda_h \times \Sigma_h$ such that

(11)
$$B_{\phi}(c_{h};\phi_{h},v_{h}) + B_{\mathbf{u}}((\mathbf{u}_{h},p_{h},\lambda_{h}),(\omega_{h},v_{h},\mu_{h})) = -(f,v_{h})_{\mathcal{T}_{h}},$$
$$\forall (\omega_{h},v_{h},\mu_{h}) \in \Pi_{h} \times \Lambda_{h} \times \Sigma_{h}.$$

2.2. DGFE method for the concentration. Due to the flexibility of the discontinuous Galerkin finite element method in constructing feasible local-shape function spaces and the advantage in capturing non-smooth or oscillatory solutions effectively, we consider it to be applied for the concentration.

Define the bilinear form

$$B_{c}(c_{h}, z_{h}) := \sum_{K \in \mathcal{T}_{h}} \int_{K} (\phi_{h} \mathbf{D}(\mathbf{u}_{h}) \nabla c_{h} - \mathbf{u}_{h} c_{h}) \nabla z_{h} dx - \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}^{k} \{(\phi_{h} \mathbf{D}(\mathbf{u}_{h}) \nabla z_{h} - \mathbf{u}_{h} z_{h}) \cdot n_{e}\} [c_{h}] ds - \sum_{e \in \mathcal{E}_{h}^{i}} \int_{e}^{k} \{(\phi_{h} \mathbf{D}(\mathbf{u}_{h}) \nabla c_{h} - \mathbf{u}_{h} c_{h}) \cdot n_{e}\} [z_{h}] ds + J_{0}^{\gamma}(c_{h}, z_{h}),$$

where $J_0^\gamma(c_h,z_h)$ denotes the penalty term defined by

$$J_0^{\gamma}(c_h, z_h) = \sum_{e \in \mathcal{E}_h^i} \int_e \frac{\gamma}{h_e^{\beta}} [c_h][z_h],$$

where γ is called penalty parameter and bounded below by a large enough constant, and β denotes some positive constant.

Now we can reach the SIPDG method for the concentration equation.

Algorithm 3 (SIPDG Algorithm). For given \mathbf{u}_h and ϕ_h , find $c_h \in \Psi_h$ such that

(12)
$$(\frac{\partial(\phi_h c_h)}{\partial t}, z_h)_{\mathcal{T}_h} + B_c(c_h, z_h) + (\kappa(1 - \phi_h)c_h, z_h)_{\mathcal{T}_h} = (f_I c_I - f_P c_h, z_h)_{\mathcal{T}_h}, \quad \forall z_h \in \Psi_h.$$

2.3. **The combined HMDG method.** Here, we will present the new combined SIPDG finite element method for incompressible wormhole problem with hybrid mixed finite element procedure.

Algorithm 4 (HMDG Algorithm). For the given initial value functions ϕ_0 and c_f^0 , find $(c_h, p_h, \mathbf{u}_h, \lambda_h) \in \Psi_h \times \Lambda_h \times \Pi_h \times \Sigma_h$ such that

(a)
$$\frac{\partial \phi_h}{\partial t} = \frac{\alpha \kappa}{\rho_s} (1 - \phi_h) \bar{c}_h, \quad \bar{c}_h = \max(0, \min(c_h, 1)),$$

(b)
$$B_{\phi}(c_h; \phi_h, v_h) + B_{\mathbf{u}}((\mathbf{u}_h, p_h, \lambda_h), (\omega_h, v_h, \mu_h)) = -(f, v_h) \tau_h,$$

(c)
$$(\frac{\partial (\phi_h c_h)}{\partial t}, z_h) \tau_h + B_c(c_h, z_h) + (\kappa(1 - \phi_h)c_h, z_h) \tau_h = (f_I c_I - f_P c_h, z_h) \tau_h.$$

Theorem 2.1. Algorithm 4 is consistent. That is, define $\lambda = p$, then the solution (p, u, λ, ϕ, c) of system (8) satisfies (13). Conversely, if $(p_h, u_h, \lambda_h, \phi_h, c_h)$ satisfies (13), then $(p_h, u_h, \lambda_h, \phi_h, c_h)$ are the solutions of problem (8).

Proof. Substituting the weak solution (p, u, ϕ, c) of problem (8) into the first equation of (13) with $\omega_h = \mu_h = 0$, we can get

$$-(\frac{\alpha\kappa}{\rho_s}(1-\phi)c_f,v_h)_{\mathcal{T}_h}-(\nabla\cdot\mathbf{u},v_h)_{\mathcal{T}_h}=-(f,v_h)_{\mathcal{T}_h}.$$

Next, we test with $v_h = \omega_h = 0$ and get the equation

(1

$$\langle \mathbf{u} \cdot n_e, \mu_h \rangle_{\mathcal{E}^i_{\mathbf{L}}} = 0,$$

so the normal component of the flux **u** is continuous at element interfaces.

Now we will prove the consistency between the model problem (8) and HMDG Algorithm (13).

Firstly, we prove that the solution of (8) also solves (13). For this, let z be an element in Ψ_h . We multiply the third equation of (8) by z and integrate on one element K:

$$\int_{K} \frac{\partial(\phi c_{f})}{\partial t} z dx + \int_{K} (\phi \mathbf{D}(\mathbf{u}) \nabla c_{f} - \mathbf{u}c_{f}) \nabla z dx - \int_{\partial K} (\phi \mathbf{D}(\mathbf{u}) \nabla c_{f} - \mathbf{u}c_{f}) \cdot n_{K} z ds$$
$$= \int_{K} (f_{I}c_{I} - f_{P}c_{f} - \kappa(1 - \phi)c_{f})z) dx.$$

Summing it over all elements and using (2), we observe that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\phi\mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e z ds = \sum_{e\in\mathcal{E}_h^i}\int_e(\phi\mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e[z]ds.$$

By the regularities of the solution ϕ , u and c, we have

(a)
$$(\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e[z] = \{(\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e\}[z],$$

(b) $[c_f] = 0.$

Therefore, we obtain the second equation of the scheme (13).

Conversely, take $z \in H^1(\Omega)$ and $c_f \in H^1(\Omega) \cap \Psi_h$. Then (13) reduces to

$$\sum_{K\in\mathcal{T}_h}\int_K \frac{\partial(\phi c_f)}{\partial t} z_h dx + \sum_{K\in\mathcal{T}_h}\int_K (\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) z dx = \sum_{K\in\mathcal{T}_h}\int_K (f_I c_I - f_P c_f - \kappa(1-\phi)c_f) z) dx.$$

So for all $K \in \mathcal{T}_h$, we can obtain

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u}c_f) = \nabla \cdot (\phi \mathbf{D}(\mathbf{u})\nabla c_f) + f_I c_I - f_P c_f - \kappa (1-\phi)c_f, \quad \text{in} \quad K.$$

Finally, let K_1 and K_2 to be two adjacent elements, and $e = \partial K_1 \cap \partial K_2$. Take $z \in C_0^{\infty}(K_1 \cup K_2)$ and extend it by zero over the rest of the domain. Integrating by parts in the second equation of (8), we can get

$$(\frac{\partial(\phi c_f)}{\partial t}, z)_{K_1 \cup K_2} + (\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f, \nabla z)_{K_1 \cup K_2} - \langle [(\phi \mathbf{D}(\mathbf{u}) \nabla c_f - \mathbf{u} c_f) \cdot n_e], z \rangle_e$$

=(f_I c_I - f_P c_f - \kappa (1 - \phi) c_f, z)_{K_1 \cup K_2}.

On the other hand, (13) reduces to

$$\left(\frac{\partial(\phi c_f)}{\partial t}, z\right)_{K_1 \cup K_2} + \left(\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f, \nabla z\right)_{K_1 \cup K_2} = (f_I c_I - f_P c_f - \kappa(1 - \phi)c_f, z)_{K_1 \cup K_2}.$$

Hence, we have

$$\langle [(\phi D(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e], z \rangle_e = 0, \quad \forall z \in C_0^\infty(K_1 \cup K_2)$$

Since this holds for all e, it implies that $(\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \cdot n_e = 0$ on $\partial \Omega$ and $\nabla \cdot (\phi \mathbf{D}(\mathbf{u})\nabla c_f - \mathbf{u}c_f) \in L^2(\Omega)$, hence we have

(14)
$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u}c_f - \phi \mathbf{D}(\mathbf{u})\nabla c_f) = f_I c_I - f_P c_f - \kappa (1 - \phi)c_f.$$

We can easily show that the discrete solution ϕ_h of ϕ satisfies the following boundedness.

Theorem 2.2 (The boundedness of porosity). For any time $t \in (0, T]$, the approximate porosity ϕ_h is bounded, that is,

(15)
$$\phi_0 \le \phi_h \le 1 - (1 - \phi_0)e^{-\eta t} < 1,$$

(16)
$$0 \le \frac{\partial \phi_h}{\partial t} \le \frac{\alpha \kappa}{\rho_s}$$

where $\eta = \frac{\alpha \kappa}{\rho_s}$ and $\phi_0 > 0$.

Proof. First, we can rewrite (9) as the following integral form

$$\int_{\phi_0}^{\phi_h} \frac{1}{1-\phi_h} d\phi = \int_0^t \eta \bar{c}_h d\tau,$$

where $\eta > 0$. We deduce that

$$\ln \frac{1-\phi_h}{1-\phi_0} = -\eta \bar{c}_h t.$$

Notice that

$$\phi_h = 1 - (1 - \phi_0)e^{-\eta \bar{c}_h t} \le 1 - (1 - \phi_0)e^{-\eta t}$$

It is easily seen that the approximate value of the porosity increases with t, and $\phi_h = \phi_0$ at t = 0, so we can get the estimate (15). The estimate (16) is reached by (15) and (9).

For HMDG Algorithm, we have the following main convergence theorem.

Theorem 2.3. Under the assumptions (1.1) and (1.2), for t > 0, there is a priori error estimate as follows:

$$(17) \begin{cases} (a) & \|\phi - \phi_h\|_{\mathcal{T}_h} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + \|c_f - c_h\|_{\mathcal{T}_h} \le Ch^s(\|c_f\|_{H^1(0,T;H^s(\mathcal{T}_h))} + \|p\|_{L^2(0,T;H^{s+1}(\mathcal{T}_h))}), \\ (b) & \|\nabla(p_h - \Pi_h p)\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\lambda_h - p\|_{\mathcal{T}_h} \le Ch^s(\|c_f\|_{H^1(0,T;H^s(\mathcal{T}_h))} + \|p\|_{L^2(0,T;H^{s+1}(\mathcal{T}_h))}), \\ (c) & \|\nabla(p_h - \Pi_h p)\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\lambda_h - p\|_{\mathcal{T}_h} \le Ch^s(\|c_f\|_{H^1(0,T;H^s(\mathcal{T}_h))} + \|p\|_{L^2(0,T;H^{s+1}(\mathcal{T}_h))}), \end{cases}$$

$$(c) \quad \|p - p_h\|_{\mathcal{T}_h} \le Ch^s(\|c_f\|_{H^1(0,T;H^s(\mathcal{T}_h))} + \|p\|_{L^2(0,T;H^{s+1}(\mathcal{T}_h))}),$$

where when $d = 2, 1 \le s \le k + 1$; when $d = 3, 3/2 \le s \le k + 1$.

3. Some important projections and lemmas

In this section, we will give some important projection operators and approximate properties, which is used to show the convergence theorem of our proposed method.

Firstly, we introduce the following norms with respect to the bilinear form B_{u} :

$$\|(\omega, v, \mu)\||_B^2 := \|\omega\|_{\mathcal{T}_h}^2 + \|\nabla v\|_{\mathcal{T}_h}^2 + \frac{1}{h_e}|\mu - v|_{\partial \mathcal{T}_h}^2$$

and

$$\|(\omega, v, \mu)\|_{B,*}^2 := \|(\omega, v, \mu)\|_B^2 + h|\omega \cdot n_e|_{\partial \mathcal{T}_h}^2$$

As in [11, 26], we can read the following stability and boundedness of the bilinear form B_u .

Lemma 3.1 (Stability and Boundedness). Assume that ϕ and ϕ_h are fixed, for all $(\mathbf{u}, p, \lambda) \in H^k(\mathcal{T}_h) \times L^2(\mathcal{T}_h) \times L^2(\partial \mathcal{T}_h)$ and $(\omega_h, v_h, \mu_h) \in \Pi_h \times \Lambda_h \times \Sigma_h$, there holds

(a)
$$|B_{\mathbf{u}}((\mathbf{u}, p, \lambda), (\omega_h, v_h, \mu_h))| \le K^* ||(\mathbf{u}, p, \lambda)||_{B,*} ||(\omega_h, v_h, \mu_h)||_B,$$

(18) (b)
$$\sup_{(\omega_h, v_h, \mu_h) \in \Pi_h \times \Lambda_h \times \Sigma_h} \frac{B_{\mathbf{u}}((\mathbf{u}_h, p_h, \lambda_h), (\omega_h, v_h, \mu_h))}{\|(\omega_h, v_h, \mu_h)\|_B} \ge K_* \|(\mathbf{u}_h, p_h, \lambda_h)\|_B,$$

where K_* and K^* denote two positive constants independent of the mesh size h.

Introduce the local L^2 -projection operators Π_h and Π_e as follows:

(19)
$$(p - \Pi_h p, v_h)_K = 0, \quad \forall v_h \in P_k(K), \\ \langle \lambda - \Pi_e p, \mu_h \rangle_e = 0, \quad \forall \mu_h \in P_k(e),$$

where $K \in \mathcal{T}_h, e \in \mathcal{E}_h^i, p \in L^2(K)$ and $\lambda \in L^2(e)$.

Lemma 3.2 ([29]). For the local L^2 -projection operators Π_h and Π_e , there exists the following approximate property

k.

(20)
$$\begin{aligned} \|p - \Pi_h p\|_K &\leq Ch^s \|p\|_{s,K}, \quad 0 \leq s \leq k+1, \\ \|\nabla (p - \Pi_h p)\|_K &\leq Ch^s \|p\|_{s+1,K}, \quad 0 \leq s \leq k, \\ \|p - \Pi_h p\|_e + \|p - \Pi_e p\|_e \leq Ch^{s+\frac{1}{2}} \|p\|_{s+1,K}, \quad 0 \leq s \leq k. \end{aligned}$$

The classical Raviart-Thomas projection operator as in [9] is also used

(21)
$$(\mathbf{u} - \Pi^{RT} \mathbf{u}, \omega_h)_K = 0, \quad \forall \omega_h \in [P_{k-1}(K)]^d,$$

(22)
$$\langle (\mathbf{u} - \Pi^{RT} \mathbf{u}) \cdot n_e, \mu_h \rangle_e = 0, \quad \forall \mu_h \in P_k(e), \quad e \in \partial K$$

We can reach the error estimate as in [9]:

Lemma 3.3. For the Raviart-Thomas interpolation Π^{RT} , the following estimate hold

(23)
$$\begin{aligned} ||\nabla \cdot (\mathbf{u} - \Pi^{RT} \mathbf{u})||_{K} &\leq Ch^{s} ||\nabla \cdot \mathbf{u}||_{s,K}, \quad 1 \leq s \leq k+1, \\ ||\mathbf{u} - \Pi^{RT} \mathbf{u}||_{K} + h^{\frac{1}{2}} ||u - \Pi^{RT} \mathbf{u}|| \leq Ch^{s} ||\mathbf{u}||_{s,K}, \quad \frac{1}{2} \leq s \leq k+1. \end{aligned}$$

Utlizing the above results element-wise, we can easily get the following error estimates.

Lemma 3.4. If $a(\phi)$ is bounded, there exists the following inequality

(24)
$$\| (\mathbf{u} - \Pi^{RT} \mathbf{u}, p - \Pi_h p, \lambda - \Pi_e p) \|_{B,*} \le Ch^s |p|_{s+1,\mathcal{T}_h}, \quad \frac{1}{2} < s \le k.$$

Remark 3.1. From (24), the following estimate holds: for any $1/2 \le s \le k+1$,

(25)
$$\|\mathbf{u} - \Pi^{RT}\mathbf{u}\|_{K} \le Ch^{s} \|p\|_{s,K}$$

For the concentration, we introduce another projection operator Π_s as follows:

(26)
$$B_c(c_f - \prod_s c_f, z_h) + \delta(c_f - \prod_s c_f, z_h) = 0, \quad \forall z_h \in \Psi_h,$$

where δ should be some sufficient large constant.

As in [17], under the following inductive hypothesis

$$\|\mathbf{u}_h\|_{L^{\infty}} \le C_{\mathbf{u}}$$

where $C_{\mathbf{u}}$ is a positive constant, we can reach the following estimates:

(28)
$$\begin{aligned} \|c_f - \Pi_s c_f\|_{s,\mathcal{T}_h} &\leq Ch^s \|c_f\|_{s,\mathcal{T}_h}, \quad 0 \leq s \leq k+1, \\ \|\frac{\partial (c_f - \Pi_s c_f)}{\partial t}\|_{s,\mathcal{T}_h} &\leq Ch^s (|c_f|_{s,\mathcal{T}_h} + |\frac{\partial c_f}{\partial t}|_{s,\mathcal{T}_h}), \quad 0 \leq s \leq k+1. \end{aligned}$$

The following trace inequalities will be also used to prove the convergence theorem (see Lemma 3.1 in [25]).

Lemma 3.5. For $\forall v \in H^1(K)$, the trace inequalities are shown below

(29)
$$\begin{aligned} \|v\|_{0,e}^{2} &\leq C(h_{e}^{-1} \|v\|_{0,K}^{2} + h_{e} \|v\|_{1,K}^{2}), \\ \|\nabla v \cdot n_{e}\|_{0,e}^{2} &\leq C(h_{e}^{-1} \|\nabla v\|_{0,K}^{2} + h_{e} \|\nabla^{2} v\|_{0,K}^{2}). \end{aligned}$$

4. The proof of convergence theorem

Now, we can complete the proof of our convergence theorem 2.3.

Proof. We firstly give the bound of $||c_h - c_f||_{\mathcal{T}_h}$. Set $\xi_c = c_h - \prod_s c_f$, $\zeta_c = c_f - \prod_s c_f$. Taking $z_h = \xi_c$ in (26), we have

$$\begin{split} \sum_{K\in\mathcal{T}_{h}} \int_{K} \frac{(\partial\phi_{h}\xi_{c})}{\partial t} \xi_{c} dx + \sum_{K\in\mathcal{T}_{h}} \int_{K} \phi_{h} \mathbf{D}(\mathbf{u}_{h}) \nabla\xi_{c} \cdot \nabla\xi_{c} dx \\ &+ \sum_{K\in\mathcal{T}_{h}} \int_{K} f_{p}\xi_{c}^{-2} dx + J_{0}^{r}(\xi_{c},\xi_{c}) \\ = \sum_{K\in\mathcal{T}_{h}} \int_{K} \frac{(\partial\phi_{h}\zeta_{c})}{\partial t} \xi_{c} dx + \sum_{K\in\mathcal{T}_{h}} \int_{K} (\phi\mathbf{D}(\mathbf{u}) - \phi_{h}\mathbf{D}(\mathbf{u}_{h})) \nabla c_{f} \cdot \nabla\xi_{c} dx \\ &+ 2\sum_{e\in\mathcal{E}_{h}^{0}} \int_{e} \{\phi_{h}\mathbf{D}(\mathbf{u}_{h})\nabla\xi_{c} \cdot n_{e}\}[\xi_{c}] ds + \sum_{K\in\mathcal{T}_{h}} \int_{K} (\mathbf{u}_{h} - \mathbf{u})c_{f}\nabla\xi_{c} dx \\ &+ \sum_{e\in\mathcal{E}_{h}^{i}} \int_{e} \{(\phi_{h}\mathbf{D}(\mathbf{u}_{h}) - \phi\mathbf{D}(\mathbf{u}))\nabla c_{f}) \cdot n_{e}\}[\xi_{c}] ds + \sum_{K\in\mathcal{T}_{h}} \int_{K} \mathbf{u}\xi_{c}\nabla\xi_{c} dx \\ &+ \sum_{e\in\mathcal{E}_{h}^{i}} \int_{e} \{(\mathbf{u} - \mathbf{u}_{h})c_{f} \cdot n_{e}\}[\xi_{c}] ds - 2\sum_{e\in\mathcal{E}_{h}^{i}} \int_{e} \{\mathbf{u}_{h}\xi_{c} \cdot n_{e}\}[\xi_{c}] ds \\ &+ \sum_{K\in\mathcal{T}_{h}} \int_{K} (f_{P} - \delta)\zeta_{c}\xi_{c} dx + \sum_{K\in\mathcal{T}_{h}} \int_{K} \kappa[(1 - \phi)c_{f} - (1 - \phi_{h})c_{h}]\xi_{c} dx \\ &+ \sum_{K\in\mathcal{T}_{h}} \int_{K} \frac{\partial(\phi - \phi_{h})}{\partial t}c_{f}\xi_{c} dx + \sum_{K\in\mathcal{T}_{h}} \int_{K} \frac{\partial c_{f}}{\partial t}(\phi - \phi_{h})\xi_{c} dx \\ &= F_{1} + F_{2} + \dots + F_{12}. \end{split}$$

Now we estimate the terms on the right hand side of (30) one by one. Using (6) and Lemma 2.2, we can get the following result

$$|F_{1}| + |F_{2}| + |F_{4}| + |F_{6}| + |F_{9}| + |F_{10}| + |F_{11}| + |F_{12}|$$

$$\leq C\{\|\zeta_{c}\|_{\mathcal{T}_{h}}^{2} + \|\xi_{c}\|_{\mathcal{T}_{h}}^{2} + \|\frac{\partial\zeta_{c}}{\partial t}\|_{\mathcal{T}_{h}}^{2} + \|\frac{\partial(\phi - \phi_{h})}{\partial t}\|_{\mathcal{T}_{h}}^{2} + \|\phi - \phi_{h}\|_{\mathcal{T}_{h}}^{2} + \|\mathbf{u} - \mathbf{u}_{h}\|_{\mathcal{T}_{h}}^{2}\} + \varepsilon\|\nabla\xi_{c}\|_{\mathcal{T}_{h}}^{2}.$$

For F_3 , using (29) we have

$$|F_3| \leq \varepsilon J_0^{\gamma}(\xi_c, \xi_c) + C \sum_{e \in \mathcal{E}_h^i} \gamma^{-1} h_e \|\nabla \xi_c\|_{L^2(e)}^2$$
$$\leq \varepsilon J_0^{\gamma}(\xi_c, \xi_c) + C_1 \gamma^{-1} \|\nabla \xi_c\|_{\mathcal{T}_h}^2.$$

Next, we estimate F_5 with (6) and (29)

$$\begin{aligned} |F_5| &\leq \|\nabla c_f\|_{L^{\infty}} \sum_{e \in \mathcal{E}_h^i} \|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)\|_{L^2(e)} \|[\xi_c]\|_{L^(e)} \\ &\leq \varepsilon J_0^{\gamma}(\xi_c, \xi_c) + C \sum_{e \in \mathcal{E}_h^i} \gamma^{-1} h_e \|\mathbf{u} - \mathbf{u}_h\|_{L^(e)}^2 \\ &\leq \varepsilon J_0^{\gamma}(\xi_c, \xi_c) + C \gamma^{-1} \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h}^2. \end{aligned}$$

Using the same technique as above, we can reach

$$|F_{7}| \leq \varepsilon J_{0}^{\gamma}(\xi_{c},\xi_{c}) + C \sum_{e \in \mathcal{E}_{h}^{i}} \gamma^{-1} h_{e} \|\mathbf{u} - \mathbf{u}_{h}\|_{L^{(e)}}^{2}$$
$$\leq \varepsilon J_{0}^{\gamma}(\xi_{c},\xi_{c}) + C \gamma^{-1} \|\mathbf{u} - \mathbf{u}_{h}\|_{\mathcal{T}_{h}}^{2},$$
$$|F_{8}| \leq \varepsilon J_{0}^{\gamma}(\xi_{c},\xi_{c}) + \|\xi_{c}\|_{\mathcal{T}_{h}}^{2},$$

where γ is large enough, ε is small enough and they satisfy

$$C_1 \gamma^{-1} \le \frac{d_m^*}{4}, \qquad \varepsilon \le \min(\frac{1}{4}, \frac{d_m^*}{4}).$$

Next, we deal with the first term on the left hand side of (30). Since $(\frac{\partial \phi_h}{\partial t}\xi_c, \xi_c)_{\mathcal{T}_h} \ge 0$, we can get

$$(\frac{\partial(\phi_h\xi_c)}{\partial t},\xi_c)_{\mathcal{T}_h} \ge \frac{1}{2}\frac{\partial}{\partial t}(\phi_h\xi_c,\xi_c)_{\mathcal{T}_h}.$$

Substituting these estimates into (30), and then integrating it on t, we get

(31)
$$\begin{aligned} \|\xi_{c}\|_{\mathcal{T}_{h}}^{2} + \int_{0}^{t} \|\nabla\xi_{c}\|_{\mathcal{T}_{h}}^{2} d\tau + \int_{0}^{t} J_{0}^{\gamma}(\xi_{c},\xi_{c}) d\tau \\ \leq C \int_{0}^{t} (\|\frac{\partial\zeta_{c}}{\partial t}\|_{\mathcal{T}_{h}}^{2} + \|\zeta_{c}\|_{\mathcal{T}_{h}}^{2} + \|\frac{\partial(\phi-\phi_{h})}{\partial t}\|_{\mathcal{T}_{h}}^{2} + \|\xi_{c}\|_{\mathcal{T}_{h}}^{2} + \|\mathbf{u}-\mathbf{u}_{h}\|_{\mathcal{T}_{h}}^{2} + \|\phi-\phi_{h}\|_{\mathcal{T}_{h}}^{2}) d\tau. \end{aligned}$$

From the above estimate, we need to estimate the bound of $\mathbf{u} - \mathbf{u}_h$. Using the definations of projection operators and $B_{\mathbf{u}}$, we have

(32)
$$B_{\mathbf{u}}((\Pi^{RT}\mathbf{u}-\mathbf{u},\Pi_hp-p,\Pi_ep-p),(\omega_h,v_h,\mu_h)) = (a(\phi_h)(\Pi^{RT}\mathbf{u}-\mathbf{u}),\omega_h)_{\mathcal{T}_h}.$$

According to the boundedness and stability of the bilinear form B_{u} , we have the estimate

$$K_* \| (\Pi^{RT} \mathbf{u} - \mathbf{u}_h, \Pi_h p - p_h, \Pi_e p - \lambda_h) \|_B$$

$$\leq \sup_{(\omega_h, v_h, \mu_h)} \frac{B_{\mathbf{u}}((\Pi^{RT} \mathbf{u} - \mathbf{u}_h, \Pi_h p - p_h, \Pi_e p - \lambda_h), (\omega_h, v_h, \mu_h))}{\|(\omega_h, v_h, \mu_h)\|_B}$$

$$\leq C(\|c_f - c_h\|_{\mathcal{T}_h} + \|\Pi^{RT} \mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\phi - \phi_h\|_{\mathcal{T}_h}).$$

Hence we get

(33) $\|(\Pi^{RT}\mathbf{u} - \mathbf{u}_h, \Pi_h p - p_h, \Pi_e p - \lambda_h)\|_B \le C\{\|c_f - c_h\|_{\mathcal{T}_h} + \|\Pi^{RT}\mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\phi - \phi_h\|_{\mathcal{T}_h}\}.$ Using (33), we get the estimate

$$\|(\mathbf{u}-\mathbf{u}_h,\Pi_h p-p_h,\Pi_e p-\lambda_h)\|_B$$

(34) $\leq \|(\Pi^{RT}\mathbf{u} - \mathbf{u}_h, \Pi_h p - p_h, \Pi_e p - \lambda_h)\|_B + \|(\mathbf{u} - \Pi^{RT}\mathbf{u}, \Pi_h p - p_h, \Pi_e p - \lambda_h)\|_B$

$$\leq C\{\|c_f - c_h\|_{\mathcal{T}_h} + \|\Pi^{RT}\mathbf{u} - \mathbf{u}\|_{\mathcal{T}_h} + \|\phi - \phi_h\|_{\mathcal{T}_h}\}.$$

Next, we estimate the boundedness of $\|\phi - \phi_h\|_{\mathcal{T}_h}$. From (9), we can get that

(35)
$$\frac{\partial(\phi - \phi_h)}{\partial t} \le \frac{\alpha \kappa}{\rho_s} [(1 - \phi_h)|c_f - c_h| + (\phi_h - \phi)c_f]$$

So we can get that

(36)
$$\left\|\frac{\partial(\phi-\phi_h)}{\partial t}\right\|_{\mathcal{T}_h}^2 \le C(\|\phi-\phi_h\|_{\mathcal{T}_h}^2 + \|c_f-c_h\|_{\mathcal{T}_h}^2)$$

Multiplying (35) by $\phi - \phi_h$ and integrating it over Ω , we will reach that

$$\frac{1}{2}\frac{\partial}{\partial t}\|\phi-\phi_h\|_{\mathcal{T}_h}^2 \le C(\|\phi-\phi_h\|_{\mathcal{T}_h}^2+\|c_f-c_h\|_{\mathcal{T}_h}^2).$$

So we obtain that

(37)
$$\|\phi - \phi_h\|_{\mathcal{T}_h}^2 \le C \int_0^t (\|\xi_c\|_{\mathcal{T}_h}^2 + \|\zeta_c\|_{\mathcal{T}_h}^2) d\tau.$$

Substituting the above estimate into (31), and using (28), (34) and Gronwall's inequality, we can get the following estimate

$$\begin{aligned} &\|c_f - c_h\|_{\mathcal{T}_h}^2 + \|\phi - \phi_h\|_{\mathcal{T}_h}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h}^2 \\ &\leq C \int_0^t (\|\zeta_c\|_{\mathcal{T}_h}^2 + \|\frac{\partial\zeta_c}{\partial t}\|_{\mathcal{T}_h}^2 + \|\mathbf{u} - \Pi^{RT}\mathbf{u}\|_{\mathcal{T}_h}^2) d\tau \\ &\leq Ch^s (\|c_f\|_{H^1(0,T;H^s(\mathcal{T}_h))}^2 + \|p\|_{L^2(0,T;H^{s+1}(\mathcal{T}_h))}^2). \end{aligned}$$

Combined the above estimate with (34), we get the second inequality of (17).

It is easily seen that our estimates are derived under the induction hypothesis (27). Now, we check it. Note that

$$\begin{aligned} \|\mathbf{u}_h\|_{L^{\infty}} &\leq \|\mathbf{u}_h - \Pi^{RT}\mathbf{u}\|_{L^{\infty}} + \|\Pi^{RT}\mathbf{u} - \mathbf{u}\|_{L^{\infty}} + \|\mathbf{u}\|_{L^{\infty}} \\ &\leq Ch^{s - \frac{d}{2}} + \|\mathbf{u}\|_{L^{\infty}} \leq C_{\mathbf{u}}. \end{aligned}$$

Thus, the hypothesis (27) holds.

Using the similar technique as in [30], we know that

$$\|\Pi_h p - p_h\|_{\mathcal{T}_h} \le C(1 + \|\mathbf{u}\|_{L^{\infty}}) \|c_h - c_f\|_{\mathcal{T}_h}.$$

Using Lemma 3.2 and (17)(a), we get (17)(c).

5. NUMERICAL EXAMPLES

In this section, we will test the efficiency of our proposed method by some numerical examples. We firstly use HMFE method for the linear elliptic problem, and then we consider SIPDG method for the convectiondiffusion equation. Next, we confirm the convergence rate of our combined method for the coupled problem. Finally, we apply the combined method to a "real" incompressible wormhole problem.

5.1. Convergence test of HMFE method. Here we will test the accuracy of the HMFE scheme. the HMFE method is considered for solving the elliptic problem with RT0 - P0, RT1 - P1 and RT2 - P2 elements. The exact solution is taken by $p = \sin \pi x \sin \pi y$ in $[0, 1] \times [0, 1]$ and $\mathbf{u} = -\nabla p$, respectively. For different mesh size h = 1/8, 1/16, 1/32, 1/64, a convergence study is presented. The L^2 -norm errors and convergence accuracies are shown in Tables 5.1-5.3. As seen in these tables, the optimal convergence rates for pressure and velocity are evaluated.

TABLE 5.1. Numerical results for p and \mathbf{u} with RT0 - P0.

h	$ p - p_h _{L^2}$	rates	$\ \mathbf{u}-\mathbf{u}_h\ _{L^2}$	rates
1/8	7.1830e-02	*	2.4473e-02	*
1/16	3.5977e-02	0.9975	1.2508e-02	0.9684
1/32	1.7992e-02	0.9997	6.2691e-03	0.9965
1/64	8.9969e-03	0.9985	3.1335e-04	1.0005

TABLE 5.2. Numerical results for p and \mathbf{u} with RT1 - P1.

h	$ p - p_h _{L^2}$	rates	$\ \mathbf{u}-\mathbf{u}_h\ _{L^2}$	rates
1/8	2.7875e-02	*	1.1113e-02	*
1/16	7.1654e-03	1.9599	2.8341e-03	1.9713
1/32	1.8225e-03	1.9752	7.1763e-04	1.9816
1/64	4.6070e-04	1.9840	1.8099e-04	1.9874

TABLE 5.3. Numerical results for p and \mathbf{u} with RT2 - P2.

h	$ p - p_h _{L^2}$	rates	$\ \mathbf{u}-\mathbf{u}_h\ _{L^2}$	rates
1/8	4.4473e-04	*	2.5267e-02	*
1/16	5.8189e-05	2.9341	2.9358e-03	3.1054
1/32	7.3831e-06	2.9785	3.3422e-04	3.1349
1/64	9.2786e-07	2.9922	3.9131e-05	3.0944

5.2. Convergence test of SIPDG method. Here we first test the convergent accuracy of the SIPDG method for convection-diffusion equation

$$\frac{\partial c_f}{\partial t} + \nabla \cdot (\mathbf{u}c_f - D\nabla c_f) = f$$

with homogeneous and nonhomogeneous boundary value conditions. For this purpose, we take the two different exact solutions respectively as

$$c_f = e^{-t} \sin \pi x \sin \pi y$$
, and $c_f = e^{-y^2 - x - t}$, $(x, y) \in [0, 1] \times [0, 1]$

The velocity function $\mathbf{u} = [-y, x]$ and the diffusion coefficient D = 1.0. The initial-boundary conditions and the right hand side term can be computed by the exact solutions. For the practical computation, the firstorder Euler backward difference scheme in time is used and L^2 -projection of the initial condition is also used. Setting T = 1.0 and time size $\Delta t = 1e - 3$, for different mesh size, we give some numerical results with P1 discontinuous finite element space in Table 5.4. These numerical results show that SIPDG method has the optimal convergence rates in L^2 -norm for both homogeneous and nonhomogeneous boundary conditions.

TABLE 5.4. Numerical results with P1 element for homogeneous and nonhomogeneous boundary cases.

h	homogenous		nonhomogenous	
	L^2 error	rates	L^2 error	rates
1/8	1.1599e-00	*	2.2602e-02	*
1/16	3.0175e-01	1.9425	5.8884e-03	1.9405
1/32	7.6512e-02	1.9796	1.5668e-03	1.9100
1/64	1.9192e-02	1.9952	4.1542e-04	1.9152

In addition, we also consider our method for the porosity and the concentration. Initial-boundary conditions can be given by the exact solutions

(38)
$$c_f(x,y,t) = \frac{2\epsilon^2}{2\epsilon^2 + 4Dt} \exp\{-\frac{(x\cos 4t + y\sin 4t + 0.2)^2 + (-x\sin 4t + y\cos 4t)^2}{2\epsilon^2 + 4Dt}\},\ \phi(x,y,t) = 0.5 + 0.4\sin(x+t)\sin(y+t), \quad \text{in} \quad \Omega = [0,1] \times [0,1].$$

The other parameters are taken as:

(39)
$$c_I = k_c = k_s = a_0 = \frac{\alpha}{\rho_s} = 1, \quad D = 0.1, \quad \epsilon = 0.1.$$

Here we still use the first-order backward Euler scheme in time, and take time step $\Delta t = 1e - 3$. The computational results at T = 1.0 are shown as in Tables 5.5 and 5.6 with the uniform triangular meshes h = 1/8, 1/16, 1/32, 1/64, 1/128. From these tables, we can get the optimal convergence rates in L^2 -norm with P1 and P2 discontinuous elements.

TABLE 5.5. Numerical results with P1 element for c_f and ϕ .

h	$\ c_f - c_h\ _{L^2}$	rates	$\ \phi - \phi_h\ _{L^2}$	rates
1/8	7.0710e-02	*	8.2931e-02	*
1/16	1.7289e-02	2.0320	2.4605e-02	1.7529
1/32	4.2388e-03	2.0281	6.7051e-03	1.8756
1/64	1.0472e-03	2.0171	1.7132e-03	1.9686
1/128	2.6026e-04	2.0085	4.3005e-04	1.9941

TABLE 5.6. Numerical results with P2 element for c_f and ϕ .

h	$ c_f - c_h _{L^2}$	rates	$\ \phi - \phi_h\ _{L^2}$	rates
1/8	5.2783e-03	*	2.7432e-02	*
1/16	6.3176e-04	3.0626	4.0521e-03	2.7591
1/32	7.5796e-05	3.0592	5.3296e-04	2.9266
1/64	9.1162e-06	3.6314	6.7731e-05	2.9761
1/128	1.0039e-06	3.1828	8.0221e-06	3.0429

5.3. Convergence test of the combined method. In this experiment, we will show the convergence of our combined method. Here the analytic solution in $\Omega = [0, 1] \times [0, 1]$ is given as in [2]

$$p(x, y, t) = t \cos \pi x \cos \pi y,$$

$$c_f(x, y, t) = tx^2 (1 - x)^2 y^2 (1 - y)^2,$$

$$\phi(x, y, t) = 1 - e^{-\frac{1}{80}t^2 x^2 (1 - x)^2 y^2 (1 - y)^2 e^{x + y + 1} - (x + y + 1)}.$$

The parameters are taken as

 $\mathbf{D} = 10^{-2}\mathbf{I}, \quad k_0 = 1, \quad a_0 = 0.5, \quad \rho_s = 10, \quad \alpha = 1, \quad k_c = k_s = 1, \quad \mu = f_I = 1,$

where I is an identity matrix. And choosing T = 1.0 and time step $\Delta t = h^2$, we give some numerical results with RT1 - P1 element and P1 discontinuous element in Tables 5.7 and 5.8. We can easily find that our combined method is of second-order accuracy in L^2 -norm, which is coincided with our theoretical analysis.

h	$\ c_f - c_h\ _{L^2}$	rates	$\ \phi - \phi_h\ _{L^2}$	rates
1/8	1.1109e-03	*	2.4572e-02	*
1/16	2.9657e-04	1.9053	6.8963e-03	1.8331
1/32	7.6954e-05	1.9463	1.7291e-03	1.9958
1/64	1.9001e-05	2.0181	4.2112e-04	2.0377
1/128	4.4123e-06	2.1065	1.0021e-04	2.0712

TABLE 5.7. Numerical results for c_f and ϕ with P1 element.

TABLE 5.8. Numerical results for u and p with RT1 - P1 element.

h	$\ \mathbf{u}-\mathbf{u}_h\ _{L^2}$	rates	$ p - p_h _{L^2}$	rates
1/8	2.4932e-03	*	6.2173e-03	*
1/16	6.2776e-04	1.9897	1.7321e-03	1.8438
1/32	1.7290e-04	1.8603	3.9021e-04	2.1502
1/64	4.2003e-05	2.0414	9.7001e-05	2.0082
1/128	1.0010e-05	2.0691	2.5231e-05	1.9428

5.4. Simulation for a "real" incompressible wormhole propagation. In this experiment, a 0.2-meter computational domain is considered, and the first-order Euler backward time discretization is used. We set a singular area on the middle of the left boundary with space size to be 0.01-meter and time size to be 1e - 4 to observe the phenomenon of wormhole propagation. The initial values and the parameters in the porous medium are taken as in Table 5.9. Initial concentration of acid and initial porosity of rock in this domain are set to be $c_0 = 0$ and $\phi_0 = 0.2$, respectively. The top and bottom boundaries of the domain are impermeable.

Properties	Value
the viscosity of fluid (μ)	$1Pa \cdot s$
the injection flow rate (f_I)	4.5
the production flow rate (f_P)	2.5
the dispersion tensor (\mathbf{D})	0.01
the local mass-transfer coefficient (k_c)	1m/s
the density of the rock (ρ_s)	$2000 kg/m^2$
the dissolving constant of the acid (α)	0.1 kg/mole
the kinetic constant for reaction (k_s)	10m/s
the initial interfacial area available for reaction (a_0)	$0.2m^{-1}$

TABLE 5.9. The properties of acid flow and porous medium.

The numerical results of the concentration and porosity at different time are shown in Figures 5.1 and 5.2. From these figures, we can observe $c_f, \phi \in [0, 1]$ and the phenomenon of wormhole propagation, which shows the effectiveness of the combined method.



FIGURE 5.1. Porosity of rock at the different time steps.



FIGURE 5.2. Concentration of acid at the different time steps.

REFERENCES

- [1] C. Zhao, Physical and chemical dissolution front instability in porous media. Cham, Switzerland: Springer, 2014.
- [2] X. Li, H. Rui, Characteristic block-centered finite difference method for simulating incompressible wormhole propagation, Comput. Math. Appl. 73 (2017) 2171-2190.
- [3] J. Kou, S. Sun, Y. Wu, Mixed finite element-based fully conservative methods for simulating wormhole propagation, Comput. Methods Appl. Mech. Engrg. 298 (2016) 279-302.
- [4] Y. Wu, A. Salama, S. Sun, Parallel simulation of wormhole propagation with the Darcy-Brinkman-Forchheimer framework, Comput. Geotech. 69 (2015) 564-577.
- [5] X. Li, H. Rui, Block-centered finite difference method for simulating compressible wormhole propagation, J. Sci. Comput. 74 (2018) 1115-1145.
- [6] J. Zhang , X. Shen, H. Guo, H. Fu, H. Han, Characteristic splitting mixed finite element analysis of compressible wormhole propagation, Appl. Numer. Math. 147 (2020) 66-87.
- [7] H. Guo, L. Tian, Z. Xu, Y. Yang, N. Qi, High-order local discontinuous Galerkin method for simulating wormhole propagation, J. Comput. Appl. Math. 350 (2019) 247-261.
- [8] D.N. Arnold, F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, ESAIM: Math. Model. Numer. Anal. 19 (1985) 7-32.
- [9] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.

- [10] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2009) 1319-1365.
- [11] H. Egger, J. Schoberl, A hybrid mixed discontinuous Galerkin finite-element method for convection-diffusion problems, IMA J. Numer. Anal. 30 (2010) 1206-1234.
- [12] J. Zhang, H. Han, H. Guo, X. Shen, A combined hybrid mixed element method for incompressible miscible displacement problem with local discontinuous Galerkin procedure, Numer. Methods Part. D. E. 36 (2020) 1629-647.
- [13] J. Zhu, H. Vargas, Robust and efficient mixed hybrid discontinuous finite element methods for elliptic interface problems, Int. J. Numer. Anal. Mod. 16 (2019) 767-788.
- [14] L. Bevilacqua, R. Feijoo, L.F. Rojas M, A variational principle for the Laplace operator with application in the torsion of composite rods, Int. J. Solids Struct. 10 (1974) 1091-1102.
- [15] B. Fraeijs de Veubeke, Displacement and equilibrium models in the finite element method, in: O.C. Zienkiewicz, G. Holister (Eds.), Stress Analysis, John Wiley and Sons, New York, 1965.
- [16] W.H. Reed, T.R. Hill, Triangular mesh methods for the neutron transport equation, Tech. Report No. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, 1973.
- [17] D.N. Arnold, An interior penalty finite element method with discontinuous element, SIAM J. Numer. Anal. 19 (1982) 742-760.
- [18] B. Rivieva, M.F. Wheeler, Discontinuous Galerkin methods for flow and transport problem in porous media, Commun. Numer. Methods. Eng. 18 (2002) 63-68.
- [19] R. Zhang, X. Yu, J. Zhu, A.F.D. Loula, Direct discontinuous Galerkin method for nonlinear reaction-diffusion systems in pattern formation, Appl. Math. Model. 38 (2014) 1612-1621.
- [20] J. Zhu, X. Yu, A.F.D. Loula, Mixed discontinuous Galerkin analysis of thermally nonlinear coupled problem, Comput. Methods Appl. Mech. Engrg. 200 (2011) 1479-1489.
- [21] X. Li, J. Zhu, R. Zhang, S. Cao, A combined discontinuous Galerkin method for the dipolar Bose-Einstein condensation, J. Comput. Phys. 275 (2014) 363-376.
- [22] Z. Li, X. Yu, J. Zhu, Z. Jia, A Runge-Kutta discontinuous Galerkin method for Lagrangian compressible Euler equations in two-dimensions, Commun. Comput. Phys. 15 (2014) 1184-1206.
- [23] S. Sun, B. Riviera, M.F. Wheeler, A combined mixed finite element and discontinuous Galerkin method for miscible displacement problem in porous media, in: Recent Progress in Computational and Applied PDES, 2002, pp. 323-351.
- [24] S. Sun, M.F. Wheeler, Discontinuous Galerkin methods for coupled flow and reactive transport problems, Appl. Numer. Math. 52 (2005) 273-298.
- [25] M.R. Cui, A combined mixed and discontinuous Galerkin method for compressible miscible displacement problem in porous media, J. Comput. Appl. Math. 198 (2007) 19-34.
- [26] J. Zhang, J. Zhu, R. Zhang, D. Yang, A.F.D. Loula, A combined discontinuous Galerkin finite element method for miscible displacement problem, J. Comput. Appl. Math. 309 (2017) 44-55.
- [27] P.A. Raviart, J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, in: Mathematical Aspects of Finite Element Methods, in: Lecture Notes in Math, vol. 606, Springer Verlag, Berlin and New York, 1977, pp. 292-315.
- [28] J.C. Nedelec, Mixed finite element in R3, Numer. Math. 35 (1980) 315-341.
- [29] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 2002.
- [30] J. Douglas Jr., R.E. Ewing, M.F. Wheeler, Approximation of the pressure by a mixed method in the simulation of miscible displacement, RAIRO Anal. Numer. 17 (1983) 17-33.

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