# AN IMMERSED WEAK GALERKIN METHOD FOR ELLIPTIC INTERFACE PROBLEMS ON POLYGONAL MESHES

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ABSTRACT. In this paper we present an immersed weak Galerkin method for solving second-order elliptic interface problems on polygonal meshes, where the meshes do not need to be aligned with the interface. The discrete space consists of constants on each edge and broken linear polynomials satisfying the interface conditions in each element. For triangular meshes, such broken linear plynomials coincide with the basis functions in immersed finite element methods [26]. We establish some approximation properties of the broken linear polynomials and the discrete weak gradient of a certain projection of the solution on polygonal meshes. We then prove an optimal error estimate of our scheme in the discrete  $H^1$ -seminorm under some assumptions on the exact solution. Numerical experiments are provided to confirm our theoretical analysis.

## 1. Introduction

There are a wide range of physical and engineering problems that are governed by partial differential equations having an interface. For example, a second-order elliptic partial differential equation with a discontinuous coefficient is often used as a model problem in material sciences and porous media involving multiple materials or media. To solve such a problem, one can use some classical numerical schemes with interface-fitted meshes, such as finite element methods (FEMs), discontinuous Galerkin (DG) methods, etc. However, it is difficult and takes a lot of time to generate such fitted meshes when the domain boundary and the interface are geometrically complicated. Even worse, when the interface is moving, one needs to generate a new fitted mesh as time evolves.

To overcome such difficulties, researchers developed and studied some numerical methods using unfitted/structured meshes, such as cut finite element methods (CutFEMs) [3,12,19,20], extended finite element methods (XFEMs) [5,6,23,28,34], immersed finite element methods (IFEMs) [22,24,26,29,30], to name just a few. In particular, the IFEMs use basis functions that are modified so that they satisfy the interface conditions. The authors in [29,30] studied IFEMs using uniform triangular or rectangular grids. In [24,31], the performance of the IFEMs was improved by adding penalty terms that are commonly used in DG methods. Linear and bilinear nonconforming IFEMs were studied in [26,32]. The IFEM was also successfully applied to other interface problems: interface elasticity problems [25], elliptic eigenvalue interface problems [27], Stokes interface problems [1], etc.

On the other hand, several numerical methods using general polygonal or polyhedral meshes have been developed, such as hybrid high-order (HHO) methods [16–18], virtual element methods (VEMs) [2,4,10], weak Galerkin (WG) methods (or weak Galerkin finite element methods) [35, 39, 40], etc. Here we explain the WG methods in some detail. In WG methods, the discrete space consists of polynomials on an element interior and polynomials on its edges, and the differential operators are replaced by the

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so-called weak differential operators. Compared to the classical FEMs, the WG methods have several advantages. For example, WG methods can handle the general polygonal and polyhedral meshes while the FEMs cannot. In addition, the WG methods can be generalized to higher orders directly. Due to such advantages, the WG methods were successfully applied to various problems: Darcy problems [40], Stokes equations [41], elasticity problems [44], Maxwell equations [36], etc. For more thorough survey, we refer to [21,33,35,39,42,43] and references therein.

In [37], an immersed WG method was proposed for the elliptic interface problems for triangular meshes. However, their method cannot be generalized to the polygonal meshes since it is impossible to define the Lagrange-type immersed finite element interpolation on polygonal elements. Besides, the discrete bilinear form formulated in their method is different from the usual WG method; they use the usual gradient and DG-type consistency terms.

In this paper, we develop a new immersed WG method for the elliptic interface problems. Our method uses general polygonal meshes which allow the interface cut through the interior. We generalize the discrete weak gradient to the case when the coefficient is discontinuous, and use it to define the bilinear form. Our weak gradient coincides with the usual one [35] when the coefficient is constant. However, they are different from each other when the coefficient is non-constant.

The rest of the paper is organized as follows. In the next section, we describe the model problem and summarize some preliminaries. In Section 3, we propose our immersed WG method for the model problem, and prove that the discrete problem is well-posed. In Section 4, we prove some technical inequalities and approximation properties of broken linear polynomials on polygonal elements. In Section 5, we derive an optimal error estimate in the discrete  $H^1$ -seminorm under some regularity assumptions on the exact solution. Finally, in Section 6, we present some numerical experiments that confirm our theoretical analysis.

### 2. Preliminaries

We follow the usual notation of Sobolev spaces, inner product, seminorms, and norms (see, for example, [15]). Let D be a bounded domain in  $\mathbb{R}$  or  $\mathbb{R}^2$ . For  $\sigma \geq 0$ , we denote by  $\|\cdot\|_{\sigma,D}$  and  $|\cdot|_{\sigma,D}$  the usual norm and seminorm of the Sobolev space  $H^{\sigma}(D)$ , respectively. We also denote by  $(\cdot, \cdot)_{0,D}$  the usual inner product in  $L^2(D)$ . We define  $H^{-1/2}(D)$  as the dual space of  $H^{1/2}(D)$  equipped with the norm given by

$$||u||_{-1/2,D} \coloneqq \sup_{v \in H^{1/2}(D)} \frac{\langle u, v \rangle_D}{||v||_{1/2,D}},$$

where  $\langle \cdot, \cdot \rangle_D$  is the duality pairing. For a nonnegative integer k, we denote by  $\mathbb{P}_k(D)$  the space of all polynomials of degree  $\leq k$  on D.

2.1. **Model problem.** Let  $\Omega$  be a convex polygonal domain in  $\mathbb{R}^2$ , which is separated into two disjoint subdomains  $\Omega^+$  and  $\Omega^-$  by an interface  $\Gamma = \partial \Omega^- \cap \partial \Omega^+$  as in Figure 1. Here we assume that  $\Gamma$  is a regular  $C^2$ -curve that is not self-intersecting. For any domain  $D \subset \Omega$  and any function  $u: D \to \mathbb{R}$ , we define its jump across the portion of the interface  $\Gamma \cap D$  as

$$[u]_{\Gamma \cap D} \coloneqq u|_{D \cap \Omega^+} - u|_{D \cap \Omega^-}.$$

We consider the following elliptic interface problem: Given  $f \in L^2(\Omega)$ , find  $u \in H^1_0(\Omega)$  such that

$$\begin{cases}
-\nabla \cdot (\beta \nabla u) = f & \text{in } \Omega^+ \cup \Omega^-, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$
(2.1)

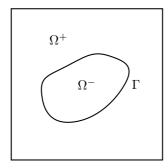


FIGURE 1. A domain  $\Omega$  with interface  $\Gamma$ .

with the jump conditions on the interface

$$[u]_{\Gamma} = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]_{\Gamma} = 0,$$
 (2.2)

where  $\beta$  is a positive and piecewise  $C^1$ -function on  $\overline{\Omega}$  bounded below and above by two positive constants  $\beta_*$  and  $\beta^*$  with  $0 < \beta^- \le \beta^+ < \infty$ . That is,

$$\beta(\boldsymbol{x}) = \left\{ \begin{array}{ll} \beta^+(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^+, \\ \beta^-(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^-, \end{array} \right.$$

for some functions  $\beta^+ \in C^1(\overline{\Omega^+}), \beta^- \in C^1(\overline{\Omega^-})$  such that  $\beta_* \leq \beta^s \leq \beta^*, s = +, -$ . A weak formulation of the model problem (2.1)-(2.2) is written as follows: Find  $u \in H_0^1(\Omega)$  such that

$$\int_{\Omega} \beta \nabla u \cdot \nabla v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega).$$
 (2.3)

For any domain  $D \subset \Omega$ , let us introduce the space

$$\widetilde{H}^2(D) := \{ u \in H^1(D) : u|_{D \cap \Omega^s} \in H^2(D \cap \Omega^s), \ s = +, - \}$$

equipped with the following norm and seminorm:

$$\begin{split} \|u\|_{\tilde{H}^2(D)}^2 & \coloneqq \|u\|_{1,D}^2 + |u|_{2,D\cap\Omega^+}^2 + |u|_{2,D\cap\Omega^-}^2, \\ |u|_{\tilde{H}^2(D)}^2 & \coloneqq |u|_{2,D\cap\Omega^+}^2 + |u|_{2,D\cap\Omega^-}^2. \end{split}$$

We also define

$$\widetilde{H}^2_{\Gamma}(D) := \left\{ u \in \widetilde{H}^2(D) : \left\lceil \beta \frac{\partial u}{\partial \boldsymbol{n}} \right\rceil_{\Gamma \cap D} = 0 \right\}.$$

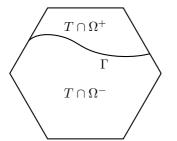
Then we have the following regularity theorem for the solution u of the variational problem (2.3); see [8,13].

**Theorem 2.1.** Suppose that  $f \in L^2(\Omega)$ . Then the variational problem (2.3) has a unique solution  $u \in H^1_0(\Omega) \cap \widetilde{H}^2_{\Gamma}(\Omega)$  satisfying

$$||u||_{\widetilde{H}^2(\Omega)} \le C||f||_{0,\Omega} \tag{2.4}$$

for some generic positive constant C.

2.2. **Mesh assumptions.** Let  $\{\mathcal{T}_h\}_h$  be a family of decompositions (meshes) of  $\Omega$  into polygonal elements T with maximum diameter h. Let  $\mathcal{E}_h$  be the set of all edges in  $\mathcal{T}_h$ . Let  $\mathcal{E}_h^i$  and  $\mathcal{E}_h^b$  denote the set of all interior and boundary edges in  $\mathcal{T}_h$ , respectively. For each  $T \in \mathcal{T}_h$ , let  $\mathcal{E}_T$  be the set of all edges of T. For each  $T \in \mathcal{T}_h$ , we denote by |T| the area of T, by  $h_T$  the diameter of T, and by  $n_T$  its exterior unit normal vector along the boundary  $\partial T$ . For each  $e \in \mathcal{E}_h$ , we denote by |e| the length of e. For  $e \in \mathcal{E}_h^i$ , we



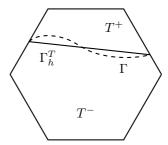


FIGURE 2. An interface element T in  $\mathcal{T}_h$ .

define  $n_e$  by a unit normal vector of e with orientation fixed once and for all. For  $e \in \mathcal{E}_h^b$ , we define  $n_e$  by a unit normal vector on e in the outward direction with respect to  $\Omega$ .

We call an element  $T \in \mathcal{T}_h$  an interface element if the interface  $\Gamma$  passes through the interior of T; otherwise we call T a noninterface element. We denote by  $\mathcal{T}_h^I$  the collection of all interface elements in  $\mathcal{T}_h$ , and by  $\mathcal{T}_h^N$  the collection of all non-interface elements in  $\mathcal{T}_h$ . For an interface element  $T \in \mathcal{T}_h$ , we denote by  $\Gamma_h^T$  the line segment connecting the intersections of  $\Gamma$  and the edges of T. This line segment divides T into two parts  $T^+$  and  $T^-$  with  $\overline{T} = \overline{T^+ \cup T^-}$  (see, for example, Figure 2). For any function  $u: T \to \mathbb{R}$ , we define its jump across  $\Gamma_h^T \cap T$  as

$$[u]_{\Gamma_t^T} \coloneqq u|_{T^+} - u|_{T^-}.$$

We assume that  $\{\mathcal{T}_h\}_h$  satisfies the following regularity assumptions [4, 26, 40].

## **Assumption 2.2.** There exists $\rho > 0$ independent of h such that

- (i) the decomposition  $\mathcal{T}_h$  consists of a finite number of nonoverlapping polygonal elements;
- (ii) for any  $T \in \mathcal{T}_h$  the diameter of any edge of T is larger than  $\rho h_T$ ;
- (iii) every element T of  $\mathcal{T}_h$  is star-shaped with respect to a ball  $B_T$  with center  $\boldsymbol{x}_T$  and radius  $\rho h_T$ ;
- (iv) if e is an edge of  $T \in \mathcal{T}_h$  then  $|e| \ge \rho h_T$ ;
- (v) the interface  $\Gamma$  meets the edges of an interface element at no more than two points;
- (vi) the interface  $\Gamma$  meets each edge in  $\mathcal{E}_h$  at most once, except possibly it passes through two vertices.

Remark 2.3. The assumptions (v) and (vi) are resonable if h is sufficiently small. Note also that the assumptions (i)-(iv) imply that the following properties [10]:

- Every  $T \in \mathcal{T}_h$  has at most N edges and vertices, where N is independent of h.
- Each element  $T \in \mathcal{T}_h$  can be decomposed as N triangles, obtained by connecting the vertices of T to  $\boldsymbol{x}_T$ , such that the minimum angle of the triangles is controlled by  $\rho$ .

Throughout this paper, C will denote a generic positive constant independent of h, not necessarily the same in each occurrence.

## 3. Immersed Weak Galerkin Method

In this section, we describe an immersed WG method for the problem (2.3).

3.1. Broken polynomial space. Let  $T \in \mathcal{T}_h$  be an interface element. We define the piecewise constant function  $\overline{\beta}_T$  on the element T as follows:

$$\overline{eta}_T(oldsymbol{x}) = \left\{ egin{array}{ll} \overline{eta}^+ & ext{if } oldsymbol{x} \in T^+, \ \overline{eta}^- & ext{if } oldsymbol{x} \in T^-, \end{array} 
ight.$$

where  $\overline{\beta}^s := \beta^s(\boldsymbol{x}^s)$  and  $\boldsymbol{x}^s$  denotes the barycenter of  $T^s$  for s = +, -. We also let  $\overline{\beta}$  be the piecewise constant function such that  $\overline{\beta}|_T = \overline{\beta}_T$  on each  $T \in \mathcal{T}_h$ . The broken polynomial space  $\widehat{\mathbb{P}}_1(T)$  of degree  $\leq 1$  is defined by

$$\widehat{\mathbb{P}}_1(T) := \left\{ q : q|_{T^+} \in \mathbb{P}_1(T^+), \ q|_{T^-} \in \mathbb{P}_1(T^-), \ [q]_{\Gamma_h^T} = 0, \ \left[ \overline{\beta}_T \frac{\partial q}{\partial \boldsymbol{n}} \right]_{\Gamma_h^T} = 0 \right\}.$$

It is easy to see that  $\dim \widehat{\mathbb{P}}_1(T) = 3$  (see, for example, [26, Theorem 2.2]), and the following piecewise polynomials form a basis of  $\widehat{\mathbb{P}}_1(T)$ :

$$\varphi_1(\boldsymbol{x}) = 1, \quad \varphi_2(\boldsymbol{x}) = \boldsymbol{t} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \varphi_3(\boldsymbol{x}) = \overline{\beta}_T^{-1} \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0),$$

where  $\boldsymbol{x}_0$  is the midpoint of the line segment  $\Gamma_h^T$ ,  $\boldsymbol{n}=(n_1,n_2)$  is a unit vector normal to  $\Gamma_h^T$  pointing from  $T^+$  to  $T^-$ , and  $\boldsymbol{t}=(-n_2,n_1)$ . Note that, since  $\widehat{\mathbb{P}}_1(T)\subset H^1(T)$ , the space  $\nabla\widehat{\mathbb{P}}_1(T)$  is well-defined, and the vector-valued functions  $\nabla\varphi_2$  and  $\nabla\varphi_3$  form a basis of  $\nabla\widehat{\mathbb{P}}_1(T)$ .

For convenience, we set  $\widehat{\mathbb{P}}_1(T) := \mathbb{P}_1(T)$  for any non-interface element  $T \in \mathcal{T}_h$ . Let

$$\widehat{\mathbb{P}}_1(\Omega) := \{ q \in L^2(\Omega) : q|_T \in \widehat{\mathbb{P}}_1(T) \ \forall T \in \mathcal{T}_h \}.$$

3.2. Weak Galerkin finite element space. We define the weak Galerkin finite element space  $V_h$  associated to  $\mathcal{T}_h$  and its subspace  $V_{h,0}$  as follows:

$$V_h := \left\{ v = \{v_0, v_{\partial}\} : v_0|_T \in \widehat{\mathbb{P}}_1(T) \ \forall T \in \mathcal{T}_h, \ v_{\partial}|_e \in \mathbb{P}_0(e) \ \forall e \in \mathcal{E}_h \right\},$$

$$V_{h,0} := \left\{ v \in V_h : v_{\partial} = 0 \text{ on } \partial \Omega \right\}.$$

Here we note that, for any  $v = \{v_0, v_{\partial}\} \in V_h$ , its second component  $v_{\partial}$  is a single-valued function on each edge  $e \in \mathcal{E}_h$ . Thus, the space  $V_h$  has 3 degrees of freedom on the interior of each element  $T \in \mathcal{T}_h$  and 1 degree of freedom on each edge  $e \in \mathcal{E}_h$ .

For each element  $T \in \mathcal{T}_h$ , let  $Q_0$  be the  $L^2$ -projection from  $L^2(T)$  onto  $\widehat{\mathbb{P}}_1(T)$ . Similarly, for each edge  $e \in \mathcal{E}_h$ , let  $Q_\partial$  the  $L^2$ -projection from  $L^2(e)$  onto  $\mathbb{P}_0(e)$ . We then define a projection operator  $Q_h: H^1(\Omega) \to V_h$  by

$$Q_h v = \{Q_0 v, Q_{\partial} v\}, \quad v \in H^1(\Omega). \tag{3.1}$$

3.3. Discrete problem and well-posedness. For each  $v_h = \{v_0, v_{\partial}\} \in V_h$ , we define a discrete weak gradient  $\nabla_w v_h$  of  $v_h$  as a vector-valued function satisfying  $\nabla_w v|_T \in \nabla \widehat{\mathbb{P}}_1(T)$  and

$$\int_{T} \overline{\beta}_{T} \nabla_{w} v_{h} \cdot \nabla q \, d\boldsymbol{x} = \int_{T} \overline{\beta}_{T} \nabla v_{0} \cdot \nabla q \, d\boldsymbol{x} - \int_{\partial T} (Q_{\partial} v_{0} - v_{\partial}) \left( \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{T} \right) ds \quad \forall q \in \widehat{\mathbb{P}}_{1}(T), \quad (3.2)$$

for each element  $T \in \mathcal{T}_h$ .

We next introduce two bilinear forms on  $V_h \times V_h$  as follows:

$$a(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w u_h \cdot \nabla_w v_h \, \mathrm{d}\boldsymbol{x},$$
  
$$s(u_h, v_h) := \lambda \sum_{T \in \mathcal{T}_h} h_T^{-1} \int_{\partial T} (Q_{\partial} u_0 - u_{\partial}) (Q_{\partial} v_0 - v_{\partial}) \, \mathrm{d}s,$$

for any  $u_h = \{u_0, u_\partial\} \in V_h$  and  $v_h = \{v_0, v_\partial\} \in V_h$ , where  $\lambda$  is an arbitrary positive constant. The stabilization  $a_s(\cdot, \cdot)$  of  $a(\cdot, \cdot)$  is defined by

$$a_s(u_h, v_h) = a(u_h, v_h) + s(u_h, v_h) \quad \forall u_h, v_h \in V_h.$$

We are now ready to formulate the immersed WG method for solving (2.3) as follows: Find  $u_h \in V_{h,0}$  such that

$$a_s(u_h, v_h) = (f, v_0)_{0,\Omega}, \quad \forall v_h = \{v_0, v_\partial\} \in V_{h,0}.$$
 (3.3)

We next analyze the well-posedness of the discrete problem (3.3). Define the energy-norm  $\|\cdot\|$  by

$$|||v_h||| := \sqrt{a_s(v_h, v_h)} \quad \forall v_h \in V_h.$$

Clearly  $\|\cdot\|$  is a seminorm on  $V_h$ . Moreover,  $\|\cdot\|$  is a norm on  $V_{h,0}$ , as shown in the following lemma.

**Lemma 3.1.**  $\|\cdot\|$  is a norm on  $V_{h,0}$ .

*Proof.* It suffices to show that  $||v_h|| = 0 \Rightarrow v_h \equiv 0$  for any  $v_h \in V_{h,0}$ . Suppose that  $v_h = \{v_0, v_\partial\} \in V_{h,0}$  satisfies  $||v_h|| = 0$ . Since

$$0 = \||v_h||^2 = \sum_{T \in \mathcal{T}_h} \int_T \overline{\beta}_T |\nabla_w v_h|^2 d\mathbf{x} + \lambda \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T} h_T^{-1} \int_e |Q_{\partial v_0} - v_{\partial}|^2 ds$$

and since  $0 < \beta_* < \overline{\beta}_T$  for any  $T \in \mathcal{T}_h$ , we obtain  $\nabla_w v_h \equiv 0$  and  $Q_{\partial} v_0 = v_{\partial}$  on each edge  $e \in \mathcal{E}_h$ . Then

$$0 = \int_{T} \overline{\beta}_{T} \nabla_{w} v_{h} \cdot \nabla v_{0} \, d\boldsymbol{x} = \int_{T} \overline{\beta}_{T} \nabla v_{0} \cdot \nabla v_{0} \, d\boldsymbol{x} + \sum_{e \subset \partial T} \int_{e} (v_{\partial} - Q_{\partial} v_{0}) \left( \overline{\beta}_{T} \frac{\partial v_{0}}{\partial \boldsymbol{n}} \right) ds$$
$$= \int_{T} \overline{\beta}_{T} |\nabla v_{0}|^{2} \, d\boldsymbol{x} \ge \int_{T} \beta_{*} |\nabla v_{0}|^{2} \, d\boldsymbol{x}$$

for any  $T \in \mathcal{T}_h$ . This shows that  $\nabla v_0 = 0$  on each  $T \in \mathcal{T}_h$ . Note that, for each  $T \in \mathcal{T}_h$ ,  $\nabla q = 0$  implies q = constant for any  $q \in \widehat{\mathbb{P}}_1(T)$ . Since  $v_0 \in \widehat{\mathbb{P}}_1(T)$  on each  $T \in \mathcal{T}_h$ , we obtain that  $v_0$  is constant on each  $T \in \mathcal{T}_h$ . Since  $Q_{\partial}v_0 = v_{\partial}$  on each  $e \in \mathcal{E}_h$  and  $v_{\partial} = 0$  on  $\partial \Omega$ , we conclude that  $v_0 = v_{\partial} = 0$ .

The well-posedness of the discrete problem (3.3) directly follows from the lemma.

# Corollary 3.2. The discrete problem (3.3) is well-posed.

*Proof.* From Lemma 3.1, the bilinear form  $a_s(\cdot, \cdot)$  on  $V_{h,0}$  is coercive and continuous with respect to the norm  $\|\cdot\|$  on  $V_{h,0}$ . The conclusion follows from the Lax-Milgram Lemma.

# 4. Some Estimates on Interface Elements

In this section, we present some inequalities for the function spaces on the interface elements, which are needed for the error analysis of the immersed WG method.

4.1. Geometric assumptions on interface elements. Let  $T \in \mathcal{T}_h$  be an interface element. Recall that  $\Gamma_h^T$  denotes the line segment connecting two intersection points of  $\Gamma$  and the edges of T. Although the analysis works for  $C^2$ -interface, we assume for the simplicity of presentation, that on each mesh element T, the portion  $\Gamma \cap T$  is a line segment so that  $\Gamma \cap T = \Gamma_h^T$  and  $T^s = T \cap \Omega^s$  for s = +, -. In addition, we assume that  $\Gamma \cap T$  aligns with the x-axis and the origin of the xy-plane is contained in T, so that

$$T^{+} = T \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \ge 0\}, \quad T^{-} = T \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 0\}$$

$$(4.1)$$

(see Figure 3). Since  $h_T = \operatorname{diam}(T)$ , we have  $T \subset [-h_T, h_T]^2$ . Since  $\beta^s \in C^1(\overline{\Omega^s})$  and  $\overline{\beta}_T = \overline{\beta}^s$  on  $T^s$  for s = +, -, we have

$$\max_{\boldsymbol{x} \in T^s} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \le Ch_T, \quad \max_{\boldsymbol{x} \in e \cap \Omega^s} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \le Ch_T, \quad s = +, -, \tag{4.2}$$

where  $e \subset \partial T$ . Let  $\mathbf{n}_{\Gamma} = (n_{1,h}, n_{2,h})$  be the unit vector normal to  $\Gamma$  pointing from  $T^+$  to  $T^-$ , and let  $\mathbf{t}_{\Gamma} = (-n_{2,h}, n_{1,h})$ .

Remark 4.1. We briefly discuss the case when the interface is not piecewise linear, that is,  $\Gamma \cap T \neq \Gamma_h^T$ . Without loss of generality we assume that  $\Gamma_h^T$  aligns with the x-axis and T is contained in the box  $I_x \times I_y$ , where  $I_x$  and  $I_y$  are intervals with length not greater than  $2h_T$ . Since  $\Gamma$  is a regular  $C^2$ -curve, there exists a parametrization  $t \mapsto (t, \gamma(t))$  of the curve  $\Gamma \cap T$  for some  $\gamma \in C^2(I_x)$ , when h is sufficiently small. Then the unit normal vector  $\mathbf{n}_\Gamma$  along  $\Gamma \cap T$  pointing from  $\Omega^+$  to  $\Omega^-$  is

$$\mathbf{n}_{\Gamma}(t,\gamma(t)) = \left(\frac{\gamma'(t)}{(1+|\gamma'(t)|^2)^{1/2}}, \frac{-1}{(1+|\gamma'(t)|^2)^{1/2}}\right), \quad t \in I_x.$$

Let us extend the vector-valued function  $\mathbf{n}_{\Gamma}$  to the box  $I_x \times I_y$  by setting  $(t, y) \mapsto \mathbf{n}_{\Gamma}(t, \gamma(t))$ . Then, since  $\gamma$  is  $C^2$ , we have

$$\sup_{\boldsymbol{x}\in T} \left| \boldsymbol{n}_{\Gamma}(\boldsymbol{x}) - \boldsymbol{n}_{\Gamma}^{h} \right| \le Ch_{T}, \tag{4.3}$$

where  $n_{\Gamma}^h$  is the unit normal vector along  $\Gamma_h^T$  pointing from  $T^+$  to  $T^-$ . In addition, one can obtain a similar result for the tangential vector of  $\Gamma \cap T$ . Next, according to Lemma 2 in [7],

$$\|\nabla u\|_{0,T_r}^2 \le Ch_T^2 \sum_{s=+,-} \left( \|(\nabla u)|_{\Omega_s}\|_{0,\Gamma\cap T}^2 + h_T^2 |\nabla u|_{1,T\cap\Omega^s}^2 \right), \quad \forall u \in \widetilde{H}^2(T), \tag{4.4}$$

where  $T_r$  is a subset of T given by

$$T_r = T - (\Omega^+ \cap T^+) - (\Omega^- \cap T^-);$$

see Figure 2. Note also that the first estimate in (4.2) is modified as follows:

$$\sup_{\boldsymbol{x}\in T^s\cap (T\cap\Omega^s)} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \le Ch_T, \quad s = +, -.$$
(4.5)

Using the estimates (4.3)-(4.5) and the standard trace inequality, all the results below can be derived with only minor modification. We leave the detailed analysis for a future investigation.

**Lemma 4.2.** If h is sufficiently small, then either  $T^+$  or  $T^-$  contains a ball with radius  $\rho h_T/8$ .

*Proof.* Recall that T is star-shaped with respect to a ball B centered at  $\mathbf{x}_T = (x_T, y_T)$  with radius  $\rho h_T$ . First, assume that  $|y_T| \leq \rho h_T/8$ . Consider the ball  $B^+$  centered at  $(x_T, y_T + \rho h_T/2)$  with radius  $\rho h_T/8$ . Then  $B^+ \subset B \cap T^+$ .

One can show that, by the same argument, for the case  $y_T \ge \rho h_T/8$  the set  $T^+$  contains the ball centered at  $(x_T, y_T + \rho h_T/2)$  with radius  $\rho h_T/8$ , and for the case  $y_T \le -\rho h_T/8$  the set  $T^-$  contains the ball centered at  $(x_T, y_T - \rho h_T/2)$  with radius  $\rho h_T/8$ .

4.2. Some inequalities for the broken polynomial space  $\widehat{\mathbb{P}}_1$ . Recall that, on each element  $T \in \mathcal{T}_h$ , the standard trace inequality holds:

$$h_T^{1/2} \|v\|_{0,\partial T} \le C(\|v\|_{0,T} + h_T \|\nabla v\|_{0,T}) \quad \forall v \in H^1(T).$$
 (4.6)

The following lemma provides a trace inequality for the space  $\nabla \widehat{\mathbb{P}}_1$ .

**Lemma 4.3.** Let  $T \in \mathcal{T}_h$  be an interface element. Then there exists a positive constant C depending only on  $\rho$  and  $\beta$  such that for any  $q \in \widehat{\mathbb{P}}_1(T)$  and any edge e of T,

$$\|\overline{\beta}_T \nabla q\|_{0,e} \le C h_T^{-1/2} \|\overline{\beta}_T^{1/2} \nabla q\|_{0,T},\tag{4.7}$$

*Proof.* Recall that the following piecewise polynomials form a basis of the space  $\widehat{\mathbb{P}}_1(T)$ :

$$\varphi_1(\boldsymbol{x}) = 1, \quad \varphi_2(\boldsymbol{x}) = \boldsymbol{t}_{\Gamma} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \varphi_3(\boldsymbol{x}) = \overline{\beta}_T^{-1} \boldsymbol{n}_{\Gamma} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \forall \boldsymbol{x} \in T,$$

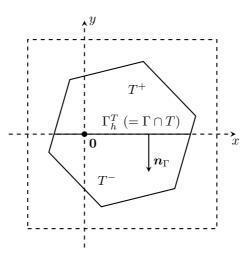


FIGURE 3. Geometric assumptions on an interface element T.

where  $x_0$  is the midpoint of  $\Gamma_h^T$ . Let  $q = a\varphi_1 + b\varphi_2 + c\varphi_3$  for  $a, b, c \in \mathbb{R}$ . Then

$$\nabla q = b \boldsymbol{t}_{\Gamma} + c \overline{\beta}_{T}^{-1} \boldsymbol{n}_{\Gamma}, \quad \nabla q \cdot \nabla q = b^{2} + c^{2} \overline{\beta}_{T}^{-2}.$$

By Assumption 2.2 (iii), we have

$$\|\overline{\beta}_{T}\nabla q\|_{0,e}^{2} = \int_{e} |\overline{\beta}_{T}\nabla q|^{2} ds \leq ((\beta^{*})^{2}b^{2} + c^{2})|e| \leq C(\beta_{*}, \beta^{*}, \rho)(b^{2} + c^{2})h_{T},$$

$$\|\overline{\beta}_{T}^{1/2}\nabla q\|_{0,T}^{2} = \int_{T} \overline{\beta}_{T}|\nabla q|^{2} dx \geq \beta_{*}(b^{2} + c^{2}(\beta^{*})^{-2})|T| \geq C(\beta_{*}, \beta^{*}, \rho)(b^{2} + c^{2})h_{T}^{2}.$$

Thus there exists a positive constant C depending only on  $\rho$  and  $\beta$  such that the inequality (4.7) holds.  $\square$ 

Note that we have the following inverse inequality holds (see, for example, (2.6) of [10]):

$$|q|_{1,T} \le Ch_T^{-1} ||q||_{0,T} \quad \forall q \in \mathbb{P}_1(T), \qquad |q|_{1,B} \le Ch_T^{-1} ||q||_{0,B} \quad \forall q \in \mathbb{P}_1(B),$$
 (4.8)

where B is a ball in  $\mathbb{R}^2$  with radius  $\rho h_T$  and C is a positive constant depending only on  $\rho$ . The following lemma shows that the inverse inequality also holds for the space  $\widehat{\mathbb{P}}_1$ .

**Lemma 4.4.** Let  $T \in \mathcal{T}_h$  be an interface element. There exists a positive constant C depending only on  $\rho$  and  $\beta$  such that

$$|q|_{1,T} \le Ch_T^{-1} ||q||_{0,T} \quad \forall q \in \widehat{\mathbb{P}}_1(T).$$

*Proof.* By Lemma 4.2, we may assume that  $T^+$  contains a ball  $B^+$  with radius  $\rho h_T/8$ . As in the proof of the previous lemma, consider the basis  $\{\varphi_1, \varphi_2, \varphi_3\}$  of  $\widehat{\mathbb{P}}_1(T)$  and let  $q = a\varphi_1 + b\varphi_2 + c\varphi_3$  for  $a, b, c \in \mathbb{R}$ , and define

$$q_+ := a + b \boldsymbol{t}_{\Gamma} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + c (\overline{\beta}^+)^{-1} \boldsymbol{n}_{\Gamma} \cdot (\boldsymbol{x} - \boldsymbol{x}_0).$$

Then  $q = q_{+}$  on  $T^{+}$ . By (4.8),

$$|q_{+}|_{1,B^{+}} \le Ch_{T}^{-1}||q_{+}||_{0,B^{+}} = Ch_{T}^{-1}||q||_{0,B^{+}} \le Ch_{T}^{-1}||q||_{0,T}.$$
 (4.9)

Since  $\boldsymbol{t}_{\Gamma} \cdot \boldsymbol{n}_{\Gamma} = 0$ ,

$$|q_{+}|_{1,B^{+}}^{2} = \int_{B^{+}} |bt_{\Gamma} + c(\overline{\beta}^{+})^{-1} n_{\Gamma}|^{2} dx = \int_{B^{+}} (b^{2} + (\overline{\beta}^{+})^{-2} c^{2}) dx$$

$$\geq \frac{\pi \rho^{2} h_{T}^{2}}{64} C(\beta_{*}, \beta^{*}) (b^{2} + c^{2}), \qquad (4.10)$$

$$|q|_{1,T}^{2} = \int_{T^{+}} |b\mathbf{t}_{\Gamma} + c(\overline{\beta}^{+})^{-1} \mathbf{n}_{\Gamma}|^{2} d\mathbf{x} + \int_{T^{-}} |b\mathbf{t}_{\Gamma} + c(\overline{\beta}^{-})^{-1} \mathbf{n}_{\Gamma}|^{2} d\mathbf{x}$$

$$= \int_{T^{+}} (b^{2} + (\overline{\beta}^{+})^{-2} c^{2}) d\mathbf{x} + \int_{T^{-}} (b^{2} + (\overline{\beta}^{-})^{-2} c^{2}) d\mathbf{x}$$

$$\leq C(\beta_{*}, \beta^{*}) h_{T}^{2} (b^{2} + c^{2}). \tag{4.11}$$

Combining the inequalities (4.9)-(4.11), we obtain

$$|q|_{1,T} \le \frac{8}{\sqrt{\pi\rho}} C(\beta_*, \beta^*) |q_+|_{1,B^+} \le C(\beta_*, \beta^*, \rho) h_T^{-1} ||q||_{0,T}.$$

This completes the proof of the lemma.

4.3. Approximation properties of the broken polynomial space  $\widehat{\mathbb{P}}_1$ . In this subsection, we derive some approximation properties of the broken linear polynomial space  $\widehat{\mathbb{P}}_1(T)$ .

It is well-known that, on each non-interface element  $T \in \mathcal{T}_h$ , for any  $u \in H^2(T)$  there exists  $q \in \mathbb{P}_1$  such that

$$||u - q||_{0,T} + h_T |u - q|_{1,T} \le C_\rho h_T^2 ||u||_{2,T}, \tag{4.12}$$

where  $C_{\rho}$  is a positive constant depending only on  $\rho$  [11, Lemma 4.3.8].

**Theorem 4.5.** Let  $u \in \widetilde{H}^{2}_{\Gamma}(\Omega)$ . Then there exists  $q \in \widehat{\mathbb{P}}_{1}(\Omega)$  such that

$$||u - q||_{0,\Omega} + h|u - q|_{1,\Omega} \le Ch^2 ||u||_{\widetilde{H}^2(\Omega)},$$

where C is a positive constant depending only on  $\rho$  and  $\beta$ .

*Proof.* Let  $T \in \mathcal{T}_h$  be an interface element. Then we have

$$\nabla u = (\nabla u \cdot \boldsymbol{t}_{\Gamma})\boldsymbol{t}_{\Gamma} + (\nabla u \cdot \boldsymbol{n}_{\Gamma})\boldsymbol{n}_{\Gamma}$$
(4.13)

on T. We note that  $\nabla u \cdot \mathbf{t}_{\Gamma} \in H^1(T)$  and  $\beta \nabla u \cdot \mathbf{n}_{\Gamma} \in H^1(T)$ . Thus, from (4.12), there exist  $c_t, c_n \in \mathbb{R}$  such that

$$\|\nabla u \cdot \mathbf{t}_{\Gamma} - c_t\|_{0,T} \le C_{\rho} h_T |\nabla u \cdot \mathbf{t}_{\Gamma}|_{1,T}, \quad \|\beta \nabla u \cdot \mathbf{n}_{\Gamma} - c_n\|_{0,T} \le C_{\rho} h_T |\beta \nabla u \cdot \mathbf{n}_{\Gamma}|_{1,T}.$$

Note that

$$|\nabla u \cdot \boldsymbol{t}_{\Gamma}|_{1,T} \le C \|u\|_{\widetilde{H}^{2}(T)}, \quad |\nabla u \cdot \boldsymbol{n}_{\Gamma}|_{1,T} \le C \|u\|_{\widetilde{H}^{2}(T)}. \tag{4.14}$$

Thus

$$\|\nabla u \cdot \boldsymbol{t}_{\Gamma} - c_t\|_{0,T} \le Ch_T \|u\|_{\widetilde{H}^2(T)}, \quad \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - c_n\|_{0,T} \le Ch_T \|u\|_{\widetilde{H}^2(T)}. \tag{4.15}$$

Let

$$\boldsymbol{r}\coloneqq c_t \boldsymbol{t}_\Gamma + \overline{\beta}_T^{-1} c_n \boldsymbol{n}_\Gamma.$$

Then  $r \in \nabla \widehat{\mathbb{P}}_1(T)$ . By (4.13), (4.15), and (4.2),

$$\|\nabla u - \boldsymbol{r}\|_{0,T} \leq \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - c_{t}\|_{0,T} + \beta_{*}^{-1} \|c_{n} - \overline{\beta}_{T} \nabla u \cdot \boldsymbol{n}_{\Gamma}\|_{0,T}$$

$$\leq Ch_{T} \|u\|_{\widetilde{H}^{2}(T)} + \beta_{*}^{-1} \|c_{n} - \beta \nabla u \cdot \boldsymbol{n}_{\Gamma}\|_{0,T} + \beta_{*}^{-1} \|(\beta - \overline{\beta}_{T}) \nabla u\|_{0,T}$$

$$\leq Ch_{T} \|u\|_{\widetilde{H}^{2}(T)}. \tag{4.16}$$

Since  $\mathbf{r} \in \nabla \widehat{\mathbb{P}}_1(T)$ , there exists  $q \in \widehat{\mathbb{P}}_1(T)$  such that  $\nabla q = \mathbf{r}$  and  $\int_T q \, d\mathbf{x} = \int_T u \, d\mathbf{x}$ . Then (4.16) and Poincaré-Friedrichs inequality (cf. [9]) imply that

$$||u - q||_{0,T} \le Ch_T|u - q|_{1,T} \le Ch_T^2||u||_{\widetilde{H}^2(T)}.$$

This completes the proof of the theorem.

As a corollary, we obtain the estimate for the  $L^2$ -projection  $Q_0$  onto the space  $\widehat{\mathbb{P}}_1$  as follows.

**Corollary 4.6.** There exists a positive constant C, depending only on  $\rho$  and  $\beta$ , such that

$$||u - Q_0 u||_{0,\Omega} + h|u - Q_0 u|_{1,\Omega} \le Ch^2 ||u||_{\widetilde{H}^2(\Omega)} \quad \forall u \in \widetilde{H}^2_{\Gamma}(\Omega).$$

*Proof.* Let  $T \in \mathcal{T}_h$  be an interface element. By Theorem 4.5, there exists  $q' \in \widehat{\mathbb{P}}_1(T)$  such that

$$||u - q'||_{0,T} + h_T |u - q'|_{1,T} \le Ch_T^2 ||u||_{\widetilde{H}^2(T)}, \tag{4.17}$$

where C is a positive constant depending only on  $\rho$  and  $\beta$ . Since  $||Q_0v||_{0,T} \leq ||v||_{0,T}$  for any  $v \in H^1(T)$  and  $Q_0q = q$  for any  $q \in \widehat{\mathbb{P}}_1(T)$ , we obtain

$$||u - Q_0 u||_{0,T} \le ||u - q'||_{0,T} + ||Q_0 q' - Q_0 u||_{0,T} \le Ch_T^2 ||u||_{\widetilde{H}^2(T)}.$$

By Lemma 4.4,

$$|u - Q_0 u|_{1,T} \leq |u - q'|_{1,T} + |Q_0 q' - Q_0 u|_{1,T} \leq |u - q'|_{1,T} + h_T^{-1} ||Q_0 q' - Q_0 u||_{0,T}$$
  
$$\leq |u - q'|_{1,T} + h_T^{-1} ||q' - u||_{0,T} \leq Ch_T ||u||_{\widetilde{H}^2(T)}.$$

This completes the proof.

The following lemma gives the  $L^2$ -norm estimate of  $\beta \nabla u - \overline{\beta}_T \nabla (Q_0 u)$  on each mesh edge (see Proposition 5.2 in [24]).

**Lemma 4.7.** There exists a positive constant C independent of h such that

$$\sum_{T \in \mathcal{T}_h} \left\| \beta \nabla u - \overline{\beta}_T \nabla(Q_0 u) \right\|_{0, \partial T}^2 \le Ch \|u\|_{\widetilde{H}^2(\Omega)}^2 \quad \forall u \in \widetilde{H}^2_{\Gamma}(\Omega).$$

*Proof.* Let  $T \in \mathcal{T}_h$  be an interface element. Let  $q = Q_0 u$ , and let  $e \subset \partial T$ . As in (4.13), we have

$$\nabla u = (\nabla u \cdot t_{\Gamma})t_{\Gamma} + (\nabla u \cdot n_{\Gamma})n_{\Gamma}, \quad \nabla q = (\nabla q \cdot t_{\Gamma})t_{\Gamma} + (\nabla q \cdot n_{\Gamma})n_{\Gamma}$$

$$(4.18)$$

on T. Since  $u \in \widetilde{H}^2(T)$ , we have  $\nabla u \cdot \boldsymbol{t}_{\Gamma} \in H^1(T)$  and  $\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} \in H^1(T)$ . Note also that  $\overline{\beta}_T \nabla q \cdot \boldsymbol{n}_{\Gamma}$  and  $\nabla q \cdot \boldsymbol{t}_{\Gamma}$  are constants on T. Then, by (4.2),

$$\begin{split} \left\| \beta \nabla u - \overline{\beta}_T \nabla q \right\|_{0,e} & \leq \left\| \beta \nabla u - \overline{\beta}_T \nabla u \right\|_{0,e} + \left\| \overline{\beta}_T \nabla u - \overline{\beta}_T \nabla q \right\|_{0,e} \\ & \leq Ch_T \|\nabla u\|_{0,e} + C\|\nabla u - \nabla q\|_{0,e}. \end{split} \tag{4.19}$$

By the trace inequality (4.6) and (4.14),

$$\|\nabla u\|_{0,e} \leq \|\nabla u \cdot \boldsymbol{t}_{\Gamma}\|_{0,e} + \beta_{*}^{-1} \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma}\|_{0,e}$$

$$\leq Ch_{T}^{-1/2} (\|\nabla u \cdot \boldsymbol{t}_{\Gamma}\|_{0,T} + h_{T} |\nabla u \cdot \boldsymbol{t}_{\Gamma}|_{1,T}) + Ch_{T}^{-1/2} (\|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma}\|_{0,T} + h_{T} |\beta \nabla u \cdot \boldsymbol{n}_{\Gamma}|_{1,T})$$

$$\leq Ch_{T}^{-1/2} \|u\|_{\tilde{H}^{2}(T)}$$

$$\leq Ch_{T}^{-1/2} \|u\|_{\tilde{H}^{2}(T)}. \tag{4.20}$$

By (4.18) and (4.2),

$$\|\nabla u - \nabla q\|_{0,e} \leq \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,e} + \beta_{*}^{-1} \|\overline{\beta}_{T} \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,e}$$

$$\leq \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,e} + \beta_{*}^{-1} \|(\overline{\beta}_{T} - \beta) \nabla u \cdot \boldsymbol{n}_{\Gamma}\|_{0,e}$$

$$+ \beta_{*}^{-1} \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,e}$$

$$\leq \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,e} + C\beta_{*}^{-1} h_{T} \|\nabla u\|_{0,e}$$

$$+ \beta_{*}^{-1} \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,e}. \tag{4.21}$$

By the trace inequality (4.6), Corollary 4.6, and (4.2),

$$\|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,e} \leq Ch_{T}^{-1/2} \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,T} + Ch_{T}^{1/2} |\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}|_{1,T}$$

$$\leq Ch_{T}^{-1/2} \|\nabla u \cdot \boldsymbol{t}_{\Gamma} - \nabla q \cdot \boldsymbol{t}_{\Gamma}\|_{0,T} + Ch_{T}^{1/2} |\nabla u \cdot \boldsymbol{t}_{\Gamma}|_{1,T}$$

$$\leq Ch_{T}^{1/2} \|u\|_{\widetilde{H}^{2}(T)}, \qquad (4.22)$$

$$\|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,e} \leq Ch_{T}^{-1/2} \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,T} + Ch_{T}^{1/2} |\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}|_{1,T}$$

$$\leq Ch_{T}^{-1/2} \|\beta \nabla u \cdot \boldsymbol{n}_{\Gamma} - \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{\Gamma}\|_{0,T} + Ch_{T}^{1/2} |\beta \nabla u \cdot \boldsymbol{n}_{\Gamma}|_{1,T}$$

$$\leq Ch_{T}^{-1/2} \|(\beta - \overline{\beta}_{T}) \nabla u\|_{0,T} + Ch_{T}^{-1/2} \|\nabla u - \nabla q\|_{0,T} + Ch_{T}^{1/2} \|u\|_{\widetilde{H}^{2}(T)}$$

$$\leq Ch_{T}^{1/2} \|u\|_{\widetilde{H}^{2}(T)}. \qquad (4.23)$$

Now the conclusion follows from the inequalities (4.19)-(4.23).

The following lemma gives the  $L^2$ -norm estimate of  $\nabla_w(Q_h u) - \nabla(Q_0 u)$  on each element in  $\mathcal{T}_h$ .

**Lemma 4.8.** There exists a positive constant C independent of h such that

$$\|\nabla_w(Q_h u) - \nabla(Q_0 u)\|_{0,\Omega} \le Ch\|u\|_{\widetilde{H}^2(\Omega)} \quad u \in \widetilde{H}^2_{\Gamma}(\Omega).$$

*Proof.* Let T be an interface element. By the definition of the discrete weak gradient (3.2), we have

$$\int_{T} \overline{\beta}_{T} \left( \nabla_{w} (Q_{h} u) - \nabla (Q_{0} u) \right) \cdot \nabla q \, d\boldsymbol{x} = -\int_{\partial T} (Q_{\partial} (Q_{0} u) - Q_{\partial} u) \left( \overline{\beta}_{T} \frac{\partial q}{\partial \boldsymbol{n}} \right) ds \quad \forall q \in \widehat{\mathbb{P}}_{1}(T).$$

Let  $q \in \widehat{\mathbb{P}}_1(T)$  satisfy  $\nabla q = \nabla_w(Q_h u) - \nabla(Q_0 u)$ . By the trace inequality (4.6), Lemma 4.3, Poincaré-Friedrichs inequality, and Corollary 4.6, we obtain

$$\|\nabla_{w}(Q_{h}u) - \nabla(Q_{0}u)\|_{0,\Omega}^{2} \leq C \sum_{T \in \mathcal{T}_{h}} \|u - Q_{0}u\|_{0,\partial T} \|\overline{\beta}_{T}\nabla q\|_{0,\partial T}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} \left(h_{T}^{-1} \|u - Q_{0}u\|_{0,T} + |u - Q_{0}u|_{1,T}\right) \|\overline{\beta}_{T}^{1/2}\nabla q\|_{0,T}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} |u - Q_{0}u|_{1,T} \|\overline{\beta}_{T}^{1/2}\nabla q\|_{0,T}$$

$$\leq C h\|u\|_{\widetilde{H}^{2}(\Omega)} \|\nabla_{w}(Q_{h}u) - \nabla(Q_{0}u)\|_{0,\Omega},$$

and this completes the proof.

## 5. Error Analysis

In this section, we present the error estimate in the discrete  $H^1$ -seminorm for the scheme (3.3).

5.1. **Discrete**  $H^1$ -seminorm. We introduce a discrete  $H^1$ -seminorm as follows: For  $v_h = \{v_0, v_\partial\} \in V_h$ ,

$$|v_h|_{1,h} := \left(\sum_{T \in \mathcal{T}_h} \|\nabla v_0\|_{0,T}^2 + \lambda h_T^{-1} \|Q_{\partial} v_0 - v_{\partial}\|_{0,\partial T}^2\right)^{1/2}.$$

The following lemma shows that two seminorms  $\|\cdot\|$  and  $|\cdot|_{1,h}$  on  $V_h$  are equivalent.

**Lemma 5.1.** There exist two positive constants  $C_1$  and  $C_2$  independent of h such that

$$C_1|v_h|_{1,h} < |||v_h||| < C_2|v_h|_{1,h} \quad \forall v_h \in V_h.$$

*Proof.* The proof is similar to the proof of Lemma 5.3 in [35]. Let  $v_h = \{v_0, v_{\partial}\} \in V_h$ . By the definition of the discrete weak gradient (3.2), we have

$$\int_{T} \overline{\beta}_{T} \nabla_{w} v_{h} \cdot \nabla q \, d\boldsymbol{x} = \int_{T} \overline{\beta}_{T} \nabla v_{0} \cdot \nabla q \, d\boldsymbol{x} + \int_{\partial T} (v_{\partial} - Q_{\partial} v_{0}) \left( \overline{\beta}_{T} \nabla q \cdot \boldsymbol{n}_{T} \right) ds \quad \forall q \in \widehat{\mathbb{P}}_{1}(T).$$
 (5.1)

Let  $q \in \widehat{\mathbb{P}}_1(\Omega)$  satisfy  $\nabla q = \nabla_w v_h$  on each  $T \in \mathcal{T}_h$ . Then, by Lemma 4.3,

$$\begin{aligned} \|\overline{\beta}^{1/2} \nabla_{w} v_{h}\|_{0,\Omega}^{2} &= \sum_{T \in \mathcal{T}_{h}} \left( \int_{T} \overline{\beta}_{T} \nabla v_{0} \cdot \nabla_{w} v_{h} \, \mathrm{d}\boldsymbol{x} + \int_{\partial T} \left( v_{\partial} - Q_{\partial} v_{0} \right) \left( \overline{\beta}_{T} \nabla_{w} v_{h} \cdot \boldsymbol{n}_{T} \right) \, \mathrm{d}\boldsymbol{s} \right) \\ &\leq C \sum_{T \in \mathcal{T}_{h}} \left( \|\nabla v_{0}\|_{0,T} \|\overline{\beta}_{T}^{1/2} \nabla_{w} v_{h}\|_{0,T} + \|Q_{\partial} v_{0} - v_{\partial}\|_{0,\partial T} \|\overline{\beta}_{T} \nabla_{w} v_{h}\|_{0,\partial T} \right) \\ &\leq C \sum_{T \in \mathcal{T}_{h}} \left( \|\nabla v_{0}\|_{0,T} \|\overline{\beta}_{T}^{1/2} \nabla_{w} v_{h}\|_{0,T} + C h^{-1/2} \|Q_{\partial} v_{0} - v_{\partial}\|_{0,\partial T} \|\overline{\beta}_{T}^{1/2} \nabla_{w} v_{h}\|_{0,T} \right) \\ &\leq C |v_{h}|_{1,h} \|\overline{\beta}^{1/2} \nabla_{w} v_{h}\|_{0,\Omega}. \end{aligned}$$

Thus we have  $\|\overline{\beta}^{1/2}\nabla_w v_h\|_{0,\Omega} \leq C|v_h|_{1,h}$ . Since  $s(v_h,v_h) \leq |v_h|_{1,h}^2$ , we have

$$\|\|v_h\|\|^2 = \|\overline{\beta}^{1/2}\nabla_w v_h\|_{0,\Omega}^2 + s(v_h, v_h) \le C|v_h|_{1,h}^2$$

On the other hand, let  $q \in \widehat{\mathbb{P}}_1(\Omega)$  satisfy  $\nabla q = \nabla v_0$  on each  $T \in \mathcal{T}_h$ . Then, by (5.1) and Lemma 4.3 we have

$$\begin{split} \|\nabla v_{0}\|_{0,\Omega}^{2} &\leq C \sum_{T \in \mathcal{T}_{h}} \int_{T} \overline{\beta}_{T} \nabla v_{0} \cdot \nabla v_{0} \, \mathrm{d}\boldsymbol{x} \\ &= C \sum_{T \in \mathcal{T}_{h}} \left( \int_{T} \overline{\beta}_{T} \nabla_{w} v_{h} \cdot \nabla v_{0} \, \mathrm{d}\boldsymbol{x} - \int_{\partial T} (v_{\partial} - Q_{\partial} v_{0}) \left( \overline{\beta}_{T} \nabla v_{0} \cdot \boldsymbol{n}_{T} \right) \, \mathrm{d}\boldsymbol{s} \right) \\ &\leq C \sum_{T \in \mathcal{T}_{h}} \left( \|\overline{\beta}_{T}^{1/2} \nabla_{w} v_{h}\|_{0,T} \|\nabla v_{0}\|_{0,T} + \|v_{\partial} - Q_{\partial} v_{0}\|_{0,\partial T} \|\overline{\beta}_{T} \nabla v_{0} \cdot \boldsymbol{n}_{T}\|_{0,\partial T} \right) \\ &\leq C \sum_{T \in \mathcal{T}_{h}} \left( \|\overline{\beta}_{T}^{1/2} \nabla_{w} v_{h}\|_{0,T} \|\nabla v_{0}\|_{0,T} + h_{T}^{-1/2} \|v_{\partial} - Q_{\partial} v_{0}\|_{0,\partial T} \|\nabla v_{0}\|_{0,T} \right) \\ &\leq C \|v_{h}\| \|\nabla v_{0}\|_{0,\Omega}. \end{split}$$

Thus  $\|\nabla v_0\|_{0,\Omega} \le C \|v_h\|$ . Since  $s(v_h, v_h) \le \|v_h\|^2$ , we obtain

$$|v_h|_{1,h}^2 = \|\nabla v_0\|_{0,\Omega}^2 + s(v_h, v_h) \le C \|v_h\|^2$$

Hence we have proved the lemma.

5.2. **Error equation.** The error equation presented in the following lemma will be used to derive the error estimate.

**Lemma 5.2.** Let  $u \in H_0^1(\Omega)$  be the solution of (2.3) with  $f \in L^2(\Omega)$ , and let  $u_h \in V_{h,0}$  be the solution of (3.3). Then we have

$$a_{s}(Q_{h}u - u_{h}, v_{h}) = s(Q_{h}u, v_{h}) + \sum_{T \in \mathcal{T}_{h}} \int_{T} (\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u) \cdot \nabla v_{0} \, d\boldsymbol{x}$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (Q_{\partial}v_{0} - v_{\partial}) \left(\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u\right) \cdot \boldsymbol{n}_{T} \, ds$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - Q_{\partial}v_{0}) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, ds, \quad \forall v_{h} \in V_{h,0}.$$

$$(5.2)$$

*Proof.* Note that, for any  $v_h \in V_{h,0}$ ,

$$\begin{split} a_s(Q_h u - u_h, v_h) &= a_s(Q_h u, v_h) - (f, v_0)_{0,\Omega} \\ &= a_s(Q_h u, v_h) - \sum_{T \in \mathcal{T}_h} \left( \int_T \beta \nabla u \cdot \nabla v_0 \, \mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \, \mathrm{d}\boldsymbol{s} \right) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w (Q_h u) \cdot \nabla_w v_h \, \mathrm{d}\boldsymbol{x} \\ &+ s(Q_h u, v_h) - \sum_{T \in \mathcal{T}_h} \left( \int_T \beta \nabla u \cdot \nabla v_0 \, \mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \, \mathrm{d}\boldsymbol{s} \right) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w (Q_h u) \cdot \nabla v_0 \, \mathrm{d}\boldsymbol{x} - \sum_{T \in \mathcal{T}_h} \int_{\partial T} (v_0 - v_\partial) \left( Q_\partial \left( \overline{\beta}_T \nabla_w (Q_h u) \cdot \boldsymbol{n}_T \right) \right) \mathrm{d}\boldsymbol{s} \\ &+ s(Q_h u, v_h) - \sum_{T \in \mathcal{T}_h} \left( \int_T \beta \nabla u \cdot \nabla v_0 \, \mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \, \mathrm{d}\boldsymbol{s} \right) \\ &= s(Q_h u, v_h) + \sum_{T \in \mathcal{T}_h} \int_T (\overline{\beta}_T \nabla_w (Q_h u) - \beta \nabla u) \cdot \nabla v_0 \, \mathrm{d}\boldsymbol{x} \\ &- \sum_{T \in \mathcal{T}_h} \int_{\partial T} (v_0 - v_\partial) \left( Q_\partial \left( \overline{\beta}_T \nabla_w (Q_h u) \cdot \boldsymbol{n}_T \right) \right) \mathrm{d}\boldsymbol{s} + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \, \mathrm{d}\boldsymbol{s}. \end{split}$$

Since  $\left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]_e = 0$  for each interior edge e and  $v_{\partial}|_e = 0$  for each boundary edge e, we obtain

$$\sum_{T \in \mathcal{T}_{h}} \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_{0} \, \mathrm{d}s = \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} (v_{0} - v_{\partial}) \, \mathrm{d}s$$

$$= \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - v_{\partial}) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s - \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - v_{\partial}) (Q_{\partial}(\beta \nabla u \cdot \boldsymbol{n}_{T})) \, \mathrm{d}s$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - v_{\partial}) (Q_{\partial}(\beta \nabla u \cdot \boldsymbol{n}_{T})) \, \mathrm{d}s$$

$$= \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - Q_{\partial} v_{0}) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s + \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - v_{\partial}) (Q_{\partial}(\beta \nabla u \cdot \boldsymbol{n}_{T})) \, \mathrm{d}s.$$

Using the above equation we obtain

$$a_{s}(Q_{h}u - u_{h}, v_{h}) = s(Q_{h}u, v_{h}) + \sum_{T \in \mathcal{T}_{h}} \int_{T} (\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u) \cdot \nabla v_{0} \, d\boldsymbol{x}$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (Q_{\partial}v_{0} - v_{\partial}) \left(\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u\right) \cdot \boldsymbol{n}_{T} \, ds$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - Q_{\partial}v_{0}) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, ds.$$

This completes the proof of the lemma.

The following lemma can be found in [14].

**Lemma 5.3.** Let  $T \in \mathcal{T}_h$  and let  $e \subset \partial T$ . Then there exists a positive constant C independent of h such that

$$||u - \overline{u}_e||_{-1/2,e} \le Ch|u|_{1,T} \quad \forall u \in H^1(T).$$

where  $\overline{u}_e = \frac{1}{|e|} \int_e u \, \mathrm{d}s$ .

5.3. Error estimate. Now we prove the error estimates in the energy norm and the discrete  $H^1$ -seminorm.

**Theorem 5.4.** Suppose that  $u \in \widetilde{H}^2(\Omega) \cap H_0^1(\Omega)$  is the solution of (2.3) with  $f \in L^2(\Omega)$ . Suppose further that  $\beta \nabla u \in H^1(\Omega)$ . Let  $u_h \in V_{h,0}$  be the solution of (3.3). Then there exists a positive constant C independent of h such that

$$|||Q_h u - u_h||| \le Ch||u||_{\widetilde{H}^2(\Omega)}.$$

*Proof.* Let  $v_h = Q_h u - u_h$ . From the error equation (5.2), we have

$$\|Q_{h}u - u_{h}\|^{2} = a_{s}(Q_{h}u - u_{h}, v_{h})$$

$$= s(Q_{h}u, v_{h}) + \sum_{T \in \mathcal{T}_{h}} \int_{T} (\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u) \cdot \nabla v_{0} \, dx$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (Q_{\partial}v_{0} - v_{\partial}) \left(\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \beta \nabla u\right) \cdot \boldsymbol{n}_{T} \, ds$$

$$+ \sum_{T \in \mathcal{T}_{h}} \int_{\partial T} (v_{0} - Q_{\partial}v_{0}) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, ds$$

$$=: I_{1} + I_{2} + I_{3} + I_{4}. \tag{5.3}$$

By the trace inequality (4.6), Poincaré-Friedrichs inequality, and Corollary 4.6,

$$|I_{1}| \leq C \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1/2} \|u - Q_{0}u\|_{0,\partial T} h_{T}^{-1/2} \|v_{\partial} - Q_{\partial}v_{0}\|_{0,\partial T}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} \left( h_{T}^{-1} \|u - Q_{0}u\|_{0,T} + |u - Q_{0}u|_{1,T} \right) h_{T}^{-1/2} \|v_{\partial} - Q_{\partial}v_{0}\|_{0,\partial T}$$

$$\leq C \sum_{T \in \mathcal{T}_{h}} |u - Q_{0}u|_{1,T} h_{T}^{-1/2} \|v_{\partial} - Q_{\partial}v_{0}\|_{0,\partial T}$$

$$\leq C h \|u\|_{\widetilde{H}^{2}(\Omega)} \|v_{h}\|. \tag{5.4}$$

From (4.2), Lemma 4.8, and Lemma 5.1,

$$|I_{2}| \leq \sum_{T \in \mathcal{T}_{h}} \left( \|\overline{\beta}_{T} \nabla_{w}(Q_{h}u) - \overline{\beta}_{T} \nabla u\|_{0,T} + \|(\overline{\beta}_{T} - \beta) \nabla u\|_{0,T} \right) \|\nabla v_{0}\|_{0,T}$$

$$\leq Ch \|u\|_{\widetilde{H}^{2}(\Omega)} \|v_{h}\|. \tag{5.5}$$

Since  $\nabla_w(Q_h u)$ ,  $\nabla(Q_0 u) \in \widehat{\mathbb{P}}_1(T)$  on each  $T \in \mathcal{T}_h$ , using Lemma 4.3, Lemma 4.7, and Lemma 4.8, we have

$$|I_{3}| \leq \sum_{T \in \mathcal{T}_{h}} h_{T}^{-1/2} \|v_{\partial} - Q_{\partial}v_{0}\|_{0,\partial T} h_{T}^{1/2} \left( \|\overline{\beta}_{T} \left( \nabla_{w}(Q_{h}u) - \nabla(Q_{0}u) \right) \|_{0,\partial T} + \|\overline{\beta}_{T} \nabla(Q_{0}u) - \beta \nabla u \|_{0,\partial T} \right)$$

$$\leq C \|v_{h}\| \left( \sum_{T \in \mathcal{T}_{h}} h_{T} \left( \|\overline{\beta}_{T} \left( \nabla_{w}(Q_{h}u) - \nabla(Q_{0}u) \right) \|_{0,\partial T}^{2} + \|\overline{\beta}_{T} \nabla(Q_{0}u) - \beta \nabla u \|_{0,\partial T}^{2} \right) \right)^{1/2}$$

$$\leq C \|v_{h}\| \left( \sum_{T \in \mathcal{T}_{h}} \left( \|\overline{\beta}_{T}^{1/2} \left( \nabla_{w}(Q_{h}u) - \nabla(Q_{0}u) \right) \|_{0,T}^{2} + h_{T} \|\overline{\beta}_{T} \nabla(Q_{0}u) - \beta \nabla u \|_{0,\partial T}^{2} \right) \right)^{1/2}$$

$$\leq C h \|u\|_{\widetilde{H}^{2}(\Omega)} \|v_{h}\|. \tag{5.6}$$

Let  $T \in \mathcal{T}_h$  and  $e \subset \partial T$ . Since  $\beta \nabla u \in H^1(\Omega)$ , by Lemma 5.3 and the trace theorem, we have

$$\left| \int_{e} (v_0 - Q_{\partial} v_0) \beta \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s \right| \leq \|\beta \nabla u\|_{1/2, e} \|v_0 - Q_{\partial} v_0\|_{-1/2, e}$$

$$\leq Ch||u||_{\widetilde{H}^2(T)}|\nabla v_0|_{1,T}$$

Thus we obtain from Remark 2.3 that

$$|I_{4}| \leq \sum_{T \in \mathcal{T}_{h}} \sum_{e \subset \partial T} \left| \int_{e} (v_{0} - Q_{\partial} v_{0}) \beta \frac{\partial u}{\partial n} \, \mathrm{d}s \right|$$

$$\leq Ch \sum_{T \in \mathcal{T}_{h}} \|u\|_{\widetilde{H}^{2}(T)} |\nabla v_{0}|_{1,T}$$

$$\leq Ch \|\|v_{h}\| \|u\|_{\widetilde{H}^{2}(\Omega)}. \tag{5.7}$$

Now combining the inequalities (5.3)-(5.7) we have

$$|||u_h - Q_h u||^2 \le Ch||u||_{\widetilde{H}^2(\Omega)}|||v_h||| = Ch||u||_{\widetilde{H}^2(\Omega)}|||u_h - Q_h u|||.$$

This concludes the proof of the theorem.

Using Lemma 5.1 and Theorem 5.4, we immediately obtain the following discrete  $H^1$ -seminorm error estimate.

Corollary 5.5. Suppose that  $u \in \widetilde{H}^2(\Omega) \cap H_0^1(\Omega)$  is the solution of (2.3) with  $f \in L^2(\Omega)$ . Suppose further that  $\beta \nabla u \in H^1(\Omega)$ . Let  $u_h \in V_{h,0}$  be the solution of (3.3). Then there exists a positive constant C independent of h such that

$$|u_h - Q_h u|_{1,h} \le Ch ||u||_{\widetilde{H}^2(\Omega)}.$$

### 6. Numerical Examples

In this section, we report several numerical results. We solve the problem (2.1)-(2.2) with  $\Omega = [0, 1]^2$  partitioned into two different families of meshes as follows:

- (i) M1: uniform square meshes with  $h = 1/2^3, 1/2^4, \dots, 1/2^8$ ,
- (ii) M2: unstructured polygonal meshes with  $h = 1/2^3, 1/2^4, \dots, 1/2^8$ .

Some examples of the meshes are shown in Figure 4. The unstructured polygonal meshes are generated from PolyMesher [38]. Let u be the exact solution and let  $u_h = \{u_0, u_\partial\}$  be the solution of our immersed WG method. We compute errors in the discrete  $H^1$ -seminorm and  $L^2$ -norm, which are given by

$$|u_h - Q_h u|_{1,h}, \quad ||u_0 - Q_0 u||_{0,\Omega},$$

respectively. For the examples below, the discrete  $H^1$ -seminorm error converges with order O(h), which agrees with our theoretical result. Moreover, the results show the  $O(h^2)$  error in discrete  $L^2$ -norm, which is optimal.

Example 6.1 (Circular interface). Take a circle centered at (0.5, 0.5) with radius  $r_0 = 0.4$  as an interface, and choose the following exact solution

$$u(x,y) = \begin{cases} \frac{1}{\beta^+} (r^2 - r_0^2)^3 & \text{if } (x,y) \in \Omega^+, \\ \frac{1}{\beta^-} (r^2 - r_0^2)^3 & \text{if } (x,y) \in \Omega^-, \end{cases}$$

where  $\beta^+$  and  $\beta^-$  are constants and  $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ . In this example, we consider two cases when  $(\beta^+, \beta^-) = (1, 10), (10, 1), (1, 1000),$  and (1000, 1). The results are reported in Tables 1 to 4.

Example 6.2 (Sharp edge). In this example, we consider an interface with sharp edge. Let  $L(x,y) = -(2y-1)^2 + ((2x-2)\tan\theta)^2(2x-1)$  be the level-set function, with  $\theta = 10^\circ$ , and

$$\Gamma = \{(x, y) \in \Omega : L(x, y) = 0\}, \quad \Omega^+ = \{(x, y) \in \Omega : L(x, y) > 0\}, \quad \Omega^- = \{(x, y) \in \Omega : L(x, y) < 0\}.$$

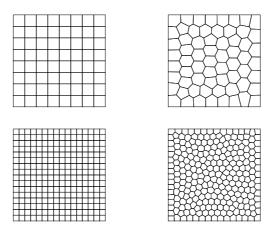


FIGURE 4. The meshes M1 (left) and M2 (right).

Table 1. The errors for Example 6.1 with  $(\beta^+, \beta^-) = (1, 10)$ .

1/h		Л1	M2					
1/11	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^{3}$	5.4587e-02		3.3027e-03		5.5979e-02		3.3820e-03	
$2^{4}$	2.6417e-02	1.0471	8.7554e-04	1.9154	2.7216e-02	1.0404	8.8878e-04	1.9280
$2^{5}$	1.3137e-02	1.0079	2.2319e-04	1.9719	1.3755e-02	0.9845	2.2742e-04	1.9665
$2^{6}$	6.5602e-03	1.0018	5.6112e-05	1.9919	6.9372e-03	0.9876	5.6936e-05	1.9979
$2^{7}$	3.2792e-03	1.0004	1.4049e-05	1.9979	3.4921e-03	0.9903	1.4224e-05	2.0011
$2^{8}$	1.6395 e-03	1.0001	3.5135e-06	1.9995	1.7500 e-03	0.9967	3.5427e-06	2.0054

Table 2. The errors for Example 6.1 with  $(\beta^+, \beta^-) = (10, 1)$ .

1/h		И1	M2					
1/11	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^{3}$	5.8617e-02		3.4872e-03		6.0526 e - 02		3.6934e-03	
$2^{4}$	2.8290e-02	1.0510	9.3409e-04	1.9004	2.9512e-02	1.0362	9.7713e-04	1.9183
$2^{5}$	1.4189e-02	0.9955	2.4360e-04	1.9391	1.5092e-02	0.9676	2.4927e-04	1.9708
$2^{6}$	7.1176e-03	0.9953	6.2122e-05	1.9713	7.7258e-03	0.9660	6.2438e-05	1.9972
$2^{7}$	3.5635e-03	0.9981	1.5643e-05	1.9896	3.9414e-03	0.9710	1.5536e-05	2.0068
$2^{8}$	1.7825e-03	0.9994	3.9192e-06	1.9969	1.9947e-03	0.9825	3.8632e-06	2.0078

Then the interface  $\Gamma$  has a sharp corner at (1,0.5) (see Figure 5). The exact solution is chosen as  $u=L/\beta$ , where  $\beta=1000$  on  $\Omega^+$  and  $\beta=1$  on  $\Omega^-$ . The results are reported in Table 5.

Example 6.3 (Variable coefficient). In this example, we take the level set of  $L(x,y) = (x-0.5)^2/r_1^2 + (y-0.5)^2/r_2^2 - 1$  with  $r_1 = 0.25$  and  $r_2 = 0.125$  as an interface, that is, we set

$$\Gamma = \{(x,y) \in \Omega : L(x,y) = 0\}, \quad \Omega^+ = \{(x,y) \in \Omega : L(x,y) > 0\}, \quad \Omega^- = \{(x,y) \in \Omega : L(x,y) < 0\}.$$

The exact solution is chosen as  $u = L/\beta$ , where

$$\beta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \Omega^+, \\ 1 + 0.5(2x - 1)^2 - (2x - 1)(2y - 1) + (2y - 1)^2 & \text{if } (x,y) \in \Omega^-. \end{cases}$$

The results are reported in Table 6.

Table 3. The errors for Example 6.1 with  $(\beta^+, \beta^-) = (1, 1000)$ .

1/h		Л1	M2					
1/11	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^3$	5.5356e-02		3.2533e-03		5.8561e-02		3.3911e-03	
$2^4$	2.6461e-02	1.0649	8.7488e-04	1.8947	2.7524 e-02	1.0893	8.8967e-04	1.9304
$2^{5}$	1.3148e-02	1.0090	2.2320e-04	1.9707	1.3777e-02	0.9984	2.2758e-04	1.9669
$2^{6}$	6.5645 e - 03	1.0021	5.6113e-05	1.9919	6.9396e-03	0.9893	5.6970e-05	1.9981
$2^{7}$	3.2800 e - 03	1.0010	1.4049e-05	1.9978	3.4929e-03	0.9904	1.4231e-05	2.0011
$2^{8}$	1.6398e-03	1.0002	3.5137e-06	1.9994	1.7504 e-03	0.9968	3.5445 e-06	2.0054

Table 4. The errors for Example 6.1 with  $(\beta^+, \beta^-) = (1000, 1)$ .

1/h		И1	M2					
	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^{3}$	1.6219e-01		3.4747e-03		7.4038e-02		3.7966e-03	
$2^{4}$	2.9224e-02	2.4725	8.7313e-04	1.9926	3.1237e-02	1.2450	9.9624e-04	1.9301
$2^{5}$	1.4104 e - 02	1.0511	2.2151e-04	1.9788	1.5075 e-02	1.0512	2.5707e-04	1.9543
$2^{6}$	7.0375e-03	1.0029	5.6761e-05	1.9644	7.5627e-03	0.9951	6.4711e-05	1.9901
$2^{7}$	3.5397e-03	0.9914	1.4787e-05	1.9406	3.7960e-03	0.9944	1.6118e-05	2.0053
$2^{8}$	1.7847e-03	0.9879	3.8351e-06	1.9470	1.9126e-03	0.9890	3.9855e-06	2.0159

Table 5. The errors for Example 6.2.

1/h		И1	M2					
	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^{3}$	6.0360e-01		3.0913e-02		5.6648e-01		3.2621e-02	
$2^{4}$	3.0771e-01	0.9720	7.1992e-03	2.1023	2.8813e-01	0.9753	8.0180e-03	2.0245
$2^{5}$	1.6788e-01	0.8742	1.7187e-03	2.0665	1.6058e-01	0.8434	1.9645e-03	2.0291
$2^{6}$	8.3267e-02	1.0116	4.2052e-04	2.0310	7.6565e-02	1.0686	4.7390e-04	2.0515
$2^{7}$	3.9446e-02	1.0779	1.0327e-04	2.0258	3.5919e-02	1.0919	1.0916e-04	2.1181
$2^{8}$	1.9396e-02	1.0241	2.5529e-05	2.0162	1.7581e-02	1.0308	2.6011e-05	2.0693

Table 6. The errors for Example 6.3.

1/h		И1	M2					
	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order	$ u_h - Q_h u _{1,h}$	Order	$  u_0-Q_0u  _{0,\Omega}$	Order
$2^{3}$	1.0067e + 01		5.8821e-01		9.7421e+00		6.1556e-01	
$2^4$	5.0872e+00	0.9847	1.4782e-01	1.9925	4.8530e+00	1.0054	1.5612e-01	1.9793
$2^{5}$	2.5514e+00	0.9956	3.7083e-02	1.9950	2.4251e+00	1.0009	3.9536e-02	1.9814
$2^{6}$	1.2772e+00	0.9983	9.2748e-03	1.9994	1.2119e+00	1.0008	9.9313e-03	1.9931
$2^{7}$	6.3883e-01	0.9995	2.3199e-03	1.9992	6.0612e-01	0.9995	2.4873e-03	1.9974
$2^{8}$	3.1948e-01	0.9997	5.8000e-04	2.0000	3.0290e-01	1.0007	6.2143e-04	2.0009

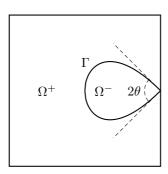


FIGURE 5. The level set of  $(2y-1)^2 = ((2x-2)\tan\theta)^2(2x-1)$ .

## 7. Conclusion

We introduce an immersed WG method for the elliptic interface problems on general unfitted polygonal meshes. The discrete space consists of constant functions on the mesh edges and piecewise linear functions in the mesh elements, satisfying the interface conditions. We prove an optimal-order convergence in the discrete  $H^1$ -seminorm under some assumptions on the exact solution.

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